Title: Protecting weak measurements against systematic errors

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Abstract: <p>Decoherence in quantum metrology may deviate the estimate of a parameter from the real value of the parameter. In this talk, we show how to suppress the systematic error of weak-measurement-based quantum metrology under decoherence. We derive the systematic error of maximum likelihood estimation in general to the first order approximation of a small deviation in the probability distribution, and compare the systematic error of standard weak measurement and postselected weak measurements, which shows that the systematic error of a postselected weak measurement with a large weak value can be significantly lower than that of a standard weak measurement when the probe undergoes decoherence.</p>

Protecting weak measurements against systematic error

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Outline

1. Review of weak value amplification
2. Maximum likelihood estimation
3. Weak measurements with decoherence
4. Numerical example

Review of weak value amplification

In a weak measurement, a typical coupling Hamiltonian between the system and the probe is

$$H_{\text{int}} = gA \otimes G\delta(t - t_0), \quad (1)$$

where g is small.

Suppose the initial system state $|\psi_i\rangle$, the initial probe state $|\phi\rangle$, then the joint state after the weak coupling is

$$|\Phi\rangle = \exp(-\mathrm{i}gA \otimes G)|\psi_i\rangle \otimes |\phi\rangle. \quad (2)$$

Review of weak value amplification

If the system is postselected to some specific state $|\psi_f\rangle$, then the probe collapses to

$$|\phi_f\rangle = \langle\psi_f| \exp(-igA \otimes G)|\psi_i\rangle|\phi\rangle. \tag{3}$$

When g is sufficiently small,

$$|\phi_f\rangle \approx \langle\psi_f|\psi_i\rangle \exp(-igA_w G)|\phi\rangle, \tag{4}$$

where A_w is the *weak value*[1]

$$A_w = \frac{\langle\psi_f|A|\psi_i\rangle}{\langle\psi_f|\psi_i\rangle}. \tag{5}$$

[1] Y. Aharonov, D. Z. Albert, and L. Vaidman, Phys. Rev. Lett. 60, 1351 (1988).

Review of weak value amplification

If one measures an observable $\hat{M}$ on the collapsed probe state, then the average measurement result is

$$\begin{aligned}\langle \Delta \hat{M} \rangle \approx\, & g\mathrm{Im}A_w(\langle\{G, \hat{M}\}\rangle_D - 2\langle G\rangle_D\langle \hat{M}\rangle_D) \\ & + \mathrm{i}g\mathrm{Re}A_w\langle[G, \hat{M}]\rangle_D.\end{aligned} \tag{6}$$

For example, if $G = \hat{p}$, and $\hat{M} = \hat{q}$, then

$$\begin{aligned}\Delta\langle\hat{q}\rangle &= g\mathrm{Re}A_w + gm\mathrm{Im}A_w \left.\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Var}(\hat{q})_{|\phi\rangle}\right|_{t\to 0}, \\ \Delta\langle\hat{p}\rangle &= 2g\mathrm{Im}A_w\, \mathrm{Var}(\hat{p})_{|\phi\rangle}\Big|_{t=0}.\end{aligned} \tag{7}$$

A_w can be very large when $|\langle\psi_f|\psi_i\rangle| \ll 1$, so $\langle\Delta\hat{M}\rangle$ can be very large.

Maximum likelihood estimation

- Given a g-dependent probability distribution $P_g : p_k(g)$, $k = 1, \cdots, d$, we want to estimate g.
- MLE is finding the most likely g conditioned on the observation results as the estimate for g.
- Suppose we observe the result k a total of N_k times in an experiment. Likelihood function: $\mathcal{L} = \prod_k p_k^{N_k}(g)$, or alternatively its logarithm $\log \mathcal{L} = \sum_k N_k \log p_k(g)$.
- Maximization of $\mathcal{L}$ leads to

$$\partial_g \log \mathcal{L} \approx \sum_k N_k \frac{\partial_g p_k(g)}{p_k(g)} = 0. \tag{8}$$

Maximum likelihood estimation

In the presence of noise, the real probability distribution observed in experiments may be $P_{g_0}^{\mathrm{exp}}: \; p_k^{\mathrm{exp}}(g), \; k = 1, \cdots, d$, which can slightly deviate from $p_k(g)$.
Thus, the estimate of g will generally deviate from the *true value* g_0, i.e., a systematic error may occur in this case.

