Title: Primitives of Nonclassicality in the N-qubit Pauli Group

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Abstract: <p>Throughout the development of quantum mechanics, the striking refusal of nature to obey classical reasoning and intuition has driven both curiosity and confusion. From the apparent inescapably probabilistic nature of the theory, to more subtle issues such as entanglement, nonlocality, and contextuality, it has always been the `nonclassicalâ€™ features that present the most interesting puzzle. More recently, it has become apparent that these features are also the primary resource for quantum information processing.
In this talk, we will introduce several types of nonclassical logical structures contained within the N-qubit Pauli group, corresponding in general to preparation and/or measurement schemes for systems of several qubits that demonstrate violation of some notion of classical reasoning. These structures are geometric in nature, and we identify the primitive elements from which more elaborate types are constructed. We will review the key types of structures that are available and explain their hierarchy. Finally we will use them to give simple and transparent proofs of entanglement correlations, quantum contextuality via the Kochen-Specker theorem, and quantum nonlocality via the Bell-GHZ theorem.
These structures have many applications in quantum information processing, but the real purpose of this talk is simply to introduce unfamiliar researchers to the simplest known logical proofs of these nonclassicality theorems, which can be understood using only the algebra of the Pauli spin matrices and simple counting arguments.</p>

Primitives of Nonclassicality in the N-qubit Pauli Group

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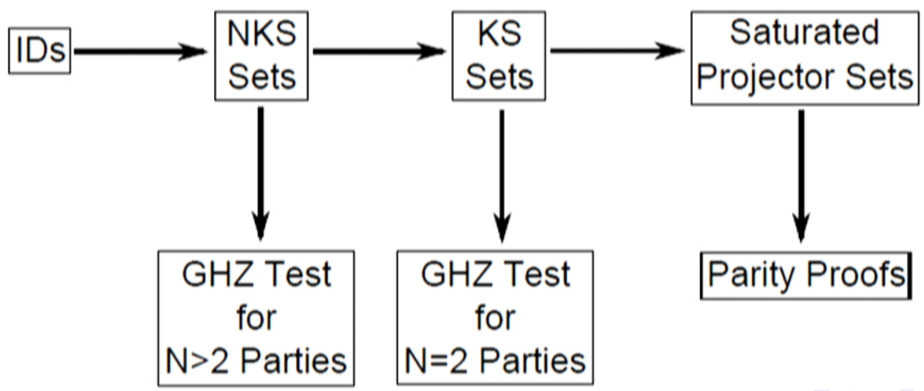
Overview

- Nonclassical structures conflicting with several notions of classical reality are introduced using the *N*-qubit Pauli Group.
- We begin with a review of the Pauli algebra for a single qubit (2-level quantum system).
- We construct primitive N-qubit structures that we call *Identity Products* (IDs), and discuss how they are related to entanglement.
- We introduce composite structures of IDs that provide transparent proofs of Bell nonlocality, Kochen-Specker contextuality, and contextual entanglement correlations.



Single-qubit Pauli Algebra

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$Z^2 = X^2 = Y^2 = I$$

$$\begin{array}{ccc} ZX = iY, & XY = iZ, & YZ = iX \\ XZ = -iY, & YX = -iZ, & ZY = -iX \end{array}$$

$$ZXY = XYZ = YZX = iI$$

$$ZYX = YXZ = XZY = -iI$$

- Any ordered product containing an even number of Z, X, and Y has product $\pm I$, and we call it an *even Single Qubit Product* (SQP).
- Any ordered product containing an odd number of Z, X, and Y has product $\pm iI$, and we call it an *odd* SQP.

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N-qubit Pauli Algebra

- We denote an SQP with M elements by the symbol SQPM.
- We define an N-qubit structure called an Identity Product (ID), as a set of N SQPMs (one for each qubit) with the property that all M different N-qubit Pauli observables they form are mutually commuting.
- The product of all M N-qubit observables in any ID is $\pm I$ (the N-qubit identity).
- We denote an ID with M observables and N qubits by the symbol IDM_O^N, where O is the number of odd SQPs in the ID.

