Title: Estimating dynamic parameters quantumly

Date: Jun 09, 2016 01:00 PM

URL: http://pirsa.org/16060094

Abstract: <p>I will give an overview of two ways to estimate parameters with a quantum system when the dynamics is nontrivial. In the first case, the parameter is changing in an irregular way, and we use consider the use of continuous measurement to track it in time. Tracking speed is of the essence for feedback purposes, and I will present our new and improved way to speed up the estimation algorithm. In the second case, I will consider the use of Hamiltonian control to estimate a fixed parameter, but of a time-dependent Hamiltonian. In this case established scaling bounds on the quantum Fisher information can be broken, and the quantum control is needed to do it. A simple spin example will be given to illustrate the general results.</p>

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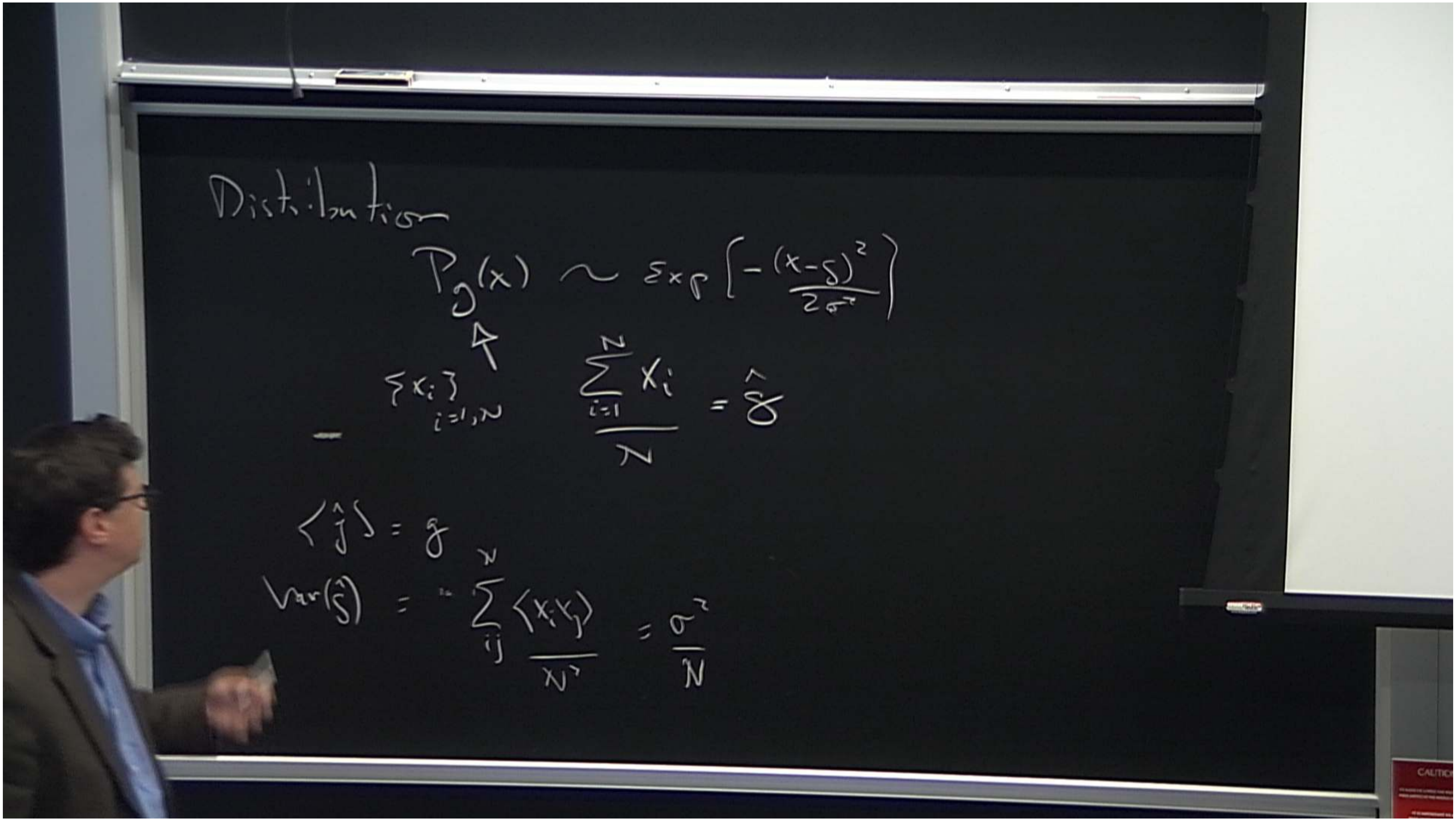
Distribution
$P_g(x) \sim \exp\left[-\frac{(x-g)^2}{2\sigma^2}\right]$
$\{x_i\}_{i=1,N}$
$\frac{\sum_{i=1}^{N} x_i}{N} = \hat{g}$
$\langle \hat{g} \rangle = g$
$\mathrm{Var}(\hat{g}) = \frac{\sum_{ij}^{N} \langle x_i x_j \rangle}{N^2} = \frac{\sigma^2}{N}$

Distribution

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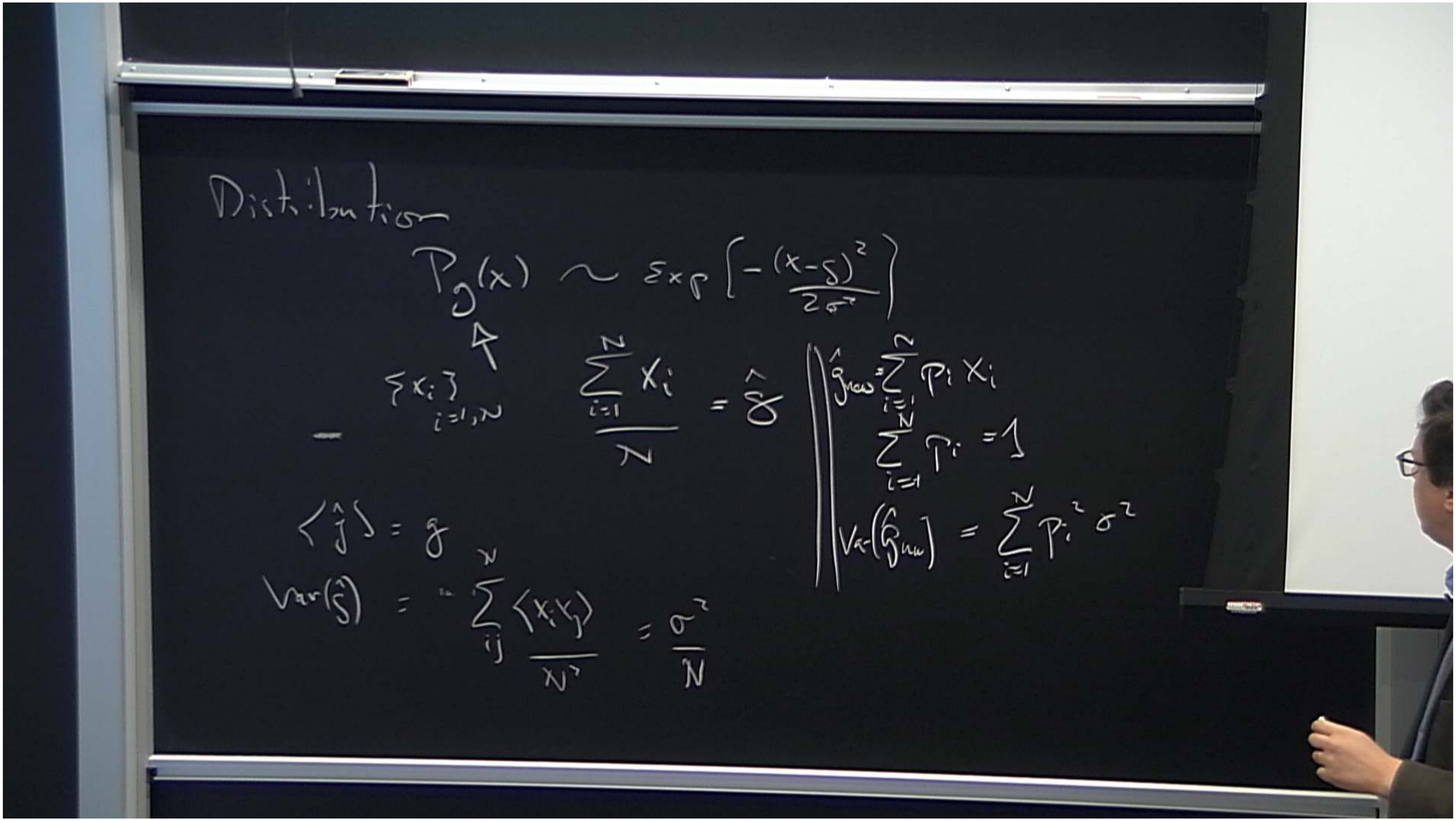
$$\{x_i\}_{i=1,N} \qquad \frac{\sum_{i=1}^{N} x_i}{N} = \hat{g}$$

$$\langle \hat{g} \rangle = g$$

$$\mathrm{Var}(\hat{g}) = \frac{\sum_{ij}^{N} \langle x_i x_j \rangle}{N^2} = \frac{\sigma^2}{N}$$


Distribution
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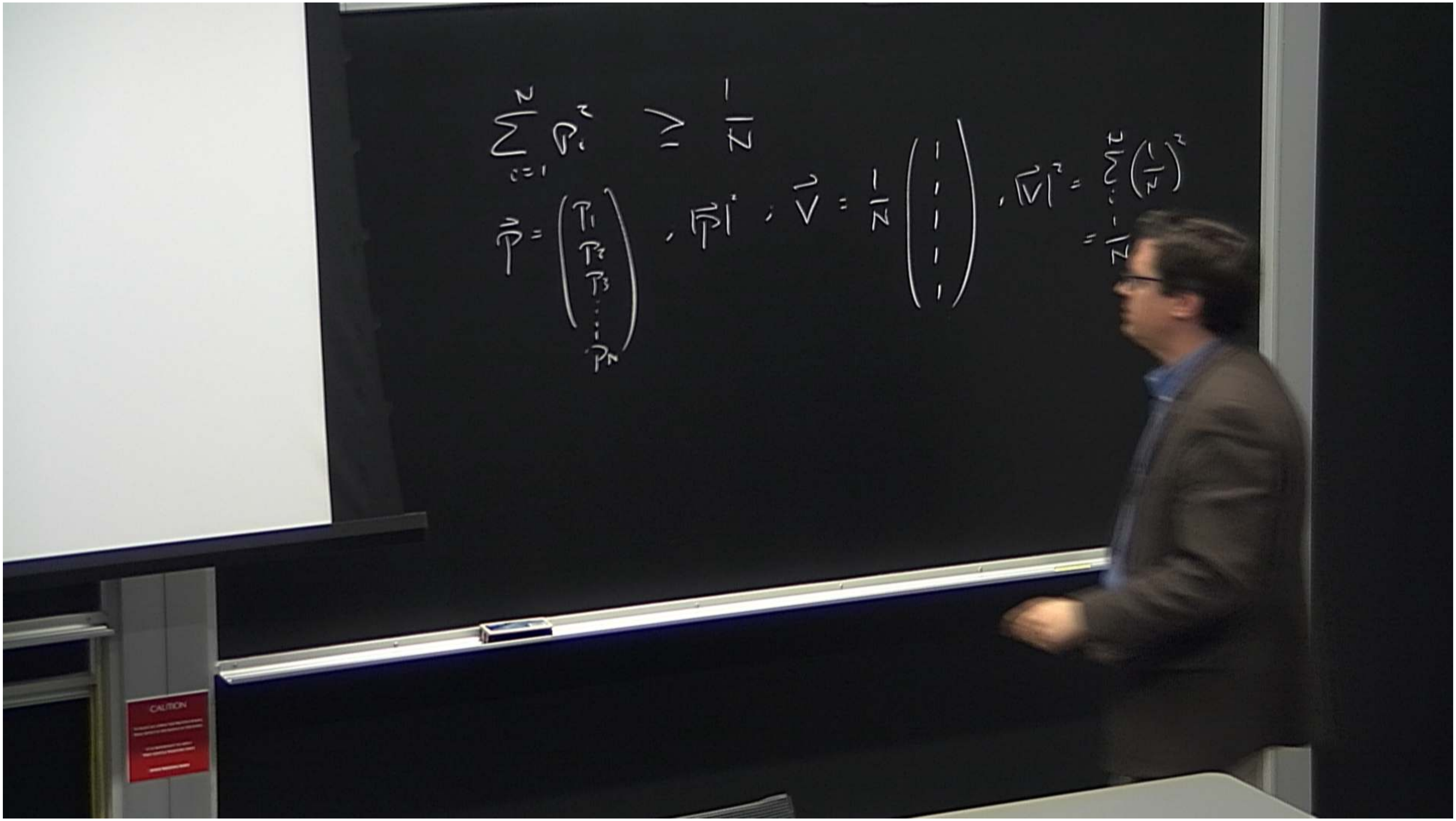
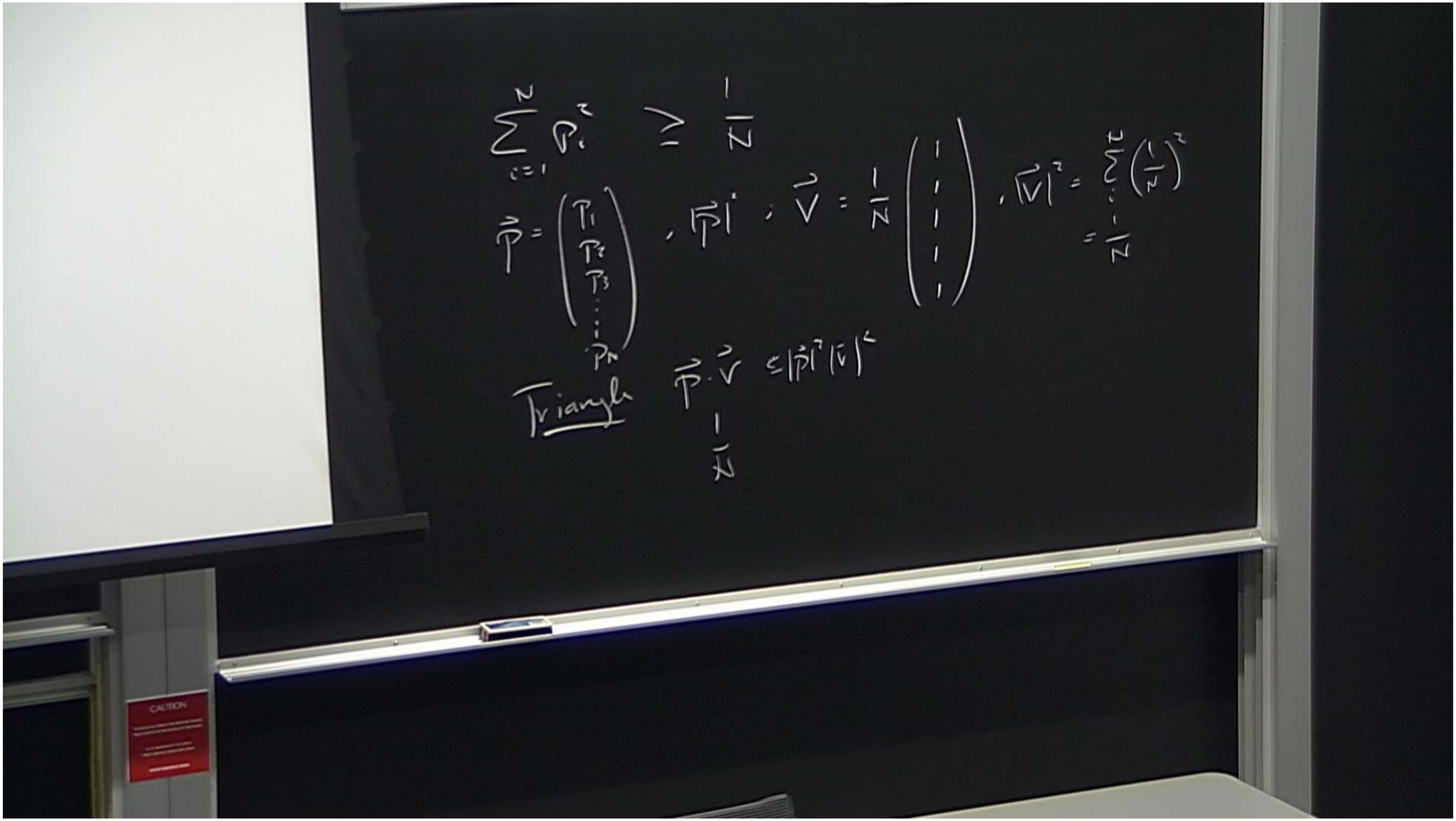
$\{x_i\}_{i=1,N}$
$\frac{\sum_{i=1}^{N} x_i}{N} = \hat{g}$
$\langle \hat{g} \rangle = g$
$Var(\hat{g}) = \frac{\sum_{ij} \langle x_i x_j \rangle}{N^2} = \frac{\sigma^2}{N}$
$\hat{g}_{new} = \sum_{i=1}^{N} p_i x_i$
$\sum_{i=1}^{N} p_i = 1$
$Var(\hat{g}_{new}) = \sum_{i=1}^{N} p_i^2 \sigma^2$