Maximum likelihood estimation

Suppose $p_k^{\text{exp}}(g) = p_k(g) + q_k(g)$, $k = 1, \cdots, d$, where $|q_k(g)| \ll 1$, and $\sum_k q_k(g) = 0$.
In this case,

$$\sum_k (p_k(g_0) + q_k(g_0)) \frac{\partial_g p_k(g)}{p_k(g)} = 0. \tag{9}$$

If we expand $\frac{\partial_g p_k(g)}{p_k(g)}$ to the first order of δg, then

$$\begin{aligned} &\sum_k (p_k(g_0) + q_k(g_0)) \left(\left. \frac{\partial_g p_k(g)}{p_k(g)} \right|_{g=g_0} \right. \\ &\left. + \delta g \left. \frac{p_k(g)\partial_g^2 p_k(g) - (\partial_g p_k(g))^2}{p_k^2(g_0)} \right|_{g=g_0} \right) = 0. \end{aligned} \tag{10}$$

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Maximum likelihood estimation

Up to the first order of $|q_k(g)|$, we get

$$\delta g = -\frac{\partial_g \mathcal{D}(P_{g_0}^{\mathrm{exp}}||P_g)|_{g=g_0}}{\mathcal{F}(P_{g_0})}, \tag{11}$$

where $\mathcal{D}(P_{g_0}^{\mathrm{exp}}||P_g)$ is the relative entropy between $P_{g_0}^{\mathrm{exp}}$ and P_g,

$$\mathcal{D}(P_{g_0}^{\mathrm{exp}}||P_g) = \sum_k p_k^{\mathrm{exp}}(g_0) \log \frac{p_k^{\mathrm{exp}}(g_0)}{p_k(g)}, \tag{12}$$

and $\mathcal{F}(P_g)$ is the Fisher information of the probability distribution P_g at $g = g_0$,

$$\mathcal{F}(P_{g_0}) = \sum_k \left. \frac{(\partial_g p_k(g))^2}{p_k(g)} \right|_{g=g_0}. \tag{13}$$

Cramér-Rao bound and systematic error

The Cramér-Rao bound tells us that

$$\langle \delta g^2 \rangle \geq \underbrace{\frac{1}{N\mathcal{F}_g}}_{\text{Fisher information}} + \underbrace{\langle \delta g \rangle^2}_{\text{Systematic error}} . \tag{14}$$

Two implications:

- Systematic errors cannot be reduced simply by increasing the number of measurements, as random noise is usually treated.
- If weak value amplification can reduce the systematic error, the low postselection probability will not affect it.

Weak measurements with decoherence

When the pointer undergoes decoherence, a typical interaction Hamiltonian is

$$H_I = gA \otimes G\delta(t - t_0) + \epsilon_D H_{DE},\ g, \epsilon_D \ll 1, \quad (15)$$

where $g, \epsilon_D \ll 1$.

Suppose the initial states of the system, the probe and the environment are ρ_S, ρ_D, ρ_E. After a short time $t \ll 1$, if $\epsilon_D = 0$, the joint state is approximately

$$\rho_{SDE}(t) = (\rho_S \otimes \rho_D - \mathrm{i}g[A \otimes G,\ \rho_S \otimes \rho_D]) \otimes \rho_E. \quad (16)$$

And in the presence of decoherence, the joint state is

$$\rho_{SDE}^{\exp}(t) = \rho_{SDE}(t) - \mathrm{i}t[\epsilon_D H_{DE},\ \rho_S \otimes \rho_D \otimes \rho_E]. \quad (17)$$