$\mathrm{ID}3_2^2$		$\mathrm{ID}4_0^3$			$\mathrm{ID}5_2^4$			
					Z	Z	Z	Z
Z	Z	Z	Z	Z	X	X	Z	Z
X	X	Z	X	X	Y	I	X	I
Y	Y	X	Z	X	I	Y	I	X
		X	X	Z	I	I	X	X

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ID3^2_2

Z	Z
X	X
Y	Y

ID4^3_0

Z	Z	Z
Z	X	X
X	Z	X
X	X	Z

ID5^4_2

Z	Z	Z	Z
X	X	Z	Z
Y	I	X	I
I	Y	I	X
I	I	X	X

Waegell Sets

- A Waegell set is a set of one or more specific IDs with the property that, 1) an odd number of the IDs in the set are negative IDs, and 2) for every qubit, there are an even number of IDs in the set with an odd SQP for that qubit.

Z	Z
X	X
Y	Y
Z	X
X	Z
Y	Y

Z	Z	Z
Z	X	X
X	Z	X
X	X	Z

Z	Z	I
X	X	I
Y	Y	I
Z	I	Z
X	I	X
Y	I	Y
I	Z	Z
I	X	X
I	Y	Y

The Waegell Theorem

- A classical hidden variable theory without entanglement correlations must assign an eigenvalue ± 1 to each single-qubit Pauli observable of the Waegell set, and in order to be consistent with experiment, the assigned values must agree with sign of each ID in the set.
- By the definition just given, such an assignment to a Waegell set is impossible. To see this consider the product β of all IDs in the set. In order to be consistent with experiment, $\beta = -1$, however because all odd SQPs are paired, every assigned eigenvalue in any complete pre-assignment is squared, and thus $\beta = +1$ for any Waegell set.
- Classical hidden variable theories without entanglement correlations are ruled out by a simple parity argument.

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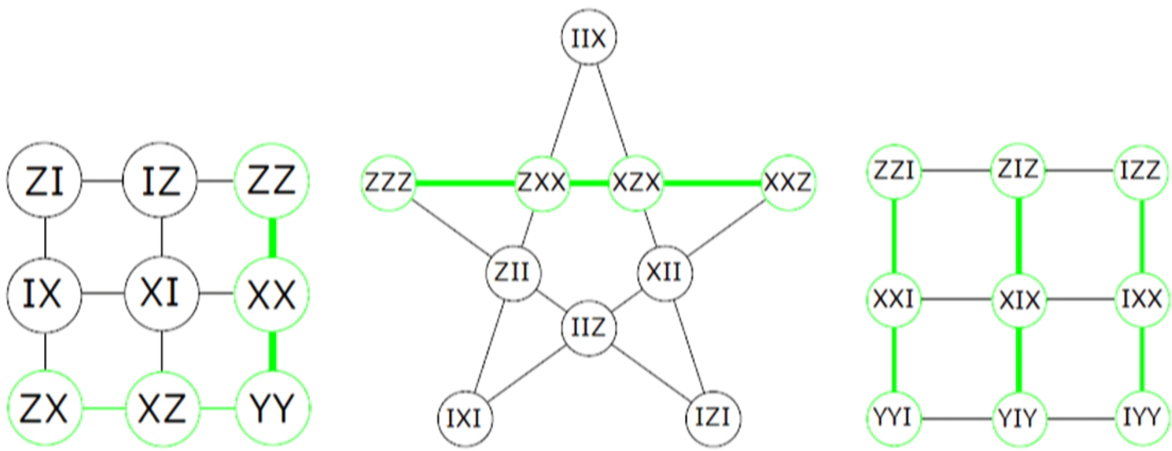
$$
\begin{array}{cc}
Z & Z \\
X & X \\
Y & Y \\
\hline
Z & X \\
X & Z \\
Y & Y
\end{array}
\quad\Bigg|\quad
\begin{array}{ccc}
Z & Z & Z \\
Z & X & X \\
X & Z & X \\
X & X & Z
\end{array}
\quad\Bigg|\quad
\begin{array}{ccc}
Z & Z & I \\
X & X & I \\
Y & Y & I \\
\hline
Z & I & Z \\
X & I & X \\
Y & I & Y \\
\hline
I & Z & Z \\
I & X & X \\
I & Y & Y
\end{array}
$$

Kochen-Specker Sets

- A Kochen-Specker (KS) set is a set of specific ID^Ns with the property that, 1) an odd number of the IDs in the set are negative IDs, and 2) each N-qubit observable appears in an even number of the IDs in the set.



