Title: Some Implications of the Aharonov Ansatz to Sensing

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Abstract: There is a common framework for the measurement problem for sensors such as radars, sonars, and optics in a common language by casting analysis of signals in the language of quantum mechanics (Rigged Hilbert Space). The use of this language can reveal a more detailed understanding of the underlying interactions of a return signal that are not usually brought out by standard signal processing design techniques. The weak measurement Ansatz first provided by the Aharonov, Albert and Vaidman paper (A2V) that introduced weak values to the world provides an explicit means to consider all interactions of a signal with an object by using what we term the Aharonov Ansatz. The Aharonov Ansatz for sensing can summarized as:

1. Any sensor measurement process, whether active or passive can be thought of as determining the mathematical operator's characteristics of a signal's interaction with a object.
2. Certain types of interaction operators can be "post-selected" for in the return signal when the broadcast signal is known for either a single or multiple operators so receiver design can be optimized.
3. In principle detectors can design can be optimized, "matched" to signal interaction for these operators (operator matched filter), so mathematical solutions to receiver (in the classical sense) design or the design of apparatus of difficult to measure quantum interactions can be improved as has been reported in the literature.
4. Matching or post-selection to a given operator, when possible, maximizes ability to detect a "signal" or the characteristics of an interaction.

Finally, in this talk we note a connection between this work and a the variational functional used in perturbation theory in quantum mechanics.

Some Implications of the Aharonov Ansatz to Sensing

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Place: Perimeter Institute,

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Date: June 25, 2016

Abstract

There is a common framework for the measurement problem for sensors such as radars, sonars, and optics in a common language by casting analysis of signals in the language of quantum mechanics (Rigged Hilbert Space). The use of this language can reveal a more detailed understanding of the underlying interactions of a return signal that are not usually brought out by standard signal processing design techniques. The weak measurement Ansatz first provided by the Aharonov, Albert and Vaidman paper (A^2V) that introduced weak values to the world provides an explicit means to consider all interactions of a signal with an object by using what we term the Aharonov Ansatz. The Aharonov Ansatz for sensing can summarized as:

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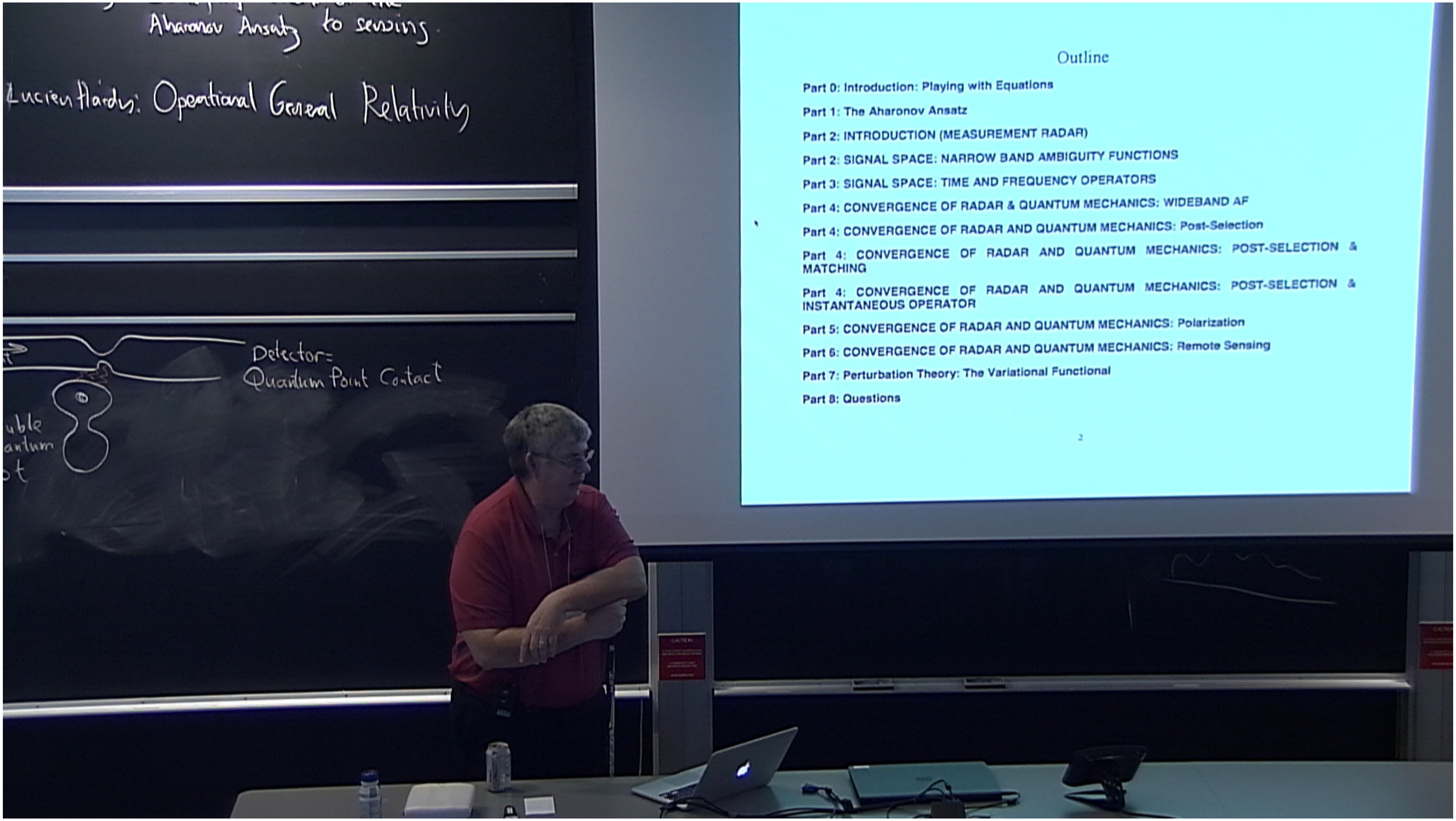
Outline


