Title: Uncertainty and Complementarity Relations with Weak values

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Abstract: The products of weak values of quantum observables have interesting implications in deriving quantum uncertainty and complementarity relations for both weak and strong measurement statistics. We show that a product representation formula allows the standard Heisenberg uncertainty relation to be derived from a classical uncertainty relation for complex random variables. This formula also leads to a strong uncertainty relation for unitary operators which displays a new preparation uncertainty relation for quantum systems.

Furthermore, the two system observables that are weakly and strongly measured in a weak measurement context are shown to obey a complementarity relation under the interchange of these observables, in the form of an upper bound on the product of the corresponding weak values.

Moreover, we derive general tradeoff relations, between weak purity, quantum purity and quantum incompatibility using the weak value formalism.

Our results may open up new ways of thinking about uncertainty and complementarity relations using products of weak values.

UNCERTAINTY AND COMPLEMENTARITY RELATIONS WITH WEAK VALUES

ARUN PATI

Quantum Information and Computation Group
Harish-Chandra Research Institute
Allahabad-211019, India
Email: akpati@hri.res.in

Harish-Chandra Research Institute
हरीश-चन्द्र अनुसंधान संस्थान

Outlines

- Introduction
- Product of weak values
- Uncertainty relation

Introduction

- Quantum theory has many counter intuitive features such as wave-particle duality, interference, entanglement and non-locality, and these features make the subject exciting even after ninety years since its initial formulation.

- The weak value can have strange properties. It is a complex number in general, and its real part can take values outside the spectrum of the observable being measured.
- The concepts of weak measurements and weak values have since been generalized in various directions and have found numerous applications. However, while the separate real and imaginary components of weak values have been given various interpretations in the literature, the weak value itself as a complex number has not.

Weak Measurement

Applications

- Strong measurement reveals one aspect of quantum system. Weak measurement can reveal incompatible aspects of quantum system.
- Measure wave function directly [Lundeen *et al* NATURE (2011)].
- Amplify weak signals [Wiseman (2003)].
- Probing average trajectory in interference setup [Kocsis *et al* Science (2011)] .
- See Aharonov-Popsecu-Tollaksen, Phys. World (2010); Steinberg, Nature (2010); Cho, Science (2011); Dressel, Malik, Miatto, Jordan and Boyd, Rev. Mod. Phys. (2014).

- It is well known that weak values respect sums, but not products. That is, the weak value of the sum of two observables is just the sum of their weak values, whereas the weak value of the product of the two observables is not equal to the product of their weak values.

What we do

- Quantum average of any product can be reconstructed from the weak values of the corresponding observables, and in this sense weak values provide a hidden variable model for the averages of a given set of quantum observables and their products.
- This formula provides a simple derivation of the Heisenberg uncertainty relation. The latter may be reinterpreted as a classical dispersion relation for complex random variables.
- Obtain a strong uncertainty relation for unitary operators which gives a new kind of preparation uncertainty relation in quantum mechanics.

- Obtain a complementarity relation for the weak values of two non-commuting projection operators, that restricts the degree to which they can take anomalous values outside the interval [0, 1].
- If we weakly measure the projection $|a\rangle\langle a|$ with postselection on projection $|b\rangle\langle b|$, and vice versa then the product of the corresponding weak values for these complementary scenarios is restricted to one for all pure initial states.
- General tradeoff relations that connect the weak joint probability distribution of two observables with their degree of incompatibility.

Weak Values

- Weak value of a Hermitian operator A

$$A_w(\phi|\psi) := \frac{\langle\phi|A|\psi\rangle}{\langle\phi|\psi\rangle}. \quad (1)$$

- Postselected state $|\phi\rangle$ may be replaced by $|m\rangle$ with $M \equiv \{|m\rangle\langle m|\}$ and $\sum_m |m\rangle\langle m| = \hat{1}$, leading to the expression

$$A_w(m|\psi) := \frac{\langle m|A|\psi\rangle}{\langle m|\psi\rangle}. \quad (2)$$

- Since the probability of measurement outcome m on state $|\psi\rangle$ is $p(m|\psi) = |\langle m|\psi\rangle|^2$, it follows that the average of the weak value

$$\langle A_w\rangle_p := \sum_m p(m|\psi)\, A_w(m|\psi) = \langle\psi|A|\psi\rangle =: \langle A\rangle_\psi, \quad (3)$$

This reconstruction formula holds for non-Hermitian operators A via linearity.

