

Title: Estimating dynamic parameters quantumly

Date: Jun 09, 2016 01:00 PM

URL: <http://pirsa.org/16060094>

Abstract: <p>I will give an overview of two ways to estimate parameters with a quantum system when the dynamics is nontrivial. In the first case, the parameter is changing in an irregular way, and we use consider the use of continuous measurement to track it in time. Tracking speed is of the essence for feedback purposes, and I will present our new and improved way to speed up the estimation algorithm. In the second case, I will consider the use of Hamiltonian control to estimate a fixed parameter, but of a time-dependent Hamiltonian. In this case established scaling bounds on the quantum Fisher information can be broken, and the quantum control is needed to do it. A simple spin example will be given to illustrate the general results.</p>

<p> </p>

Distribution

$$P_g(x) \sim \exp\left[-\frac{(x-g)^2}{2\sigma^2}\right]$$

$$\sum_{i=1}^N x_i$$

$$\frac{\sum_{i=1}^N x_i}{N} = \hat{g}$$

$$\langle \hat{g} \rangle = g$$

$$\text{var}(\hat{g}) = \frac{\sum_{i,j} \langle x_i x_j \rangle}{N^2} = \frac{\sigma^2}{N}$$

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Distribution

$$P_g(x) \sim \exp\left[-\frac{(x-g)^2}{2\sigma^2}\right]$$

$$\{x_i\}_{i=1, \dots, N}$$

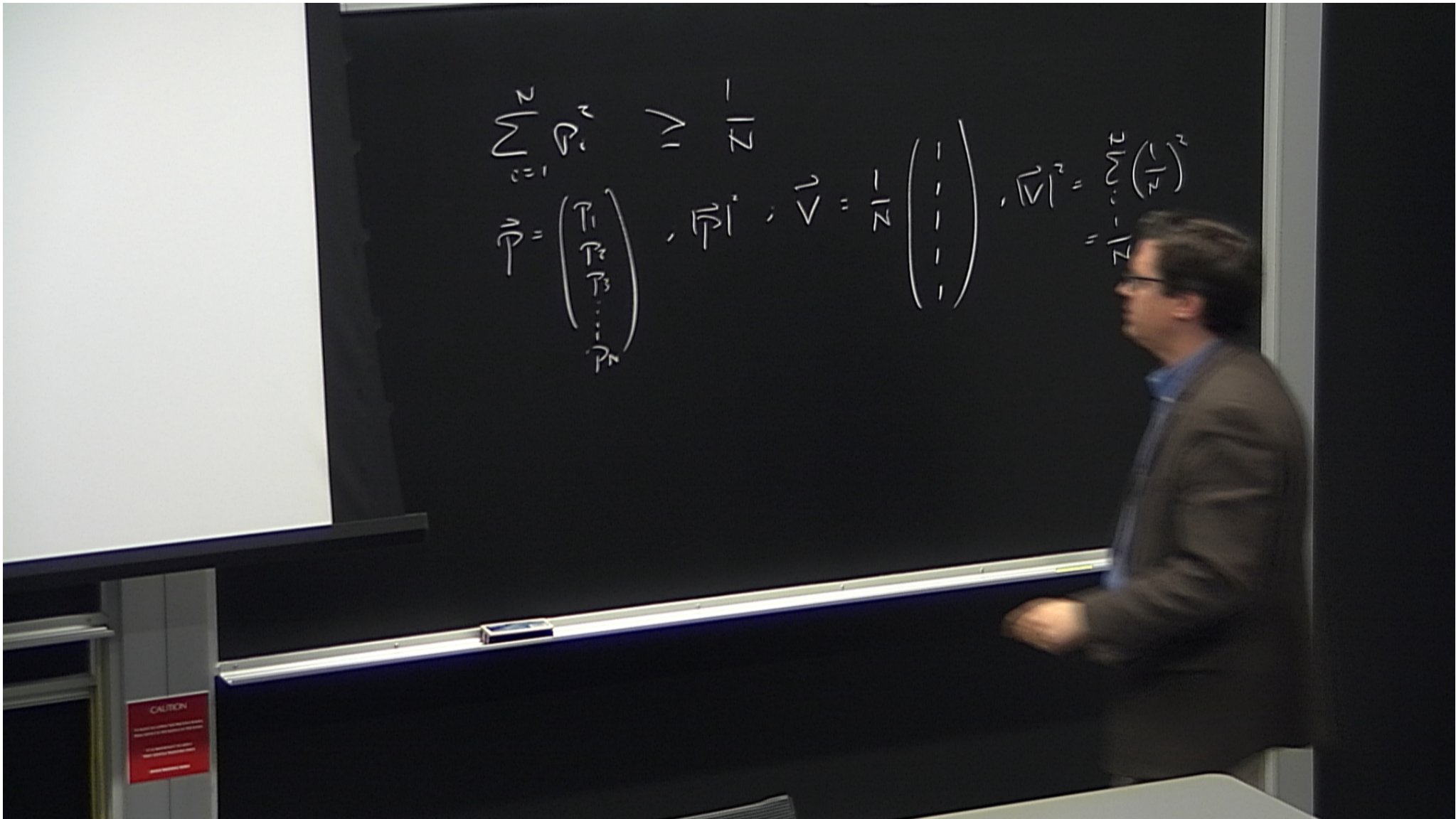
$$\frac{\sum_{i=1}^N x_i}{N} = \hat{g}$$

$$\hat{g}_{new} = \sum_{i=1}^N p_i x_i$$
$$\sum_{i=1}^N p_i = 1$$

$$\text{Var}(\hat{g}_{new}) = \sum_{i=1}^N p_i^2 \sigma^2$$

$$\langle \hat{g} \rangle = g$$

$$\text{Var}(\hat{g}) = \frac{\sum_{i,j} \langle x_i x_j \rangle}{N^2} = \frac{\sigma^2}{N}$$



$$\sum_{i=1}^n p_i^2 \quad \sqrt{-1/2}$$

$$\vec{p} = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ \vdots \\ p_n \end{pmatrix} \quad \vec{p} \cdot \vec{v} = -1/2 \quad \left(\begin{array}{c} - \\ - \\ - \\ - \\ - \end{array} \right)$$

$$\sqrt{|\vec{v}|^2} = \sqrt{\sum_{i=1}^n \left(\frac{-1}{2}\right)^2} = \sqrt{-1/2}$$

$$\vec{p} \cdot \vec{v} \leq |\vec{p}| |\vec{v}|^2$$

-1/2

Triangle

Fisher information

$$\tilde{I}_g = \int dx p(x) \left(\partial_g \ln p(x) \right)^2$$

$$\text{Var } \hat{g} \geq \frac{1}{\tilde{I}_g}$$

Fisher information

$$\tilde{I}_g = \int dx p(x) \left(\partial_g \ln p(x) \right)^2$$

$$\text{Var } \hat{g} \geq \frac{1}{N \tilde{I}_g}$$

$$H = H(g)$$

u_g is unitary

$$| \psi_F(g) \rangle = u_g | \psi_{in} \rangle$$

$$\tilde{I}_g^{(4)} = 4 \left(\langle \partial_g \psi_F | \partial_g \psi_F \rangle - \langle \psi_F | \partial_g \psi_F \rangle^2 \right)$$

$$\hat{H} = g \hat{P} \mapsto u = e^{-igt \hat{P}}$$

$\underline{\phi}(x) = \text{Gaussian (mean } 0, \text{ Variance } \frac{\sigma^2}{2})$