Weak measurements with decoherence

Standard weak measurement

When there is no postselection on the system in the weak measurement, the probe state after time t is

$$\rho_D(t) \approx \rho_D - \mathrm{i}g\langle A\rangle_i[G, \rho_D], \quad (18)$$

and in the presence of decoherence,

$$\rho_D^{\mathrm{exp}}(t) = \rho_D(t) - \mathrm{i}t\epsilon_D[H'_D, \rho_D], \quad (19)$$

where

$$H'_D = \mathrm{Tr}_E(H_{DE}\rho_E). \quad (20)$$

If we measure an orthonormal basis $\{|k\rangle\}$ on the probe, the probability distribution in the decoherence-free case is

$$p_k(g) = \langle k|\rho_D|k\rangle - \mathrm{i}g\langle A\rangle_i\langle k|[G, \rho_D]|k\rangle, \quad (21)$$

and in the presence of decoherence,

$$p_k^{\mathrm{exp}}(g) = p_k(g) - \mathrm{i}\epsilon_D t\langle k|[H'_D, \rho_D]|k\rangle. \quad (22)$$

Weak measurements with decoherence

Standard weak measurement

Then,

$$\partial_g \mathcal{D}(P_{g_0}^{\exp}||P_g)|_{g=g_0} \approx -4\epsilon_D t\langle A\rangle_i \sum_k \langle k|\rho_D|k\rangle \mathrm{Im} H_{Dw}^{\prime(k)} \mathrm{Im}\, G_w^{(k)},$$
$$\mathcal{F}(P_{g_0}) \approx 4\langle A\rangle_i^2 \sum_k \langle k|\rho_D|k\rangle \mathrm{Im}^2 G_w^{(k)}, \quad (23)$$

where

$$G_w^{(k)} = \frac{\langle k|G\rho_D|k\rangle}{\langle k|\rho_D|k\rangle},\ H_{Dw}^{\prime(k)} = \frac{\langle k|H_D'\rho_D|k\rangle}{\langle k|\rho_D|k\rangle}, \quad (24)$$

Therefore, the systematic error δg_n of the standard weak measurement is approximately

$$\delta g_n \approx \frac{\epsilon_D t \sum_k \langle k|\rho_D|k\rangle \mathrm{Im} H_{Dw}^{\prime(k)} \mathrm{Im}\, G_w^{(k)}}{\langle A\rangle_i \sum_k \langle k|\rho_D|k\rangle \mathrm{Im}^2 G_w^{(k)}}. \quad (25)$$