$\sum_{i=1}^{N} p_i^2 \geq \frac{1}{N}$
$\vec{p} = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ \vdots \\ p_N \end{pmatrix}$, $|\vec{p}|^2$; $\vec{v} = \frac{1}{N}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$, $|\vec{v}|^2 = \sum^{N} \left(\frac{1}{N}\right)^2 = \frac{1}{N}$


$\sum_{i=1}^{N} p_i^2 \geq \frac{1}{N}$
$\vec{p} = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ \vdots \\ p_N \end{pmatrix}$, $|\vec{p}|^2$, $\vec{V} = \frac{1}{N}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$, $|\vec{V}|^2 = \sum_{i}^{N}\left(\frac{1}{N}\right)^2 = \frac{1}{N}$
Triangle $\vec{p}\cdot\vec{V} \leq |\vec{p}|^2|\vec{V}|^2$
$\frac{1}{N}$

Fisher information

$$I_g = \int dx\, p(x) \left(\partial_g \ln p_g(x) \right)^2$$

$$\mathrm{Var}\,\hat{g} \geq \frac{1}{\sqrt{I_g}}$$

Fisher information

$$\underset{\sim}{I}_g = \int dx\, p(x) \left(\partial_g \ln p_g(x) \right)^2$$

$$\mathrm{Var}\, \hat{g} \geq \frac{1}{\sqrt{\underline{I}_g}}$$

$$H = H(g)$$

U_g is unitary

$$|\psi_f(g)\rangle = U_g |\psi_{in}\rangle$$

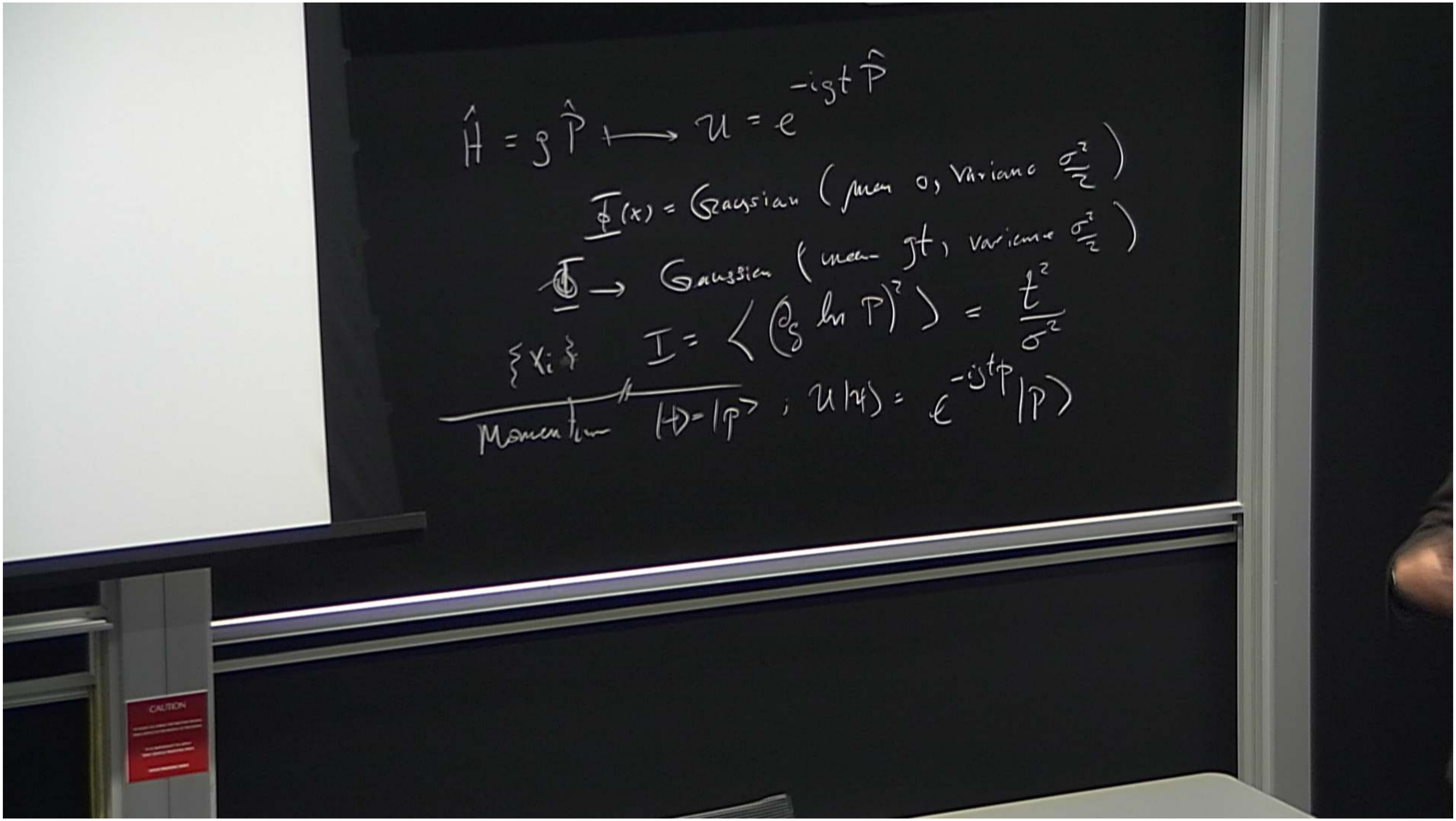
$$\underset{\sim}{I}_g^{(Q)} = 4\left(\langle \partial_g \psi_g | \partial_g \psi_g \rangle - |\langle \psi_g | \partial_g \psi_g \rangle|^2 \right)$$

$$\hat{H} = g\hat{P} \longmapsto \mathcal{U} = e^{-igt\hat{P}}$$

$\underline{\Phi}(x)$ = Gaussian (mean 0, variance $\frac{\sigma^2}{2}$)

$\underline{\Phi} \to$ Gaussian (mean gt, variance $\frac{\sigma^2}{2}$)

$\{x_i\}$ $\quad$ $\underline{I} = \left\langle (\partial_g \ln P)^2 \right\rangle = \frac{t^2}{\sigma^2}$


$\hat{H} = g\hat{P} \longmapsto \mathcal{U} = e^{-igt\hat{P}}$
$\Phi(x) = \text{Gaussian}(\text{mean } 0, \text{ variance } \frac{\sigma^2}{2})$
$\Phi \rightarrow \text{Gaussian}(\text{mean } gt, \text{ variance } \frac{\sigma^2}{2})$
$\{x_i\}$ $I = \langle (\partial_g \ln P)^2 \rangle = \frac{t^2}{\sigma^2}$
Momentum $|\psi\rangle = |p\rangle$; $\mathcal{U}|\psi\rangle = e^{-igtp}|p\rangle$

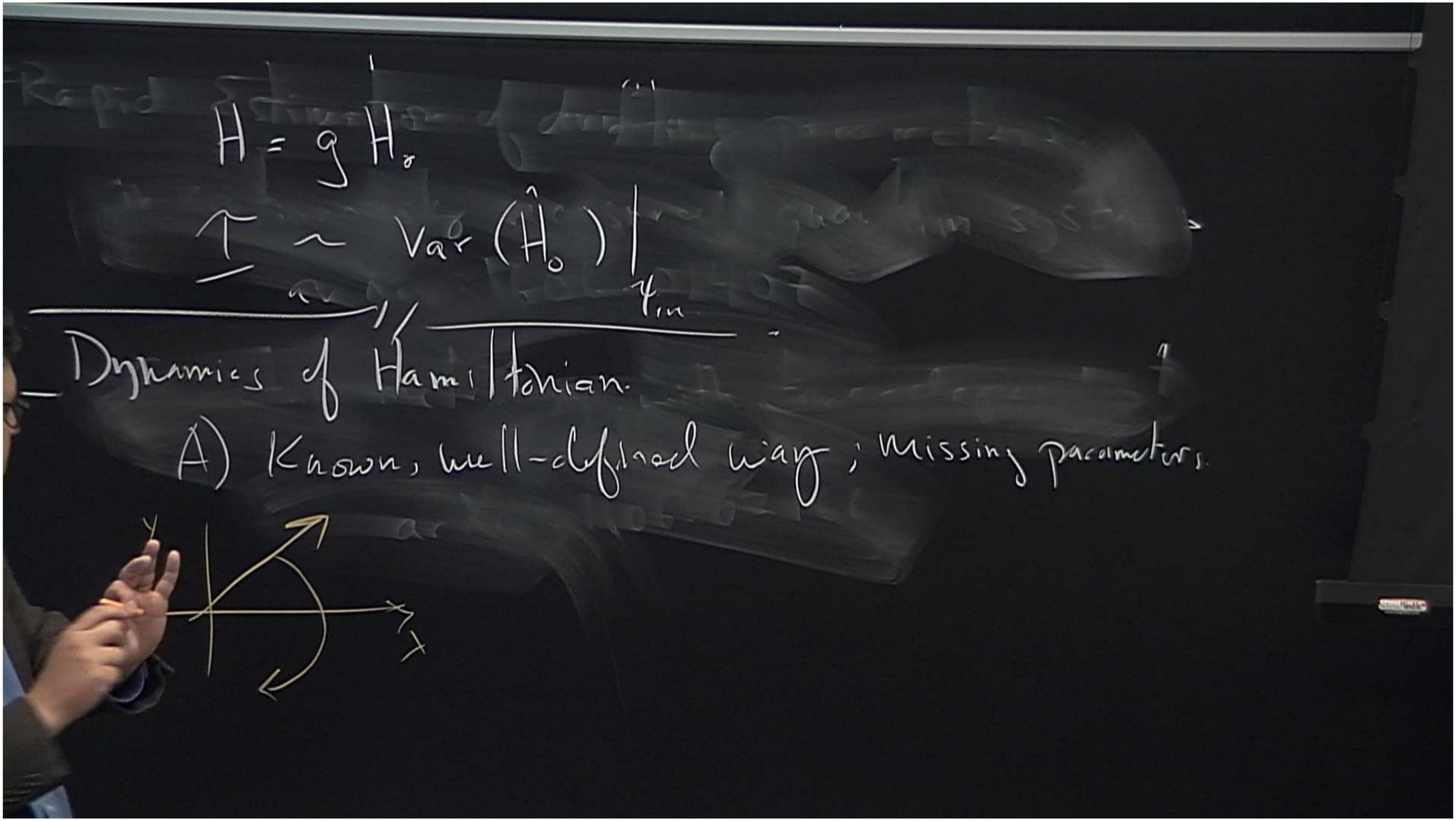
$$\hat{H} = g\,\hat{H}_0$$

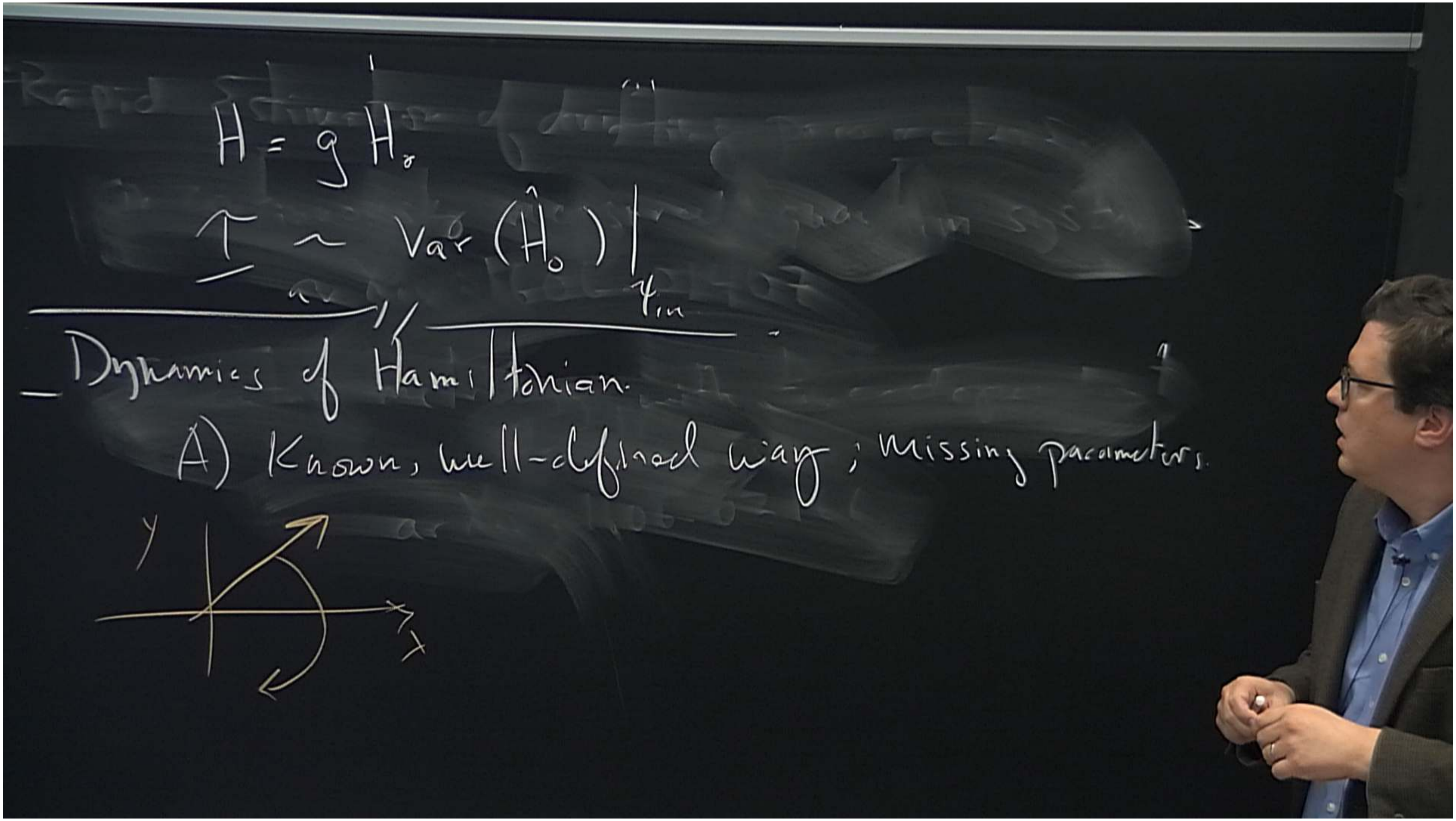
$$\underline{T} \sim \mathrm{Var}(\hat{H}_0)\Big|_{\psi_{in}}$$

$$H = g H_0$$

$$T \sim \mathrm{Var}(\hat{H}_0)\big|_{\psi_{in}}$$

— Dynamics of Hamiltonian.