A Zoo of KS Sets

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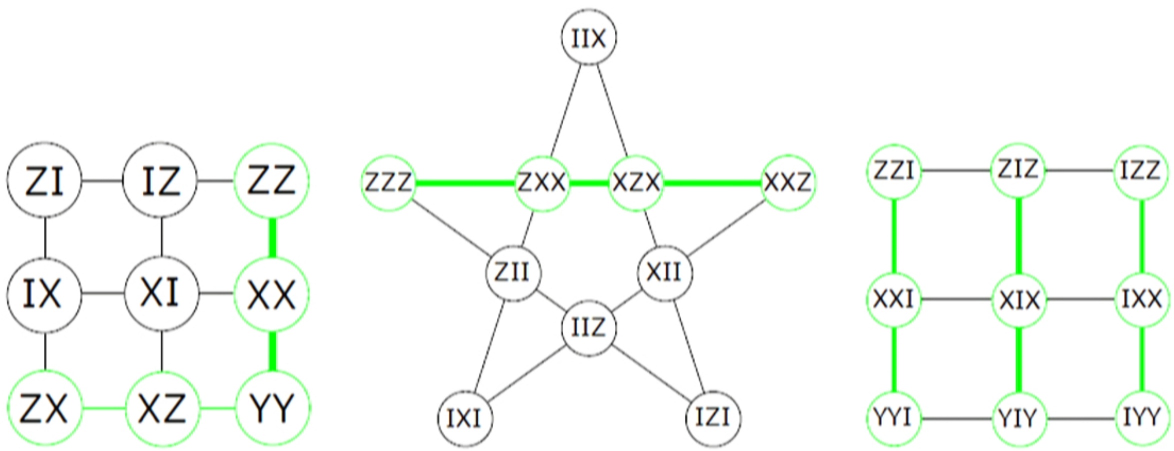