Weak measurements with decoherence

Standard weak measurement

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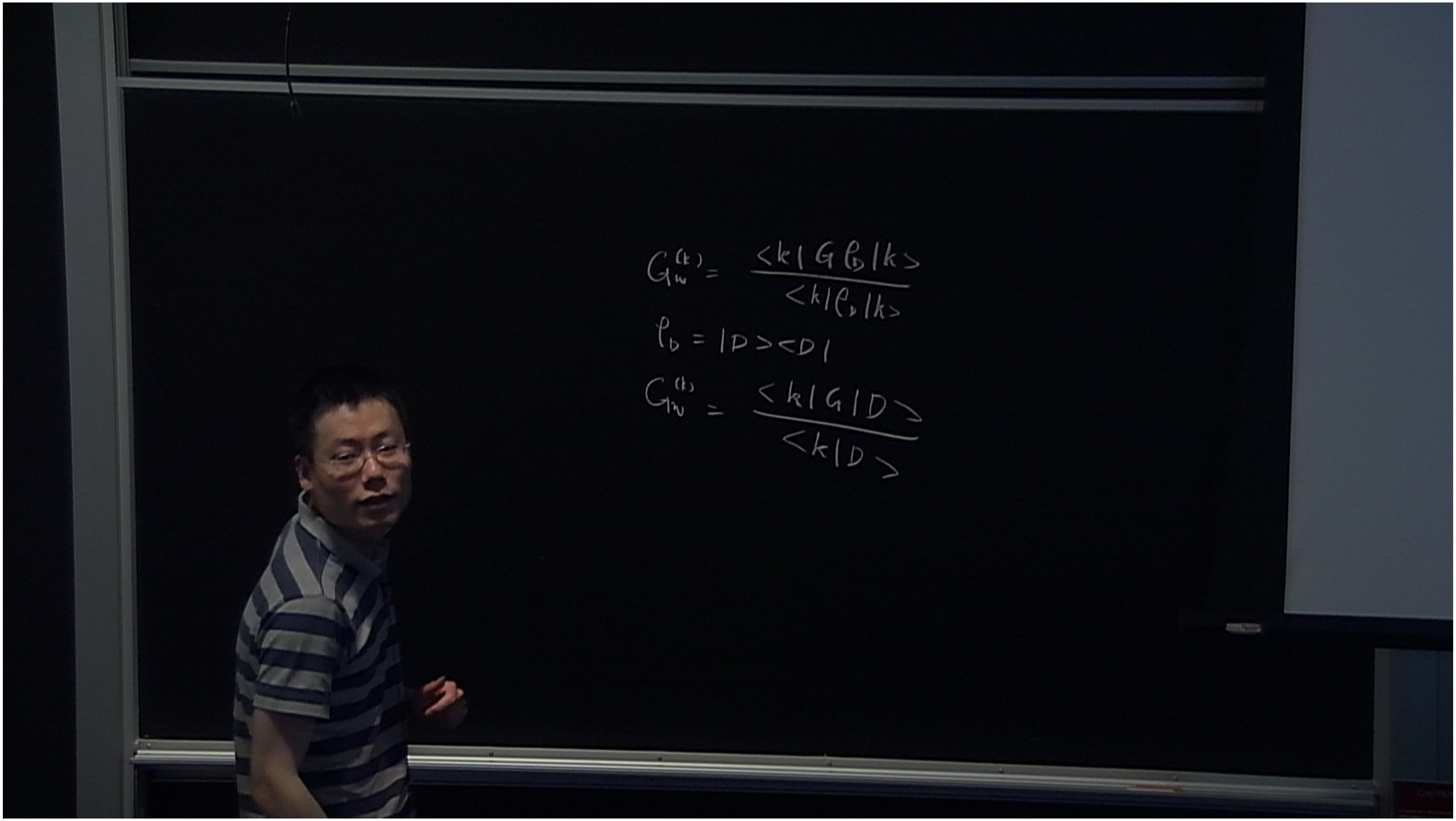

$G_w^{(k)} = \frac{\langle k|G\rho_D|k\rangle}{\langle k|\rho_D|k\rangle}$
$\rho_D = |D\rangle\langle D|$
$G_w^{(k)} = \frac{\langle k|G|D\rangle}{\langle k|D\rangle}$

Weak measurements with decoherence

Postselected weak measurement

Suppose the system is postselected to $|\psi_f\rangle$.
The pointer state after the postselection without decoherence is

$$\rho_D(t) \propto \rho_D - \mathrm{i}g\mathrm{Re}A_w^{G,\rho_D} + g\mathrm{Im}A_w\{G, \rho_D\}. \quad (26)$$

In the presence of decoherence on the probe,

$$\rho_D^{\exp}(t) \propto \rho_D - \mathrm{i}g\mathrm{Re}A_w[G, \rho_D] + g\mathrm{Im}A_w\{G, \rho_D\} - \mathrm{i}\epsilon_D t[H'_D, \rho_D]. \quad (27)$$

If we measure along an orthonormal basis $\{|k\rangle\}$ on the probe, the probability distribution of the measurement results in the decoherence-free case is

$$p_k(g) = \langle k|\rho_D|k\rangle[1 + 2g\mathrm{Im}(A_w G_w^{(k)})]. \quad (28)$$

And in the presence of decoherence,

$$p_k^{\exp}(g) = p_k(g) + 2\langle k|\rho_D|k\rangle t\epsilon_D \mathrm{Im}H_{Dw}'^{(k)}. \quad (29)$$