H = g H_0
T ~ Var(Ĥ_0) | ψ_in
Dynamics of Hamiltonian.
A) Known, well-defined way; missing parameters.
y
x

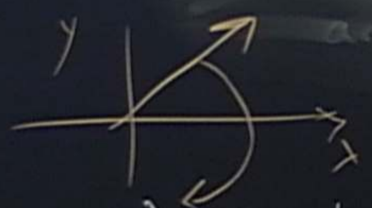

H = g H_0
T ~ Var(Ĥ_0) | ψ_in
Dynamics of Hamiltonian.
A) Known, well-defined way; missing parameters.
y
x

$$\hat{H} = g \hat{H}_0$$

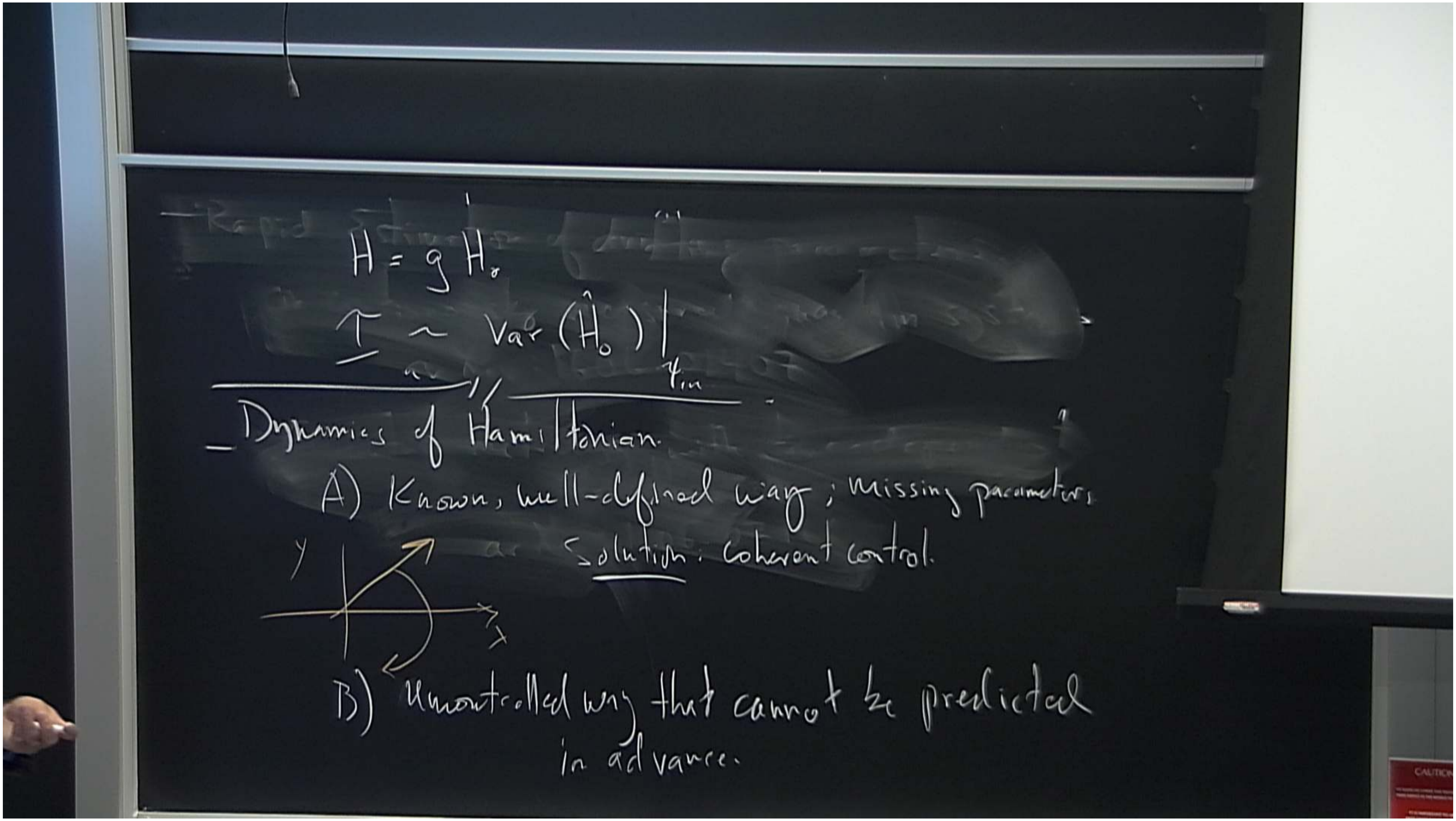
$$T \sim \mathrm{Var}(\hat{H}_0)\Big|_{\psi_{in}}$$

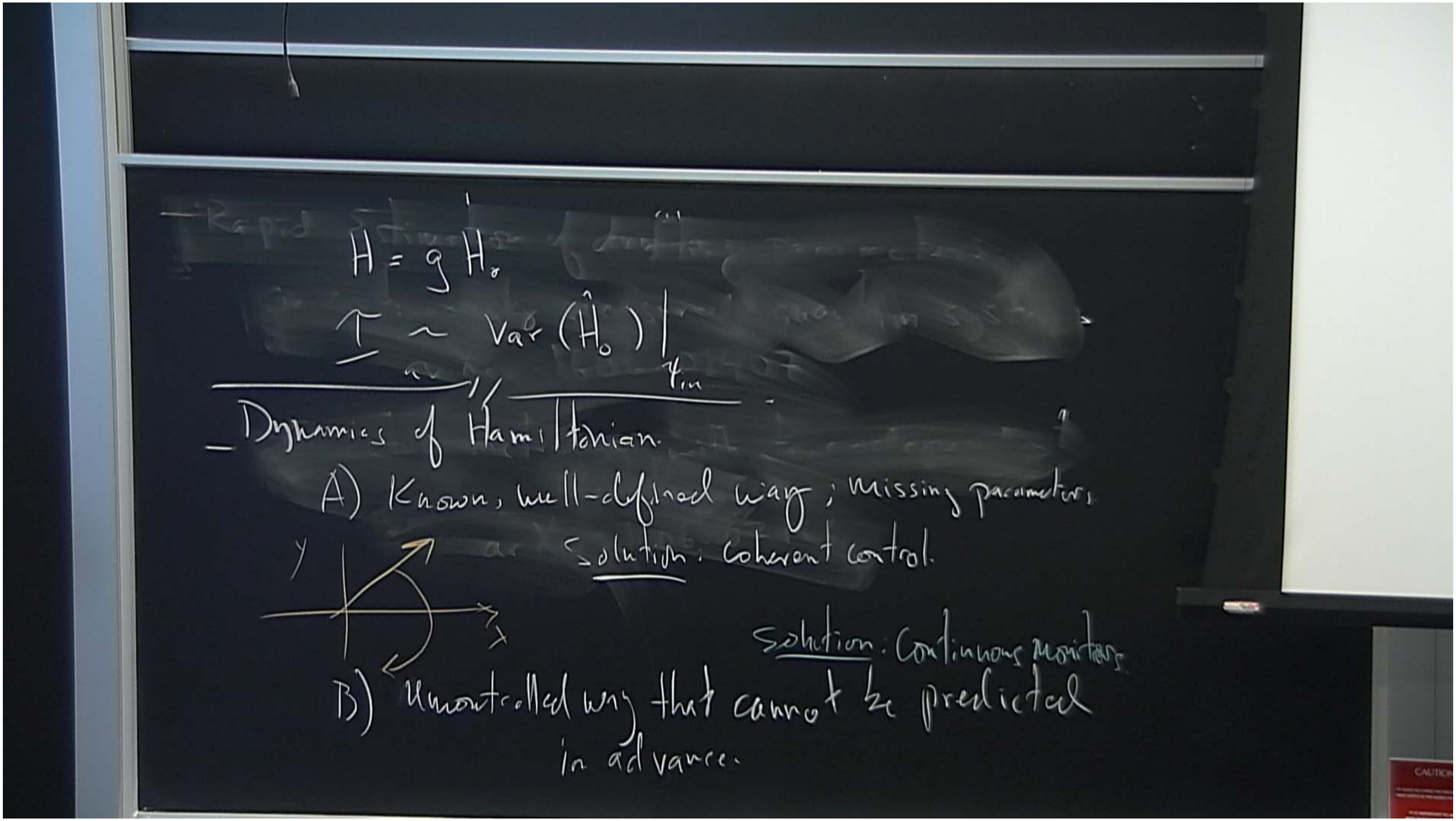
– Dynamics of Hamiltonian.

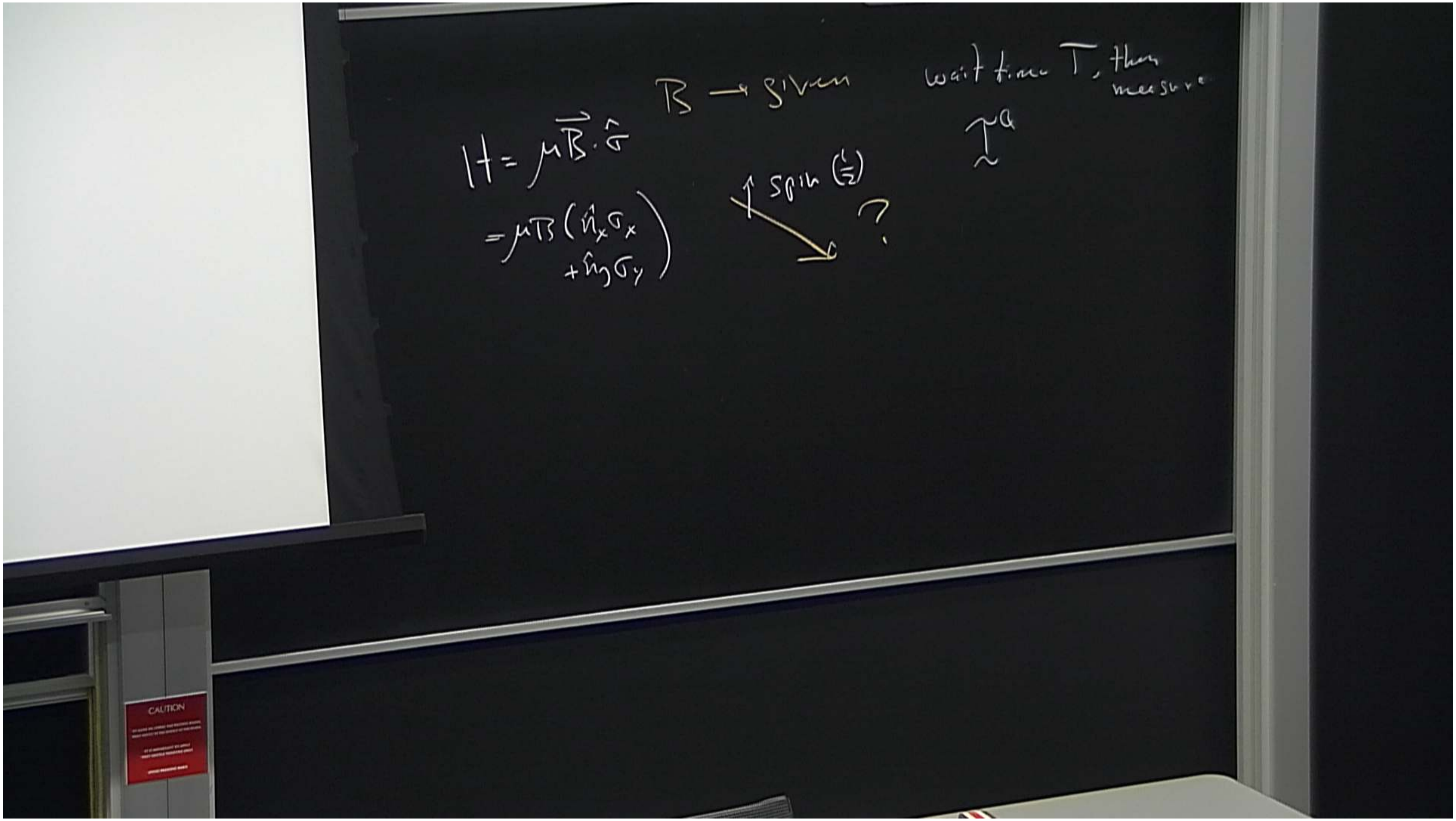
A) Known, well-defined way; missing parameters.