- The above reconstruction formula for the average of an observable may be extended to a similar formula for products [Shikano-Hosoya (2010)] (generalised here to arbitrary non-Hermitian operators):

$$\sum_m p(m|\psi)\, A_w(m|\psi)^*\, B_w(m|\psi) = \langle\psi|A^\dagger B|\psi\rangle$$

$$\langle A_w^* B_w\rangle_p = \langle A^\dagger B\rangle_\psi, \qquad (4)$$

- Average of an operator product, with respect to a quantum state $|\psi\rangle$, can be replaced by a average of a product of weak values, with respect to the classical probability distribution $p(m|\psi)$.

- The product representation formula has a clear operational significance. For example, if the weak values of two Hermitian operators A and B, postselected on measurement M, are determined experimentally, then one can immediately recover not only the averages of the observables A and B, but also the averages of A^2, B^2, and AB—and, hence, the variances and covariances of A and B.

- One can also experimentally recover the average of the operator $(A-B)^2$ from the weak values of A and B, where this average appears in various error-disturbance and joint-measurement uncertainty relations.

- Weak values also provide a (complex) hidden variable model for the averages of a given set of quantum observables and their pairwise products.

- For example, the average values of all linear and quadratic functions of the annihilation and creation operators a and $a^\dagger$ of a single mode field, including the quadrature observable $X_\theta = ae^{i\theta} + a^\dagger e^{-i\theta}$ and the number operator $a^\dagger a$, can be modelled for any state $|\psi\rangle$, via the corresponding weak values $a_w(m|\psi)$ and $a^\dagger_w(m|\psi)$ and classical probability distribution $p(m|\psi)$ (and for any measurement $M \equiv \{|m\rangle\langle m|\})$.

Uncertainty relations from weak values

- Complex random variables are standard tools in classical signal processing and information theory.
- A complex random variable $\alpha = \alpha_1 + i\alpha_2$ is described by some real and positive probability density $p(\alpha)$. The expectation value of function $f(\alpha)$ is then given by $\langle f(\alpha)\rangle := \int d\alpha\, p(\alpha) f(\alpha)$, where the integral is over the complex plane with respect to the uniform measure.
- A well known example in quantum mechanics: The measurement described by the coherent state POVM $\{\pi^{-1}|\alpha\rangle\langle\alpha|\}$ that results the probability density for field state ρ given by the Husimi Q-function $p(\alpha) = \pi^{-1}\langle\alpha|\rho|\alpha\rangle$.

- The variance of α is just the average mean square distance between α and its mean value, i.e.,

$$Var\,\alpha := \langle\, |\alpha - \langle\alpha\rangle|^2 \rangle = \langle\, |\alpha|^2 \rangle - |\langle\alpha\rangle|^2. \tag{5}$$

- Similarly, the covariance of two such random variables, α and β, with respect to a joint probability distribution $p(\alpha, \beta)$, is defined by

$$Cov(\alpha, \beta) := \langle\, (\alpha - \langle\alpha\rangle)^*(\beta - \langle\beta\rangle)\rangle = \langle\, \alpha^*\beta\rangle - \langle\alpha^*\rangle\langle\beta\rangle \tag{6}$$

with $\langle f(\alpha, \beta)\rangle := \int d\alpha d\beta\, p(\alpha, \beta)\, f(\alpha, \beta)$. Thus, $Var\,\alpha = Cov(\alpha, \alpha)$.

- One can have the *classical* uncertainty relation [F. D. Nesser and J. L. Massey, IEEE Trans. Inf. Theory, **39**, 1293 (1993); B. Hajek, *Random Processes for Engineers*, (Cambridge University Press, 2015)].

- Classical uncertainty relation for complex random variables

$$Var\,\alpha\;Var\,\beta \geq |Cov(\alpha,\beta)|^2 \tag{7}$$

from the Schwarz inequality for complex numbers.

- Choosing $\alpha = A_w(m|\psi)$, $\beta = B_w(m|\psi)$, and probability distribution $p(m|\psi)$, this classical uncertainty relation reduces to

$$Var_p A_w\;Var_p B_w \geq |Cov_p(A_w, B_w)|^2\,. \tag{8}$$

for the case of weak values.

Weak Values Classicalize Heisenberg uncertainty relation

- The standard Heisenberg uncertainty relation for two non-commuting observables follows directly from the product representation formula and the classical uncertainty relation.

- For two Hermitian operators A and B (decomposing the right hand side into real and imaginary parts) yields

$$Var_{\psi}A \, Var_{\psi}B \geq |\langle AB\rangle_{\psi} - \langle A\rangle_{\psi}\langle B\rangle_{\psi}|^2$$

$$= Cov_{\psi}(A,B)^2 + \frac{1}{4}\,|\langle [A,B]\rangle_{\psi}|^2\,,$$

  with the quantum covariance defined by $Cov_{\psi}(A,B) := \frac{1}{2}\langle AB + BA\rangle_{\psi} - \langle A\rangle_{\psi}\langle B\rangle_{\psi}$.