Weak measurements with decoherence

Postselected weak measurement

In the weak interaction limit $g \ll 1$,

$$\begin{aligned} \partial_g \mathcal{D}(P_{g_0}^{\exp}||P_g)|_{g=g_0} &\approx 4t\sum_k \langle k|\rho_D|k\rangle \epsilon_D \mathrm{Im} H_{Dw}^{\prime(k)} \mathrm{Im}(A_w G_w^{(k)}) \\ \mathcal{F}(P_{g_0}) &\approx 4\sum_k \langle k|\rho_D|k\rangle \mathrm{Im}^2(A_w G_w^{(k)}). \end{aligned} \tag{30}$$

Therefore, the systematic error δg_p is

$$\delta g_p \approx \frac{\epsilon_D t \sum_k \langle k|\rho_D|k\rangle \mathrm{Im} H_{Dw}^{\prime(k)} \mathrm{Im}(A_w G_w^{(k)})}{\sum_k \langle k|\rho_D|k\rangle \mathrm{Im}^2(A_w G_w^{(k)})}. \tag{31}$$

Weak measurements with decoherence

Postselected weak measurement

- If we know H'_D, we can choose a basis $\{|k\rangle\}$ for the measurement on the probe such that all $H'^{(k)}_{Dw}$ are real, thus $\mathrm{Im} H'^{(k)}_{Dw} = 0$ for all k, and δg_n (and δg_p) would be approximately zero. This is a simpler way to suppress the systematic error.
- However, in practice, one generally does not have complete information about the decoherence. The method proposed above, which is based on the weak value amplification, only requires a large weak value A_w, regardless of the details of the decoherence. This method is universal.

Numerical example

Suppose the total Hamiltonian for the system, probe, and environment is

$$H = g\sigma_S^z \otimes \sigma_D^z \delta(t - t_0) + \epsilon_D \sigma_D^y \otimes b^\dagger b,\ g,\ \epsilon_D \ll 1 \tag{32}$$

Suppose the system and the probe are initially in $|\psi_i\rangle$ and $|D\rangle$, and that the environment is initially in the thermal equilibrium state ρ_E,

$$\rho_E = \frac{1}{Z}\exp(-\beta b^\dagger b),\ \beta = \frac{\omega}{kT}, \tag{33}$$

After a short time t, the joint state of the system and probe evolves to

$$\rho_{SD} = \frac{1}{Z}\sum_n \mathrm{e}^{-\mathrm{i}\beta n}|\Phi_f^{(n)}\rangle\langle\Phi_f^{(n)}|, \tag{34}$$

where $|\Phi_f^{(n)}\rangle$ is

$$|\Phi_f^{(n)}\rangle = \exp[-\mathrm{i}(g\sigma_S^z \otimes \sigma_D^z + tn\epsilon_D\sigma_D^y)]|\psi_i\rangle|D\rangle. \tag{35}$$

Numerical example

Let $|\psi_i\rangle = |+\rangle$ and $|D\rangle = |+\rangle$ as the initial states for the system and the probe. The postselected state of the system is

$$|\psi_f\rangle = \exp\left(-\mathrm{i}\delta\sigma_S^y\right)|-\rangle,\ \delta \ll 1. \tag{36}$$

The weak value of σ_z is

$$(\sigma_S^z)_w = \frac{\langle\psi_f|\sigma_S^z|\psi_i\rangle}{\langle\psi_f|\psi_i\rangle} = \cot\delta, \tag{37}$$

which is approximately $1/\delta$ when $\delta \ll 1$. The measurement basis that we choose on the probe is

$$|k'\rangle = \mathrm{e}^{-\mathrm{i}\theta\sigma^x}|k\rangle,\ k = 0, 1, \tag{38}$$

where θ is a parameter to adjust.