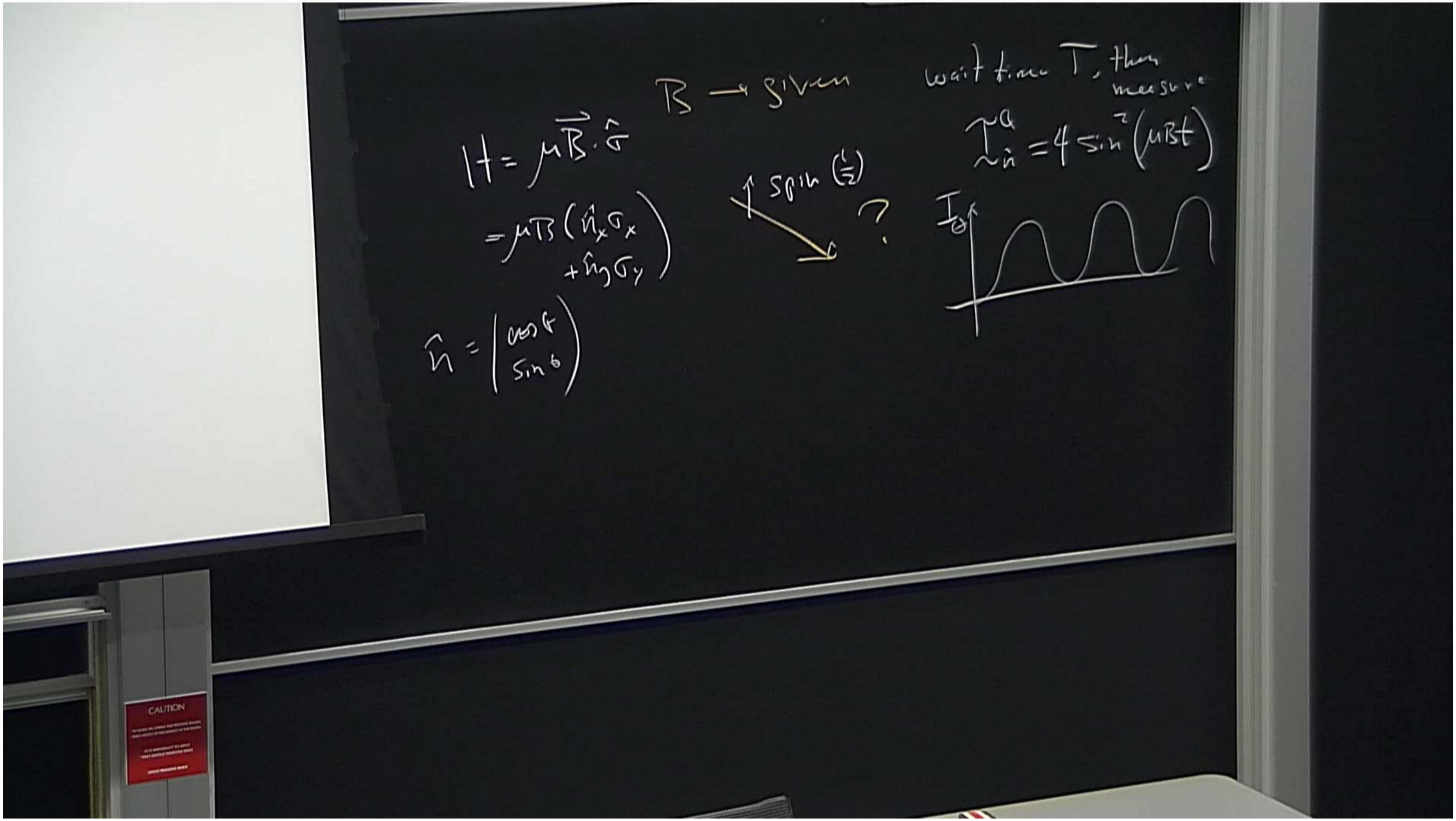


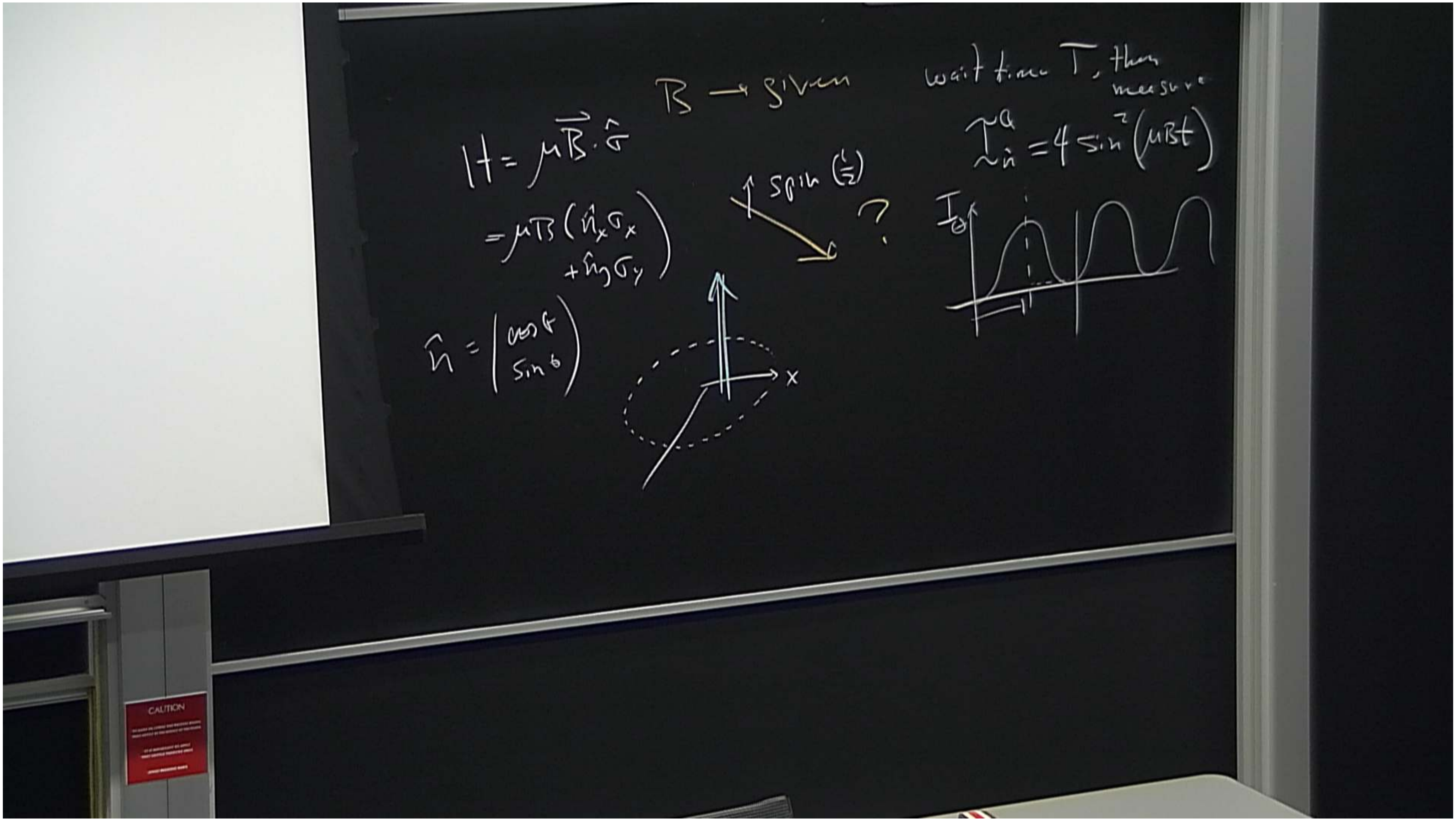
B) Uncontrolled way that cannot be predicted in advance.


H = g H0
T ~ Var(Ĥ0) | ψin
Dynamics of Hamiltonian.
A) Known, well-defined way; missing parameters
Solution: coherent control.
y
x
B) Uncontrolled way that cannot be predicted
in advance.

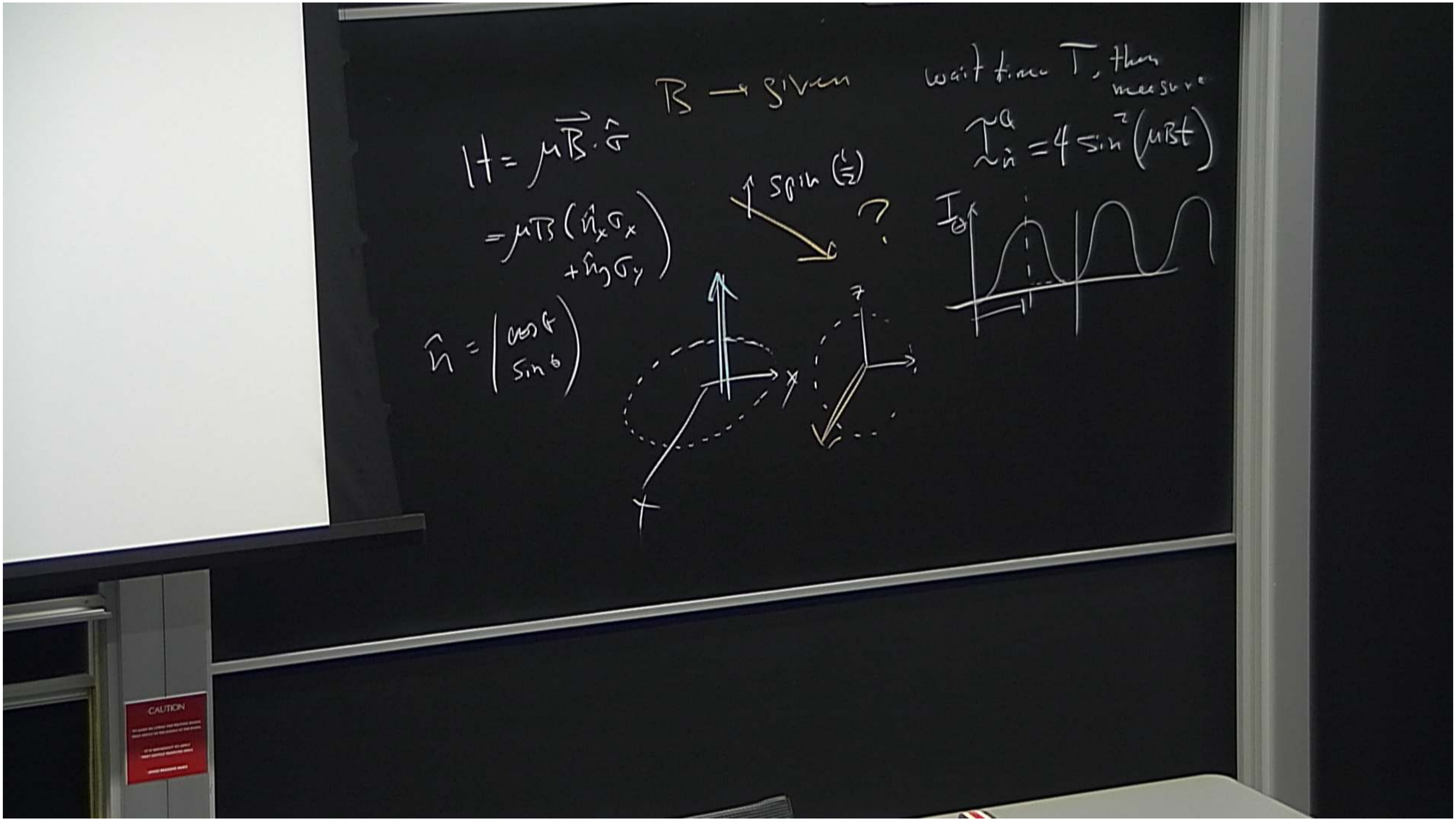

H = g H0
T ~ Var(Ĥ0) | ψin
Dynamics of Hamiltonian.
A) Known, well-defined way ; missing parameters.
Solution: coherent control.
y
x
Solution: Continuous monitoring.
B) Uncontrolled way that cannot be predicted
in advance.

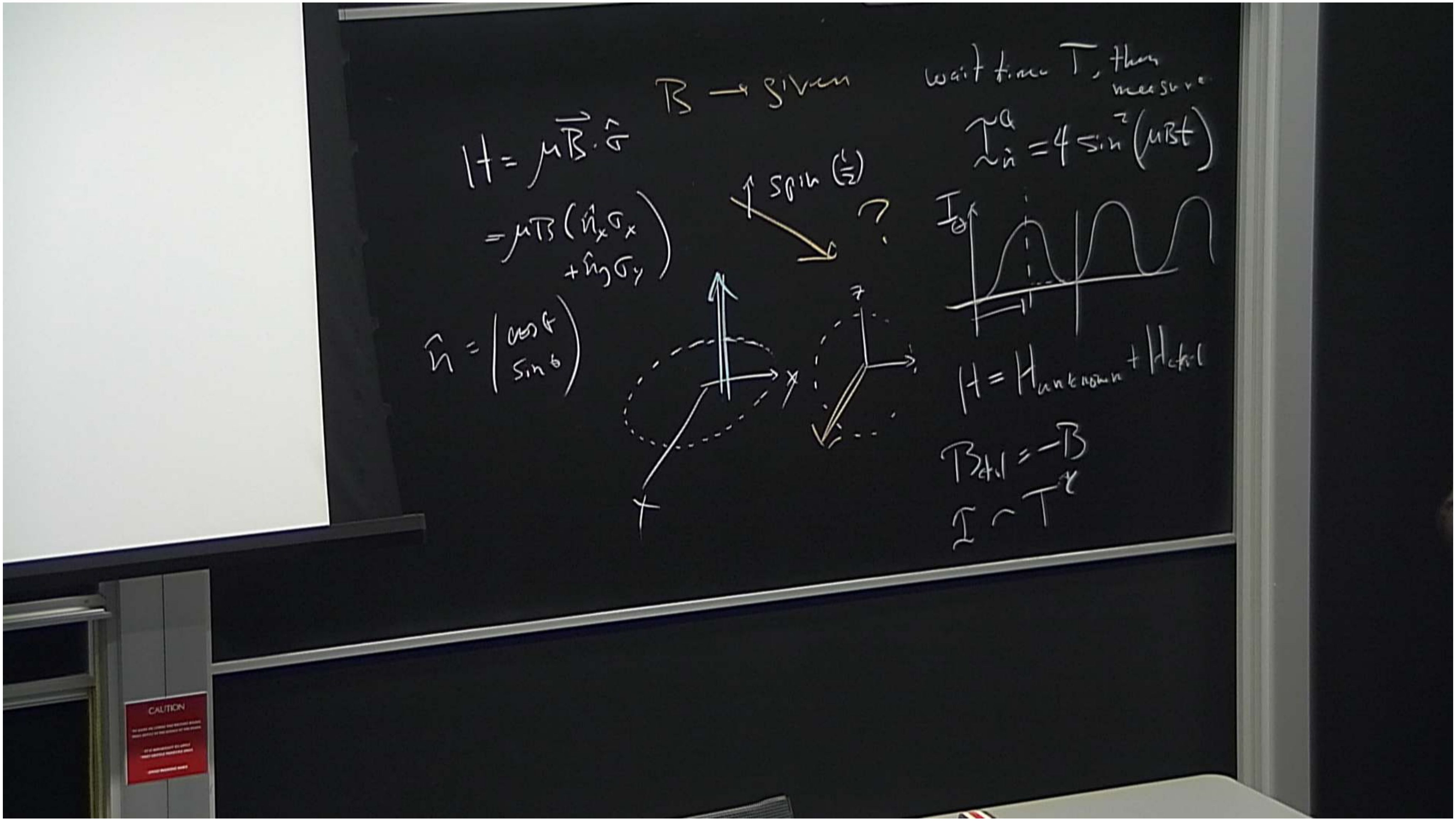

H = μB⃗ · σ̂
= μB (n̂ₓσₓ + n̂ᵧσᵧ)
B → given
spin (1/2)
?
wait time T, then measure
σᵃ
CAUTION


B → given
wait time T, then measure
H = μB·σ
= μB(n_x σ_x + n_y σ_y)
n = (cos θ, sin θ)
spin (1/2)
?
= 4 sin²(μBt)
CAUTION



$H = \mu \vec{B} \cdot \hat{\sigma}$
$= \mu B (\hat{n}_x \sigma_x + \hat{n}_y \sigma_y)$
$\hat{n} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$
B → given
spin ($\frac{1}{2}$)
?
x
wait time T, then measure
$\tilde{I}^{a}_{\hat{n}} = 4 \sin^2(\mu B t)$


B → given
wait time T, then
measure
H = μB⃗·σ̂
= μB (n̂x σx
+ n̂y σy)
spin (1/2)
?
n̂ = (cos θ
sin θ)
z
x
y


$B \to$ given
wait time T, then measure
$H = \mu \vec{B} \cdot \hat{\sigma}$
$= \mu B (\hat{n}_x \sigma_x + \hat{n}_y \sigma_y)$
$\hat{n} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$
spin $(\frac{1}{2})$
?
$\tilde{I}^{a}_{\hat{n}} = 4 \sin^2(\mu B t)$
$H = H_{unknown} + H_{ctrl}$
$B_{ctrl} = -B$
$I \sim T^{\gamma}$

in advance.

—Accuracy
—Speed.

$H(t)$

$$|\psi(t)\rangle = \mathcal{U}|\psi(0)\rangle$$

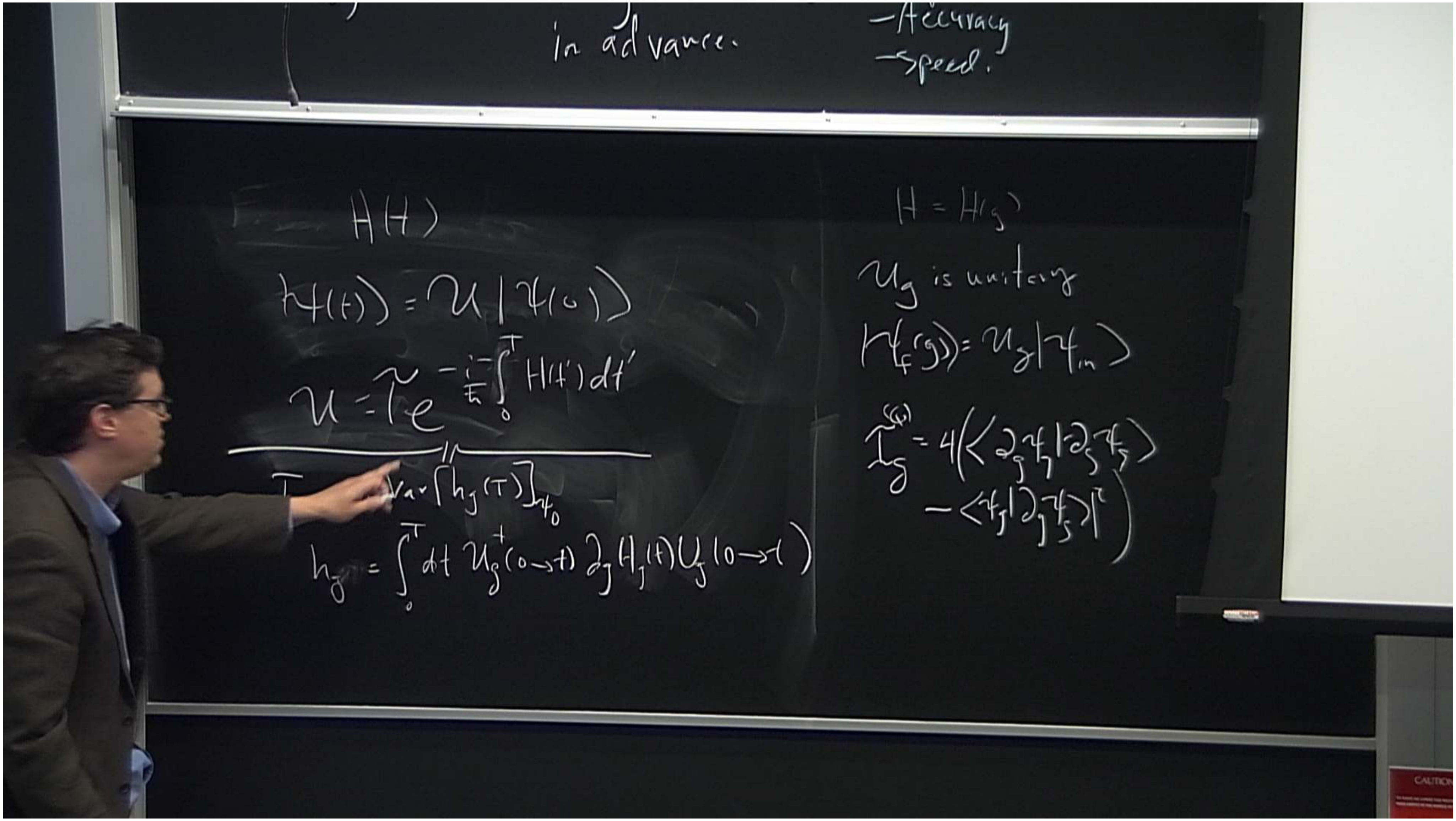
$$\mathcal{U} = \tilde{T} e^{-\frac{i}{\hbar}\int_0^T H(t')dt'}$$

$H = H(g)$

$\mathcal{U}_g$ is unitary

$$|\psi_f(g)\rangle = \mathcal{U}_g|\psi_{in}\rangle$$

$$I_g^{(Q)} = 4\left(\langle\partial_g\psi_g|\partial_g\psi_g\rangle - |\langle\psi_g|\partial_g\psi_g\rangle|^2\right)$$


in advance.
—Accuracy
—speed.
$H(t)$
$|\psi(t)\rangle = \mathcal{U}|\psi(0)\rangle$
$\mathcal{U} = \tilde{T} e^{-\frac{i}{\hbar}\int_0^T H(t')dt'}$
$\mathrm{Var}[h_g(T)]_{\psi_0}$
$h_g = \int_0^T dt\, \mathcal{U}_g^{\dagger}(0\to t)\, \partial_g H_g(t)\, \mathcal{U}_g(0\to t)$
$H = H(g)$
$\mathcal{U}_g$ is unitary
$|\psi_f(g)\rangle = \mathcal{U}_g|\psi_{in}\rangle$
$I_g^{(Q)} = 4\left(\langle\partial_g\psi_g|\partial_g\psi_g\rangle - |\langle\psi_g|\partial_g\psi_g\rangle|^2\right)$

in advance.

