Title: An algebraic classification of entangled states

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Abstract: <p>We provide a classification of entangled states that uses new discrete entanglement invariants. The invariants are defined by algebraic properties of linear maps associated with the states. We prove a theorem on a correspondence between the invariants and sets of equivalent classes of entangled states. The new method works for an arbitrary finite number of finite-dimensional state subspaces. As an application of the method, we considered a large selection of cases of three subspaces of various dimensions. We also obtain an entanglement classification of four qubits, where we find 27 fundamental sets of classes.

</p>

Thumbnails

An algebraic classification of entangled states

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The phenomenon of quantum entanglement

- The fundamental and counterintuitive feature of QM
- The ever increasing stream of research
- Advanced and extensive applications
- Still incomplete fundamental understanding
- Well-known and simple to formulate qualitative features
- Very complex and not completely understood quantitative features

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Thumbnails

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Thumbnails

Quantitative features of entanglement

- Distinguishing entangled states
- Entanglement measures
- The unsolved problem of entanglement classification
- The classical theory of invariants as standard method of finding entanglement measures
- Partial or complete classification
- The known continuous measures
- We propose a new method of entanglement classification whose measures are discrete algebraic invariants.

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Key properties of entangled states

- A state is disentangled if it can be transformed into a factorizable state; any other state is entangled.
- A state is disentangled if and only if each subsystem is in a definite state
- Is the smallest number of linearly independent factorizable terms representing a state an appropriate characteristic of its entanglement?
- Yes for 2 subsystems
- No for ≥ 3 subsystems (since it depends on a choice of bases)
- Studying invariant properties of states of composite systems

Thumbnails

Tensor product space

$I = \{1, \ldots, n\}$	a set
F	a field (we take $\mathbb{R}$, but $\mathbb{C}$ is just as easy)
S	a composite quantum system
$\{S_i\}_{i \in I}$	subsystems of S
$\{V_i\}_{i \in I}$	quantum states of $\{S_i\}_{i \in I}$ (finite-dimensional vector spaces over F)
$V \subseteq \otimes_{i \in I} V_i$	a space of quantum states of S

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Transformation groups

- Properties of the system S related to its composition in terms of the subsystems $\{S_i\}_{i\in I}$
- These are equivalent to properties of V related to its composition in terms of $\{V_i\}_{i\in I}$
- Manifest themselves in their transformations under an appropriate group

Which transformation group?
From the tensor structure of V: $\times_{i\in I}\,\mathrm{GL}(V_i)$, not $\mathrm{GL}(\otimes_{i\in I} V_i)$

More generally:

$$G_i \subseteq \mathrm{GL}(V_i)$$
$$G = \times_{i\in I} G_i$$

Equivalence classes

$\sim_V$	equivalence relation on V induced by G
$v' \sim_V v$	iff $\exists g \in G : v' = gv$

$C(v) = \{v' \in V : v' \sim_V v\}$	the equivalence class of v
$\tilde{v} \in C(v)$	a representative element of $C(v)$
$C = \cup_{v \in V} \{C(v)\}$	the set of equivalence classes
$\tilde{V} = \{\tilde{v} \in C(v) : C(v) \in C\}$	the quotient set $\tilde{V} = V / \sim_V$