$\underline{\psi} \rightarrow \text{Gaussian (mean } gt, \text{ Variance } \frac{\sigma^2}{2})$

$$\{x_i\} \quad I = \left\langle \left(\frac{\partial}{\partial g} \ln P \right)^2 \right\rangle = \frac{t^2}{\sigma^2}$$

$$\hat{H} = g \hat{P} \rightarrow u = e^{-igt \hat{P}}$$

$$\underline{\phi}(x) = \text{Gaussian (mean } 0, \text{ variance } \frac{\sigma^2}{2})$$

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$$\{x_i\} \quad I = \left\langle \left(\frac{\partial}{\partial g} \ln P \right)^2 \right\rangle = \frac{t^2}{\sigma^2}$$

Momentum $|D=1\rangle$; $u|y\rangle = e^{-igtP} |P\rangle$

Rapid
 $H = g H_0$
 $\hat{T} \sim \text{Var}(\hat{H}_0) | \hat{T}_m$

~~Rapid~~

$$H = g H_0$$

$$\tau \sim \frac{\text{Var}(\hat{H}_0)}{\hbar \omega}$$

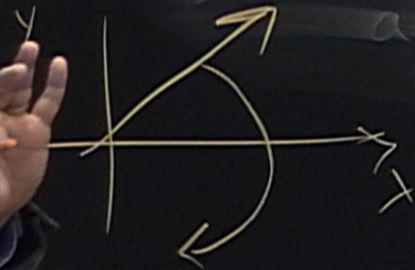
Dynamics of Hamiltonian.

$$H = g H_0$$

$$\Gamma \sim \text{Var}(\hat{H}_0) \Big|_{\psi_{in}}$$

Dynamics of Hamiltonian.

A) Known, well-defined way; missing parameters.



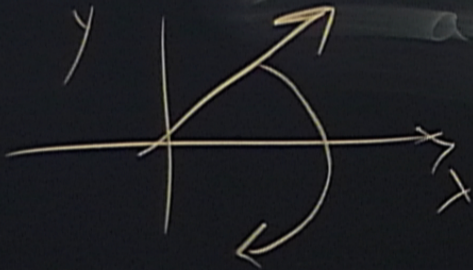
Rapid

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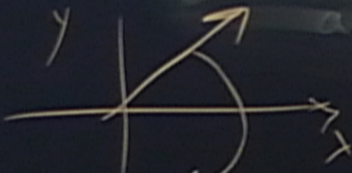
Rapid

$$H = g H_0$$

$$\Gamma \sim \text{Var}(\hat{H}_0) / \tau_{\text{int}}$$

Dynamics of Hamiltonian.

A) Known, well-defined way; missing parameters.



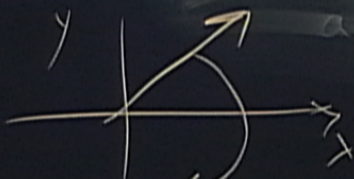
B) Uncontrolled way that cannot be predicted in advance.

$$H = g H_0$$

$$\underline{\Gamma} \sim \text{Var}(\hat{H}_0) \Big|_{\psi_{in}}$$

Dynamics of Hamiltonian.

A) Known, well-defined way; missing parameters.



Solution: coherent control.

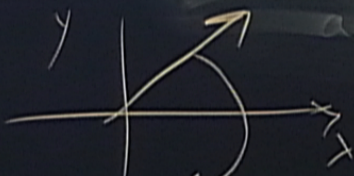
B) Uncontrolled way that cannot be predicted in advance.

$$H = g H_0$$

$$\underline{I} \sim \text{Var}(\hat{H}_0) \Big|_{t_{in}}$$

Dynamics of Hamiltonian.

A) Known, well-defined way; missing parameters.



Solution: Coherent control.

B) Solution: Continuous monitoring.
Uncontrolled way that cannot be predicted in advance.

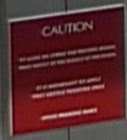
$$H = \mu \vec{B} \cdot \hat{\sigma}$$
$$= \mu B (\hat{n}_x \sigma_x + \hat{n}_y \sigma_y)$$

$B \rightarrow$ given

spin $(\frac{1}{2})$?

wait time T , then measure

$\sim a$



$H = \mu \vec{B} \cdot \hat{n}$ $B \rightarrow$ given

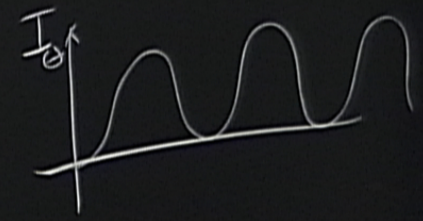
$= \mu B (\hat{n}_x \sigma_x + \hat{n}_y \sigma_y)$

$\hat{n} = \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}$

spin $(\frac{1}{2})$?

wait time T , then measure

$\frac{P_a}{P_i} = 4 \sin^2(\mu B T)$



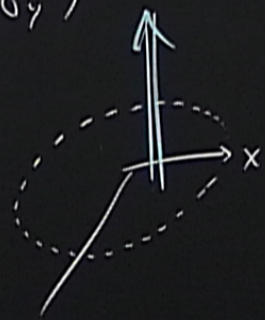
CAUTION
DO NOT TOUCH THE EQUIPMENT
IF IT IS DAMAGED BY YOU
CONTACT THE STAFF

$H = \mu \vec{B} \cdot \hat{n}$ $B \rightarrow$ given

$= \mu B (\hat{n}_x \sigma_x + \hat{n}_y \sigma_y)$

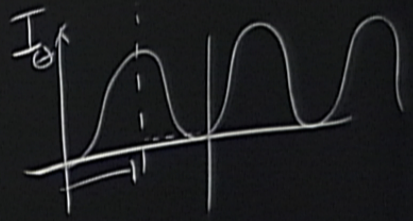
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spin ($\frac{1}{2}$) ?



wait time T , then measure

$\overline{I_a} \sim \overline{I_b} = 4 \sin^2(\mu B T)$



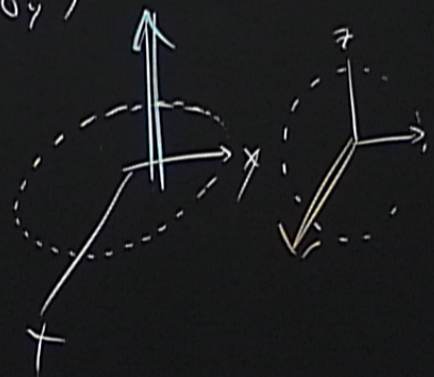
CAUTION
DO NOT TOUCH THE LASER BEAM
OR THE LASER HEAD OR THE MIRROR
IF IT IS NEARLY TO YOU
OR ANYONE ELSE
PLEASE REPORT TO THE
LAB ASSISTANT

$H = \mu \vec{B} \cdot \hat{n}$ $B \rightarrow$ given

$= \mu B (\hat{n}_x \sigma_x + \hat{n}_y \sigma_y)$

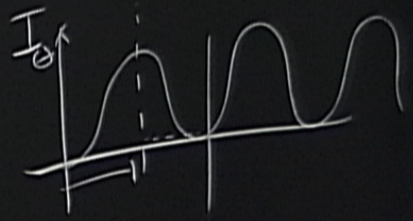
$\hat{n} = \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}$

spin ($\frac{1}{2}$)



wait time T , then measure

$\frac{P_a}{P_b} = 4 \sin^2(\mu B T)$



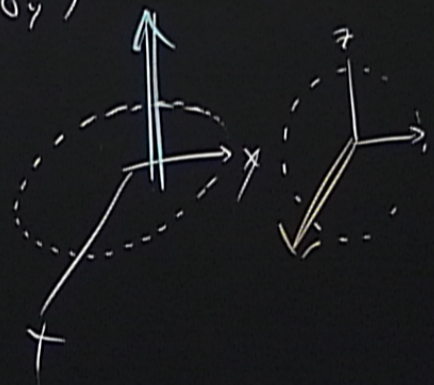
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$H = \mu \vec{B} \cdot \hat{n}$ $B \rightarrow$ given