Aharonov Ansatz to sensing
Lucien Hardy: Operational General Relativity
Detector=
Quantum Point Contact
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PART 0: PLAYING WITH EQUATIONS

"*Causality applies only to a system which is left undisturbed. If a system is small, we cannot observe it without producing a serious disturbance and hence we cannot expect to find any causal connection between the results of our observations. Causality will still be assumed to apply to undisturbed systems and the equations which will be set up to describe an undisturbed system will be differential equations expressing a causal connection between conditions at one time and conditions at a later time. These equations will be in close correspondence with the equations of classical mechanics, but they will be connected only indirectly with the results of observations."* I. The Principle of Superposition - 1. The Need for a Quantum Theory **P.A.M. Dirac**

"*Nevertheless it has been found possible to set up a new scheme, called quantum mechanics, which is more suitable for the description of phenomena on the atomic scale and which is in some respects more elegant and satisfying than the classical scheme. This possibility is due to the changes which the new scheme involves being of a very profound character and not clashing with the features of the classical theory that make it so attractive, as a result of which all these features can be incorporated in the new scheme."* I. The Principle of Superposition - 1. The Need for a Quantum Theory **P.A.M. Dirac**

Aharonov and others have deepened our understanding what can be found buried within the framework of quantum mechanics by playing with paradoxes.

"*I think it's a peculiarity of myself that I like to play about with equations, just looking for beautiful mathematical relations which maybe don't have any physical meaning at all. Sometimes they do."* Interview with Dr. P. A. M. Dirac by Thomas S. Kuhn at Dirac's home, Cambridge, England, May 7, 1963

PART 1: INTRODUCTION (AHARONOV ANSATZ)

The **Aharonov Ansatz** for sensing can summarized as:

1. Any (sensor) measurement process, whether active or passive can be thought of as determining the mathematical operator's characteristics based on a signal's interaction with an object.

2. Certain types of interaction operators can be "post-selected" for in the return signal when the broadcast signal is known for either a single or multiple operators, thus a receiver or measurement apparatus design can be optimized for the these operators.

3. In principle, detectors can design can be matched" to signal interaction or optimized so that mathematical solutions to receiver (in the classical sense) design or the design of apparatus in the quantum mechanical sense for difficult to measure quantum interactions can be improved, (Examples of such improvements have been reported in the literature).

4. Matching or post-selection to a given operator, when possible, maximizes ability to detect a "signal" or the characteristics of an interaction, which changes are ability to find the hard to find, and possibly to detect the new.

PART 2: INTRODUCTION (MEASUREMENT IN RADAR)

- The radar measurement problem is to design a waveform to be broadcast by a radar or sonar, so as to maximize the receiver response to the signal which has interacted with an object.
- The theoretical solution to measurement problem of radar has influenced radar waveform design to this day in terms of what may be implemented.
- The means to accomplish this was drawn from work in WW II, the "matched filter". A matched filter is obtained by correlating a known signal template with an unknown signal to detect the presence or absence of the template in the unknown signal.
- Woodward introduced the ambiguity function in the seminal book on radar as the means to solve the radar measurement problem.
- This is exactly equivalent to convolving the unknown signal with the complex conjugate of the time-reversed version of the known signal template which; is termed cross-correlation.
- The matched filter is the optimal linear filter for maximizing the signal to noise ratio (SNR) in the presence of additive.
- With a new interpretation of Woodward's ambiguity function as the expected value of an operator it is possible to cast measurement with active sensors in the same framework.

PART 3: SIGNAL SPACE: NARROW BAND AMBIGUITY FUNCTIONS

- The inner product of two waveforms, $\langle \circ | \circ \rangle$, is defined to be

$$\langle \Psi_p(t) | \Psi_s(t) \rangle = \int_{-\infty}^{\infty} \Psi_p{}^*(t) \Psi_s(t) dt,$$

  where * denotes complex conjugate.

Definition: The *Fourier Transform (FT) of the signal* is defined as (symmetric form)

$$\mathcal{F}\Psi_s(t) = \Psi_s(\omega) = \frac{\langle e^{i\omega t} | \Psi_s(t) \rangle}{\sqrt{2\pi}},$$

while the *Inverse Fourier Transform (IFT) of the signal* is defined as

$$\mathcal{F}^{-1}\Psi_s(\omega) = \Psi_s(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega t}\, \Psi_s(\omega) d\omega = \frac{\langle e^{-i\omega t} | \Psi_s(\omega) \rangle}{\sqrt{2\pi}}.$$

- We assume the Fourier transform always exists for signals (see Papoulis), at least in the distribution sense.
- The Ambiguity Function (AF) is the "best" possible approach to the design of radar receiver, though in practice they are difficult to implement in a real receiver until recently when digital technology has changed what is possible.
- Woodward, in one of the first theoretical books on the foundation of radar based his foundation for extraction of data from radars on two principles: Bayesian probability and the notion of the ambiguity function.

