Title: Phase Space Methods in Quantum Mechanics and Weak Values.

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Abstract: Phase space methods are ubiquitous in quantum mechanics. From the Weyl-Wigner Moyal formalism to coherent states and discrete phase spaces we see the imprints of the classical world again and again. In this presentation, we address one of two major developments introduced by Aharonov and his collaborators: The concept of weak values that stems from a time-symmetric view of quantum physics. We look at the weak measurement through two distinct geometric frames: The geometry of the measuring apparatus and the geometry of the measured system. We Â…nalize with some brief comments on the second major conceptual due to Aharonov and collaborators: the theory of modular variables and how SchwingerÂ´s discrete phase space structure helps to shed light on it. We conclude then with the following mantra: ItÂ´ s all about phase space.

Phase Space Methods in Quantum Mechanics and Weak Values

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June 2016

Weak Values and Modular Variables

Back in the sixties, Aharonov and collaborators introduced the concept of *modular variables* [1] to explain *non-local* dynamical quantum effects as the (AB) effect.

He also introduced the *time-symmetric* [2] formulation of quantum mechanics.

In the late eighties, the time-symmetric formulation led to the concept of *weak values* [3].

- Our main theme is the mathematical structure of weak values.
- Our mantra is "*look at phase space*".
- In the end of the presentation, we will briefly discuss the concept of modular variables also in the domain of (discrete) phase spaces.

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modular variables also in the domain of (discrete) phase spaces.

[1] Aharonov, Y., Pendleton, H., and Petersen, A. International Journal of Theoretical Physics 2(3), 213–230 (1969)

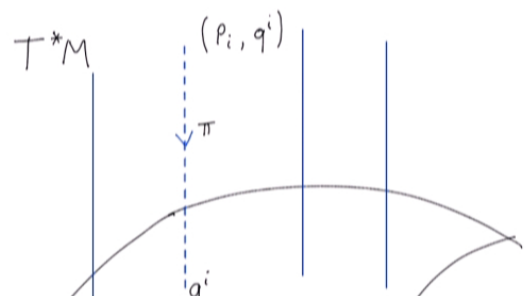
[2] Y. Aharonov, P. G. Bergmann, J. L. Lebowitz, Phys. Rev.134 B1410 (1964)

[3] Y. Aharonov, D. Albert, L. Vaidman, Phys. Rev. Lett. 60, 1351 (1988).

The Symplectic Structure of Phase Space

Classical phase spaces are the *cotangent bundle* of the configuration spaces of particles submitted to holonomic constraints (non-compact spaces).

$$\theta = p_i dq^i \qquad \Omega = -d\theta = -dp_i \wedge dq^i \qquad d\Omega = 0$$



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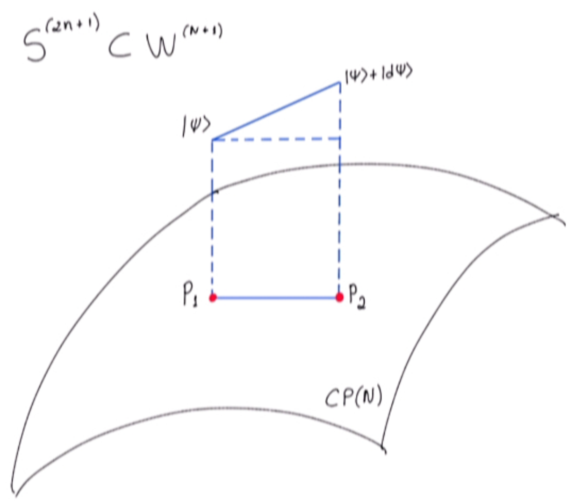
Quantum phase spaces

Quantum phase spaces are the *space of rays* or *projective spaces* $CP(N)$ of the $N+1$-dimensional Hilbert spaces $W^{(N+1)}$.



Quantum projective spaces are *always* compact.

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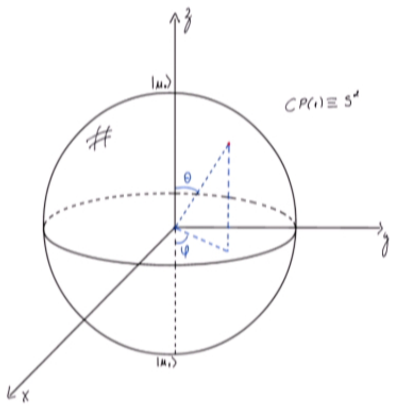
The projective space $CP(N)$ inherits the following structures from the Hilbert space $W^{(N+1)}$:

- A symplectic structure given by a closed 2-form Ω, that is $d\Omega = 0$.
- A Riemannian structure given by a non-degenerate positive definite metric g.