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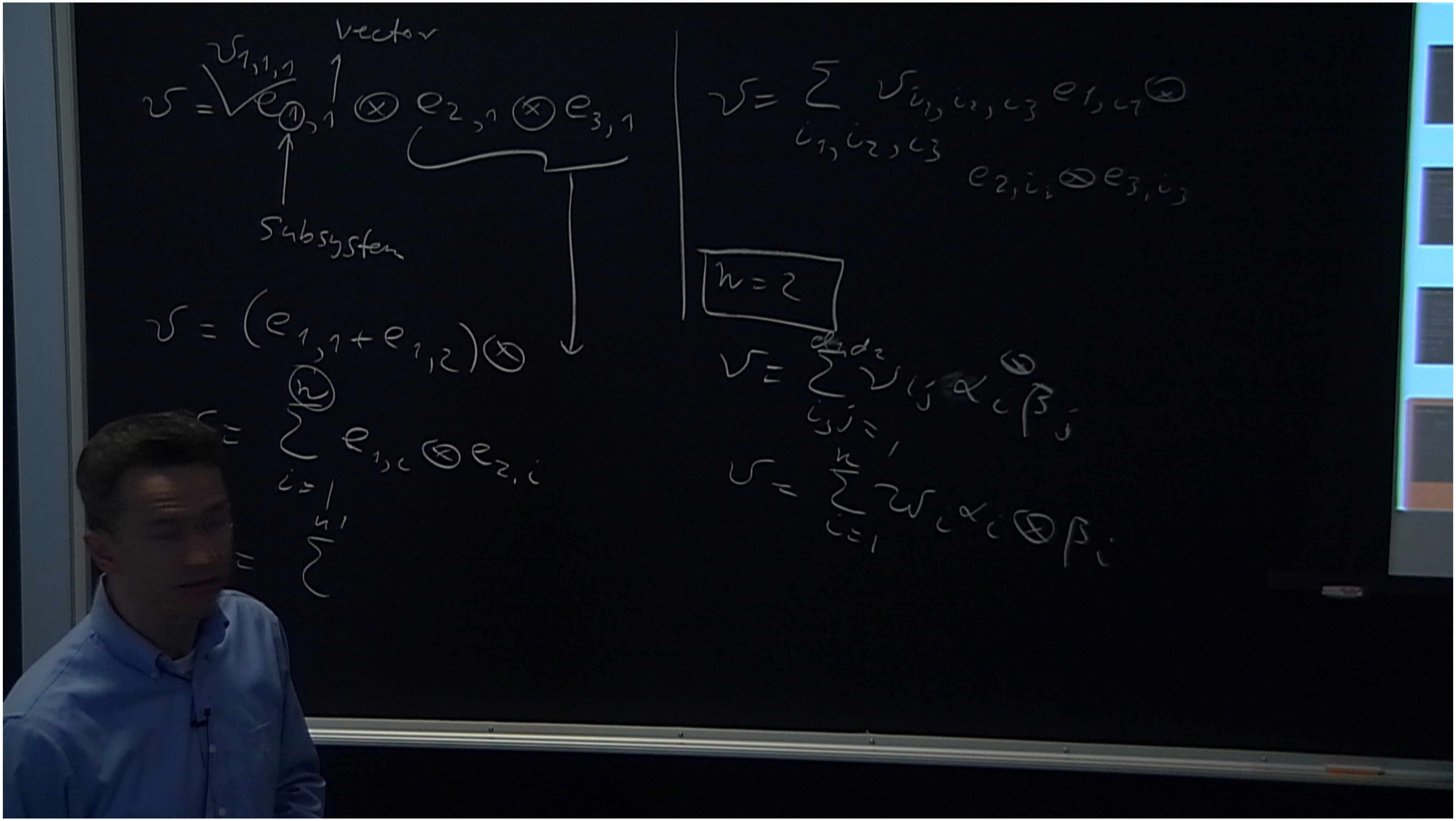
vector
subsystem
n = 2

Properties of entangled states

The ultimate goal: understanding the structure of $\tilde{V}$

Types of vectors in $\tilde{V}$:

- the zero vector
- decomposable vectors ($v = \otimes_{i \in I} v_i$, where $0 \neq v_i \in V_i$)
- nondecomposable vectors

Types of quantum states in S:

- the vacuum state
- disentangled states
- entangled states

One-to-one correspondence

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Properties of entangled states

- The zero vector (the vacuum state) and decomposable vectors (disentangled states) are the simplest elements of V; although they comprise only a small part of V, they span all of it.
- By contrast, nondecomposable vectors (entangled states) are more complex and difficult to categorize.
- The difficulty is combinatorial because decomposable vectors from V that enter the linear combination representing a nondecomposable vector differ by ways in which linearly independent vectors from $\{V_i\}_{i \in I}$ enter the expression.

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Properties of entangled states

How large is $\tilde{V}$?

From $\tilde{V} = V / \sim_V$: $\dim \tilde{V} \geq \dim V - \dim G$
(the system of linear equations $v' = gv$ is linearly dependent)

Two cases:

$\dim V - \dim G \leq 0$: no information on the size of $\tilde{V}$

$\dim V - \dim G > 0$: $|\tilde{V}| = \infty$

For $n \to \infty$:

$\dim V = O(\exp(cn))$, $c = \text{const}$

$\dim G = O(n^2)$

$\Rightarrow n$ does not need to be very large for the set $\tilde{V}$ to be infinite
$\Rightarrow \tilde{V}$ is typically infinite

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Properties of entangled states

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$\Rightarrow$ $\tilde{V}$ is typically infinite

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The trivial example: 2 qubits

spaces: V_1, V_2

$V = V_1 \otimes V_2$

transformations: $G_1 : V_1 \to V_1$

$G_2 : V_2 \to V_2$

$G_1 \times G_2 : V \to V$

states: $v_{1,j} \in V_1$

$v_{2,j} \in V_2$

$v \in V$

$v = \sum_{j=1}^{l} v_{1,j} \otimes v_{2,j}$, rank $v = l$, $0 \leq l \leq \min\{\dim V_1, \dim V_2\}$