PART 3: SIGNAL SPACE: NARROW BAND AMBIGUITY FUNCTIONS

- When there is an interaction of a signal with an object, it is natural to associate a parameter τ with the interaction. We denote by $\chi(\omega,\tau)$ the Fourier transform of the product: $s(t+\tau)s^*(t) = s(t,\tau)$, with respect to t as

$$\chi_s(\omega,\tau) = \langle e^{-i\omega t} | s(t,\tau)\rangle$$

- Since the Fourier transform of $s(t+\tau)$ is the transform pair: $s(t+\tau) \Leftrightarrow e^{i\omega\tau}S(\omega)$, and $s^*(t) \Leftrightarrow S^*(-\omega)$, we have

$$\chi_s(\omega,\tau) \Leftrightarrow \frac{1}{2\pi} e^{i\omega\tau} S(\omega) S^*(-\omega) = \frac{1}{2\pi} \langle S(y-\omega) | e^{i\tau y} S(\omega)\rangle.$$

- The correlation of a signal is defined as

$$\rho(\tau) = \int_{-\infty}^{\infty} s(t+\tau)s^*(t)\, dt = \chi_s(0,\tau)$$

or with $\omega = 0$ in the second expression for $\chi_s(\omega,\tau)$ is

$$\rho(\tau) = \frac{1}{2\pi} \langle S(y) | e^{i\tau y} S(y)\rangle.$$

- It follows that $\rho(\tau) \Leftrightarrow S(\omega)S^*(\omega)$ and $\rho(\tau) = s(t)s^*(-t)$. It is also useful to note that

$$|\rho(\tau)| \le \frac{1}{2\pi} \langle S(y) | S(y)\rangle = \rho(0) = \langle s(t) | s(t)\rangle = E$$

- Because of this we always have an upper bound that $|\rho(\tau)| \le \rho(0)$ on the correlation of a signal with itself. The notion of auto-correlation provides both a mathematical and physical connection between the concept of an AF. Auto-correlation is a special case of cross correlation which is the basis for discussions of post-selection from a classical point of view.

PART 3: SIGNAL SPACE: TIME AND FREQUENCY OPERATORS

Since we can interpret $\rho_s(\omega)$ as a density of energy in frequency or time, it makes sense to define means, spread, standard deviation, etc. with respect to the density.

Definition: The *average* or *mean frequency* of the spectral density is

$$\langle\omega\rangle = \int_{-\infty}^{\infty} \omega\, \rho_s(\omega) d\omega = \langle S(\omega)|\omega S(\omega)\rangle.$$

Definition: The n^{th} *frequency* moment of the spectral density is

$$\langle\omega^n\rangle = \int_{-\infty}^{\infty} \omega^n\, \rho_s(\omega) d\omega = \langle S(\omega)|\omega^n S(\omega)\rangle.$$

Definition: The bandwidth of the spectral density is

$$B^2 = \sigma_\omega^2 = \int_{-\infty}^{\infty} (\omega - \langle\omega\rangle)^2\, \rho_s(\omega) d\omega = \langle S(\omega)|(\omega - \langle\omega\rangle)^2 S(\omega)\rangle = \langle\omega^2\rangle - \langle\omega\rangle^2.$$

Definition: The expected value of a frequency function $g(\omega)$ is

$$\langle g(\omega)\rangle = \int_{-\infty}^{\infty} g(\omega)\, \rho_s(\omega) d\omega = \langle S(\omega)|g(\omega) S(\omega)\rangle.$$

Because of this, we can do a calculation in one domain or the other depending on the integral.

Definition: If $\widehat{\mathcal{W}} = -i\frac{d}{dt}$, then if g is an analytic function, ($g(\omega) = \sum g_n \omega^n$), so

$$\langle g(\omega)\rangle = \left\langle s(t) \middle| \sum g_n\, \widehat{\mathcal{W}}^n s(t) \right\rangle = \langle s(t)|g(\widehat{\mathcal{W}}) s(t)\rangle.$$

Note: Because of Parseval's theorem, we have $\langle\omega^n\rangle = \langle S(\omega)|\omega^n S(\omega)\rangle = \langle s(t)|\mathcal{W}^n s(t)\rangle$.

PART 3: SIGNAL SPACE: TIME & FREQUENCY OPERATORS

Note: It is sometimes easier to do the frequency integral evaluations in the time domain or vice versa because of the derivatives.

- Recalling the complex representation of a signal is $s(t) = A(t)e^{i\varphi(t)}$ we have $\widehat{\mathcal{W}}s(t) = \left(\varphi'(t) - i\frac{A'(t)}{A(t)}\right)s(t)$, so
$$\langle\omega\rangle = \langle s(t)|\widehat{\mathcal{W}}s(t)\rangle = \int_{-\infty}^{\infty}\left(\varphi'(t) - i\frac{A'(t)}{A(t)}\right)A^2(t)\,dt \;= \int_{-\infty}^{\infty}\varphi'(t)A^2(t)\,dt\,.$$

- Thus, the average frequency is the average of the derivative of the phase over all time with respect to the signal amplitude.

- The instantaneous frequency, ω_i, begs to be identified with the derivative of the phase from the above expression, so we do so $\omega_i = \varphi'(t)$.

- Note we express the second moment, the bandwidth, the AM and FM contributions, and the AM/FM contribution expressing them using the complex signal notation.

- The operator $\mathbb{W} = e^{i\tau\widehat{\mathcal{W}}}$ can be interpreted as a *time translation operator* on a function s(t):
$$\mathbb{W}\mathrm{s}(\mathrm{t}) = e^{i\tau\widehat{\mathcal{W}}}\mathrm{s}(\mathrm{t}) = \sum_{m=0}^{\infty}\frac{\tau^m\frac{d^m s(t)}{dt^m}}{m!} = \mathrm{s}(\mathrm{t}+\tau).$$

- The *frequency translation operator* $\mathbb{T} = e^{i\theta\hat{\mathcal{T}}}$ has exactly the same effect: $\mathbb{T}\mathrm{S}(\omega) = S(\omega + \theta)$.

PART 3: SIGNAL SPACE: COMMUTATORS & OPERATORS

- The commutator of two operators $\mathcal{A}$ *and* $\mathcal{B}$ is defined as $[\hat{\mathcal{A}},\hat{\mathcal{B}}] = \hat{\mathcal{A}}\hat{\mathcal{B}} - \hat{\mathcal{B}}\hat{\mathcal{A}}$, while the anti-commutator of the two operators is $[\hat{\mathcal{A}},\hat{\mathcal{B}}]_+ = \hat{\mathcal{A}}\hat{\mathcal{B}} + \hat{\mathcal{B}}\hat{\mathcal{A}}$.

