Title: The physical meaning of Tsirelsonâ€™s bound

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Abstract:

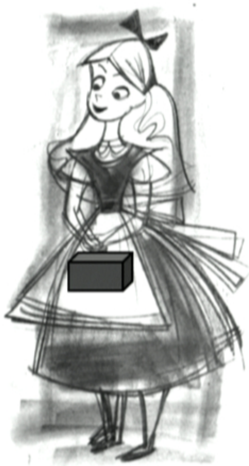
Bell-CHSH inequality:

$$|C_L(a,b)+C_L(a,b')+C_L(a',b)-C_L(a',b')| \leq 2$$

for any "local" correlations $C_L(a,b)$ etc.
(Measurements of a, a', b and b' yield ± 1.)

Bob
measures
b or b'

J. S. Bell, *Physics* **1**, 195 (1964);
J. F. Clauser, M. A. Horne, A. Shimony, and
R. A. Holt, *Phys. Rev. Lett.* **23**, 880 (1969)



Alice
measures
a or a'

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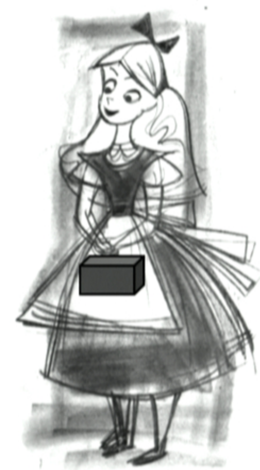
Tsirelson's bound:

$$|C_Q(a,b)+C_Q(a,b')+C_Q(a',b)-C_Q(a',b')| \leq 2\sqrt{2}$$

Bob measures b or b'

B. S. Tsirelson, *Lett. Math. Phys.* **4**, 93 (1980)



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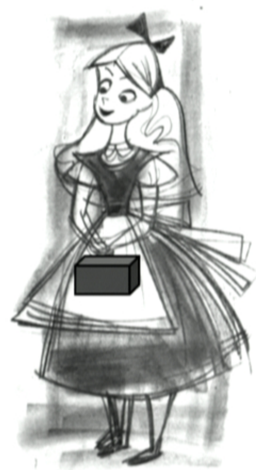
Tsirelson's bound

$$|C_Q(a,b)+C_Q(a,b')+C_Q(a',b)-C_Q(a',b')| \leq 2\sqrt{2}$$

is a *theorem* of quantum mechanics.

Bob measures b or b'

B. S. Tsirelson, *Lett. Math. Phys.* **4**, 93 (1980)



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drawings by Tom Oreb © Walt Disney Co.

“Maximally nonlocal” or “PR-box”correlations:

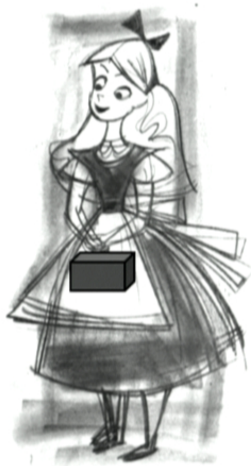
$$|C_{PR}(a,b)+C_{PR}(a,b')+C_{PR}(a',b)-C_{PR}(a',b')| \leq 4$$

•Take $C_{PR}(a,b) = C_{PR}(a,b') = C_{PR}(a',b) = 1$ and $C_{PR}(a',b') = -1$.

•For any measurement of a, a', b, and b', outcomes ± 1 are equally likely.

Bob measures b or b'

S. Popescu and D. Rohrlich, *Found. Phys.* **24**, 379 (1994)



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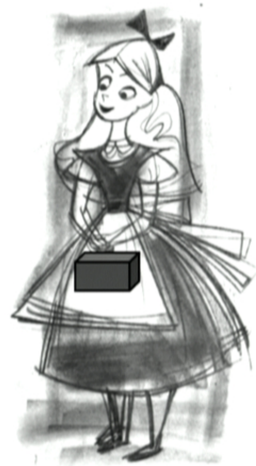
So why aren’t quantum correlations *more* nonlocal than they are?



Alice measures a or a'

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PR-box correlations:

Suppose Alice measures a.

She knows that $b = b'$.

Suppose Alice measures a'.

She knows that $b = -b'$.

Alice can even prepare an ensemble (e.g. by measuring a and postselecting $a = 1$) in which $b = 1 = b'$ and $\Delta b = 0 = \Delta b'$.

PR-box correlations:

All that stops Alice from signalling to Bob is *complementarity* between Bob's measuring b and his measuring b' – Bob cannot measure both – even though (from Alice's point of view) no uncertainty principle governs b and b'.

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Complementarity is a  for the PR box.

But quantum mechanics has a classical limit. In this limit there are no non-commuting observables; there are only jointly measurable macroscopic observables. This classical limit – our direct experience – is an inherent constraint, a kind of boundary condition, on quantum mechanics and on any generalization of quantum mechanics. Thus stronger-than-quantum correlations, too, must have a classical limit.