$= \mu B (\hat{n}_x \sigma_x + \hat{n}_y \sigma_y)$

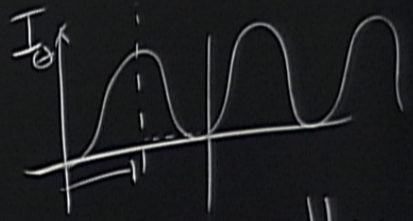
$\hat{n} = \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}$

spin $(\frac{1}{2})$



wait time T , then measure

$\frac{P_a}{P_i} = 4 \sin^2(\mu B T)$



$H = H_{unknown} + H_{ctrl}$

$B_{ctrl} = -B$

$I \sim T^2$

CAUTION
DO NOT TOUCH THE EQUIPMENT WHILE THE SYSTEM IS RUNNING OR THE RESULT OF THE EXPERIMENT WILL BE INVALID.
IT IS RESPONSIBLE BY YOUR OWNERSHIP TO SAFELY OPERATE THE EQUIPMENT.

in advance.

- Accuracy
- Speed.

$$H(t)$$

$$\psi(t) = U |\psi(0)\rangle$$

$$U = \tilde{U} e^{-\frac{i}{\hbar} \int_0^t H(t') dt'}$$

$$H = H(g)$$

U_g is unitary

$$|\psi_f(g)\rangle = U_g |\psi_i\rangle$$

$$\hat{I}_g^{(S)} = 4 \left(\begin{array}{c} \langle \psi_f | \psi_i \rangle \\ \langle \psi_f | \psi_f \rangle \\ \langle \psi_i | \psi_i \rangle \end{array} \right)$$

in advance.

- Accuracy
- Speed.

$H(t)$

$$|\psi(t)\rangle = U |\psi(0)\rangle$$

$$U = \tilde{U} e^{-\frac{i}{\hbar} \int_0^T H(t') dt'}$$

$$\tilde{U} = \frac{1}{\hbar} \text{Var}[h_j(t)]_{t_0}$$

$$h_j = \int_0^T dt U_j^\dagger(0 \rightarrow t) \partial_j H_j(t) U_j(0 \rightarrow t)$$

$H = H(t)$

U_j is unitary

$$|\psi_f(t)\rangle = U_j |\psi_{in}\rangle$$

$$\tilde{I}_j^{(S)} = 4 \left(\langle \partial_j \psi_f | \partial_j \psi_f \rangle - \langle \psi_f | \partial_j \psi_f \rangle^2 \right)$$

in advance.

- Accuracy
- Speed.

$H(t)$

$$|\psi(t)\rangle = U |\psi(0)\rangle$$

$$U \approx 1 - \frac{i}{\hbar} \int_0^T H(t') dt'$$

$$\bar{I} = 4 \text{Var}[h_j(\tau)]_{t_0}$$

$$h_j = \int_0^T dt U_j^\dagger(0 \rightarrow t) \partial_j H_j(t) U_j(0 \rightarrow t)$$

$H = H(t)$

U_j is unitary

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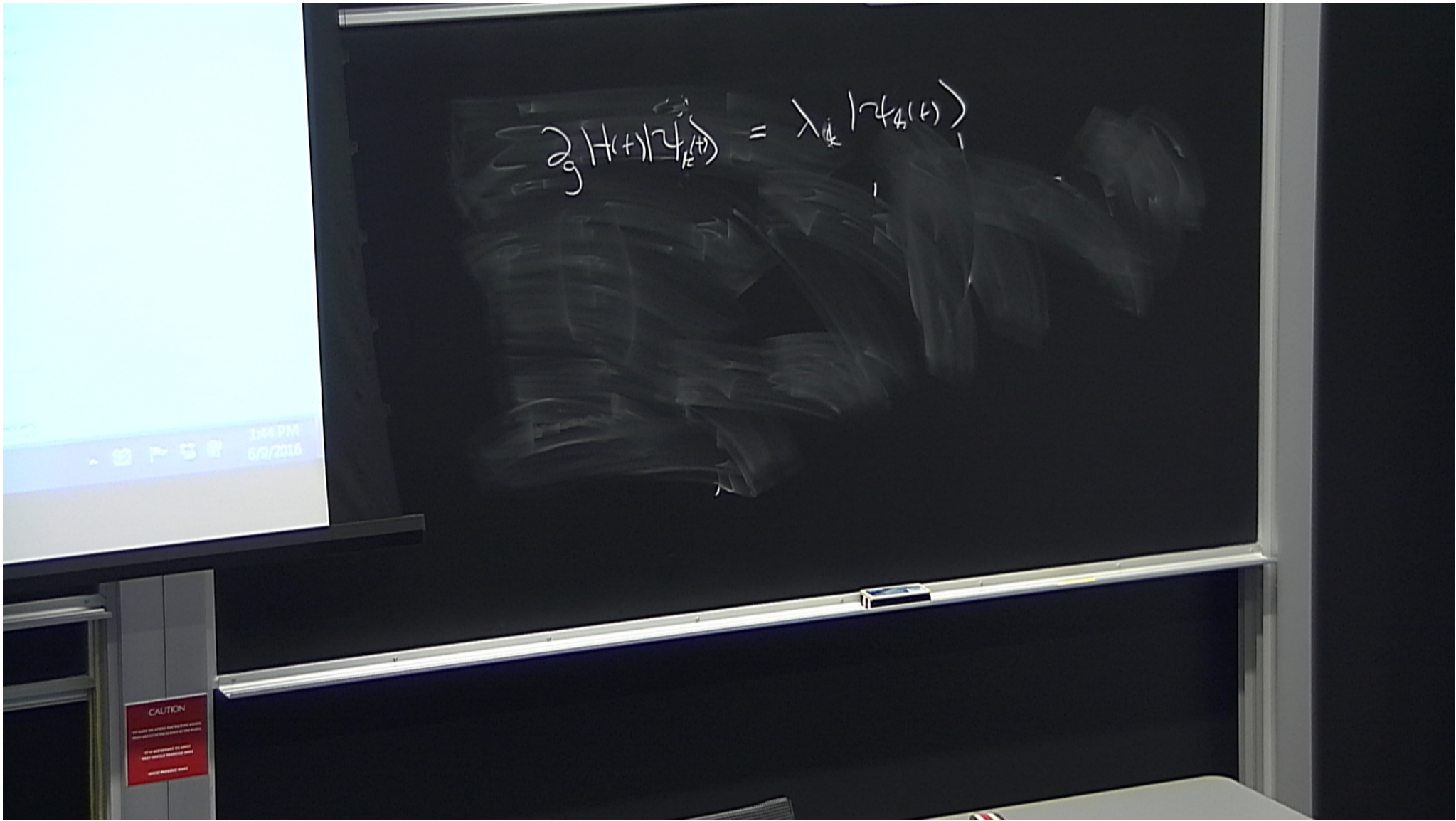
$$h_j = \int_0^T dt U_j^\dagger(0 \rightarrow t) \left[\frac{\partial}{\partial \theta_j} H_j(t) \right] U_j(0 \rightarrow t)$$