- Recall that in the time domain $\hat{\mathcal{T}} = t$ while in the frequency domain, $\hat{\mathcal{W}} = \omega$.

- Thus in the time domain, we have
$$[\hat{\mathcal{T}},\hat{\mathcal{W}}]_t = -i\left(t\frac{d}{dt} - \frac{d}{dt}t\right) = i,$$
while in the frequency domain, we have
$$[\hat{\mathcal{T}},\hat{\mathcal{W}}]_\omega = -i\left(\frac{d}{d\omega}\omega - \omega\frac{d}{d\omega}\right) = -i.$$

- The commutator of the time operator and the frequency operator is
$$[\hat{\mathcal{T}},\hat{\mathcal{W}}]\mathrm{s(t)} = (\hat{\mathcal{T}}\hat{\mathcal{W}} - \hat{\mathcal{W}}\hat{\mathcal{T}})\mathrm{s(t)} = \mathrm{is(t)},$$
so
$$[\hat{\mathcal{T}},\hat{\mathcal{W}}] = i;$$
while the anti-commutator is defined as the scale operator $\hat{\mathcal{C}}$
$$\hat{\mathcal{C}} = \frac{1}{2}[\hat{\mathcal{T}},\hat{\mathcal{W}}]_+ = \hat{\mathcal{T}}\hat{\mathcal{W}} + \frac{i}{2} = \frac{1}{2}\left(t\frac{d}{dt} + \frac{d}{dt}t\right).$$

- The operator $\hat{\mathbb{C}} = e^{i\sigma\hat{\mathcal{C}}}$ has the property that it transforms a signal s(t) according to (for an arbitrary scale factor σ) the rule:
$$\hat{\mathbb{C}}\mathrm{s(t)} = e^{i\sigma\hat{\mathcal{C}}}\mathrm{s(t)} = \mathrm{e}^{\frac{\sigma}{2}}\mathrm{s}\left(\mathrm{e}^{\frac{\sigma}{2}}\mathrm{t}\right).$$

PART 3: SIGNAL SPACE: COMMUTATORS & OPERATORS

- Properties of $\hat{\mathcal{C}}$ acting on a function of time: $\hat{\mathcal{C}}t^n = -i\left(n+\frac{1}{2}\right)t^n$, $\hat{\mathcal{C}}^k t^n = (-i)^k\left(n+\frac{1}{2}\right)^k t^n$,

$$\hat{\mathbb{C}}t^n = e^{\sigma\left(n+\frac{1}{2}\right)^k} t^n,$$

and

$$\hat{\mathbb{C}}s(t) = e^{\frac{\sigma}{2}} s\left(e^{\frac{\sigma}{2}} t\right).$$

- Six properties of $\hat{\mathcal{C}}'s$ commutator with other principal ones (c is a constant):

$$[\hat{\mathcal{T}}, \hat{\mathcal{C}}] = \left[\hat{\mathcal{T}}, \hat{\mathcal{T}}\widehat{\mathcal{W}} + \frac{i}{2}\right] = [\hat{\mathcal{T}}, \hat{\mathcal{T}}\widehat{\mathcal{W}}] = [\hat{\mathcal{T}}, \hat{\mathcal{T}}]\widehat{\mathcal{W}} + \hat{\mathcal{T}}[\hat{\mathcal{T}}, \widehat{\mathcal{W}}] = 0 + \hat{\mathcal{T}}i = i\hat{\mathcal{T}},$$

$$[\widehat{\mathcal{W}}, \hat{\mathcal{C}}] = \left[\widehat{\mathcal{W}}, \hat{\mathcal{T}}\widehat{\mathcal{W}} + \frac{i}{2}\right] = -[\hat{\mathcal{T}}\widehat{\mathcal{W}}, \widehat{\mathcal{W}}] = -\widehat{\mathcal{W}},$$

$$\left[\hat{\mathcal{T}}, [\hat{\mathcal{T}}, \hat{\mathcal{C}}]\right] = 0,$$

$$\left[c, [\widehat{\mathcal{W}}, \hat{\mathcal{C}}]\right] = 0,$$

$$\left[\hat{\mathcal{C}}, [\hat{\mathcal{T}}, \hat{\mathcal{C}}]\right] = 0,$$

$$\left[\hat{\mathcal{C}}, [\widehat{\mathcal{W}}, \hat{\mathcal{C}}]\right] = 0.$$

- Note the average of an operator $\langle\hat{A}\rangle$ is defined to be

$$\langle\hat{A}\rangle = \int s^*(t)\hat{A}s(t)dt = \int \left(\frac{\hat{A}s(t)}{s(t)}\right)|s(t)|^2\, dt.$$