$l = 2$: the EPR state

The less trivial example: 3 qubits

Vacuum state: $v = 0$

Disentangled state: $v = e_{1,1} \otimes e_{2,1} \otimes e_{3,1}$

Entangled states:

Type 1:
$$v = (e_{1,1} \otimes e_{2,1} + e_{1,2} \otimes e_{2,2}) \otimes e_{3,1}$$
$$v = (e_{1,1} \otimes e_{3,1} + e_{1,2} \otimes e_{3,2}) \otimes e_{2,1}$$
$$v = (e_{2,1} \otimes e_{3,1} + e_{2,2} \otimes e_{3,2}) \otimes e_{1,1}$$

Type 2 (W state):
$$v = e_{1,1} \otimes e_{2,1} \otimes e_{3,2} + e_{1,1} \otimes e_{2,2} \otimes e_{3,1} + e_{1,2} \otimes e_{2,1} \otimes e_{3,1}$$

Type 3 (GHZ state):
$$v = e_{1,1} \otimes e_{2,1} \otimes e_{3,1} + e_{1,2} \otimes e_{2,2} \otimes e_{3,2}$$

Complete solution with 7 equivalence classes

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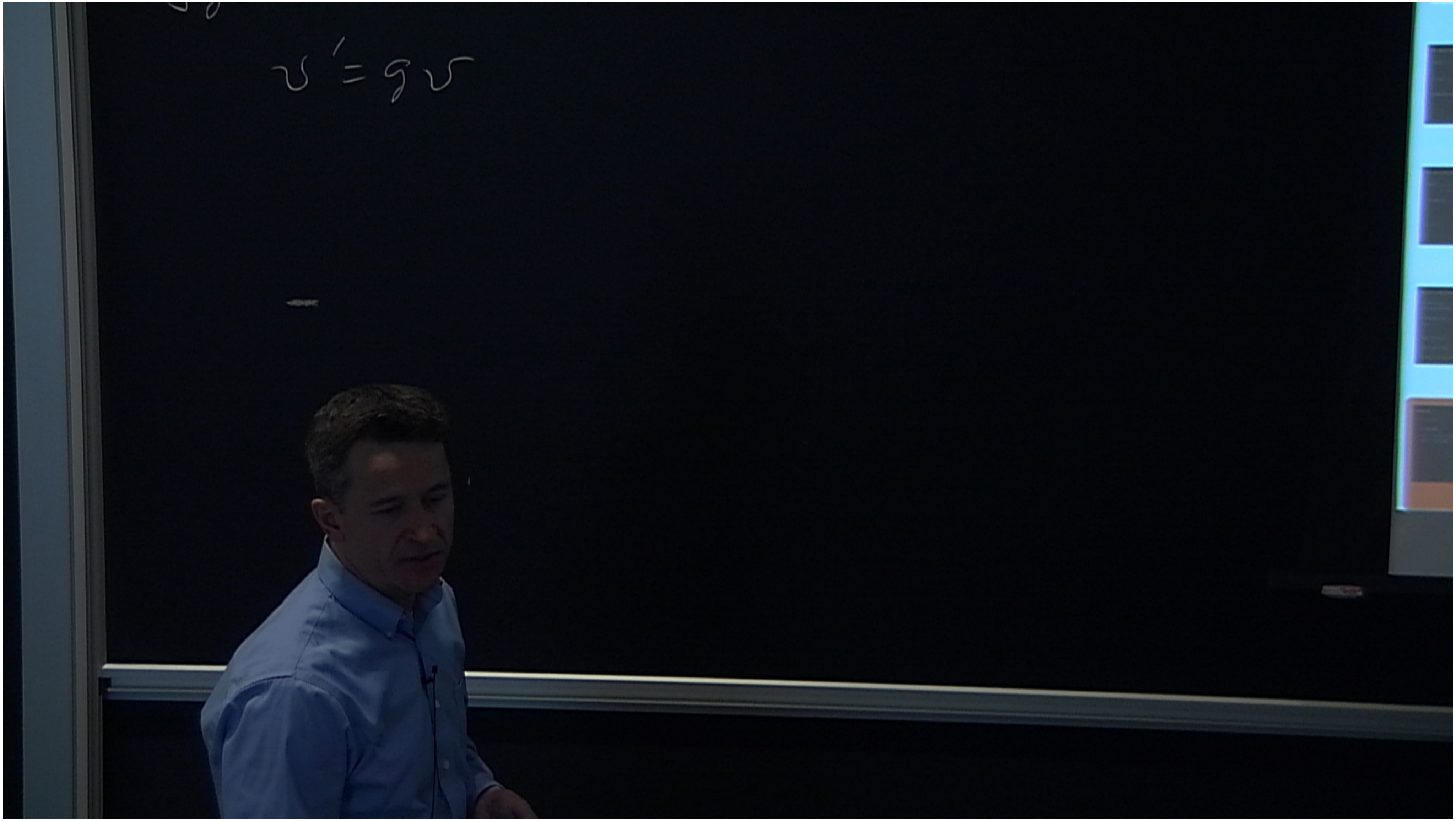
Methods