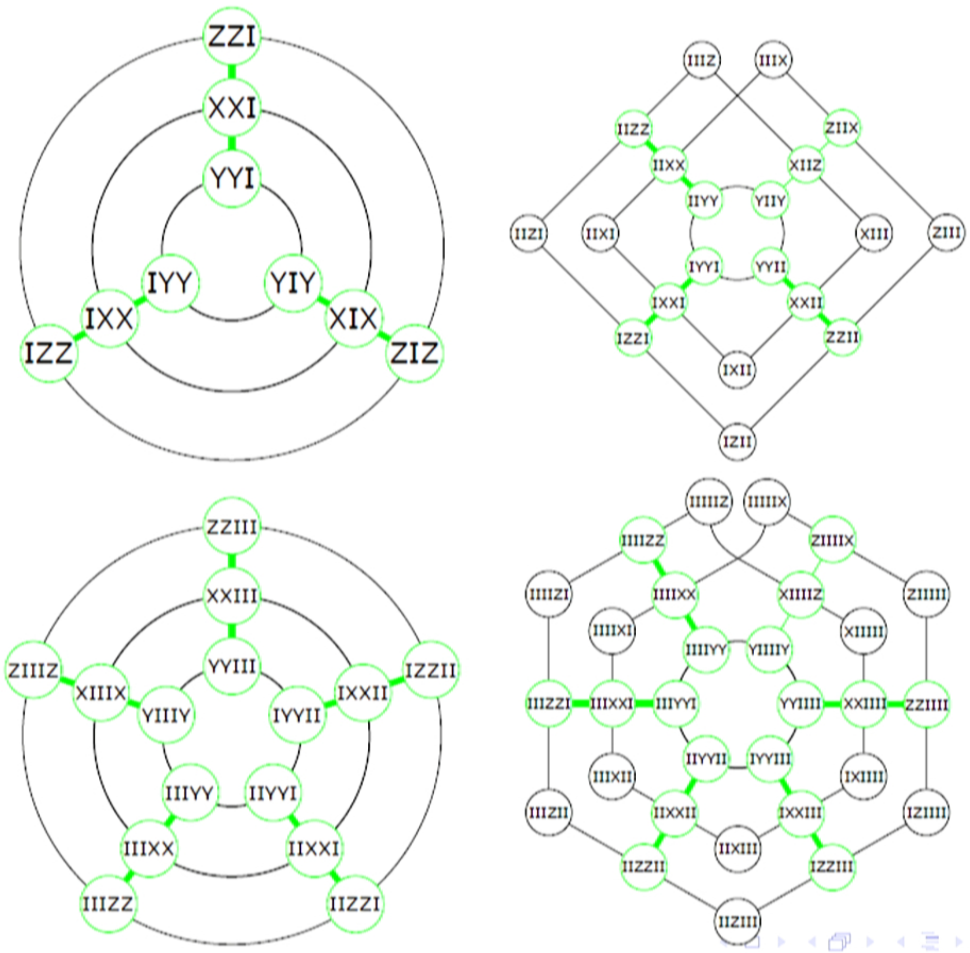
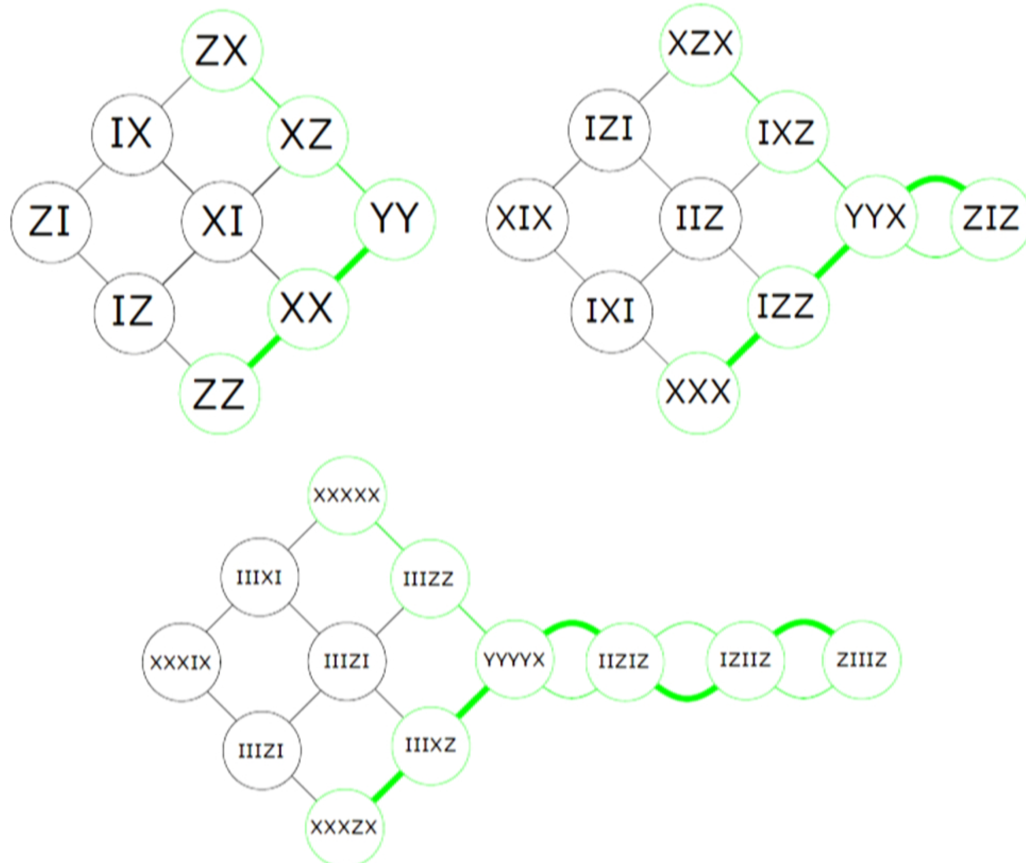