PART 3: SIGNAL SPACE: COMMUTATORS & OPERATORS

- The expression $\left(\frac{\hat{A}s(t)}{s(t)}\right)$ can be written as $\left(\frac{\hat{A}s(t)}{s(t)}\right) = \left(\frac{\hat{A}s(t)}{s(t)}\right)_R + i\left(\frac{\hat{A}s(t)}{s(t)}\right)_I$, so
$$\langle \hat{A} \rangle = \langle a \rangle = \int \left[\left(\frac{\hat{A}s(t)}{s(t)}\right)_R + i\left(\frac{\hat{A}s(t)}{s(t)}\right)_I\right] |s(t)|^2\, dt;$$
when $\hat{A}$ is Hermitian.
- This becomes
$$\langle a \rangle = \langle \hat{A} \rangle = \int \left(\frac{\hat{A}s(t)}{s(t)}\right)_R |s(t)|^2\, dt;$$
since the imaginary part is zero.
- When $\hat{A} = \widehat{\mathcal{W}}$, we have $\left(\frac{\hat{A}s(t)}{s(t)}\right)_R = \varphi'(t)$ and $\left(\frac{\hat{A}s(t)}{s(t)}\right)_I = -\frac{A'(t)}{A(t)}$ so
$$\langle \widehat{\mathcal{W}} \rangle = \langle \omega \rangle = \int \varphi'(t)|s(t)|^2\, dt.$$
Likewise, the bandwidth of an operator is
$$\sigma_a^2 = \langle a^2 \rangle - \langle a \rangle^2 = \int \left[\left(\frac{\hat{A}s(t)}{s(t)}\right)_I\right]^2 |s(t)|^2\, dt + \int \left[\left(\frac{\hat{A}s(t)}{s(t)}\right)_R - \langle \hat{A} \rangle\right]^2 |s(t)|^2\, dt;$$
- For $\widehat{\mathcal{W}}$, it becomes
$$\sigma_\omega^2 = \langle \omega^2 \rangle - \langle \omega \rangle^2 = \int \left(\frac{A'(t)}{A(t)}\right)^2 |s(t)|^2\, dt + \int (\varphi'(t) - \langle \omega \rangle)^2 |s(t)|^2\, dt,$$
which is the same expression we have already presented.

Part 4: Convergence of Radar & Quantum Mechanics: Post-Selection

- In QM, the selection of the post-selection wave function allows one to produce anomalous eigenvalues of the spin state of an electron that were well outside the normal range of an operator $\hat{A}$ acting on a wave function $|\Psi\rangle$.
- This is the traditional range of the expected value of $\langle\hat{A}\rangle$ which is defined as

$$\langle\hat{A}\rangle = \frac{\langle\Psi|\hat{A}|\Psi\rangle}{\langle\Psi|\Psi\rangle}.$$

- Note, when there is a set of $\{|\Psi_i\rangle\}$ which are the eigenfunctions of the solution to the Schrodinger equation, there is a maximum eigenvalue, so $\hat{A}\,|\Psi_i\rangle \leq \lambda_{max}$, so for all eigenfunctions $\langle\hat{A}\rangle \leq \lambda_{max}$.
- Using post-selection and "weak measurement", Aharonov, Albert, Vaidman [11] were able to define a new type of observable for an operator, the weak value of $\hat{A}$. The weak value is defined as

$$\langle\hat{A}\rangle_w = \frac{\langle\Psi_p|\hat{A}|\Psi_s\rangle}{\langle\Psi_p|\Psi_s\rangle}.$$

- Another way of interpreting post-selection in the waveform language is that the selection of a particular waveform $\langle\Psi_p|$, is to interpret it as cross-correlation of the return signal $\hat{A}\,|\Psi_s\rangle$ with a post-selected signal.

Part 4: Convergence of Radar and Quantum Mechanics: Post-Selection

- In the classical setting, cross-correlation is not conceptually difficult, it is however difficult to implement in many receiver systems.
- Digital receivers allow more freedom to select the correlation waveform $\langle\Psi_p|$. However a mathematical criteria for selecting it is not obvious as a general procedure.

- A judicious choice of $\langle\Psi_p|$ has two characteristics:
  - $\langle\Psi_p|$ should be nearly orthogonal to $|\Psi_s\rangle$ which means $\langle\Psi_p|\Psi_s\rangle = \mu \ll 1$.
  - The constant $\epsilon = \langle\Psi_p| \ \hat{A}|\Psi_s\rangle$ must not be of the same order as μ, the ratio $w = \frac{\epsilon}{\mu}$ must not be $\mathcal{O}(\mu)$ and the choice of $\langle\Psi_p|$ must meet the criteria $\langle\Psi_s|\hat{A}|\Psi_s\rangle < \frac{1}{\mu}\langle\Psi_p|\hat{A}|\Psi_s\rangle$.
- When this is true, some interactions can be made intelligible relative to the background noise they are embedded within.
- What happens if one considers a cross ambiguity function which we define as:

$$ {}^{r}_{s}\chi_s{}'(0,\tau) = \langle r(t)|\hat{O}_1 \ldots \hat{O}_n \mathbb{CW}|s(t)\rangle = \chi^r_s\langle\hat{O}_1 \ldots \hat{O}_n \mathbb{CW}\rangle? $$

- For a given operator set of operators, $\mathcal{O}_n$, determine whether there is a post-selection waveform $r(t)$ or $r_n(t)$ that does the equivalent to what the matched filter does for $\mathbb{CW}$.

PART 4: CONVERGENCE OF RADAR AND QUANTUM MECHANICS: INSTANTANEOUS OPERATOR

- Any operator can be written as an *instantaneous operator*:

$$\frac{\hat{O}\vartheta(x)}{\vartheta(x)} = \left(\frac{\hat{O}\vartheta(x)}{\vartheta(x)}\right)_R + i\left(\frac{\hat{O}\vartheta(x)}{\vartheta(x)}\right)_I ;$$

  so with a post-selection waveform $\mathrm{P}(x)$, we have

$${}_P\langle\hat{O}\rangle_T = \langle \mathrm{P}(x)|\hat{O}\mathrm{T}(x)\rangle = \left\langle \mathrm{P}(x)\middle|\mathrm{T}(x)\left(\frac{\hat{O}\mathrm{T}(x)}{\mathrm{T}(x)}\right)\right\rangle$$

- We can now compute the *bandwidth of this operator* to be

$$\sigma^2_{{}_P\langle\hat{O}\rangle_T} = \left\langle P(x)\middle|T(x)\left(\hat{O} - {}_P\langle\hat{O}\rangle_T\right)^2\right\rangle = \left\langle P(x)\middle|T(x)\left(\frac{\hat{O}T(x)}{T(x)} - {}_P\langle\hat{O}\rangle_T\right)^2\right\rangle$$

- When $\hat{O}$ is not Hermitian, this expression cannot be simplified easily, but it can be deconstructed into real and imaginary components.