How to find $\tilde{V}$?

Two methods:

direct:	general form of $\tilde{v} \in \tilde{V}$ from definition	(inefficient)
indirect:	invariants of $\tilde{v} \in \tilde{V}$ under G	(efficient)


v' = gv

Equivalence classes

Projective spaces in quantum mechanics:
$v \sim f\,v, v \in V, f \in F, f \neq 0$ ($F = \mathbb{R}$ or $F = \mathbb{C}$)

Invariants are homogeneous polynomials
$A(v) = 0$ is the most important value

Equivalence for rescaled invariants:
$a' \sim_F a$ iff $\exists f \in F, f \neq 0 \colon a' = f\,a$
$(a'_k)_{k \in K} \sim_F (a_k)_{k \in K}$ iff $\exists f_k \in F, f_k \neq 0 \colon a'_k = f_k a_k$

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Equivalence classes of equivalence classes

Additional equivalences on V:

$v' \sim'_V v$ iff $A(v') \sim_F A(v)$ (for old invariants)

$v' \sim''_V v$ iff $\tilde{N}(v') = \tilde{N}(v)$ (for new invariants)

$\sim_V, C(v), \tilde{v}, C, \tilde{V}$	complete classification
$\sim'_V, C'(v), \tilde{v}', C', \tilde{V}'$	restricted old classification
$\sim''_V, C''(v), \tilde{v}'', C'', \tilde{V}''$	restricted new classification

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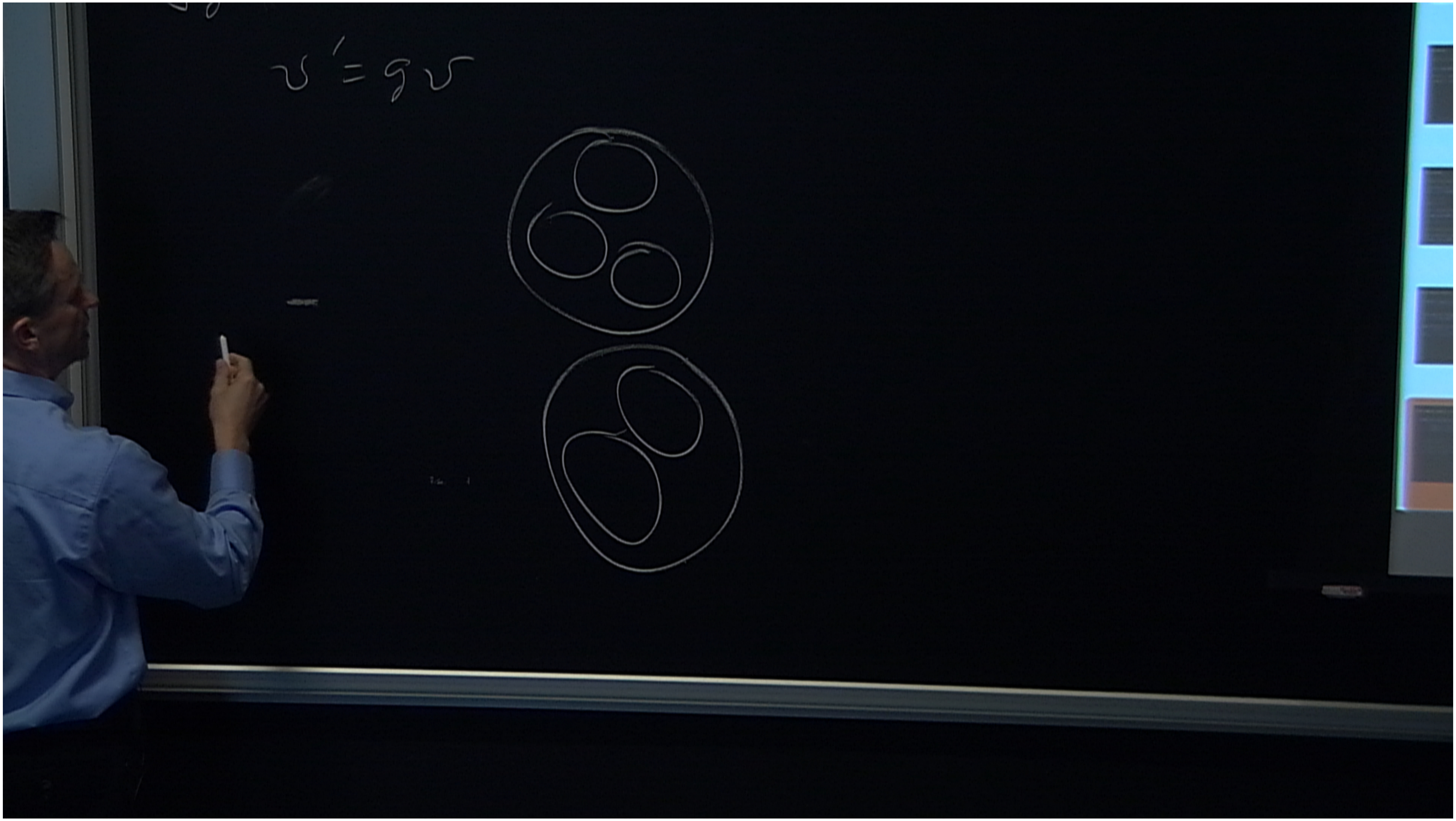
v' = gv