—Accuracy

—Speed.

$H(t)$

$|\psi(t)\rangle = \mathcal{U}|\psi(0)\rangle$

$$\mathcal{U} = \tilde{T} e^{-\frac{i}{\hbar}\int_0^T H(t')\,dt'}$$

$$I = 4\,\mathrm{Var}[h_g(T)]_{\psi_0}$$

$$h_g(T) = \int_0^T dt\, \mathcal{U}_g^\dagger(0\to t)\, \partial_g H_g(t)\, \mathcal{U}_g(0\to t)$$

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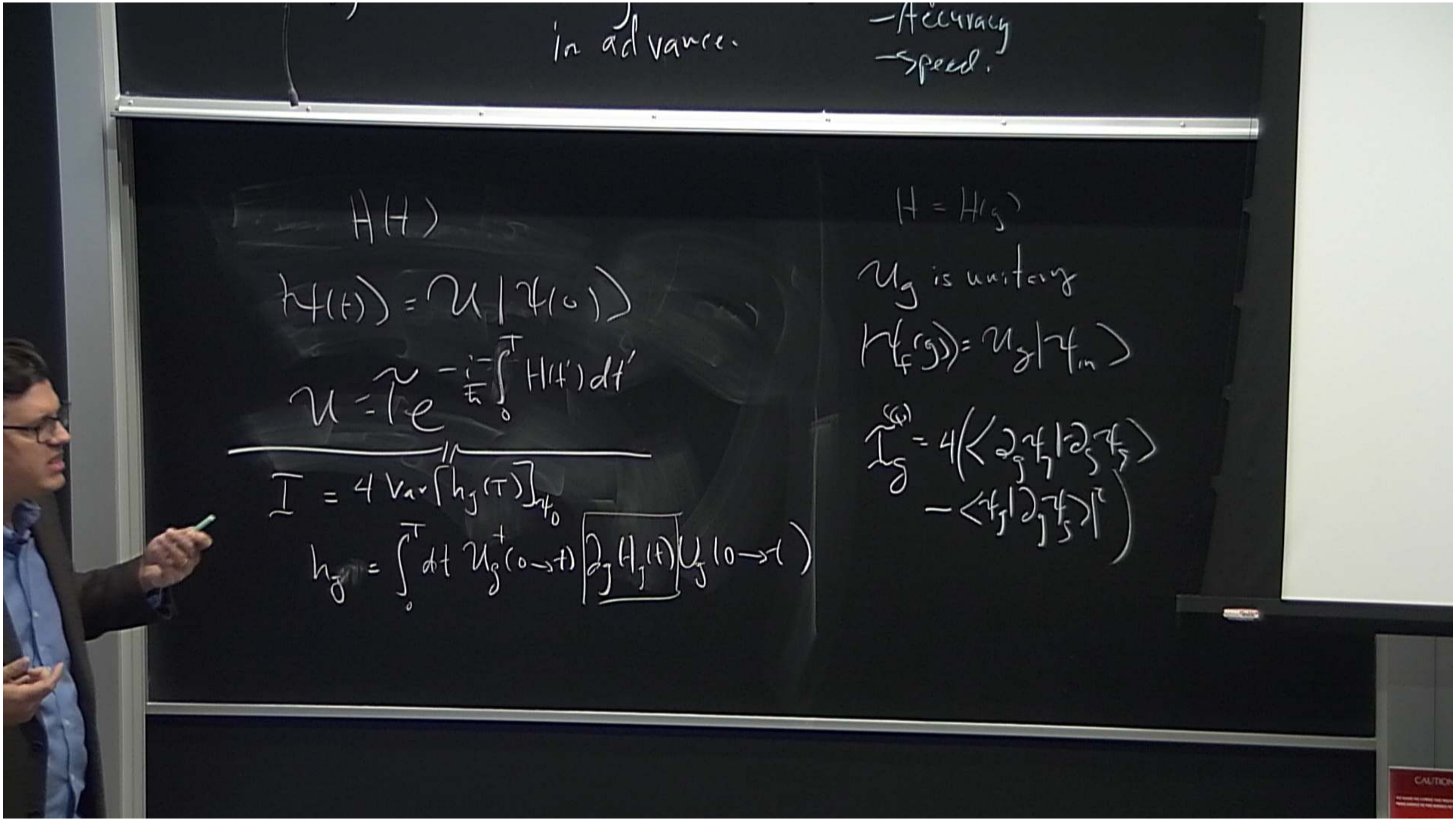
in advance.
—Accuracy
—Speed.
$H(t)$
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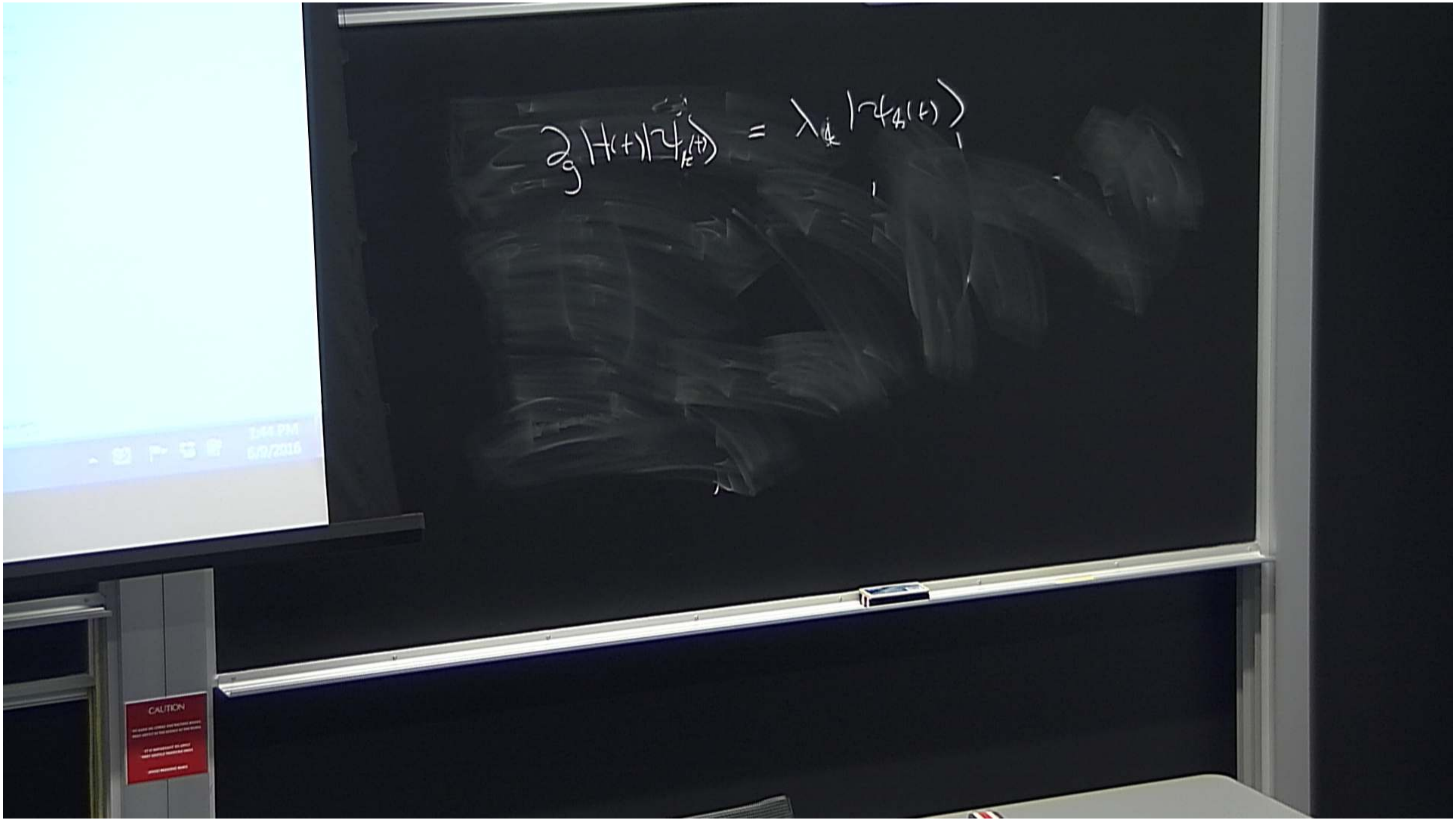
$h_g = \int_0^T dt\, \mathcal{U}_g^\dagger(0\to t)\, \boxed{\partial_g H_g(t)}\, \mathcal{U}_g(0\to t)$
$H = H(g)$
$\mathcal{U}_g$ is unitary
$|\psi_f(g)\rangle = \mathcal{U}_g|\psi_{in}\rangle$
$I_g^{(Q)} = 4\left(\langle \partial_g\psi_g|\partial_g\psi_g\rangle - |\langle\psi_g|\partial_g\psi_g\rangle|^2\right)$


$\partial_g H(t)|\psi_k(t)\rangle = \lambda_k |\psi_k(t)\rangle$
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CAUTION

to $|\psi_k(t+\Delta t)\rangle$ by $U_g(t \to t+\Delta t)$. When Δt is sufficiently small, both $U_g(t \to t+\Delta t)$ and $|\psi_k(t)\rangle$ can be expanded to the first order of Δt, and the Hamiltonian $H_{\text{tot}}(t)$ is approximately constant within Δt, then we can find $H_{\text{tot}}(t)|\psi_k(t)\rangle = \mathrm{i}|\partial_t \psi_k(t)\rangle$ by equating $U_g(t \to t+\Delta t)|\psi_k(t)\rangle$ with $|\psi_k(t+\Delta t)\rangle$. This has the form of a Schrödinger equation. But unlike the usual situations where we know the Hamiltonian and must find the solution to the state, here we know the solution of the state, $|\psi_k(t)\rangle$, and need to find the appropriate Hamiltonian $H_{\text{tot}}(t)$ instead. A direct solution to this equation is obviously $H_{\text{tot}}(t) = \mathrm{i}\sum_k |\partial_t \psi_k(t)\rangle\langle\psi_k(t)|$. (Note this solution is Hermitian because $\sum_k |\partial_t \psi_k(t)\rangle\langle\psi_k(t)|$ is skew-Hermitian.) Considering every eigenstate $|\psi_k(t)\rangle$ satisfies the $U(1)$ symmetry (i.e., multiplying $|\psi_k(t)\rangle$ by an arbitrary phase $\mathrm{e}^{-\mathrm{i}\theta_k(t)}$ does not change that state), $H_{\text{tot}}(t)$ can be generalized to include an additional term $\sum_k \dot\theta_k(t)|\psi_k(t)\rangle\langle\psi_k(t)|$. $\dot\theta_k(t)$ can be replaced by arbitrary real functions $f_k(t)$, and $\theta_k(t) = \int_0^t f_k(t')\mathrm{d}t'$. Thus, the control Hamiltonian $H_c(t)$ finally turns out to be

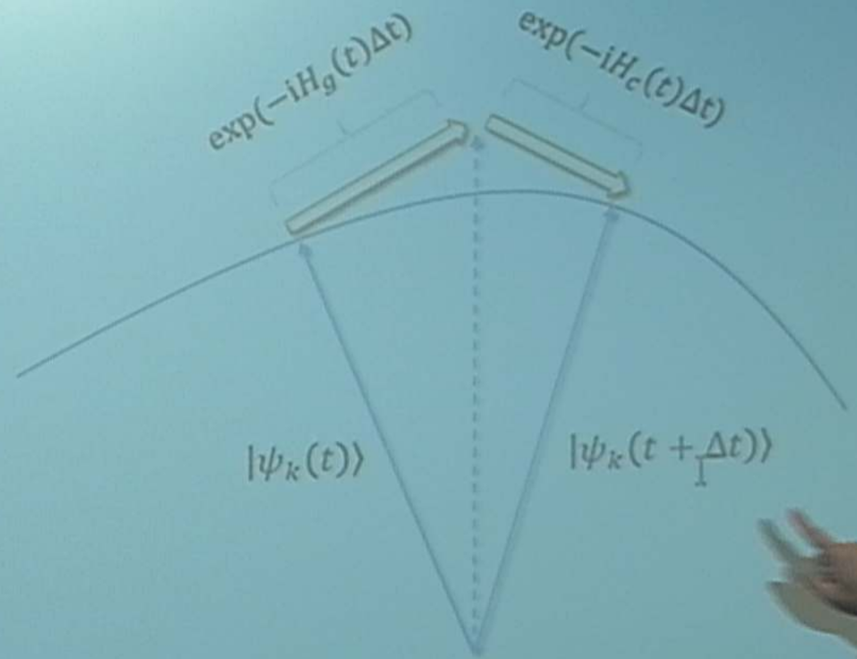


Figure 2. Optimal Hamiltonian control scheme to achieve the maximum Fisher information. The evolution of the system under the total Hamiltonian $H_g(t) + H_c(t)$ for a short time Δt can be approximated as $\exp(-\mathrm{i}H_c(t)\Delta t)\exp(-\mathrm{i}H_g(t)\Delta t)$.

to $|\psi_k(t+\Delta t)\rangle$ by $U_g(t \to t+\Delta t)$. When Δt is sufficiently small, both $U_g(t \to t+\Delta t)$ and $|\psi_k(t)\rangle$ can be expanded to the first order of Δt, and the Hamiltonian $H_{\rm tot}(t)$ is approximately constant within Δt, then we can find $H_{\rm tot}(t)|\psi_k(t)\rangle = \mathrm{i}|\partial_t\psi_k(t)\rangle$ by equating $U_g(t \to t+\Delta t)|\psi_k(t)\rangle$ with $|\psi_k(t+\Delta t)\rangle$. This has the form of a Schrödinger equation. But unlike the usual situations where we know the Hamiltonian and must find the solution to the state, here we know the solution of the state, $|\psi_k(t)\rangle$, and need to find the appropriate Hamiltonian $H_{\rm tot}(t)$ instead. A direct solution to this equation is obviously $H_{\rm tot}(t) = \mathrm{i}\sum_k |\partial_t\psi_k(t)\rangle\langle\psi_k(t)|$. (Note this solution is Hermitian because $\sum_k |\partial_t\psi_k(t)\rangle\langle\psi_k(t)|$ is skew-Hermitian.) Considering every eigenstate $|\psi_k(t)\rangle$ satisfies the $U(1)$ symmetry (i.e., multiplying $|\psi_k(t)\rangle$ by an arbitrary phase $\mathrm{e}^{-\mathrm{i}\theta_k(t)}$ does not change that state), $H_{\rm tot}(t)$ can be generalized to include an additional term $\sum_k \dot{\theta}_k(t)|\psi_k(t)\rangle\langle\psi_k(t)|$. $\dot{\theta}_k(t)$ can be replaced by arbitrary real functions $f_k(t)$, and $\theta_k(t) = \int_0^t f_k(t')\mathrm{d}t'$. Thus, the control Hamiltonian $H_c(t)$ finally turns out to be