The Wheel Family of KS Sets



The Kite Family of KS Sets



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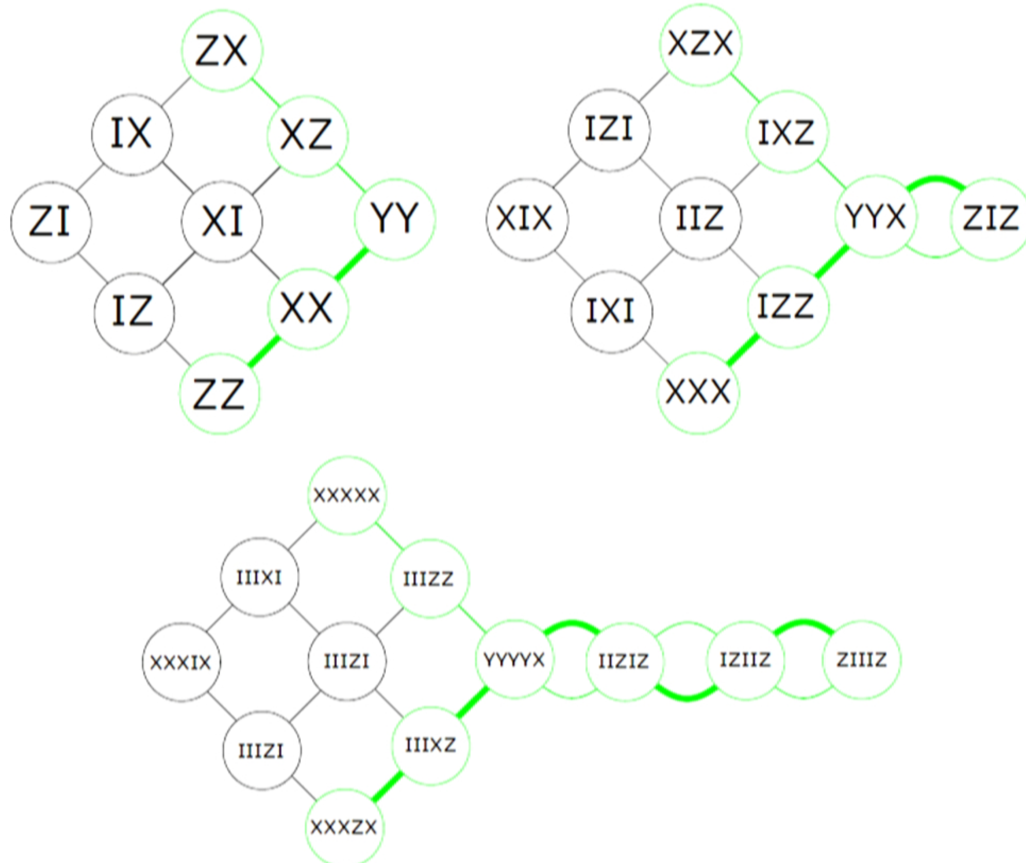


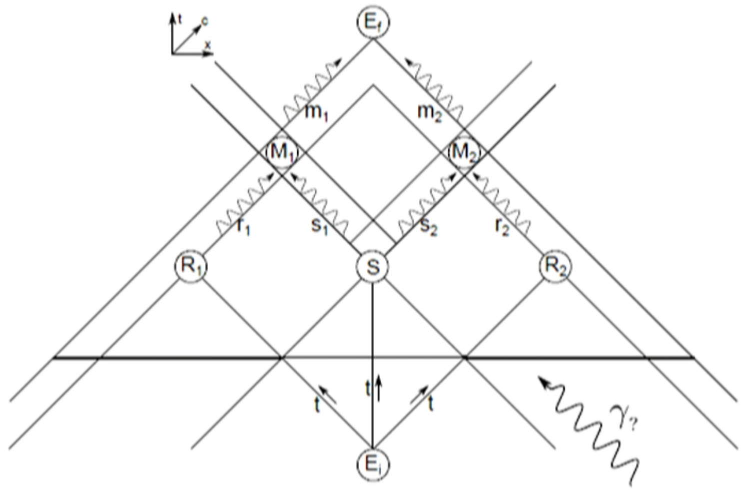
Bell-GHZ Theorem for Three or More Parties (1)