Maps

A partition:

$$S = T \cup T'$$
$$V = W \otimes W'$$

A linear map:

$$f(v) : W \to W'$$
$$f(v)(w) = v \otimes w^*$$

Fundamental spaces of $f(v)$:

$$\ker f(v) = \{w \in W : f(v)(w) = 0\} \subseteq W$$
$$\operatorname{im} f(v) = \{w' \in W' : w' = f(v)(w), w \in W\} \subseteq W'$$

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Maps

$$v = \sum_{i=1}^{\dim W} \sum_{j=1}^{\dim W'} v_{i,j} e_i \otimes e'_j$$

$$f(v)(w) = \sum_{i=1}^{\dim W} \sum_{j=1}^{\dim W'} v_{i,j} w_i e'_j$$

$$\ker f(v) = \left\{ w \in W : \sum_{i=1}^{\dim W} v_{i,j} w_i = 0,\ j \in \{1, \ldots, \dim W'\} \right\}$$

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Tensor products of maps

Old maps: $\{f_J(v)\}_{J \in P(I)}$

New maps: $\{\tilde{f}_J(v)\}_{J \in P(I)}$

(linearity in v): the only additional maps must be identities

(comparisons): common domains; V is a natural choice

$$\tilde{f}_J(v): V \to V_{I\setminus J} \otimes V_{I\setminus J}$$

$$\tilde{f}_J(v) = f_J(v) \otimes \mathrm{id}_{I\setminus J}$$

$$\tilde{K}_J(v) = \ker \tilde{f}_J(v), \quad \tilde{n}_J(v) = \dim \tilde{K}_J(v)$$

3 qubits

New maps:

$$\tilde{f}_{\{1\}}(v): V \to V_2 \otimes V_3 \otimes V_2 \otimes V_3$$

$$\tilde{f}_{\{1\}}(v)(w) = \sum_{j_1=1}^{2}\sum_{j_2=1}^{2}\sum_{j_3=1}^{2}\sum_{k_2=1}^{2}\sum_{k_3=1}^{2} v_{j_1,j_2,j_3} w_{j_1,k_2,k_3} e_{2,j_2} \otimes e_{3,j_3} \otimes e_{2,k_2} \otimes e_{3,k_3}$$

$$\tilde{f}_{\{2\}}(v): V \to V_1 \otimes V_3 \otimes V_1 \otimes V_3$$

$$\tilde{f}_{\{2\}}(v)(w) = \sum_{j_1=1}^{2}\sum_{j_2=1}^{2}\sum_{j_3=1}^{2}\sum_{k_1=1}^{2}\sum_{k_3=1}^{2} v_{j_1,j_2,j_3} w_{k_1,j_2,k_3} e_{1,j_1} \otimes e_{3,j_3} \otimes e_{1,k_1} \otimes e_{3,k_3}$$

$$\tilde{f}_{\{3\}}(v): V \to V_1 \otimes V_2 \otimes V_1 \otimes V_2$$

$$\tilde{f}_{\{3\}}(v)(w) = \sum_{j_1=1}^{2}\sum_{j_2=1}^{2}\sum_{j_3=1}^{2}\sum_{k_1=1}^{2}\sum_{k_2=1}^{2} v_{j_1,j_2,j_3} w_{k_1,k_2,j_3} e_{1,j_1} \otimes e_{2,j_2} \otimes e_{1,k_1} \otimes e_{2,k_2}$$