- Note, from this expression for $\sigma^2_{{}_\mathrm{P}\langle\hat{O}\rangle_\mathrm{T}}$, we can write: $B_{AM}{}^2\left({}_P\langle\hat{O}\rangle_T\right) = \left\langle P(x)\middle|T(x)\left(\left(\frac{\hat{O}T(x)}{T(x)}\right)_I\right)^2\right\rangle$,

  and $B_{FM}{}^2\left({}_P\langle\hat{O}\rangle_T\right) = \left\langle P(x)\middle|T(x)\left(\left(\frac{\hat{O}T(x)}{T(x)}\right)_R - {}_P\langle\hat{O}\rangle_T\right)^2\right\rangle$ for the post-selection AM and FM components.

PART 4: CONVERGENCE OF RADAR AND QUANTUM MECHANICS: INSTANTANEOUS OPERATOR

- The fractional components are

$$\tau^2{}_{AM}\left({}_P\langle\hat{O}\rangle_T \right) = \frac{B^2{}_{AM}\left({}_P\langle\hat{O}\rangle_T \right)}{\sigma^2_{\;{}_P\langle\hat{O}\rangle_T}},$$

  and

$$\tau^2{}_{FM}\; {}_P\langle\hat{O}\rangle_T = \frac{B_{FM}{}^2\left({}_P\langle\hat{O}\rangle_T \right)}{\sigma^2_{\;{}_P\langle\hat{O}\rangle_T}}.$$

  Note, in addition, that we have

$$\tau^2{}_{AM}\left({}_P\langle\hat{O}\rangle_T \right) + \tau^2{}_{FM}\; {}_P\langle\hat{O}\rangle_T = 1$$

- The instantaneous operator has real and imaginary parts like a weak value, so they contain useful information about what we are measuring from a quantum mechanics perspective.

Part 4: Convergence of Radar & Quantum Mechanics: Post-Selection & Matching

- Cross-correlation measures the similarity of two different signals while autocorrelation is a means to find repeated periodic patterns within the signal.
- A non-zero auto-correlation tells us that there is an underlying pattern in the components of a signal $s(t)$.
- For the scattering operators, the post-selection process is straightforward, the proper ratio of the fields allows us to choose an angle to mix with the return signal tuned to a particular scattering matrix.
- As we have discussed in this document, we can always view the return signal as an operator $\hat{H}$ acting on the transmitted signal s(t).
- The operator can assumed to be Hermitian under the circumstances that arise in radar and other remote sensing applications.

- It can be written explicitly as $\hat{M} = e^{i\alpha\hat{A}}$, so the return signal can be written in terms of the transmitted signal as $s_r(t) = \hat{M}s(t) = e^{i\alpha\hat{A}}s(t)$ where s(t) is the transmitted signal.

- If we have a return signal that can be represented as $e^{i\alpha\hat{A}}s(t)$ where the energy of the signal $s(t)$ is finite; then the problem that we want solve is to maximize the system response at a time t_0 of the post-selection system $h(t)$, then to obtain the system response subject to the set of constraint relations i:

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\alpha\hat{A}} s(t)\Phi_i(t)dt = \vartheta_i.$$

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Part 4: Convergence of Radar and Quantum Mechanics: Post-Selection & Matching

- For radar tracking a moving object, the operator could be deconstructed as $e^{i\alpha\hat{A}} = e^{i\alpha\hat{H}} e^{t_0\frac{d}{d\tau}}$ where τ_0 is the delay time to the target and back.
- In the parlance of normal radar, α would be taken to be zero and this operator would be the standard solution like the matched filter.
-
- Equality is achieved if $h(t_0) = k\left[e^{i\alpha\hat{A}} s(\tau_0) + \sum_{i=1}^{n} \beta_i u_i(\tau_0)\right]$ where $\hat{A} = \hat{A}(\tau_0)$.
- This gives a method for choosing the waveform to achieve maximum response for a given set of constraints.
- The traditional matched filter for radar is equivalent to this expression with $\beta_i = 0$ and $\hat{A} = \hat{\mathcal{W}}$, so $h(t_0) = ks^*(\tau_0 + \alpha)$ where k is an arbitrary constant.
- Take $f_p(\tau) = h(t_0)$, we have one approach to obtaining the waveform.
- The same idea could be applied to design of quantum mechanical "receivers" to maximize response for a given operator $\hat{A}$.

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Part 4: Convergence of Radar and Quantum Mechanics: Post-Selection & Matching

- For radar tracking a moving object, the operator could be deconstructed as $e^{\iota\alpha\hat{A}} = e^{\iota\alpha\hat{H}} e^{t_0 \frac{d}{d\tau}}$ where τ_0 is the delay time to the target and back.
- In the parlance of normal radar, α would be taken to be zero and this operator would be the standard solution like the matched filter.
-
- Equality is achieved if $h(t_0) = k\left[e^{\iota\alpha\hat{A}} s(\tau_0) + \sum_{i=1}^{n} \beta_i u_i(\tau_0)\right]$ where $\hat{A} = \hat{A}(\tau_0)$.
- This gives a method for choosing the waveform to achieve maximum response for a given set of constraints.
- The traditional matched filter for radar is equivalent to this expression with $\beta_i = 0$ and $\hat{A} = \hat{\mathcal{W}}$, so $h(t_0) = ks^*(\tau_0 + \alpha)$ where k is an arbitrary constant.
- Take $f_p(\tau) = h(t_0)$, we have one approach to obtaining the waveform.
- The same idea could be applied to design of quantum mechanical "receivers" to maximize response for a given operator $\hat{A}$.