$$H = H(\theta)$$

U_θ is unitary

$$|\psi_f(\theta)\rangle = U_\theta |\psi_{in}\rangle$$

$$\bar{I}_\theta = 4 \left(\langle \frac{\partial \psi_f}{\partial \theta_j} | \frac{\partial \psi_f}{\partial \theta_j} \rangle - \langle \psi_f | \frac{\partial \psi_f}{\partial \theta_j} \rangle^2 \right)$$



to $|\psi_k(t + \Delta t)\rangle$ by $U_g(t \rightarrow t + \Delta t)$. When Δt is sufficiently small, both $U_g(t \rightarrow t + \Delta t)$ and $|\psi_k(t)\rangle$ can be expanded to the first order of Δt , and the Hamiltonian $H_{\text{tot}}(t)$ is approximately constant within Δt , then we can find $H_{\text{tot}}(t)|\psi_k(t)\rangle = i|\partial_t \psi_k(t)\rangle$ by equating $U_g(t \rightarrow t + \Delta t)|\psi_k(t)\rangle$ with $|\psi_k(t + \Delta t)\rangle$. This has the form of a Schrödinger equation. But unlike the usual situations where we know the Hamiltonian and must find the solution to the state, here we know the solution of the state, $|\psi_k(t)\rangle$, and need to find the appropriate Hamiltonian $H_{\text{tot}}(t)$ instead. A direct solution to this equation is obviously $H_{\text{tot}}(t) = i \sum_k |\partial_t \psi_k(t)\rangle \langle \psi_k(t)|$. (Note this solution is Hermitian because $\sum_k |\partial_t \psi_k(t)\rangle \langle \psi_k(t)|$ is skew-Hermitian.) Considering every eigenstate $|\psi_k(t)\rangle$ satisfies the $U(1)$ symmetry (i.e., multiplying $|\psi_k(t)\rangle$ by an arbitrary phase $e^{-i\theta_k(t)}$ does not change that state), $H_{\text{tot}}(t)$ can be generalized to include an additional term $\sum_k \dot{\theta}_k(t) |\psi_k(t)\rangle \langle \psi_k(t)|$. $\dot{\theta}_k(t)$ can be replaced by arbitrary real functions $f_k(t)$, and $\theta_k(t) = \int_0^t f_k(t') dt'$. Thus, the control Hamiltonian $H_c(t)$ finally turns out to be

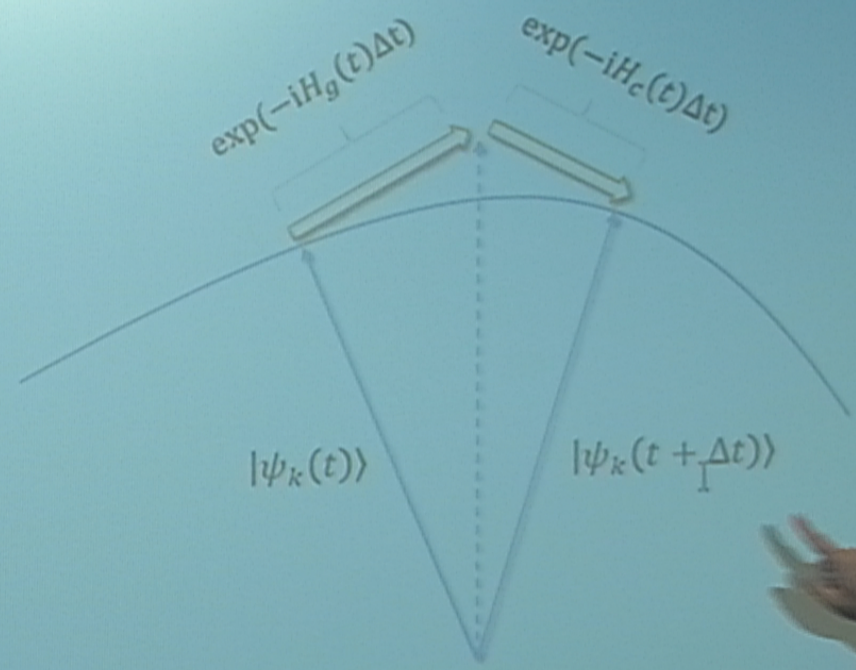


Figure 2. Optimal Hamiltonian control scheme to achieve the maximum Fisher information. The evolution of the system under the total Hamiltonian $H_g(t) + H_c(t)$ for a short time Δt can be approximated as $\exp(-iH_c(t)\Delta t) \exp(-iH_g(t)\Delta t)$.

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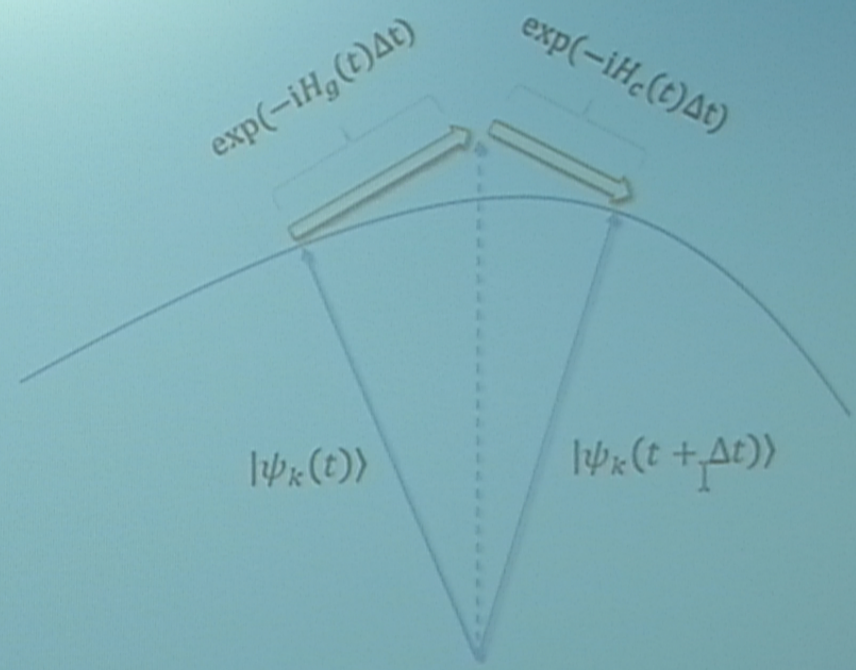
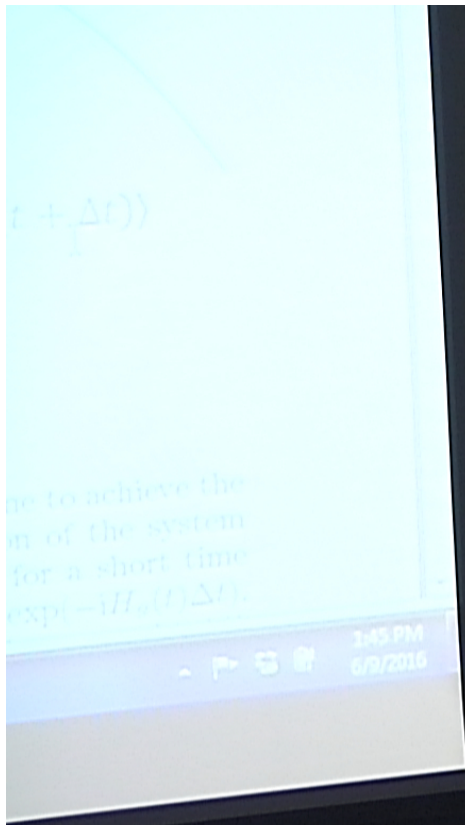