Numerical example

Ratio of systematic error for different δ

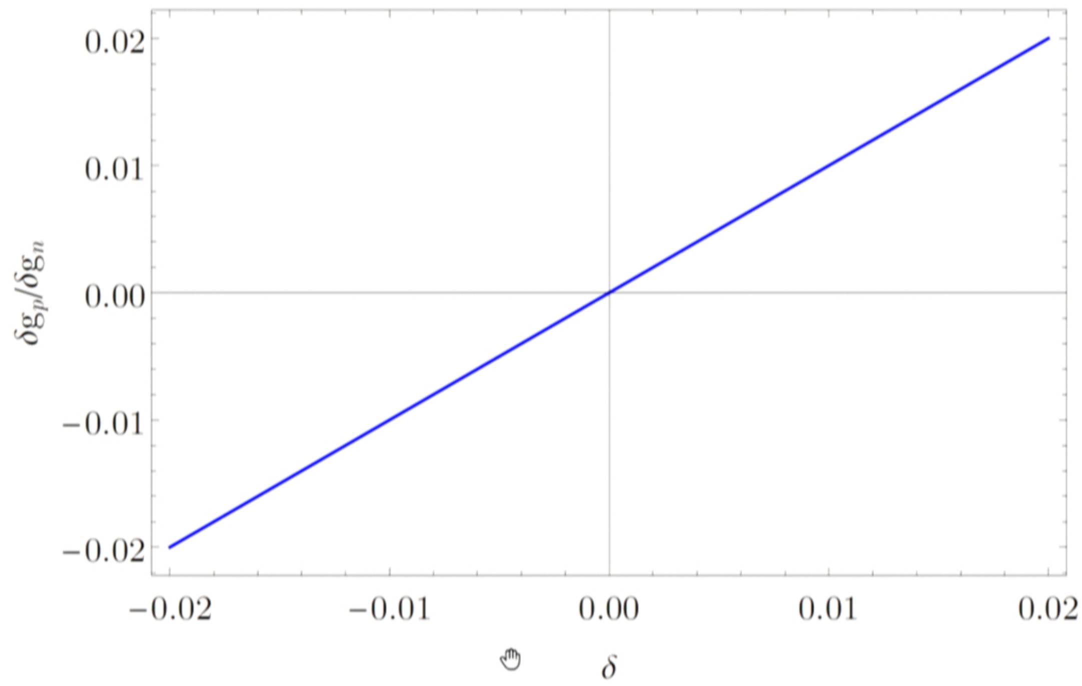


Figure: Ratio between the systematic error with and without postselection with varying δ. $\beta = 1.0$, $\theta = \frac{\pi}{8}$, $g = 1.0 \times 10^{-5}$, $\epsilon_D^i = 1.0 \times 10^{-5}$.