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Spaces and their intersections

$K_J(v) \otimes V_{I\backslash J} \subseteq \tilde{K}_J(v)$ (general property)

$K_J(v) \otimes V_{I\backslash J} \subset \tilde{K}_J(v)$ (nontrivial invariants)

The full information about a set of linear subspaces is given by the dimensions of the subspaces and of all their intersections.

$Q \in P(P'(I))$

$\tilde{K}_Q(v) = \cap_{J\in Q} \tilde{K}_J(v)$

$\tilde{n}_Q(v) = \dim \tilde{K}_Q(v)$

New invariants: $\tilde{N}(v) = \{\tilde{n}_Q(v)\}_{Q\in P(P'(I))}$

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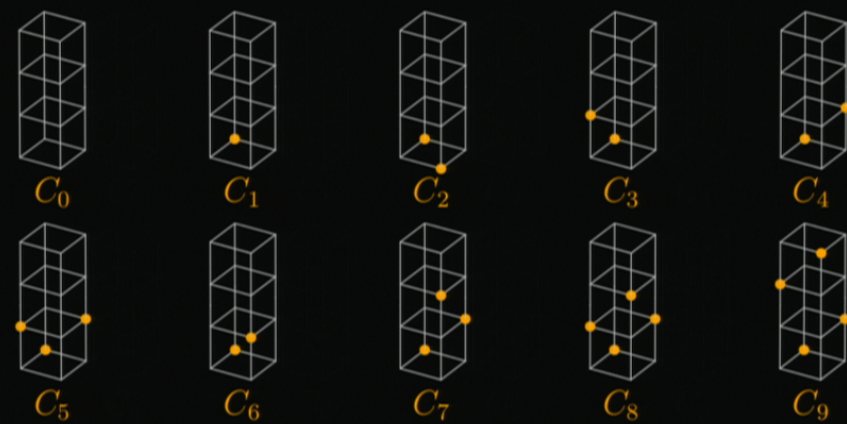
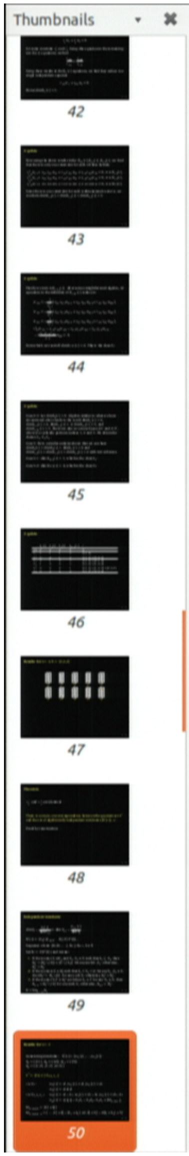
3 qubits

	$k_1(v)$	$k_2(v)$	$k_3(v)$	$k_{1,2,3}(v)$	v
C_0	2	2	2	8	0
C_1	1	1	1	4	$[1,1,1]$
C_2	0	0	1	3	$[1,1,1]+[2,2,1]$
C_3	0	1	0	3	$[1,1,1]+[2,1,2]$
C_4	1	0	0	3	$[1,1,1]+[1,2,2]$
C_5	0	0	0	1	$[1,1,1]+[1,2,2]+[2,1,2]$
C_6	0	0	0	0	$[1,1,1]+[2,2,2]$

Results for $n = 3$, $D = (2, 2, d)$





Results for $n = 3$

Generating invariants: $\tilde{N}(v) = (n_{Q_1}(v), \ldots, n_{Q_4}(v))$

$Q_1 = \{\{1\}\}, Q_2 = \{\{2\}\}, Q_3 = \{\{3\}\}$
$Q_4 = \{\{1,2\},\{1,3\},\{2,3\}\}$

$C'' = \{C_0\} \cup \{C_{k_1,k_2,k_3,j}\}$

$v \in C_0$: $\quad n_{Q_1}(v) = d_1, n_{Q_2}(v) = d_2, n_{Q_3}(v) = d_3$
$\quad n_{Q_4}(v) = d_1 d_2 d_3$