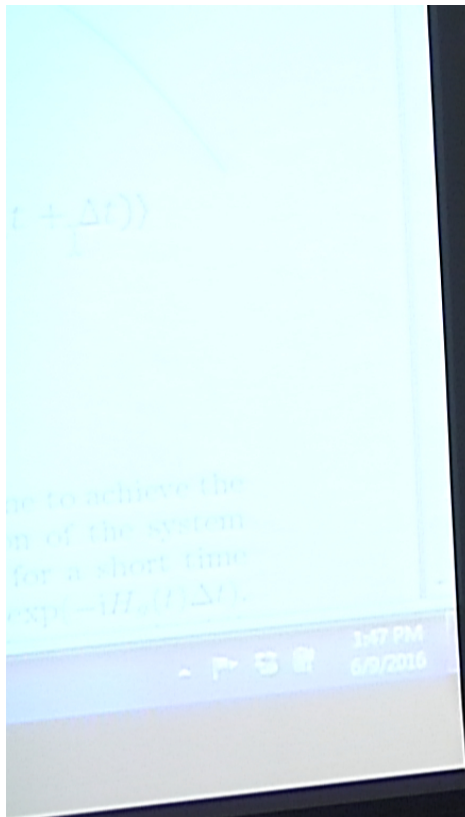
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$2g |H(+)|^2 \Delta t = \lambda_{\pm} |r_{\pm}(t)|$

frequency ω

$\vec{B} = B \begin{pmatrix} \cos \omega t \\ \sin \omega t \end{pmatrix}$

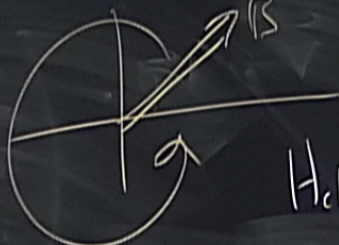


$2g |H(+)| \gamma_k(t) = \lambda_k | \gamma_k(t) |$

$H_\omega = -B (\cos \omega t \hat{G}_x + \sin \omega t \hat{G}_z)$

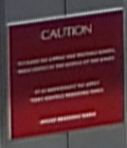
frequency ω

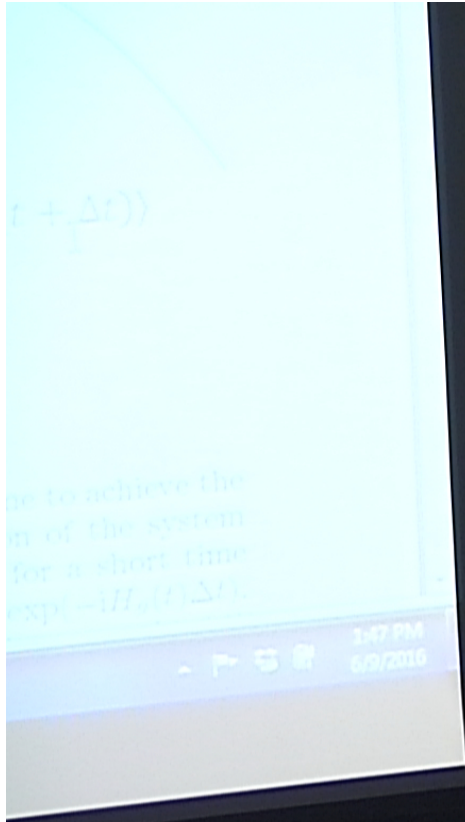
$\vec{B} = B \begin{pmatrix} \cos \omega t \\ \sin \omega t \end{pmatrix}$



$H_{\text{total}} = -H_\omega - \frac{\omega}{2} \hat{G}_y$

$\Gamma_\omega = B^2 T^4$





$\omega = \omega?$

$2g |H(+)|^2 |H(-)|^2 = \lambda_4 |r_4(t)|$

$H_\omega = -B (\cos \omega t \hat{G}_x + \sin \omega t \hat{G}_z)$

frequency ω $\vec{B} = B \begin{pmatrix} \cos \omega t \\ \sin \omega t \end{pmatrix}$

$H_{ctrl} = -H_\omega - \frac{\omega}{2} \hat{G}_y$

$\Gamma_\omega = B^2 T^4$

The quantum Fisher information $I_B^{(Q)}$ is plotted for different rotation frequencies ω without the control Hamiltonian and compared to that with the control Hamiltonian $H_c(t)$ in Fig. 3. It can be observed that when ω is large compared to the amplitude of the magnetic field B , the Fisher information becomes small. And the Fisher information with the optimal Hamiltonian control is the highest, which verifies the advantage of Hamiltonian control in increasing Fisher information.

Estimation of field rotation frequency. Now, we turn to estimating the rotation frequency ω of the magnetic field. Since $\partial_\omega H(t)$ is $tB(\sin\omega t\sigma_X - \cos\omega t\sigma_Z)$, the eigenvalues of $\partial_\omega H(t)$ are $\mu(t) = \pm tB$, and the maximum and minimum eigenvalues of $h_\omega(T)$ are therefore $\int \mu(t)dt = \pm \frac{1}{2}BT^2$. Therefore, the maximum Fisher information of estimating ω is

$$I_\omega^{(Q)} = B^2 T^4. \quad (14)$$

The eigenstates of $\partial_\omega H(t)$ are $|\psi_+(t)\rangle = \sin \frac{\omega t}{2}|0\rangle + \cos \frac{\omega t}{2}|1\rangle$ and $|\psi_-(t)\rangle = \cos \frac{\omega t}{2}|0\rangle - \sin \frac{\omega t}{2}|1\rangle$. If we choose $f_c(t) = 0$ in the control Hamiltonian, then