- Any single-ID Waegell set for N qubits provides a proof of the Bell-GHZ theorem for N space-like separated parties.
- The smallest single-ID Waegell set is the standard proof of the GHZ theorem, and has $N = 3$.
- To demonstrate the proof, we prepare a state $|\psi\rangle$ that is any joint eigenstate of this ID (a GHZ state), and then send each qubit to a space-like separated detector, where Alice, Bob, and Charlie each randomly choose and perform a single-qubit Pauli measurement Z or X.

$$\begin{matrix} Z & Z & Z \\ Z & X & X \\ X & Z & X \\ X & X & Z \end{matrix}$$

Bell-GHZ Theorem for Two Parties (1)

- Any KS set for N qubits provides a proof of the Bell-GHZ theorem for two parties using $2N$ qubits.
- To demonstrate the proof, a source prepares N copies of the correlated Bell state, $|\psi\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$, and sends one qubit from each Bell pair to two space-like separated detectors where Alice and Bob each randomly choose and perform an N-qubit measurement in the eigenbasis of an ID in the KS.



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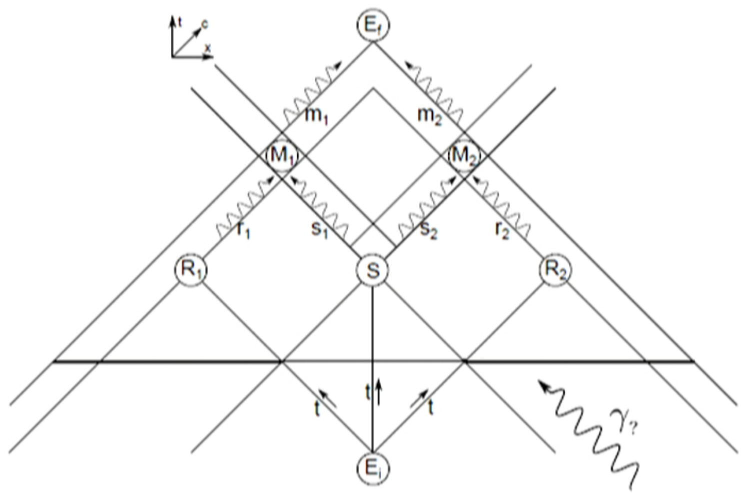