- Thus, the standard quantum uncertainty relation may be reinterpreted as a *classical* uncertainty relation for weak values.

- It is a curious fact that one of the fundamental relations of quantum mechanics can be understood as a classical uncertainty relation for complex random variables.
- Recently, strongre uncertainty relations have been proved which go beyond the Robertson-Schrödinger uncertainty relation [Maccone-Pati (2014)].

$$\Delta A^2 + \Delta B^2 \geq \pm i\langle[A, B]\rangle + \left|\langle\psi||(A \pm iB)||\psi^\perp\rangle\right|^2 \tag{9}$$

which is valid for arbitrary states $|\psi^\perp\rangle$ orthogonal to the state of the system $|\psi\rangle$, where the sign should be chosen so that $\pm i\langle[A, B]\rangle$ (a real quantity) is positive.

Uncertainty relations for General operators

- Variance of a general operator [Levy-Leblond (1976), Anandan (1990), Pati-Singh-Sinha (2015)]

$$Var_\psi A := \langle A^\dagger A\rangle_\psi - |\langle A\rangle_\psi|^2 = Var_p A_w, \tag{10}$$

  where the second equality follows from the product representation formula.

- This has the desirable properties of vanishing if and only if $|\psi\rangle$ is an eigenstate of A, and of reducing to the usual variance in the Hermitian case $A = A^\dagger$.

- Annihilation and creation operators a and $a^\dagger$ of a single-mode bosonic field satisfy $[a, a^\dagger] = 1$, their variances are related by

$$Var\ a^\dagger = \langle aa^\dagger\rangle - |\langle a\rangle|^2 = Var\ a + 1 \geq 1, \tag{11}$$

  implying immediately that $a^\dagger$ has no eigenstates.

Uncertainty Relations for General Operators

- The classical uncertainty relation yields the generalization

$$Var_{\psi}A\, Var_{\psi}B \geq |Cov_{p}(A_{w}, B_{w})|^{2}$$

$$= |\langle A^{\dagger}B\rangle_{\psi} - \langle A^{\dagger}\rangle_{\psi}\langle B\rangle_{\psi}|^{2}$$

- Second line follows via the product representation formula.

Uncertainty Relation for Unitary Operators

- Uncertainty in unitary operator $\Delta U^2 = 1 - |\langle\psi|U|\psi\rangle|^2$.
- If we send a particle in Mach-Zehnder interferometer and apply U in one arm of the interferometer, visibility in interference $|\langle\psi|U|\psi\rangle|^2$.

Uncertainty Relation for Unitary Operators

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- If we send a particle in Mach-Zehnder interferometer and apply U in one arm of the interferometer, visibility in interference $|\langle\psi|U|\psi\rangle|^2$.
- Uncertainty in U places restriction on the interference visibility.
- A new preparation UR: Uncertainty relation for two unitary operators suggests that quantum states cannot be prepared such that sum of interference visibility due to two non-commuting unitary operators will be arbitrary, i.e., there is a non-trivial upper bound.

- As an application of the generalised classical uncertainty relation consider two unitary operators U and V. Define

$$u := |\langle U \rangle_\psi|, \qquad v := |\langle V \rangle_\psi|,$$

- Uncertainty Relation for two Unitaries [Bagchi-Pati (2016)]:

$$u^2 + v^2 - 2uv|\langle U^\dagger V \rangle_\psi| \cos\Phi \leq 1 - |\langle U^\dagger V \rangle_\psi|^2. \qquad (12)$$

  Here Φ denotes the phase of the complex number $\langle U \rangle_\psi \langle U^\dagger V \rangle_\psi \langle V^\dagger \rangle_\psi$, where the latter is the Bargman invariant associated with $|\psi\rangle$, $U|\psi\rangle$ and $V|\psi\rangle$.
- Note that $|\langle U^\dagger V \rangle_\psi|^2$ is the overlap of the states $U|\psi\rangle$ and $V|\psi\rangle$. Hence, the overlap plays a role analogous to the commutator in the Heisenberg uncertainty relation.

Complementarity of weak values

- In quantum theory, complementarity imposes limitations on our ability to unambiguously define and measure aspects of quantum systems in a single measurement setup.
- Bohr "...it is only the mutual exclusion of any two experimental procedures, permitting the unambiguous definition of complementary physical quantities, which provides room for new physical laws, the coexistence of which might at first sight appear irreconcilable with the basic principles of science. ..."
- In the context of weak measurements, it is possible that one can probe two complementary aspects of a quantum system at some price (e.g. introducing noise), as the apparatus interacts with the system weakly, allowing a gentle observation without disturbing the system too much.