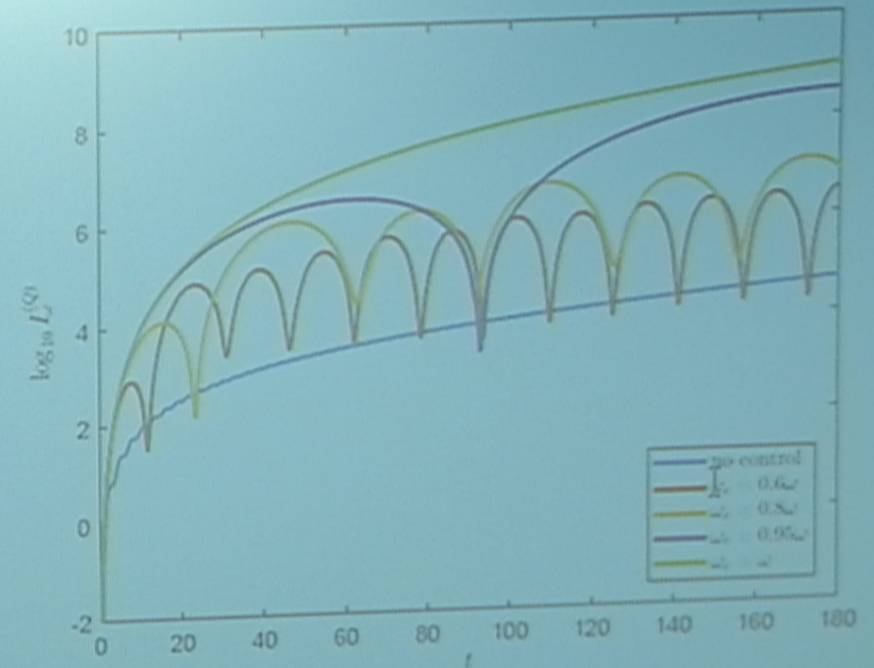
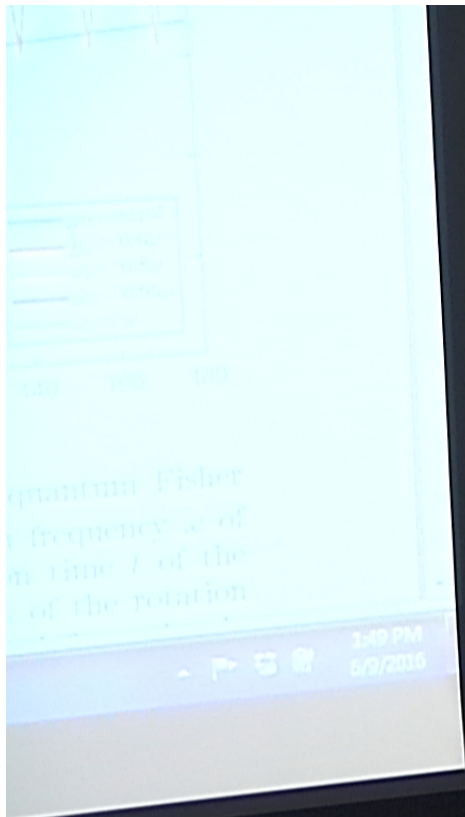
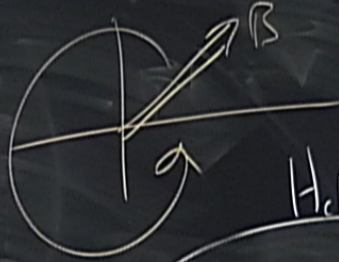


Figure 4. The logarithm (base 10) of the quantum Fisher information $I_\omega^{(Q)}$ for estimating the rotation frequency ω of the magnetic field $B(t)$ versus the evolution time t of the qubit is plotted for different trial values ω_c of the rotation



$$2g |H(+)\rangle | \chi_{\pm}(t) \rangle = \chi_{\pm} | \chi_{\pm}(t) \rangle$$



frequency ω

$\omega = \omega?$

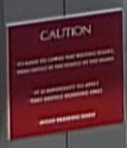
$$H_{\omega} = -B (\cos \omega t \hat{\sigma}_x + \sin \omega t \hat{\sigma}_z)$$

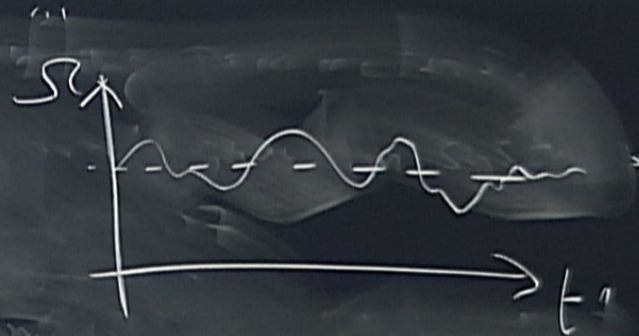
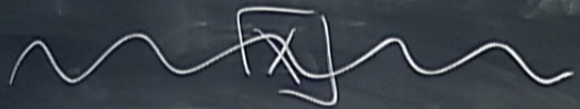
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$$H_{\text{total}} = -H_{\omega} - \frac{\omega}{2} \hat{\sigma}_y$$