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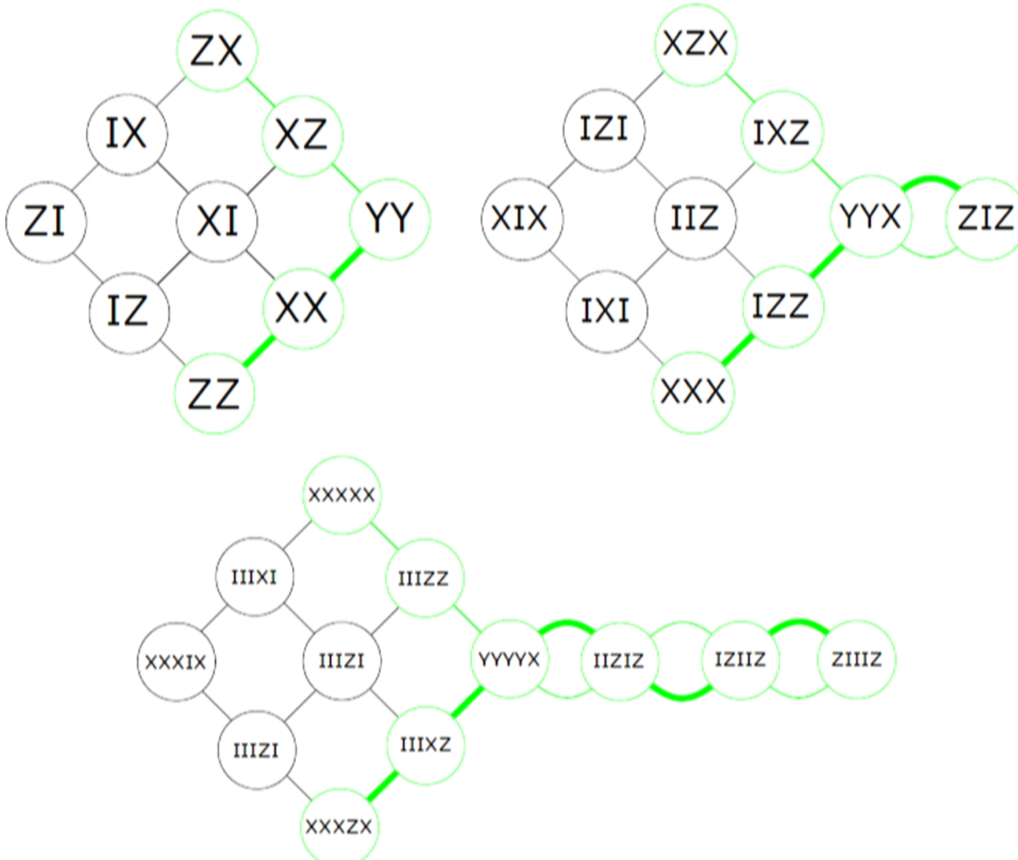
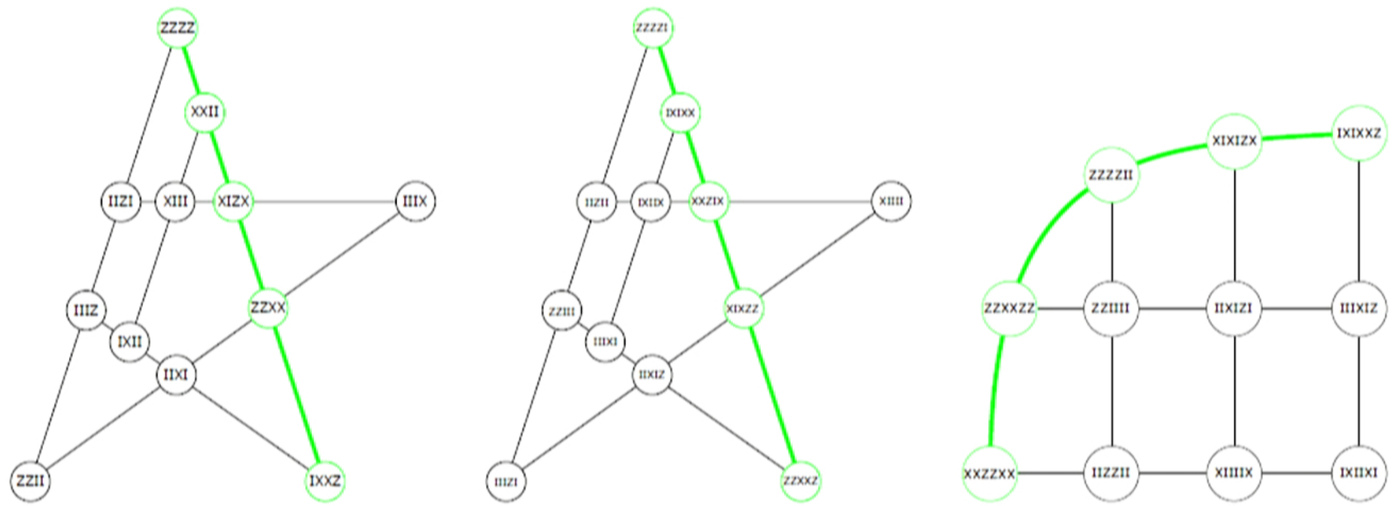