- However, there is a strong type of complementarity between a given weak measurement procedure, and the mutually exclusive procedure obtained by interchanging the weak and the strong components.

- Weak measurements involving projection operators

$$A^a = |a\rangle\langle a|, \quad B^b = |b\rangle\langle b|,$$

corresponding to the eigenvalue decompositions of two nondegenerate observables $A = \sum_a aA^a$ and $B = \sum_b bB^b$.

- For a given initial state $|\psi\rangle$, consider a weak measurement of projector A^a postselected on state $|b\rangle$, i.e, on $B = b$, and next a weak measurement of projector B^b postselected on state $|a\rangle$, i.e., on $A = a$. The corresponding weak values

$$A^a_w(b|\psi) = \frac{\langle b|a\rangle\langle a|\psi\rangle}{\langle b|\psi\rangle},$$

$$B^b_w(a|\psi) = \frac{\langle a|b\rangle\langle b|\psi\rangle}{\langle a|\psi\rangle}.$$

- Next, we ask can these two weak values be arbitrarily large at the same time? Strangely, not.

- First, note that the weak values for the projectors A^a and B^b can be expressed as the sum of the average of the projectors in the state $|\psi\rangle$ plus an anomalous part [Pati-Wu (2014)]

$$A^a_w(b|\psi) = \langle A^a\rangle_\psi + \Delta_\psi A^a \frac{\langle b|\bar{\psi}_a\rangle}{\langle b|\psi\rangle},$$
$$B^b_w(a|\psi) = \langle B_b\rangle_\psi + \Delta_\psi B^b \frac{\langle a|\bar{\psi}_b\rangle}{\langle a|\psi\rangle}, \tag{14}$$

where $\Delta_\psi A^a := (Var_\psi A^a)^{1/2}$ is the uncertainty of the projector in the state $|\psi\rangle$, $|\bar{\psi}_a\rangle$ is a state orthogonal to $|\psi\rangle$, and similar definitions hold for the other projector Π_b [Vaidman (1992)].

- This shows that the weak values of these projectors can be large, and lie outside the eigenvalue range $[0, 1]$ of the projectors.

- However, both weak values cannot be large at the same time.
- The product of these weak values satisfies

$$A_w^a(b|\psi)\, B_w^b(a|\psi) = |\langle a|b\rangle|^2 \leq 1. \quad (15)$$

- Thus, even though individually each of these weak values can be complex, with arbitrarily large moduluses, their product is real, independent of the pre-selected state, and bounded by unity.

Weak joint probabilities and weak purity

- The weak values $A^a(b|\psi)$ and $A^a_w(a|\rho)$ for the projection operator A^a postselected on measurement result $B = b$, are sometimes referred to as 'weak probabilities'.

- However, here we will follow the common practice of identifying the real part of this quantity as a weak probability. More generally, for two arbitrary POVM observables $A = \{A_a\}$ and $B = \{B_b\}$ we define the weak conditional probability of $A = a$ postselected on outcome $B = b$ by

$$p_w(a|b) := Re\frac{Tr\rho B_b A_a}{Tr\rho B_b}. \tag{17}$$

Note that it is not assumed that A_a and B_b are rank-1 projection operators.

- The weak conditional probability distribution is physically measurable via suitable weak measurements, as is the corresponding weak *joint* probability distribution.

$$p_w(a, b) := p_w(a|b)\, p(b) = \frac{1}{2}\langle A_a B_b + B_b A_a \rangle_\rho, \qquad (18)$$

- Here $p(b) = Tr \rho B^b$ is the probability of measurement outcome $B = b$. The weak joint probability distribution may also be recognised as the Margeneau-Hill quasiprobability distribution, and satisfies

$$\sum_b p_w(a, b) = p(a), \sum_a p_w(a, b) = p(b), \qquad (19)$$

just as for classical joint distributions. However, the weak joint probabilities $p_w(a, b)$ can take anomalous values.

Incompatibility

- Incompatibility between two POVM elements A_a and B_b in a quantum state ρ is given by

$$I(a,b) := \frac{1}{4}|\langle [A_a, B_b]\rangle_\rho|^2 \geq 0.$$

- Note that $I(a,b)$ vanishes if and only if $[A_a, B_b]\rho = 0$. It follows from a result of Busch-Heinosaari (2008) that if one of A_a and B_b is a projector, then $I(a,b) = 0$ is equivalent to $p_w(a,b) \geq 0$. Thus there is a connection between anomalous weak probabilities and incompatibility.