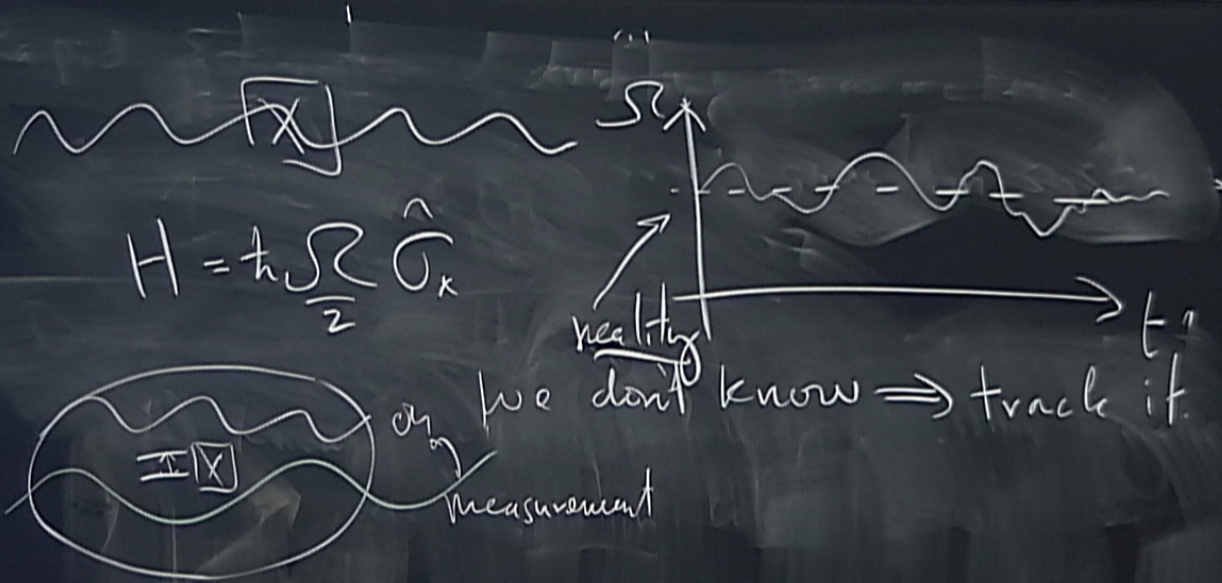
$$\Gamma_{\omega} = B^2 T^4$$

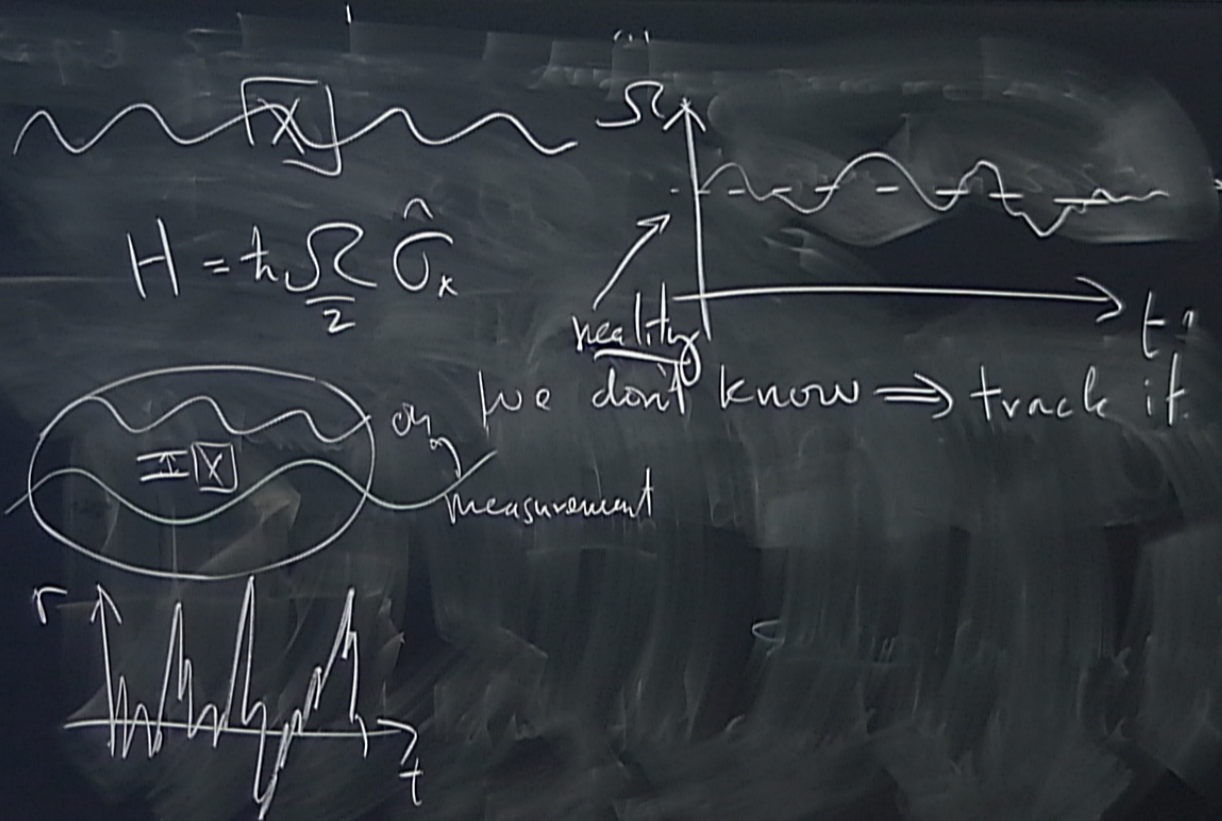
Basis scaling of time-independent Hamiltonians