Part 5: Post-Selection & Polarization

- If the scattering matrix $\widehat{\mathrm{M}}_S$ represents the target object-broadcast waveform interaction, then the scattered signal is $|E^R(t)\rangle = \widehat{\mathrm{M}}_S|E^T(t)\rangle$ and the associated cross-correlation amplification of $\widehat{\mathrm{M}}_S$ is $(\langle E^R(t)|E^T(t)\rangle \neq 0)$

$$M_{post} = \frac{\langle E^R(t)|\widehat{M}_S|E^T(t)\rangle}{\langle E^R(t)|E^T(t)\rangle}.$$

Here, $\langle E(t)|$ denotes the Hermitian conjugate transpose of $|E(t)\rangle$, i.e. $\langle E(t)| = [E_1^* \quad E_2^*]$. Note, that since $(\langle E^R(t)|$and $|E^T(t)\rangle$ are assumed to be normalized, we have $|\langle E^R(t)|E^T(t)\rangle| \geq 1$, then $|E^T(t)\rangle$ and $\langle E^R(t)|$ an be chosen to produce a large relocation of $\langle E^R(t)|\widehat{\mathrm{M}}_S|E^T(t)\rangle$.

- Even without normalization, post-selection we can still measure the effect of operators on a signal which are not evident by basing measurement on the mean of the scale or translation operators.
- Polarization interactions can be written in terms of the Pauli matrix operators

$$\widehat{\Phi}_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \hat{I}, \quad \widehat{\Phi}_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \widehat{\Phi}_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \widehat{\Phi}_3 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix},$$

  where $\hat{I}$ is the identity matrix.
- These simple operators can be used to construct scattering matrices for specific target geometries of interest).
- The associated cross correlation amplifications assume the usual form

$$\Phi_{k,post} = \frac{\langle E^R(t)|\widehat{\Phi}_k|E^T(t)\rangle}{\langle E^R(t)|E^T(t)\rangle}, \qquad k = 0,1,2,3.$$

  In the following, it is assumed that $|E^T(t)\rangle$ and $\langle E^R(t)|$ are real valued.

Part 5: Post-Selection & Polarization

Example: When $k = 1$, then we have

$$\Phi_{1,p} = \frac{1 - tan\theta_R tan\theta_T}{1 + tan\theta_R tan\theta_T} = \frac{cos(\theta_R + \theta_T)}{cos(\theta_R - \theta_T)}$$

where

$$tan\theta_{R,T} = \frac{E_y^{R,T}}{E_x^{R,T}}.$$

Observe that $\Phi_{1,p}$ can be relocated to an arbitrarily large negative location when $\theta_R = \frac{\pi}{4} + \epsilon$ and $\theta_T = -\frac{\pi}{4}$ since $\Phi_{1,p} = -cot\epsilon \approx -\frac{1}{\epsilon}$, where $0 < \epsilon \ll 1$.

Example: Similarly, when $k = 2$, then

$$\Phi_{2,p} = \frac{tan\theta_R + tan\theta_T}{1 + tan\theta_R tan\theta_T} = \frac{sin(\theta_R + \theta_T)}{cos(\theta_R - \theta_T)}$$

so that $\Phi_{2,cross}$ can be relocated to an arbitrarily large positive location when $\theta_R = \frac{\pi}{2} - \epsilon$ and $\theta_T = 0$ since $\Phi_{2,p} = cot\epsilon \approx \frac{1}{\epsilon}$, where $0 < \epsilon \ll 1$.

Example: When $k = 3$, we have

$$\Phi_{3,p} = -i\left[\frac{tan\theta_R - tan\theta_T}{1 + tan\theta_R tan\theta_T}\right] = -i\left[\frac{sin(\theta_R + \theta_T)}{cos(\theta_R - \theta_T)}\right]$$

so that $\Phi_{2,p}$ can be relocated to an arbitrarily large positive location when $\theta_R = \frac{\pi}{2} - \epsilon$ and $\theta_T = 0$ since $\Phi_{3,cross} = -icot\epsilon \approx -\frac{i}{\epsilon}$, where $0 < \epsilon \ll 1$.

Part 5: Post-Selection & Polarization

- It is also convenient at this point to evaluate for use in the next section the cross correlation amplifications

$$\Phi_{k,p} = \frac{\langle E^R(t)|\widehat{\mathrm{P}}_k|E^T(t)\rangle}{\langle E^R(t)|E^T(t)\rangle}, \qquad k = 0,1,2,3.$$

where $\widehat{\mathrm{P}}_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $\widehat{\mathrm{P}}_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $\widehat{\mathrm{P}}_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $\widehat{\mathrm{P}}_3 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

- The electric field component selection matrices:

$$\mathrm{P}_{0,p} = \frac{1}{1 + tan\theta_R tan\theta_T},$$
$$\mathrm{P}_{1,p} = \frac{tan\theta_T}{1 + tan\theta_R tan\theta_T},$$
$$\mathrm{P}_{2,p} = \frac{tan\theta_R}{1 + tan\theta_R tan\theta_T},$$

and

$$\mathrm{P}_{3,p} = \frac{tan\theta_R tan\theta_T}{1 + tan\theta_R tan\theta_T},$$

- In a manner analogous to the above development, it is easily shown that $\mathrm{P}_{0,p}$ and $\mathrm{P}_{3,p}$ are relocated to 1 , and that $\mathrm{P}_{1,p}$ and $\mathrm{P}_{2,p}$ are relocated to arbitrarily large locations when $\theta_R = \frac{\pi}{4}$ and $\theta_T = \frac{\pi}{4} + \epsilon$ where $0 < \epsilon \ll 1$.

Part 5: Post-Selection, Polarization & Scattering

- These scattering operators are special cases (two dimensional matrices) of the more general operators (four dimensional matrices).
- The scattering operators for spheres, planes, or horizontally (^{h}) or vertically (^{v}) oriented triangular corner reflectors are $\hat{S}(h) = \hat{I}$ and $\hat{S}(v) = i\hat{\Phi}_2$ respectively.
- Consequently, $\hat{S}_{cross}(h)$ is relocated to 1 and since $\Phi_{2,cross}$ can be relocated to an arbitrarily large location, $\hat{S}_{cross}(v)$ can be relocated to an arbitrarily large imaginary location.