But quantum mechanics has a classical limit. In this limit there are no non-commuting observables; there are only jointly measurable macroscopic observables. This classical limit – our direct experience – is an inherent constraint, a kind of boundary condition, on quantum mechanics and on any generalization of quantum mechanics. Thus stronger-than-quantum correlations, too, must have a classical limit.

And now begins the fun…!

PR-box correlations in the classical limit:

Now suppose Alice measures just a or just a' on N pairs.

Define macroscopic observables B and B':

$$B = \frac{b_1 + b_2 + \ldots + b_N}{N} \quad , \quad B' = \frac{b_1' + b_2' + \ldots + b_N'}{N} \quad .$$

There must be "weak" measurements that Bob can make to obtain partial information about *both* B and B', because *there is no complementarity in the classical limit!* On average both B and B' vanish, but if Alice measures a, B and B' will of order $1/\sqrt{N}$ and *correlated*; if she measures a', B and B' will of order $1/\sqrt{N}$ and *anti-correlated*.

Ultimately, Alice will be able to signal to Bob by consistently measuring a or a'. What matters is only that when Bob detects a correlation, it is more likely that Alice measured a than when he detects an anti-correlation. If not, Bob's measurements yield zero information about B or about B', contradicting the axiom of a classical limit in which B and B' are jointly measurable.

Alice and Bob can measure exponentially many pairs (in groups of N). Their expenses and exertions don't concern us. For example, if Alice measures a consistently, then the probability for Bob to obtain $B = 1$ is 2^{-N}. But the probability for Bob to obtain $B = 1$ *and* $B' = 1$ is *also* 2^{-N} and *not* 2^{-2N}. The probability for Bob to obtain $B = 1$ *and* $B' = -1$ vanishes.

Conclusion (interim):

The requirement of a classical limit is a *natural* and *minimal* axiom that, together with relativistic causality, rules out PR-box correlations.

D. R., PR-box correlations have no classical limit, in *Quantum Theory: A Two-Time Success Story* [Yakir Aharonov Festschrift], eds. D. C. Struppa and J. M. Tollaksen (Milan: Springer), 2013, pp. 205-211..

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Alternative proof:

If Alice measures a consistently, then $b - b' = 0$ identically, and $b + b'$ is distributed binomially. If she measures a' consistently, then $b + b' = 0$ identically, and $b - b'$ is distributed binomially.

Thus Bob can detect Alice's signal by measuring the variances

$[\Delta(B + B')]^2$ and $[\Delta(B - B')]^2$. Since $\langle B \rangle = 0 = \langle B' \rangle$, we have

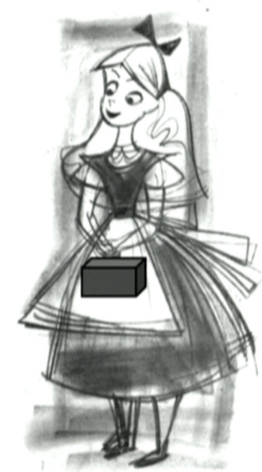
$[\Delta(B + B')]^2 = \langle (B + B')^2 \rangle$ and $[\Delta(B - B')]^2 = \langle (B - B')^2 \rangle$.

“Stronger than quantum” correlations:

$$2\sqrt{2} < C_{SQ}(a,b)+C_{SQ}(a,b')+C_{SQ}(a',b)-C_{SQ}(a',b')| \leq 4$$

•For any measurement of a, a', b, and b', outcomes ± 1 are equally likely.

Bob measures b or b'



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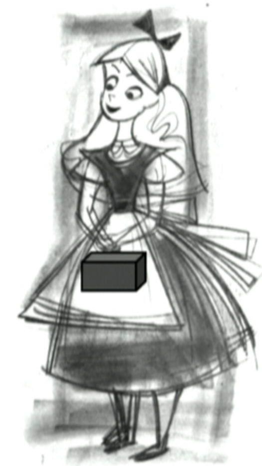

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drawings by Tom Oreb © Walt Disney Co.

Relativistic causality imposes a constraint:

If Alice measures a consistently, then the standard deviation in $B + B'$ that Bob observes is $\Delta_a(B + B')$.

If Alice measures a' consistently, then the standard deviation in $B + B'$ that Bob observes is $\Delta_{a'}(B + B')$.

Bob must not be able to detect what Alice measures, hence relativistic causality implies that

$$\Delta_a(B + B') = \Delta_{a'}(B + B') \ .$$

But how do we calculate $\Delta_a(B \pm B')$ and $\Delta_{a'}(B \pm B')$?