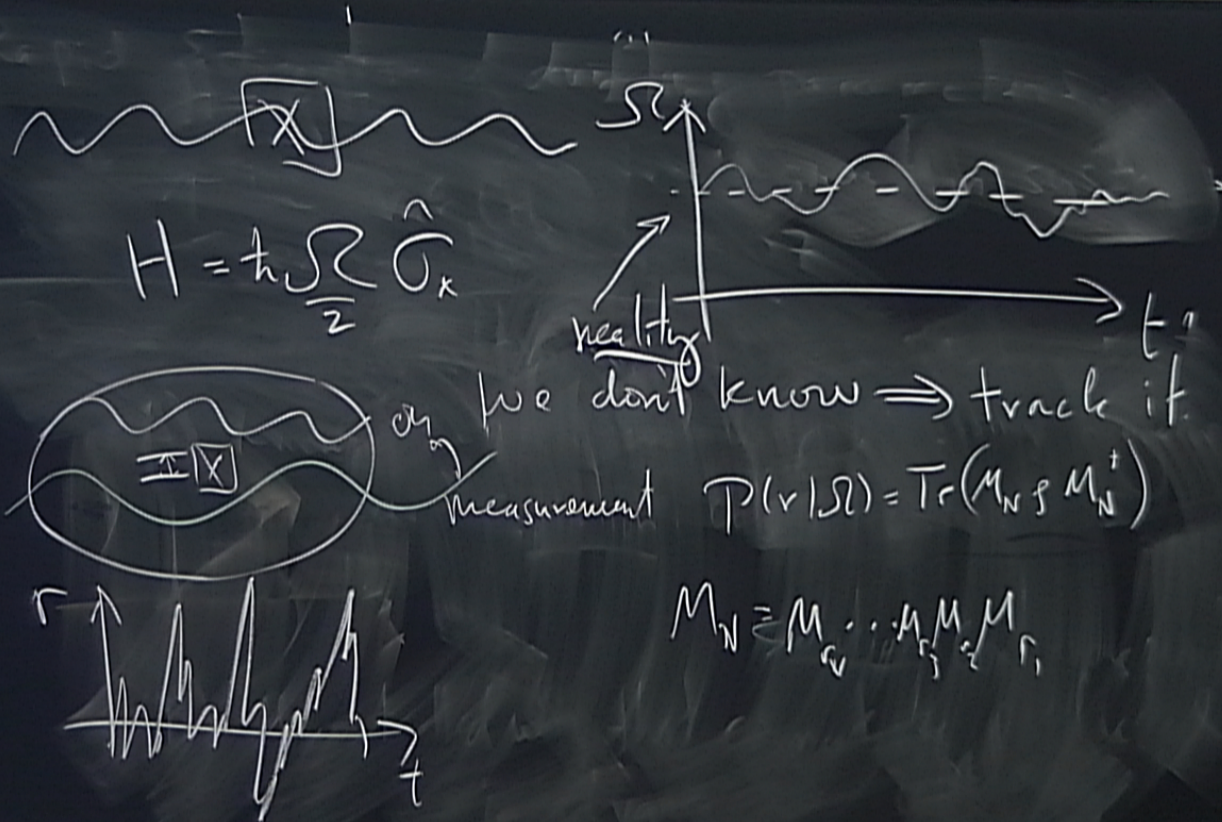


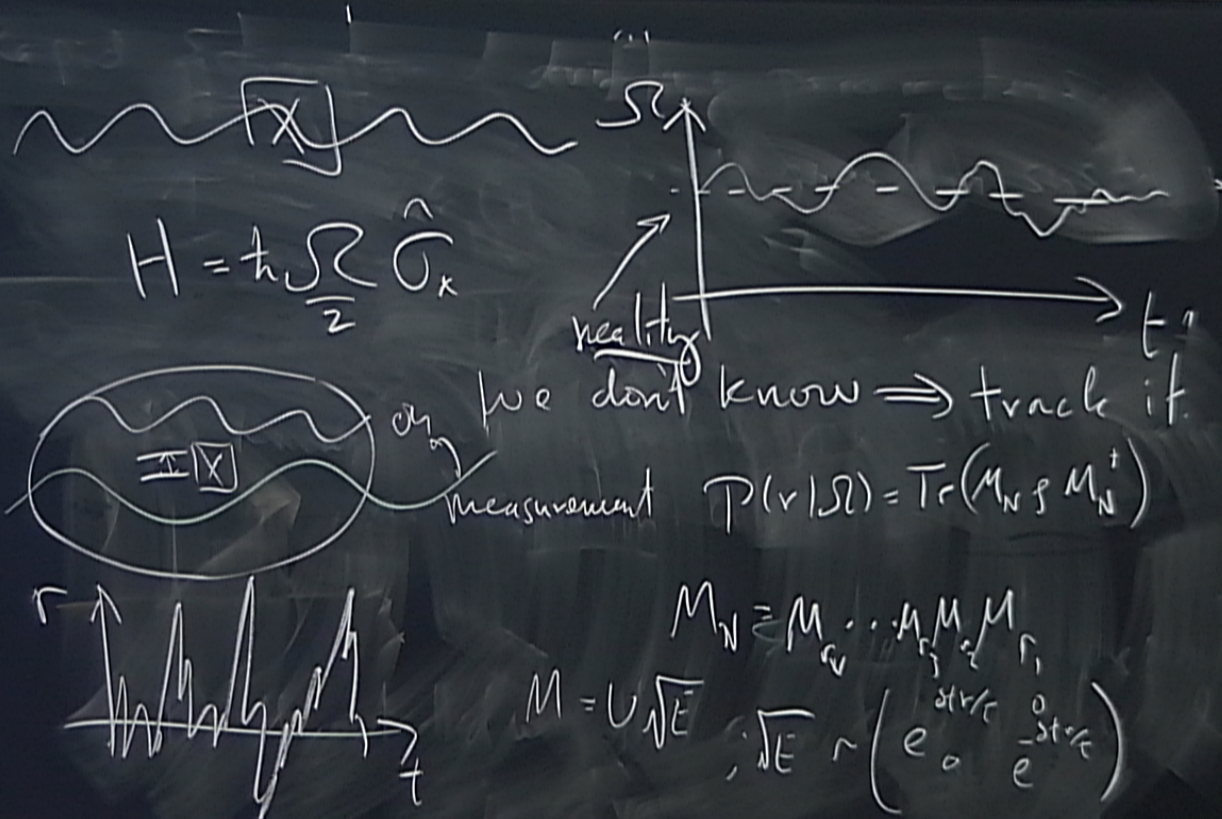


$$H = \hbar \int \frac{\hat{O}_x}{2}$$











That is, given an observed data set $\{y_i\}_{i=1}^N$ from an experiment, we are interested in finding the maximum likelihood estimate $\hat{\theta}_{ML}$ such that the likelihood function $L(\theta)$ is maximized. Equivalently, it is not necessary to maximize the likelihood since it is not necessary to compute the constant term in $L(\theta)$ (e.g., by using Eq. (1)) or writing it as a product of matrix exponentials, since we are only concerned with maximizing the parameter θ .

Consequently, only the functional form of the maximum likelihood estimate is needed, and only the non-constant part of each $L_i(\theta)$ will be relevant for the MLE optimization. As such, the non-constant likelihood function may be conveniently abstracted as a measurement y_i which follows an unknown measurement operator $H_i(\theta)$ using an unobserved parameter θ_i that defines the measurement process:

$$y_i = H_i(\theta) \theta_i \quad (10)$$

The maximum likelihood algorithm will later help numerical algorithms avoid problems of very small numbers.

The Fisher information about the parameter θ to be estimated is computed directly from the log-likelihood for all possible realizations:

$$F(\theta) = -\mathbb{E} \left[\frac{\partial^2 \ln L(\theta)}{\partial \theta^2} \right]$$

FIG. 1. The likelihood $L(\theta)$ is a function of possible field frequencies θ . A long noise Z measurement record $\{y_i\}_{i=1}^N$ of duration $T = 100$ s was simulated with $N = 10^7$ discrete time steps $\Delta t = 10$ ns. A characteristic measurement time τ_m is indicated a true field oscillation frequency $\theta = 1000$ Hz (green curve). The maximum likelihood estimate (red dashed) is accurate to ± 0.1 Hz ($\approx 10^{-5}$ relative error). The precision σ_{θ} is 0.001 Hz.

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$$\hat{I}(t) = \text{Gaussian (mean } g, \text{ variance } \sigma_g^2)$$

$$\hat{g} \rightarrow \text{Gaussian (mean } \hat{g}, \text{ variance } \sigma_g^2)$$

$$F(\theta) = \mathbb{E} \left[\left(\frac{\partial \ln L(\theta)}{\partial \theta} \right)^2 \right] = \frac{1}{\sigma^2}$$

$$\text{Minimum } \frac{1}{\sigma^2} \rightarrow \sigma = \frac{1}{\sqrt{F(\theta)}}$$

$$H = \hat{g} \hat{P} \hat{G}$$

$$P = \frac{1}{2} (\hat{g} + \hat{g}^\dagger)$$

$$G = \frac{1}{2} (\hat{g} - \hat{g}^\dagger)$$

$$H = \frac{1}{2} (\hat{g} + \hat{g}^\dagger) \hat{P} \frac{1}{2} (\hat{g} - \hat{g}^\dagger)$$

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$$\omega = \omega'$$

$$2 \langle H_{11} \rangle = \lambda_k \langle \hat{g} \rangle$$

$$H_{11} = -\frac{1}{2} (\hat{g} + \hat{g}^\dagger)$$

$$I_{\omega} = B^T T^V$$

Partial scaling of trace-independent Hamiltonians



$P_{\text{prior}}(\Omega)$ may be computed either from the RMS error, or from the uncertainty of the previous measurement, plus the typical expected drift uncertainty from the time-varying parameter. In these cases we can improve MLE by incorporating this information.

The maximum, *a posteriori*, probability of the parameter, taking into account the prior is given by Bayes rule, given the measurement record $r(t)$,

$$P(\Omega|r(t)) = \frac{P(r(t)|\Omega)P_{\text{prior}}(\Omega)}{P(r(t))} \propto P(r(t)|\Omega)P_{\text{prior}}(\Omega). \quad (24)$$

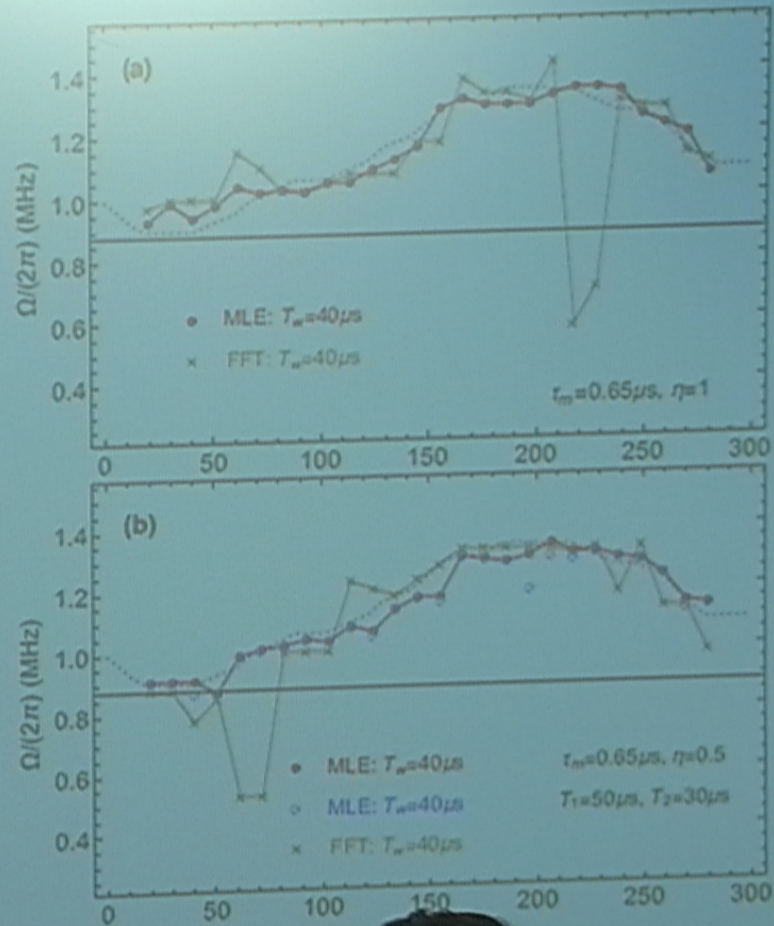
We may then modify our log-likelihood function as

$$\mathcal{L} \propto \ln P(r(t)|\Omega) + \ln P_{\text{prior}}(\Omega), \quad (25)$$

where the first term is our previous log-likelihood, and the second term takes into account the information from the prior experiments. Note that the denominator in Bayes rule can be dropped given that it is independent of Ω , the quantity over which we are maximizing.

B. Time-dependent frequency tracking

We now illustrate the method of time-dependent parameter tracking of a slowly drifting Rabi frequency. We



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