A Zoo of KS Sets

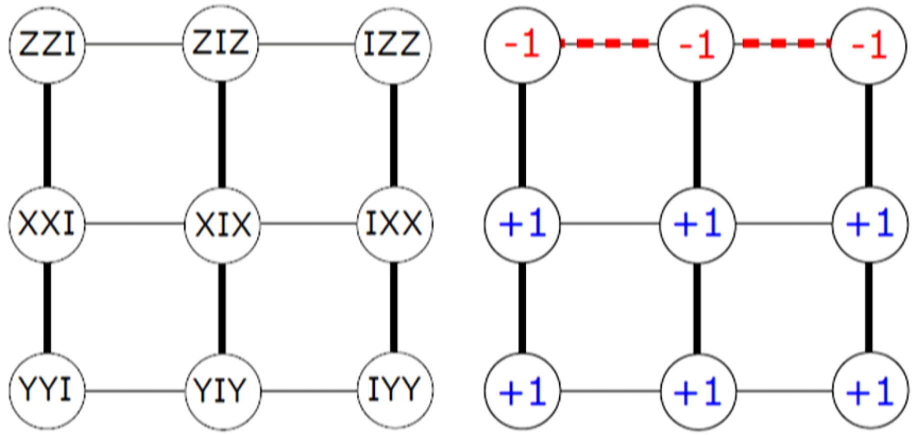


Pre- and Post-Selected States

- The KS theorem is a state-independent proof of contextuality.
- By choosing specific pre-selected and post-selected (PPS) states, we can assign definite values to some of the observables of a KS set.
- The PPS states also force some additional value assignments according to the ABL formulation, and this allows the violation of the product rule to be localized in a specific ID. The forced values can be certified by weak measurements.
- In the eigenbasis of the ID, the projector with all eigenvalues opposite to the forced values in that ID has an anomalous weak value. This further localizes the nonclassical behavior of the system, and provides an alternative proof of contextuality (M. Pusey, PRL, 2015).

The Quantum Pigeonhole Effect (1)

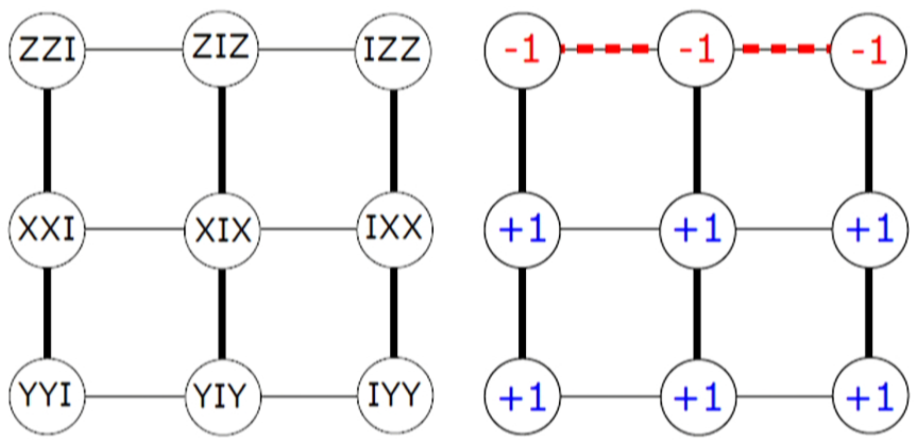
- We choose pre-selected state $|\psi\rangle = |+X^{\otimes 3}\rangle$ and post-selected state $|\phi\rangle = |+Y^{\otimes 3}\rangle$ for this KS set.
- The PPS assigns eigenvalues $+1$ to six observables of the KS set as shown in blue.
- The PPS forces additional assignments -1 to the remaining three observables as shown in red, and these values violate the product rule for the top row.



The Quantum Pigeonhole Effect (2)

- The top row is the 'classical basis,' which contains product states without superposition, and the forced values do not agree with any projector in this basis.
- The weak value of the rank-2 projector onto $(ZZI = +1, ZIZ = +1, IZZ = +1)$ is given by,

$$\frac{\langle\phi|\left(|+Z^{\otimes 3}\rangle\langle +Z^{\otimes 3}| + |-Z^{\otimes 3}\rangle\langle -Z^{\otimes 3}|\right)|\psi\rangle}{\langle\phi|\psi\rangle} = -0.5. \qquad (1)$$



Conclusion

- The N-qubit Pauli group contains many elegant and simple proofs of important nonclassical effects in quantum mechanics, which provides a wonderfully rich pedagogical tool for teaching about foundations.
- The nonclassical structures here present many clear protocols for experiments to test nonclassical features of nature as predicted by quantum mechanics.
- Nonclassical effects are the fundamental resource for quantum computational advantage, and these structures appear in a wide array of applications in quantum information processing.