- The incompatibility of two measurements A and B is naturally defined as the total incompatibility of their POVM elements, i.e.,

$$I(A,B) := \sum_{a,b} I(a,b).$$

Incompatibility

- Note that this measure is independent of the particular values assigned to the outcomes of A and B, and hence characterizes incompatibility in an invariant manner. The incompatibility ranges between 0 and 1, i.e.,

$$0 \leq I(A, B) \leq 1.$$

- The lower bound is is reached if and only and if $[A_a, B_b]\rho = 0$ for all a and b. (Schwarz inequality and $0 \leq A_a, B_b \leq 1$ gives $I(a, b) \leq p(a)p(b)$.)

Tradeoff relations for purity and incompatibility

- We have the relation

$$p_w(a,b)^2 + I(a,b) = |\,Tr(\rho A_a B_b)|^2.$$

- Consider the special case of a pure state and nondegenerate (or maximal) observables, i.e.,

$$\rho = |\psi\rangle\langle\psi|, \quad A_a = |a\rangle\langle a|, \quad B_b = |b\rangle\langle b|.$$

Here the states $\{|a\rangle\}$ and $\{|b\rangle\}$ are not assumed to be orthonormal.

$$p_w(a,b)^2 + I(a,b) = |\langle a|b\rangle|^2\, p(a)\, p(b).$$

- It may be shown that this is equivalent to the equality in the complementarity relation. Thus, the latter relation may also be interpreted as relating weak joint probabilities and incompatibility. Further, recalling $I(a,b) \geq 0$, one has the weaker complementarity relation

$$p_w(a|b)\, p_w(b|a) \leq |\langle a|b\rangle|^2 \leq 1,$$

relating weak probabilities for a weak measurement of A postselected on a strong measurement of B and vice versa.

- Thus, if the weak probability of $A = a$, postselected on $B = b$, is greater than 1, then the converse weak probability must be less than 1.

- Summing over a and b yields the tradeoff relation

$$P_W + I(A,B) \leq c_{AB} := \max_{a,b} |\langle a|b\rangle|^2 \leq 1,$$

between the weak purity and the total incompatibility of A and B.

- For the general case of arbitrary states one has from the Schwarz inequality

$$
\begin{aligned}
|Tr\rho A_a B_b|^2 &= |Tr\rho(A_a B_b)|^2 \\
&\leq Tr\rho^2 \; Tr[(A_a)^2 (B_b)^2] \\
&\leq Tr\rho^2 \; Tr[A_a B_b], \qquad (20)
\end{aligned}
$$

where the final inequality makes use of the property $0 \leq A_a, B_b \leq 1$.

- Substitution into then gives

$$p_W(a,b)^2 + I(a,b) \leq Tr\rho^2 \; Tr[A_a B_b],$$

and summation over a and b yields the tradeoff relation

$$P_W + I(A,B) \leq Tr\rho^2 =: P_Q,$$

where P_Q denotes the quantum purity. It also follows, recalling $I(A,B) \geq 0$, that $P_W \leq P_Q$, i.e., the weak purity can never exceed the quantum purity.

- The tradeoff relation generalises to all states and observables, and has several physical implications.
- First, the weak purity is never greater than the classical maximum value of 1, even when some of the weak probabilities are negative, thus restricting the degree to which the weak probabilities can take anomalous values.
- Second, the greater the incompatibility of A and B, the smaller the weak purity, and vice versa.
- Third, the incompatibility of A and B is upper-bounded by the difference between the quantum purity and the weak purity. This difference can, therefore, be considered a resource for incompatibility, analogous to the manner in which the difference between a quantum purity and a classical purity acts as a resource for quantum coherence.

Conclusions

- Weak values of quantum observables have many physical applications.
- Product representation formula can be used to recover various quantum averages from experimentally determined weak values.
- Heisenberg UR is equivalent to a classical uncertainty relation for complex random variables.
- Prove strong uncertainty relation for a pair of unitary operators.

Conclusions

- Weak values of quantum observables have many physical applications.
- Product representation formula can be used to recover various quantum averages from experimentally determined weak values.
- Heisenberg UR is equivalent to a classical uncertainty relation for complex random variables.
- Prove strong uncertainty relation for a pair of unitary operators.
- In weak measurement scenario if the weakly and strongly measured observables are interchanged, there is a complementarity relation for weak values.

Conclusions

- Trade-off relation between weak purities, quantum purities and the degree of incompatibility of two observables.
- Quantify the extent to which weak probabilities can take anomalous values.
- We hope that our results will open up new ways of thinking about uncertainty and complementarity relations using weak values.

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