Whatever Alice measures, we have

$$\left\langle (B+B')^2 \right\rangle + \left\langle (B-B')^2 \right\rangle = 2\left\langle B^2 \right\rangle + 2\left\langle (B')^2 \right\rangle = \frac{4}{N} \quad ,$$

since

$$\left\langle B^2 \right\rangle = \frac{\left\langle b_1{}^2 \right\rangle + \ldots + \left\langle b_N{}^2 \right\rangle}{N^2} = \frac{1}{N} = \frac{\left\langle (b_1{}')^2 \right\rangle + \ldots + \left\langle (b_N{}')^2 \right\rangle}{N^2} = \left\langle (B')^2 \right\rangle \quad .$$

Therefore $[\Delta_a(B + B')]^2 = [\Delta_{a'}(B + B')]^2 = \dfrac{4}{N} - [\Delta_{a'}(B - B')]^2$,

i.e. $[\Delta_a(B + B')]^2 + [\Delta_{a'}(B - B')]^2 = \dfrac{4}{N}$.

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Let us now define $A = (a_1 + a_2 + \ldots + a_N)/N$ and compare the distribution of A as measured by Alice with the distribution of $B + B'$ as measured by Bob.

The distribution of A is a simple binomial. The possible values of A are 1, $1–2/N,\ldots, 1–2n/N,\ldots, –1+2/N$, –1 with probabilities $N!/2^N n!(N–n)!$ respectively.

In contrast, we can only estimate the values of $B + B'$: for example, when Alice obtains 1, then Bob obtains approximately the value $B + B' = C_{SQ}(a,b) + C_{SQ}(a,b')$. In general, if Alice obtains $1–2n/N$, then Bob obtains

$$B + B' \approx (1–2n/N)[C_{SQ}(a,b) + C_{SQ}(a,b')]$$

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The fact that

$$B + B' \approx (1-2n/N)[C_{SQ}(a,b) + C_{SQ}(a,b')]$$

(with ≈ and not =) means that we can only write

$$\Delta_a(B + B') \geq [C_{SQ}(a,b) + C_{SQ}(a,b')]\ \Delta A$$
$$= [C_{SQ}(a,b) + C_{SQ}(a,b')]/\sqrt{N} \quad .$$

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$$\begin{aligned}\Delta_a(B + B') &\geq [C_{SQ}(a,b) + C_{SQ}(a,b')]\,\Delta A \\ &= [C_{SQ}(a,b) + C_{SQ}(a,b')]/\sqrt{N} \quad ,\end{aligned}$$

and similarly

$$\begin{aligned}\Delta_{a'}(B - B') &\geq [C_{SQ}(a',b) - C_{SQ}(a',b')]\,\Delta A' \\ &= [C_{SQ}(a',b) - C_{SQ}(a',b')]/\sqrt{N} \quad .\end{aligned}$$

Remember

$$[\Delta_a(B + B')]^2 + [\Delta_{a'}(B - B')]^2 = 4/N \quad ,$$

hence, given

$$[\Delta_a(B + B')]^2 \geq [C_{SQ}(a,b) + C_{SQ}(a,b')]^2/N$$

and

$$[\Delta_{a'}(B - B')]^2 \geq [C_{SQ}(a',b) - C_{SQ}(a',b')]^2/N \quad ,$$

we get

$$4 \geq [C_{SQ}(a,b) + C_{SQ}(a,b')]^2 + [C_{SQ}(a',b) - C_{SQ}(a',b')]^2 \quad .$$

But how do we calculate $\Delta_a(B \pm B')$ and $\Delta_{a'}(B \pm B')$?

Whatever Alice measures, we have

$$\left\langle (B+B')^2 \right\rangle + \left\langle (B-B')^2 \right\rangle = 2\left\langle B^2 \right\rangle + 2\left\langle (B')^2 \right\rangle = \frac{4}{N} \quad ,$$

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Remember

$$[\Delta_a(B + B')]^2 + [\Delta_{a'}(B - B')]^2 = 4/N \quad ,$$

hence, given

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Applying the inequality

$$(x^2 + y^2)^{1/2} \geq |x + y| \,/\sqrt{2}$$

to

$$4 \geq [C_{SQ}(a,b) + C_{SQ}(a,b')]^2 + [C_{SQ}(a',b) - C_{SQ}(a',b')]^2 \quad ,$$

we finally obtain Tsirelson's bound:

$$2\sqrt{2} \geq |C_{SQ}(a,b) + C_{SQ}(a,b') + C_{SQ}(a',b) - C_{SQ}(a',b')| \quad .$$

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we finally obtain Tsirelson's bound

$$2\sqrt{2} \geq |C_{SQ}(a,b) + C_{SQ}(a,b') + C_{SQ}(a',b) - C_{SQ}(a',b')| \quad ,$$

as a consequence of relativistic causality in the classical limit.

Conclusion:

The requirement of a classical limit is a *natural* and *minimal* axiom that, together with relativistic causality, rules out PR-box correlations.

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Further analysis of these two axioms yields a theorem of quantum mechanics: Tsirelson's bound. It also points to the Hilbert-space structure of quantum mechanics.

D.R., Stronger-than-quantum bipartite correlations violate relativistic causality in the classical limit, arXiv:1408.3125.