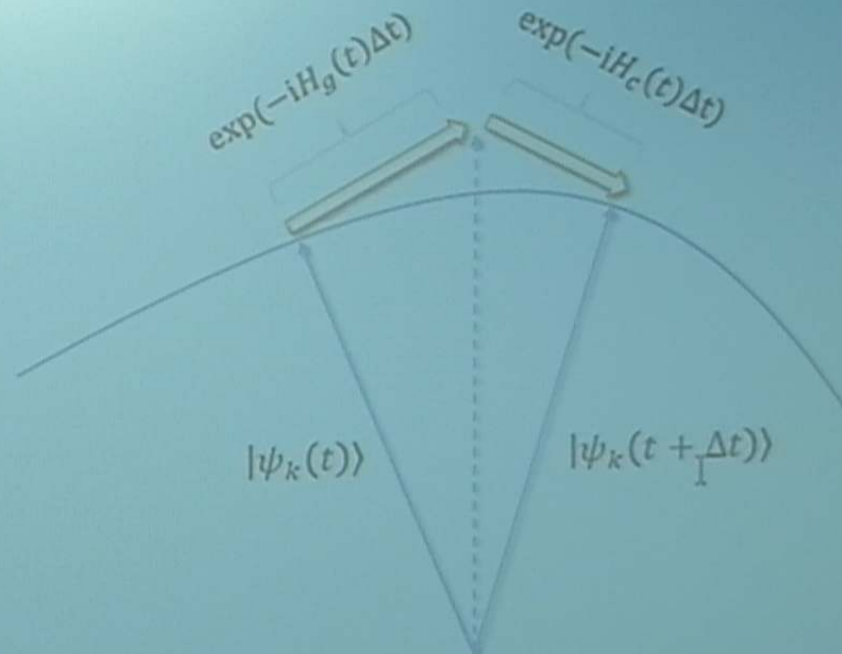
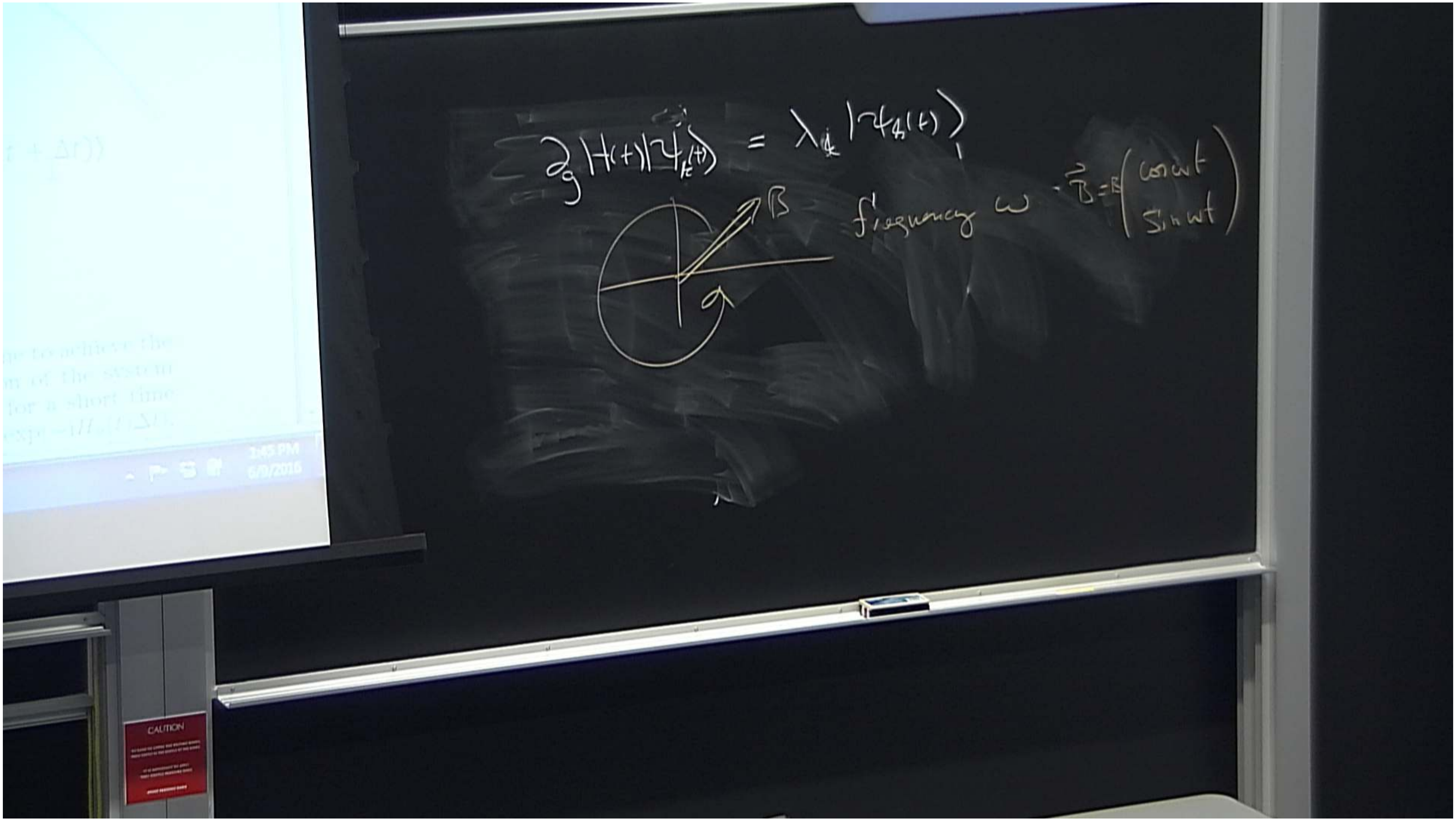
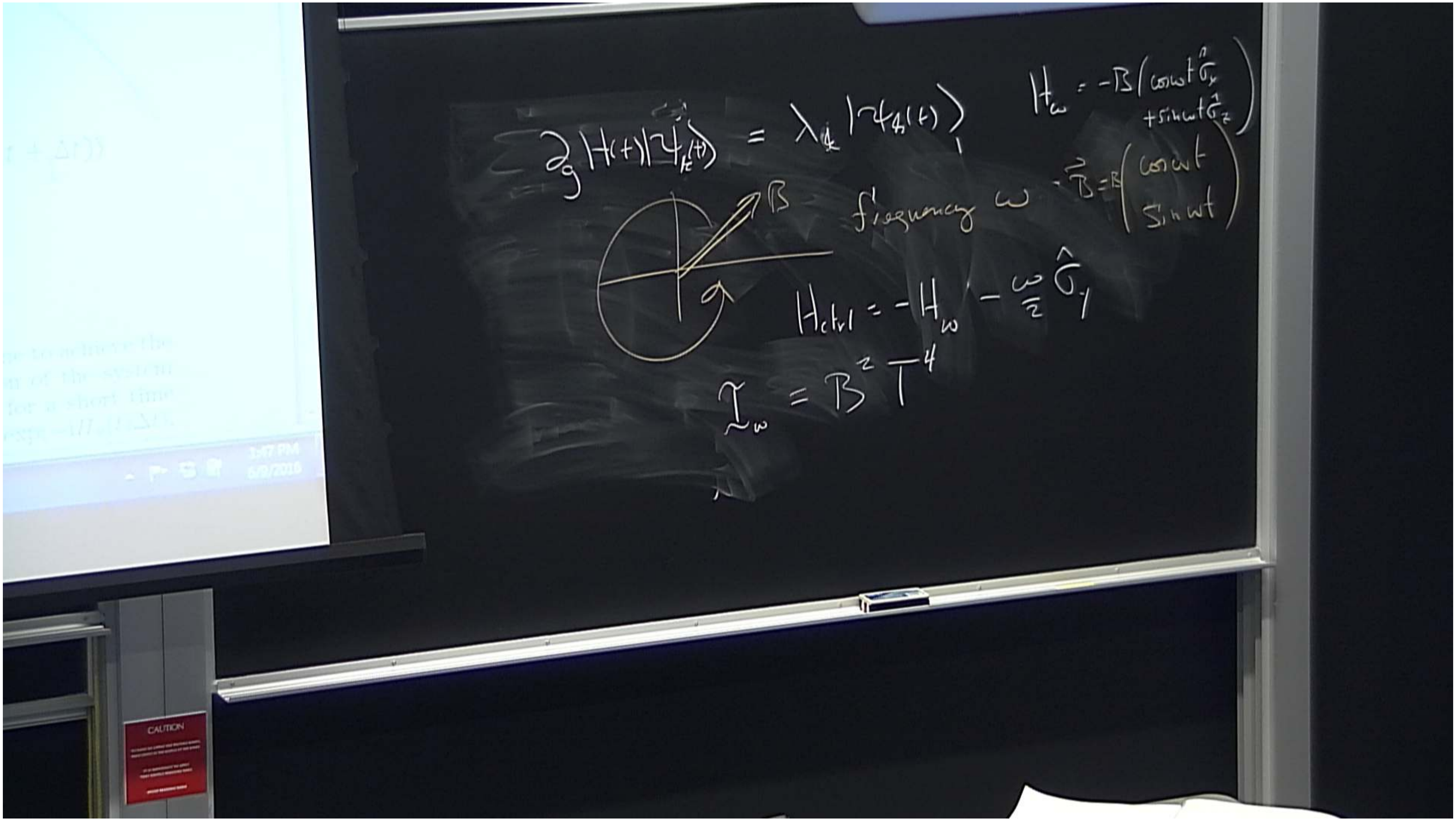
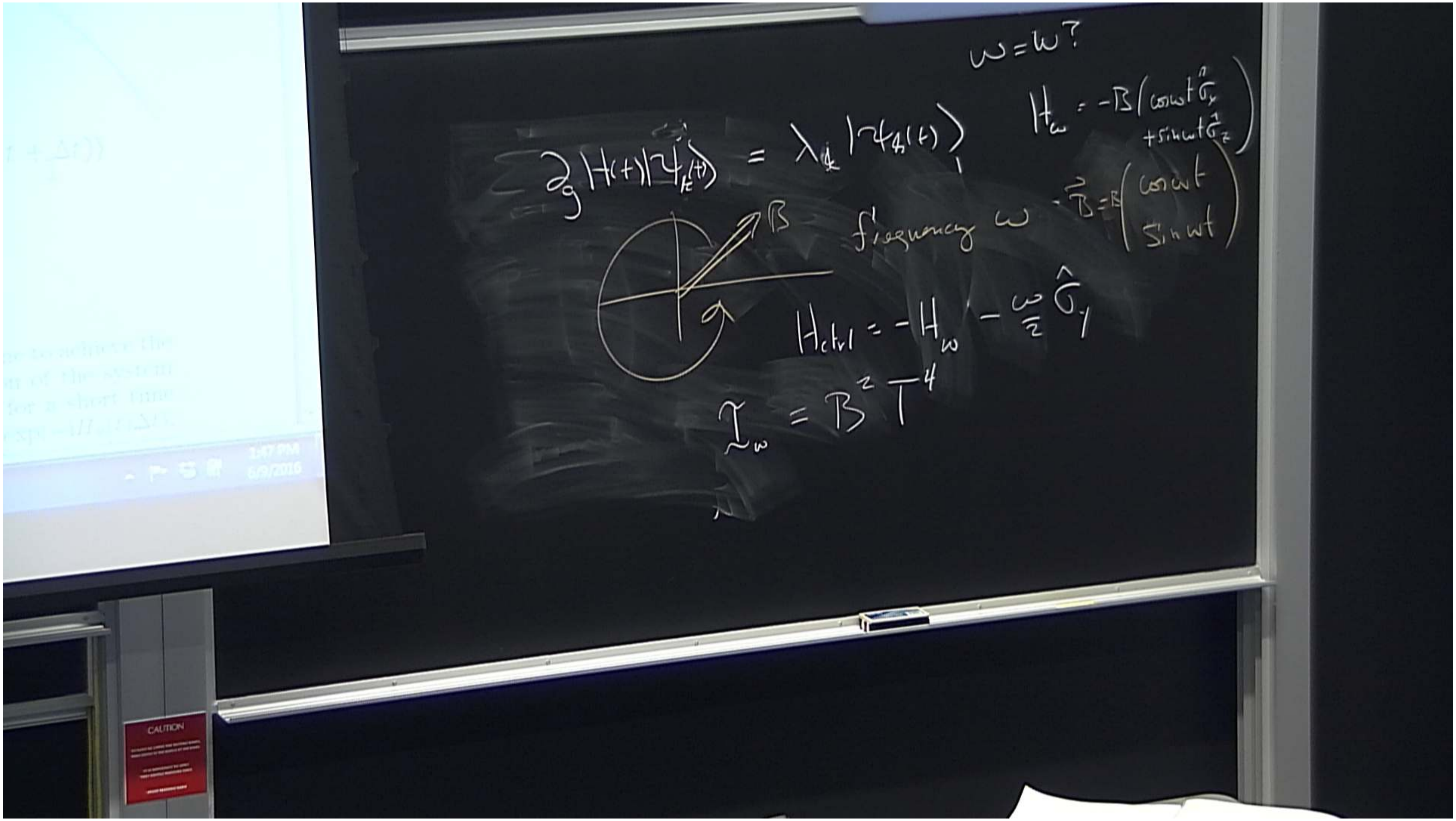


Figure 2. Optimal Hamiltonian control scheme to achieve the maximum Fisher information. The evolution of the system under the total Hamiltonian $H_g(t)+H_c(t)$ for a short time Δt can be approximated as $\exp(-\mathrm{i}H_c(t)\Delta t)\exp(-\mathrm{i}H_g(t)\Delta t)$.


frequency ω


∂g H(t)|ψk(t)⟩ = λk |ψk(t)⟩
Hω = -B(cosωt σx + sinωt σz)
frequency ω
B = B(cosωt, sinωt)
Hctrl = -Hω - ω/2 σy
Iω = B² T⁴


ω = ω?
frequency ω
CAUTION

The quantum Fisher information $I_B^{(Q)}$ is plotted for different rotation frequencies ω without the control Hamiltonian and compared to that with the control Hamiltonian $H_c(t)$ in Fig. 3. It can be observed that when ω is large compared to the amplitude of the magnetic field B, the Fisher information becomes small. And the Fisher information with the optimal Hamiltonian control is the highest, which verifies the advantage of Hamiltonian control in increasing Fisher information.