Numerical example

Ratio of systematic error for different g

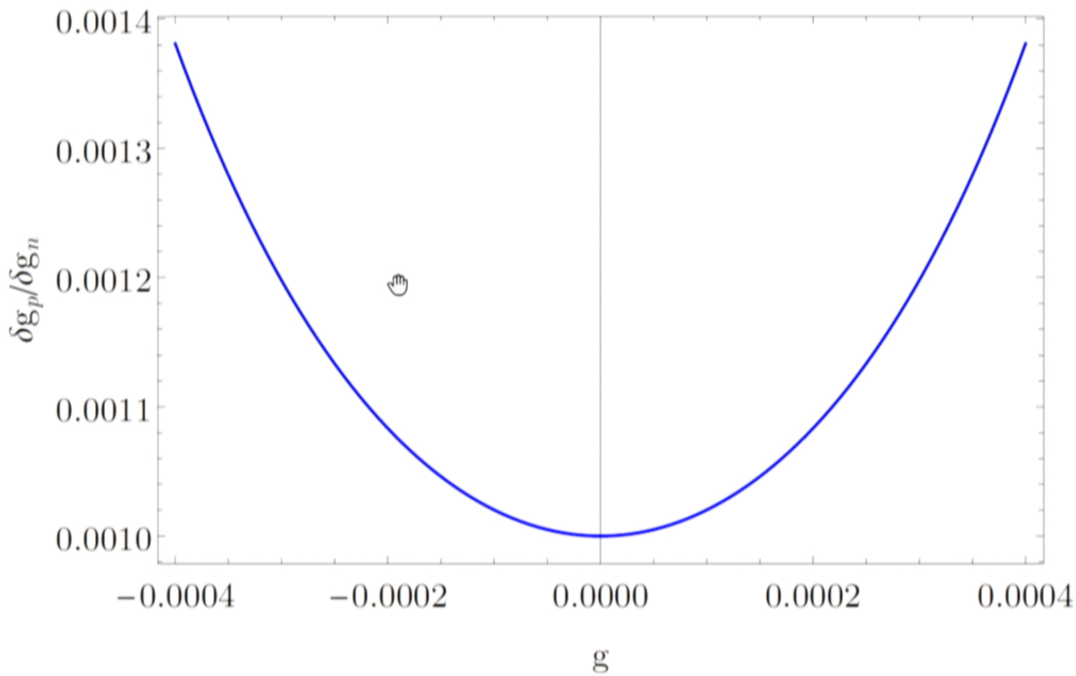


Figure: Ratio between the systematic error with and without postselection for different g. $\beta = 1.0$, $\theta = \frac{\pi}{8}$, $\delta = 1.0 \times 10^{-3}$, $\epsilon_D^i = 1.0 \times 10^{-5}$.