$v \in C_{k_1,k_2,k_3,j}$: $\quad n_{Q_1}(v) = d_1 - k_1, n_{Q_2}(v) = d_2 - k_2, n_{Q_3}(v) = d_3 - k_3$
$\quad n_{Q_4}(v) = d_1 d_2 d_3 - k_1 d_1 - k_2 d_2 - k_3 d_3 + (M_{k_1,k_2,k_3})_j$

$M_{k_1,k_2,k_1 k_2} = (k_1^2 + k_2^2)$
$M_{k_1,k_2,k_1 k_2 - 1} = (\ldots, k_1^2 + k_2^2 - (k_1 + k_2) + 2, k_1^2 + k_2^2 - 2(k_1 + k_2) + 5)$

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Results for $n = 3$

(k_1, k_2, k_3)	M_{k_1,k_2,k_3}
$(1,1,1)$	(2)
$(1,2,2)$	(5)
$(1,3,3)$	(10)
$(1,4,4)$	(17)
$(1,5,5)$	(26)
$(2,2,2)$	$(5,4)$
$(2,2,3)$	$(6,5)$
$(2,2,4)$	(8)
$(2,3,3)$	$(8,\dots,4)$
$(2,3,4)$	$(10,8,\dots,5)$
$(2,3,5)$	$(10,8)$
$(2,3,6)$	(13)
$(2,4,4)$	$(13,\dots,5)$
$(2,4,5)$	$(16,13,\dots,5)$
$(2,4,6)$	$(15,13,12,10,9,8)$
$(2,4,7)$	$(16,13)$
$(2,4,8)$	(20)
$(2,5,5)$	$(20,19,16,\dots,6)$
$(2,5,6)$	$(24,18,\dots,5)$
$(2,5,7)$	$(22,20,18,\dots,10,8)$
$(2,5,8)$	$(22,20,19,16,15,13)$
$(2,5,9)$	$(24,20)$
$(2,5,10)$	(29)

(k_1, k_2, k_3)	M_{k_1,k_2,k_3}
$(3,3,3)$	$(10,8,\dots,2)$
$(3,3,4)$	$(11,\dots,2)$
$(3,3,5)$	$(14,11,\dots,2)$
$(3,3,6)$	$(12,\dots,2)$
$(3,3,7)$	$(14,11,\dots,4)$
$(3,3,8)$	$(14,11,10)$
$(3,3,9)$	(18)
$(3,4,4)$	$(14,12,\dots,2)$
$(3,4,5)$	$(16,\dots,2)$
$(3,4,6)$	$(20,16,\dots,2)$
$(3,4,7)$	$(17,\dots,2)$
$(3,4,8)$	$(19,17,\dots,2)$
$(3,4,9)$	$(20,17,\dots,2)$
$(3,4,10)$	$(20,19,16,\dots,5)$
$(3,4,11)$	$(20,16,14)$
$(3,4,12)$	(25)

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Results for $n = 3,\ D = (2, 2, d)$

Canonical form: $v = \sum_{(j_1, j_2, j_3) \in J(v)} e_{1,j_1} \otimes e_{2,j_2} \otimes e_{3,j_3}$

	$\tilde{N}(v)$	$J(v)$
C_0	$(2, 2, d, 4d)$	$\varnothing$
C_1	$(1, 1, d-1, 3d-2)$	$\{(1,1,1)\}$
C_2	$(0, 0, d-1, 3d-3)$	$\{(1,1,1), (2,2,1)\}$
C_3	$(0, 1, d-2, 2d-1)$	$\{(1,1,1), (2,1,2)\}$
C_4	$(1, 0, d-2, 2d-1)$	$\{(1,1,1), (1,2,2)\}$
C_5	$(0, 0, d-2, 2d-3)$	$\{(1,1,1), (1,2,2), (2,1,2)\}$
C_6	$(0, 0, d-2, 2d-4)$	$\{(1,1,1), (2,2,2)\}$
C_7	$(0, 0, d-3, d-2)$	$\{(1,1,1), (1,2,2), (2,2,3)\}$
C_8	$(0, 0, d-3, d-3)$	$\{(1,1,1), (1,2,2), (2,1,2), (2,2,3)\}$
C_9	$(0, 0, d-4, 0)$	$\{(1,1,1), (1,2,2), (2,1,3), (2,2,4)\}$

Classical invariants for new classes for $n = 3$, $D = (2, 2, 2)$

	h_1	h_2	h_3	h_4
C_0	0	0	0	0
C_1	0	0	0	0
C_2	0	0	$\neq 0$	0
C_3	0	$\neq 0$	0	0
C_4	$\neq 0$	0	0	0
C_5	$\neq 0$	$\neq 0$	$\neq 0$	0
C_6	0	0	0	$\neq 0$