Estimation of field rotation frequency. Now, we turn to estimating the rotation frequency ω of the magnetic field. Since $\partial_\omega H(t)$ is $tB(\sin\omega t\sigma_X - \cos\omega t\sigma_Z)$, the eigenvalues of $\partial_\omega H(t)$ are $\mu(t) = \pm tB$, and the maximum and minimum eigenvalues of $h_\omega(T)$ are therefore $\int \mu(t)\mathrm{d}t = \pm\frac{1}{2}BT^2$. Therefore, the maximum Fisher information of estimating ω is

$$I_\omega^{(Q)} = B^2T^4. \quad (14)$$

The eigenstates of $\partial_\omega H(t)$ are $|\psi_+(t)\rangle = \sin\frac{\omega t}{2}|0\rangle + \cos\frac{\omega t}{2}|1\rangle$ and $|\psi_-(t)\rangle = \cos\frac{\omega t}{2}|0\rangle - \sin\frac{\omega t}{2}|1\rangle$. If we choose $f_c(t) = 0$ in the control Hamiltonian, then

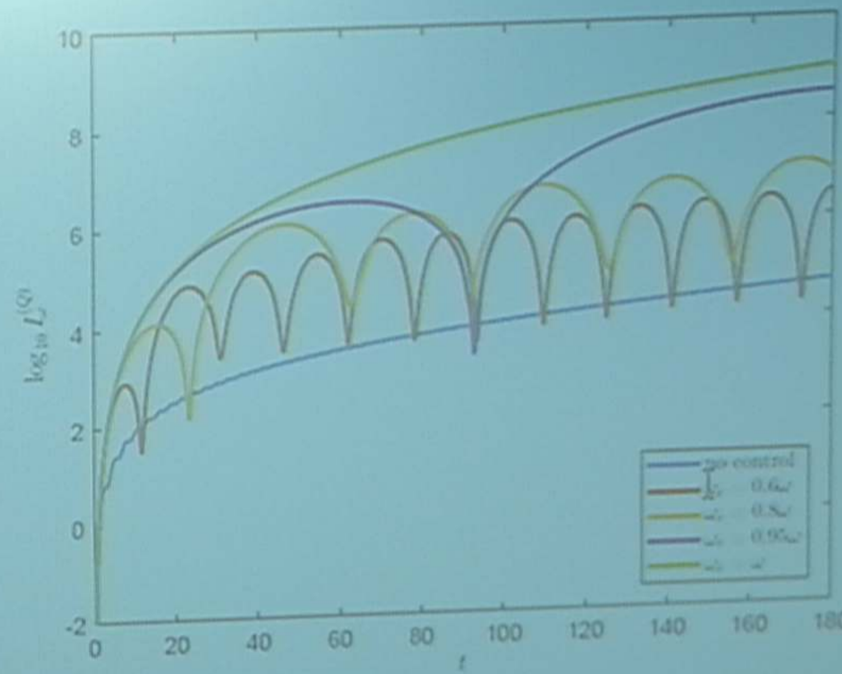
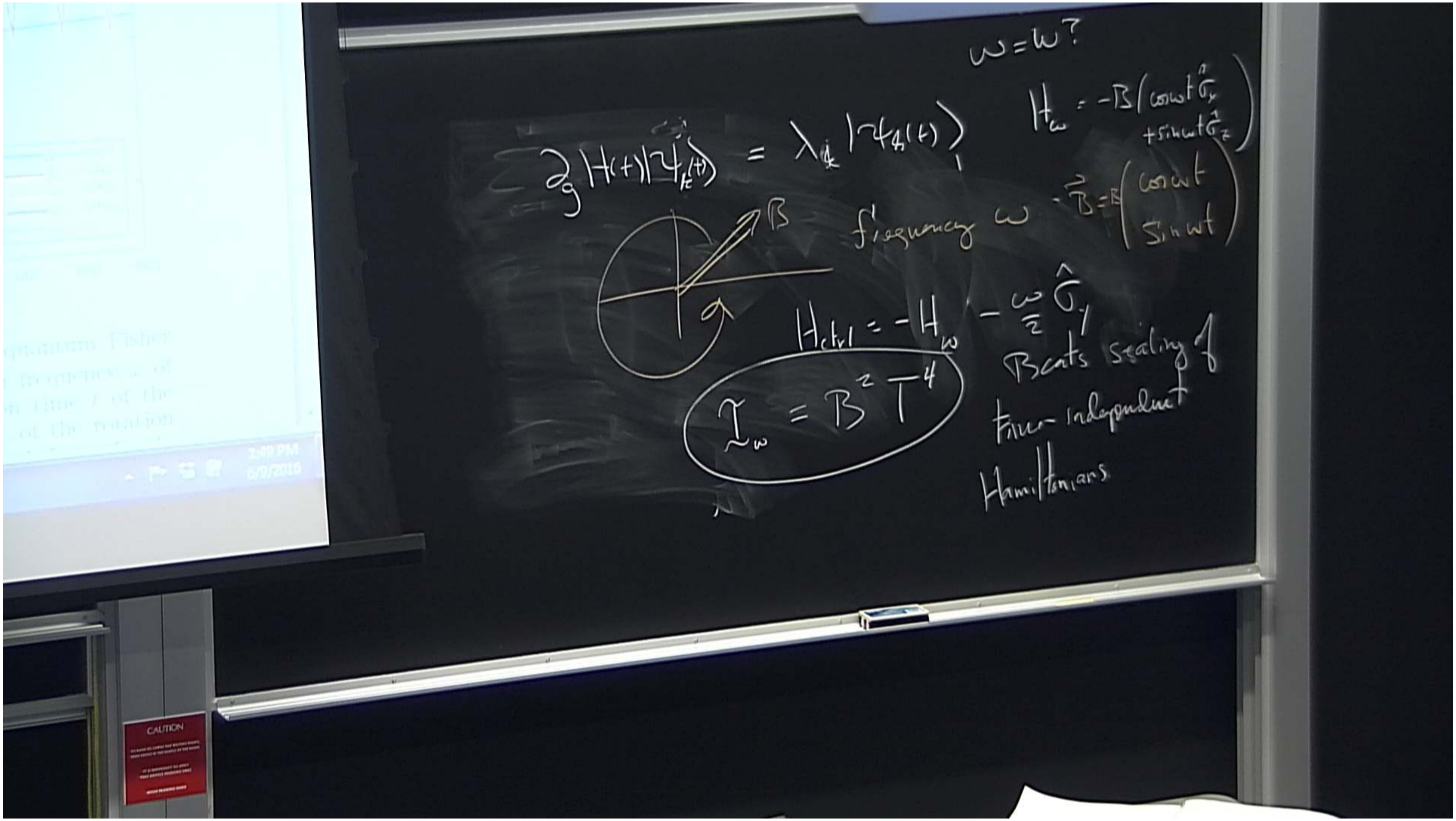
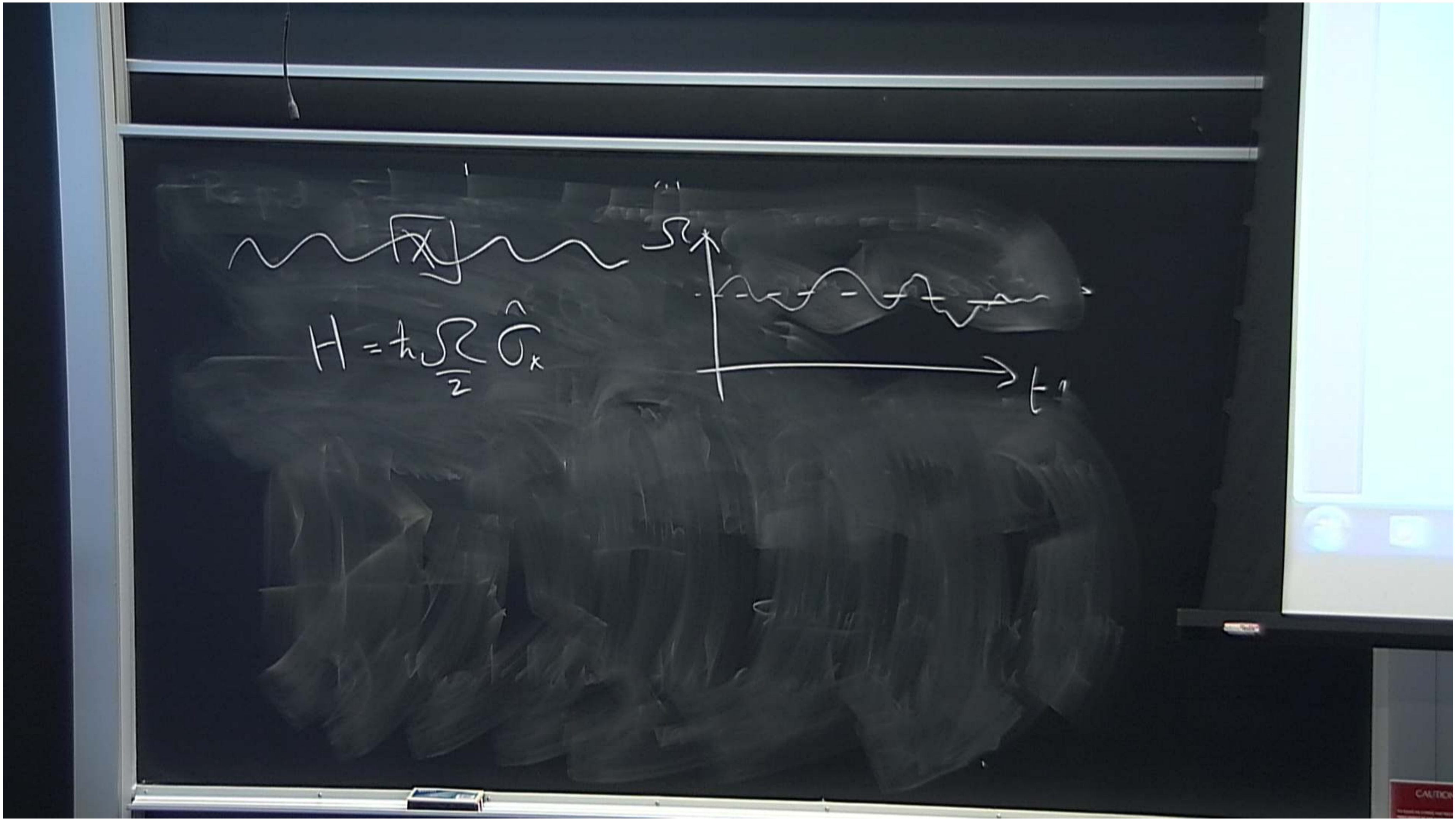
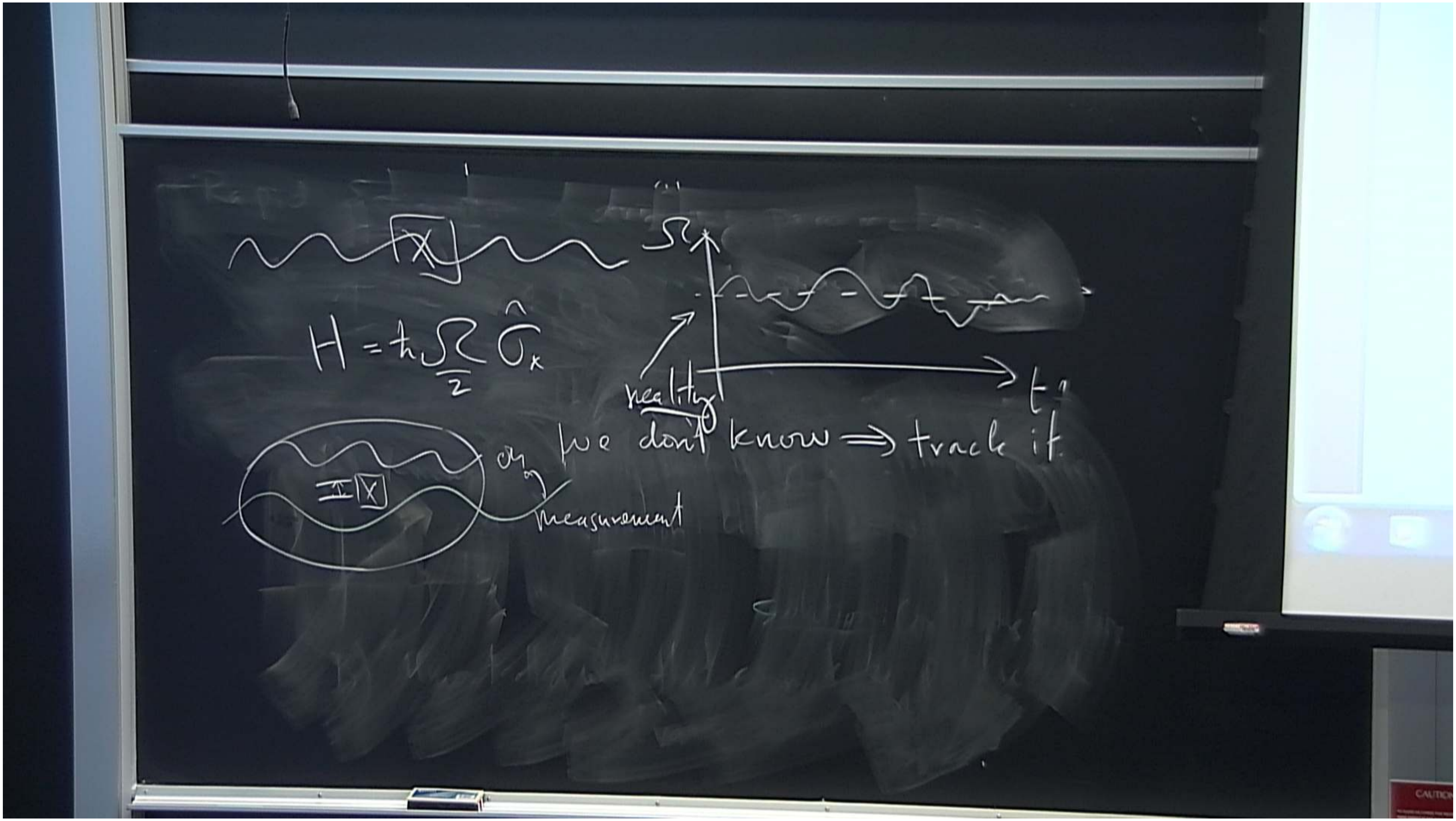


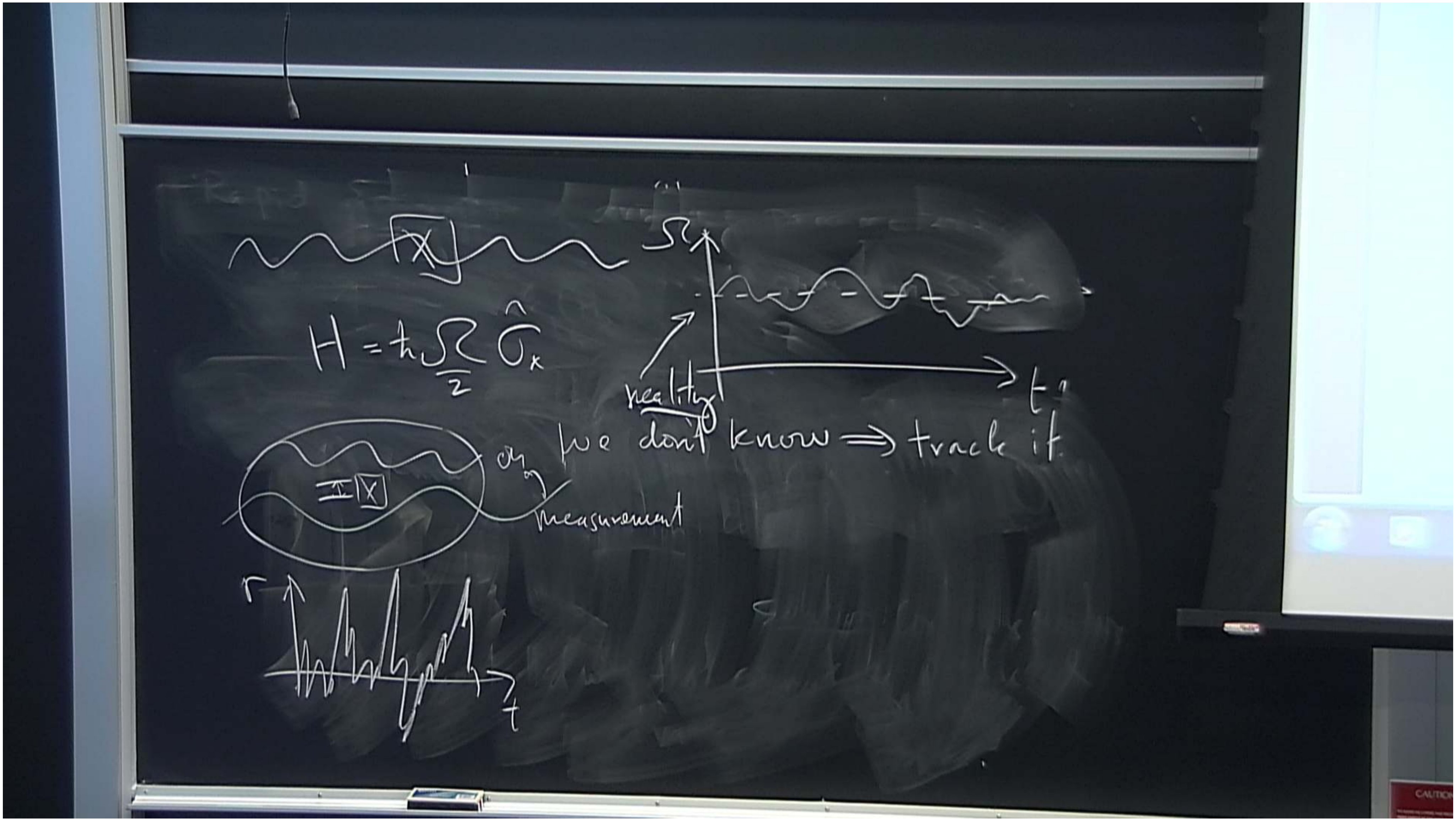
Figure 4. The logarithm (base 10) of the quantum Fisher information $I_\omega^{(Q)}$ for estimating the rotation frequency ω of the magnetic field $\mathbf{B}(t)$ versus the evolution time t of the qubit is plotted for different trial values ω_c of the rotation