Example: The scattering operators for dipoles oriented at an angle α measured from the positive horizontal axis are

$$\hat{d}(h,\alpha) = \begin{bmatrix} cos^2\alpha & \frac{1}{2}sin2\alpha \\ \frac{1}{2}sin2\alpha & sin^2\alpha \end{bmatrix} = \left(\frac{1}{2}sin2\alpha\right)\hat{\Phi}_2 + (cos^2\alpha)\hat{P}_0 + (sin^2\alpha)\hat{P}_3$$

and

$$\hat{d}(v,\alpha) = \begin{bmatrix} cos^2\alpha & \frac{1}{2}sin2\alpha \\ \frac{1}{2}sin2\alpha & sin^2\alpha \end{bmatrix} = -\frac{i}{2}\hat{\Phi}_2 + e^{2i\alpha}\hat{P}_0 + e^{-2i\alpha}\hat{P}_3$$

Since $\Phi_{2,cross}$ can be relocated to an arbitrarily large location, then so can $\hat{d}_{cross}(h,\alpha)$ and $\hat{d}_{cross}(v,\alpha)$.

PART 6: REMOTE SENSING

- Remote Sensing can be cast into the language post-selection, particularly when one is getting a return signal from a multi-static sensor network.
- LIGO and radio astronomy are examples where one assumes one knows the return signal, but not the broadcast signal, so one needs to pre-select the signal in a manner that determines $\odot$ in $\langle E^R(t)|\hat{M}_s|\odot\rangle$ we can assume to have measured the remainder.
- Neutrino physics is naturally formulated as a two state system, so post-selection remains a viable option if a Mach-Zhender interferometer or a polarizer technique for neutrinos could be found.
- Polarization radar exists and is starting to be used in a variety platforms including satellites. Post-selection for specific scattering attributes of a landscape that can be formulated in terms of the polarization operators.
- In general, any sensing problem that can be cast in terms of the polarization matrices, or their higher dimensional analogs, is subject to post-selection and its potential benefits.

PART 7: VARIATIONAL FUNCTIONAL

- If we just play with an equation that was defined long before weak measurement was proposed, the Variational Functional one encounters quantum mechanical perturbation theory:

$$\Lambda = \frac{\langle\varphi|\hat{A}|\psi\rangle}{\langle\vartheta|\psi\rangle}$$

  what does it tell us?

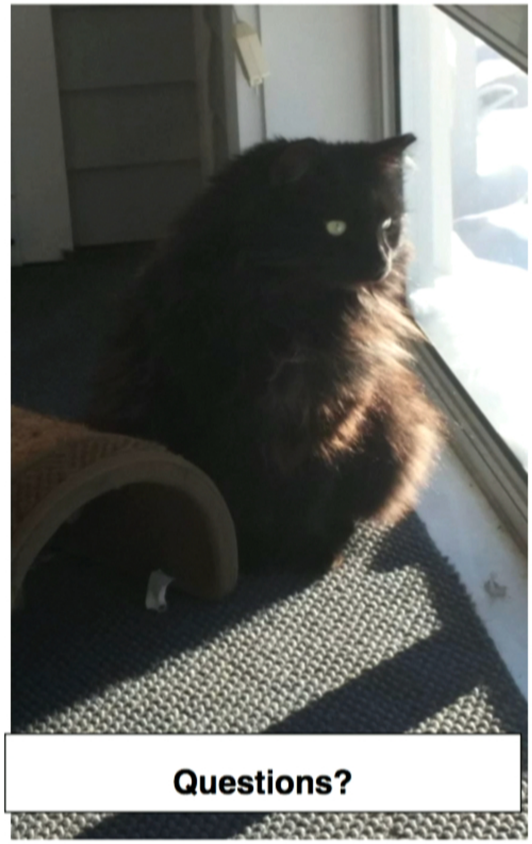
- Normally, we just use it to get the lowest energy eigenvalue when we can't find an exact solution (see Ballintine for numerous examples).

- By varying Λ with respect to $|\vartheta\rangle$ and $|\psi\rangle$, then one obtains two Schrodinger like equations for $|\psi\rangle$ and the complex conjugate of $|\varphi\rangle$. When $\hat{A} = \hat{A}^{\#}$, the operator is Hermitian operator and we are back to normal quantum mechanics. But what if we don't?

- Use Λ as the defintion of what the measurement of the operator $\hat{A}$ is subject to constraints such as $\langle\psi|\psi\rangle = 1$ and $\langle\vartheta|\vartheta\rangle = 1$, $1 > |\langle\vartheta|\psi\rangle| = \varepsilon > 0$, noise, and the interpretation of the inner product ("New Interpretation of the Scalar Prouct in Hilbert Space", A[3], PRL, Vol 47 # 15.). What does that tell us about state evolution?

- There are a variety questions that a functional starting point for weak measurement leaves open that are worth considering.

- Landau and Lifshitz *Quantum Mechanics* (pp 56-8): "*Schrodinger's equation can be obtained from the variational principle* $\delta\int\psi^*(\hat{H}-E)\psi dq = 0$. *Since* ψ *is complex, we can vary* ψ *and* ψ^* *independently. ..., we obtain the required solution* $(\hat{H} - E)\psi = 0$. *The variation of* ψ *gives nothing different.*" (This observation is where I first encounter post-selection when I was an undergraduate taking a summer class on quantum mechanics with a bunch of graduate students from Gordon Baird, though I did not know it at the time.)


Questions?


3:30 Lucien Hardy: Operational General Relativity
Detector:
Quantum Point Contact
Double
Quantum
Dot
Questions?
Majorana
Dirac