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Results for $n = 4$

Generating invariants: $\tilde{N}(v) = (n_{Q_1}(v), \ldots, n_{Q_{19}}(v))$

$Q_1 = \{\{1\}\}$
$Q_2 = \{\{2\}\}$
$Q_3 = \{\{3\}\}$
$Q_4 = \{\{4\}\}$
$Q_5 = \{\{1,2\},\{1,3\},\{2,3\}\}$
$Q_6 = \{\{1,2\},\{1,4\},\{2,4\}\}$
$Q_7 = \{\{1,3\},\{1,4\},\{3,4\}\}$
$Q_8 = \{\{2,3\},\{2,4\},\{3,4\}\}$

$Q_9 = \{\{1,2\},\{1,3,4\},\{2,3,4\}\}$
$Q_{10} = \{\{1,3\},\{1,2,4\},\{2,3,4\}\}$
$Q_{11} = \{\{1,4\},\{1,2,3\},\{2,3,4\}\}$
$Q_{12} = \{\{2,3\},\{1,2,4\},\{1,3,4\}\}$
$Q_{13} = \{\{2,4\},\{1,2,3\},\{1,3,4\}\}$
$Q_{14} = \{\{3,4\},\{1,2,3\},\{1,2,4\}\}$
$Q_{15} = \{\{1,2\},\{1,3\},\{1,4\},\{2,3,4\}\}$
$Q_{16} = \{\{1,2\},\{2,3\},\{2,4\},\{1,3,4\}\}$
$Q_{17} = \{\{1,3\},\{2,3\},\{3,4\},\{1,2,4\}\}$
$Q_{18} = \{\{1,4\},\{2,4\},\{3,4\},\{1,2,3\}\}$
$Q_{19} = \{\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\}\}$

Classical invariants for $n = 4$, $D = (2, 2, 2, 2)$

$$v = \sum_{j_1=1}^{2} \cdots \sum_{j_4=1}^{2} v_{j_1,\ldots,j_4} e_{1,j_1} \otimes \cdots \otimes e_{4,j_4}$$

From the Hilbert series:

$$h_1 = v_{1,1,1,1}v_{2,2,2,2} - v_{1,1,1,2}v_{2,2,2,1} - v_{1,1,2,1}v_{2,2,1,2} + v_{1,1,2,2}v_{2,2,1,1} - v_{1,2,1,1}v_{2,1,2,2} + v_{1,2,1,2}v_{2,1,2,1} + v_{1,2,2,1}v_{2,1,1,2} - v_{1,2,2,2}v_{2,1,1,1}$$

$$h_2 = \begin{vmatrix} v_{1,1,1,1} & v_{1,2,1,1} & v_{2,1,1,1} & v_{2,2,1,1} \\ v_{1,1,1,2} & v_{1,2,1,2} & v_{2,1,1,2} & v_{2,2,1,2} \\ v_{1,1,2,1} & v_{1,2,2,1} & v_{2,1,2,1} & v_{2,2,2,1} \\ v_{1,1,2,2} & v_{1,2,2,2} & v_{2,1,2,2} & v_{2,2,2,2} \end{vmatrix}$$

$$h_3 = \begin{vmatrix} v_{1,1,1,1} & v_{2,1,1,1} & v_{1,1,2,1} & v_{2,1,2,1} \\ v_{1,1,1,2} & v_{2,1,1,2} & v_{1,1,2,2} & v_{2,1,2,2} \\ v_{1,2,1,1} & v_{2,2,1,1} & v_{1,2,2,1} & v_{2,2,2,1} \\ v_{1,2,1,2} & v_{2,2,1,2} & v_{1,2,2,2} & v_{2,2,2,2} \end{vmatrix}$$

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Summary of our method and results

- Mathematical structure of entangled states → New entanglement invariants → A new method of entanglement classification.
- Quantum states → Linear maps → Algebraic invariants
- Each invariant takes a value from a finite set of integers → The set of entanglement classes is finite (for finite-dimensional spaces)
- Different values of the new discrete invariants correspond to certain standard continuous invariants being zero or nonzero
- The new method works for an arbitrary finite number of spaces of finite dimensions
- We believe that our classification is the most refined restricted classification possible.

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Outlook

Present and future work:

- Full classification for $n = 3$; $D = \{d_1, d_2, d_3\}$
- Full classification for $n = 4, 5, \ldots$; $D = \{2, \ldots, 2\}$

Applications:

- Continuous measures of entanglement (entanglement entropy)
- Discrete measures of entanglement (topological charges)
- Black hole entropy, holography, AdS/CFT
- Extension to superspaces

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