ω = ω?
frequency ω
Beats scaling of
time-independent
Hamiltonians

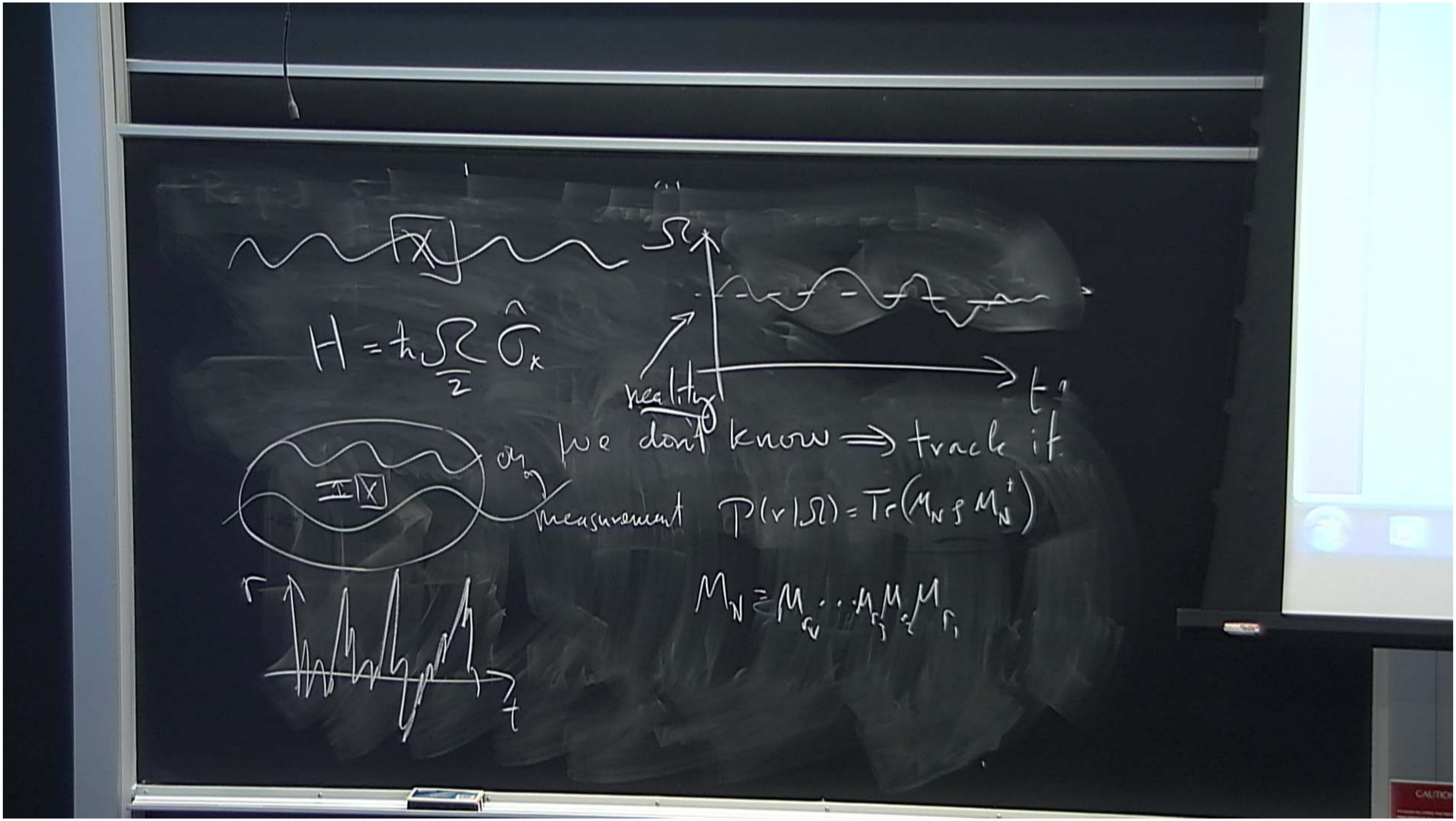

Fx
H = ħ Ω/2 σ̂x
Ω
t

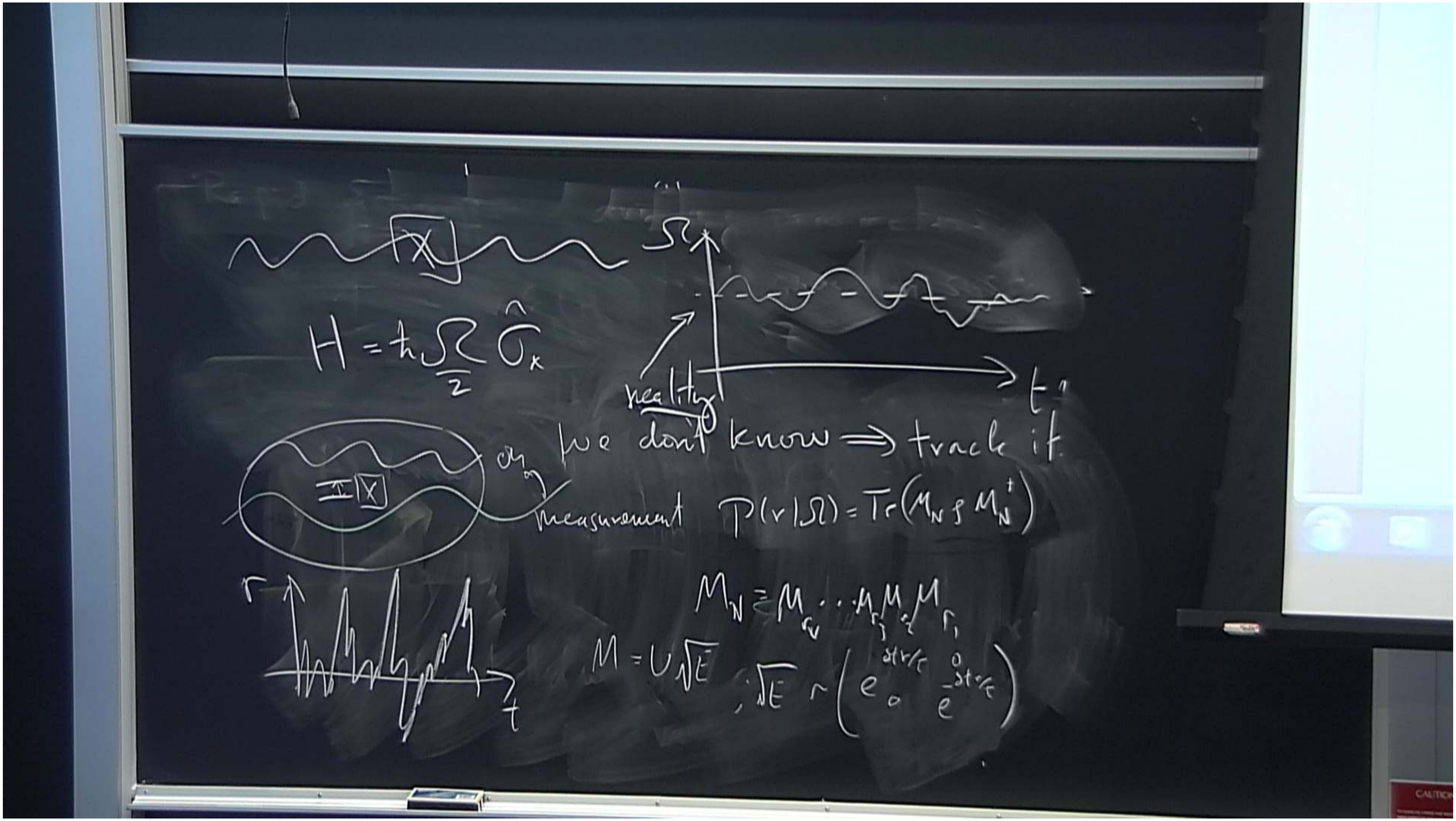


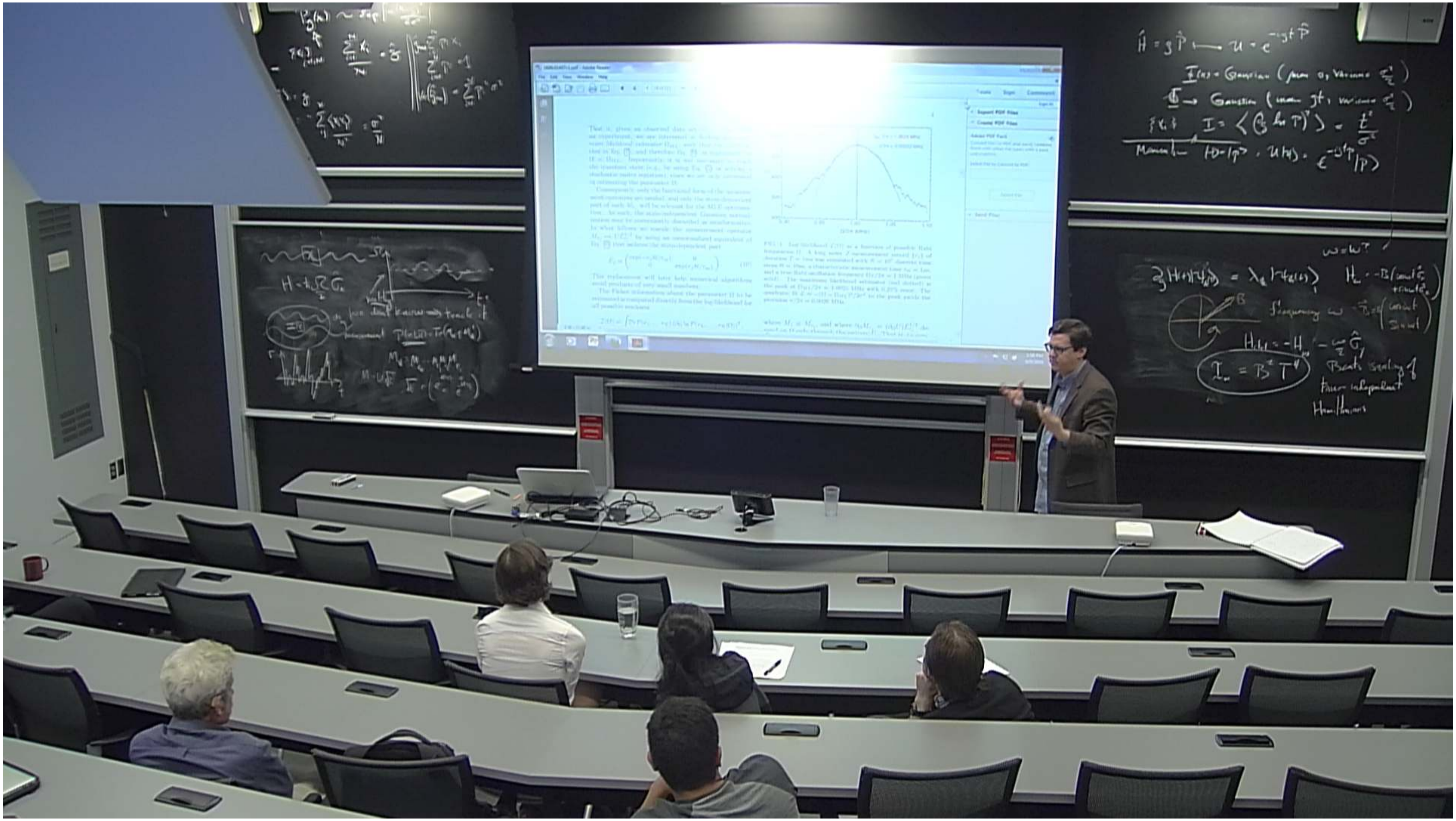
$H = \hbar \frac{\Omega}{2} \hat{\sigma}_x$
Ω
t
reality
we don't know ⟹ track it.
measurement



H = ħ Ω/2 σ̂ₓ
Ω
reality
t
we don't know ⟹ track it.
measurement
r
t


$H = \hbar \frac{\Omega}{2} \hat{\sigma}_x$
Ω
reality
t
we don't know ⇒ track it
measurement
$P(r|\Omega) = Tr(M_N \rho M_N^\dagger)$
$M_N = M_{r_N} \cdots M_{r_3} M_{r_2} M_{r_1}$
r
t


H = ħΩ/2 σ̂x
reality
we don't know ⇒ track it.
measurement P(r|Ω) = Tr(MN ρ MN†)
MN = MrN · · · Mr3 Mr2 Mr1
M = U√E ; √E ~ (e^{δt r/τ} 0 ; 0 e^{-δt r/τ})
r
t



$P_{\text{prior}}(\Omega)$ may be computed either from the RMS error, or from the uncertainty of the previous measurement, plus the typical expected drift uncertainty from the time-varying parameter. In these cases we can improve MLE by incorporating this information.

The maximum, *a posteriori*, probability of the parameter, taking into account the prior is given by Bayes rule, given the measurement record $r(t)$,

$$P(\Omega|r(t)) = \frac{P(r(t)|\Omega)P_{\text{prior}}(\Omega)}{P(r(t))} \propto P(r(t)|\Omega)P_{\text{prior}}(\Omega). \tag{24}$$

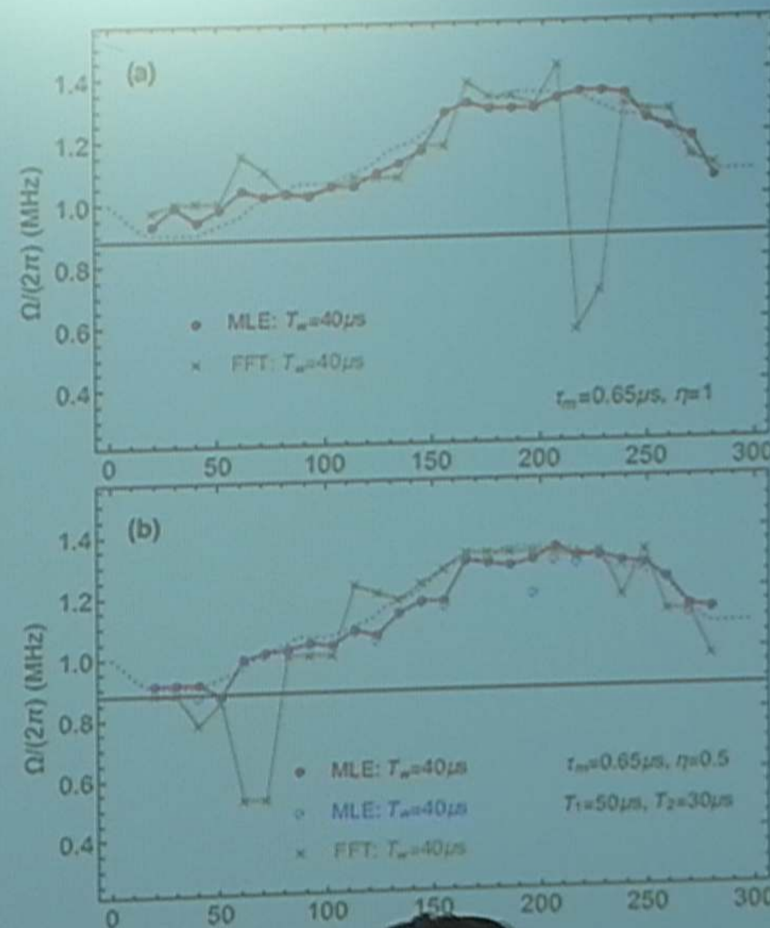
We may then modify our log-likelihood function as

$$\mathcal{L} \propto \ln P(r(t)|\Omega) + \ln P_{\text{prior}}(\Omega), \tag{25}$$

where the first term is our previous log-likelihood, and the second term takes into account the information from the prior experiments. Note that the denominator in Bayes rule can be dropped given that it is independent of Ω, the quantity over which we are maximizing.

B. Time-dependent frequency tracking

We now illustrate the method of time-dependent parameter tracking of a slowly drifting Rabi frquency. We



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