Numerical example

Ratio of systematic error for different ϵ_D

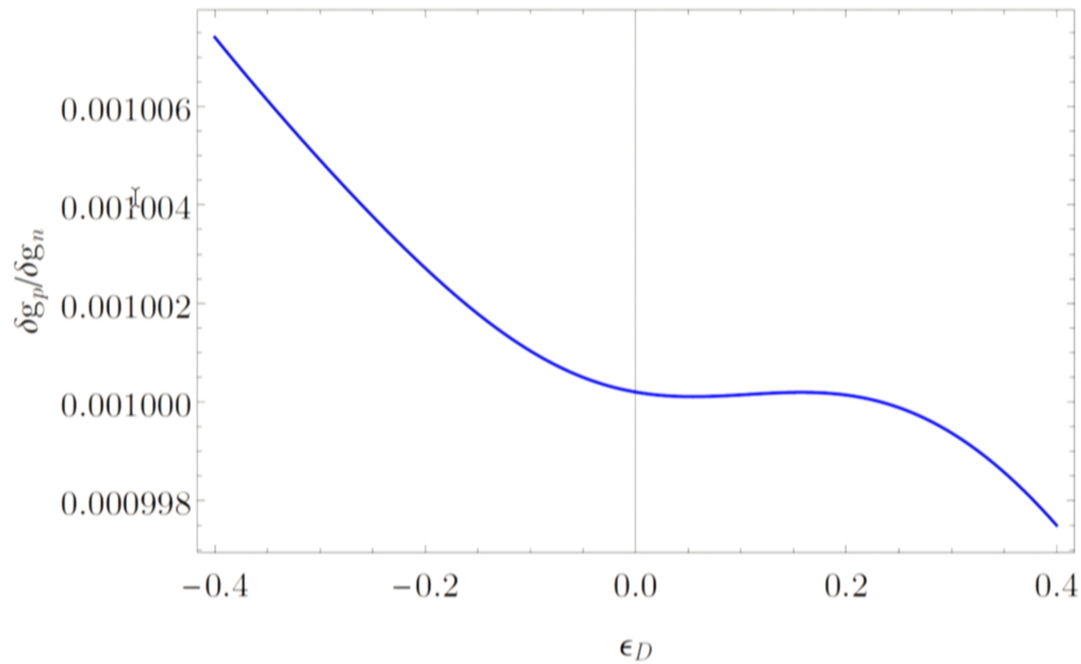
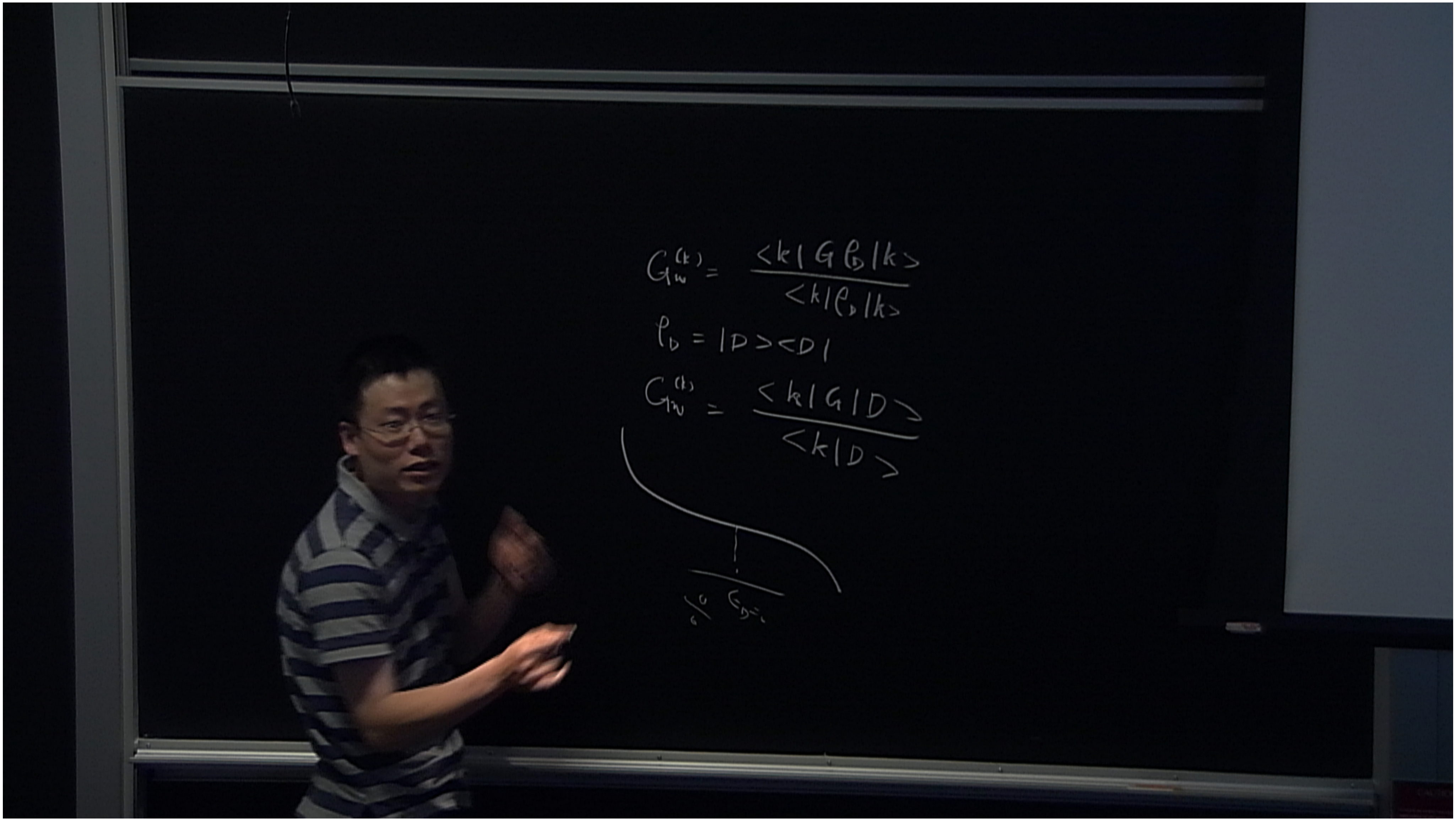


Figure: Ratio between the systematic error with and without postselection for different ϵ_D. $\beta = 1.0$, $\theta = \frac{\pi}{8}$, $\delta = 1.0 \times 10^{-3}$, $g = 1.0 \times 10^{-5}$.

Thank you!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