A good example is the $CP(1)$ Bloch sphere for a single qubit:



A general point of the Bloch sphere can be represented by the state

$$|\theta, \varphi\rangle = \cos(\theta/2)\,|u_0\rangle + e^{i\varphi}\sin(\theta/2)\,|u_1\rangle$$

The Fourier Transform Operator and the Quantum Symplectic Group

Consider the quantum motion of a single non-relativistic particle in one dimension (defined below with the following "unusual" notation)

$$\hat{Q}|q(x)\rangle = x|q(x)\rangle \quad \text{and} \quad \hat{P}|p(x)\rangle = x|p(x)\rangle \quad (-\infty < x < \infty)$$

The position and momentum basis are complete and normalized in the sense that

$$\hat{I} = \int_{-\infty}^{+\infty} dx|q(x)\rangle\langle q(x)| = \int_{-\infty}^{+\infty} dx|p(x)\rangle\langle p(x)| \qquad (\hbar = 1)$$

$$\langle q(x)|q(x\prime)\rangle = \langle p(x)|p(x\prime)\rangle = \delta\left(x - x\prime\right)$$

$$\text{with} \qquad \langle q(x)|p(x\prime)\rangle = \frac{1}{\sqrt{2\pi}}e^{ixx\prime}$$

We denote the quantum space generated by these generalized eigenvectors as $W^{(\infty)}$ for reasons that we shall soon see.

The Fourier Transform Operator

The above notation allows us to define the Fourier transform operator as:

$$\hat{F}|q(x)\rangle = |p(x)\rangle$$

The *Weyl-Wigner* operator can be defined in a compact and elegant manner as [4]

$$\hat{\Delta}(p,q) = 2\hat{V}_q^{\dagger}\hat{U}_{2p}\hat{V}_q^{\dagger}\hat{F}^2$$

with translation operators given by

$$\hat{V}_q = e^{iq\hat{P}} \qquad \text{and} \qquad \hat{U}_p = e^{ip\hat{Q}}$$

Actually, $\hat{\Delta}(p,q)$ is a family of hermitian operators parametrized by the phase space points (p,q)

- The Weyl-Wigner operator is *complete* in operator space:

$$\frac{1}{2\pi}\int\int dpdq\hat{\Delta}(p,q)\hat{O}\hat{\Delta}(p,q) = (tr\hat{O})\hat{I}$$

  for all quantum observables $\hat{O}$.
- A *classical* observable is a *real-valued* function defined on phase space.

$$O(p,q) = tr\left(\hat{O}\hat{\Delta}(p,q)\right)$$

- The inner product between two operators $\hat{A}$ and $\hat{B}$ is

$$tr\left(\hat{A}^{\dagger}\hat{B}\right) = \frac{1}{2\pi}\int dxdy\bar{a}(p,q)b(p,q)$$

The transform of a product of two arbitrary operators is the well-known *Moyal star-product*

$$
\begin{aligned}
a(p,q) \star b(p,q) &= tr(\hat{\Delta}(p,q)\hat{A}\hat{B}) = \\
&= a(p,q)\exp\left[-\frac{i}{2}\left(\frac{\overleftarrow{\partial}}{\partial p}\frac{\overrightarrow{\partial}}{\partial q} - \frac{\overleftarrow{\partial}}{\partial q}\frac{\overrightarrow{\partial}}{\partial p}\right)\right] b(p,q)
\end{aligned}
$$

The star-product establishes a *non-commutative algebra* over the classical observables on phase space and it forms the basis of the *non-commutative* geometric point of view for quantization.

Coherent States and the Quantum Symplectic Group

The spectrum of the Fourier Transform operator

Let $\hat{a}$ and $\hat{a}^\dagger$ be respectively the usual *annihilation* and *creation* operators defined as

$$\hat{a} = \frac{1}{\sqrt{2}}(\hat{Q} + i\hat{P}) \quad \text{and} \quad \hat{a}^\dagger = \frac{1}{\sqrt{2}}(\hat{Q} - i\hat{P})$$

such that

$$[\hat{a}, \hat{a}^\dagger] = \hat{I} \quad \text{such that} \quad [\hat{N}, \hat{a}] = -\hat{a} \quad \text{and} \quad [\hat{N}, \hat{a}^\dagger] = \hat{a}^\dagger$$

with

$$\hat{N} = \hat{a}^\dagger \hat{a} \quad \text{and} \quad \hat{N}|n\rangle = n|n\rangle$$

with also

$$\hat{a}|n\rangle = \sqrt{n}||n\rangle \qquad \text{and} \qquad \hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$$

$$\hat{a}|0\rangle = 0 \qquad \langle q(x)|0\rangle = \langle p(x)|0\rangle = \pi^{-1/4}e^{-x^2/2}$$

which allows us to conclude that the *vacuum state* is a *fixed point* of the Fourier Transform operator

$$\hat{F}|0\rangle = |0\rangle$$

The translation operators $\hat{U}_p = e^{ip\hat{Q}}$ and $\hat{V}_q = e^{iq\hat{P}}$ obey the following important relations:

$$\hat{F}\hat{V}_q\hat{F}^\dagger = \hat{U}_q^\dagger \qquad \text{and} \qquad \hat{F}\hat{U}_p\hat{F}^\dagger = \hat{V}_p$$

which implies for the *parity operator* $\hat{F}^2$:

$$\hat{F}^2\hat{U}_p\hat{F}^2 = \hat{U}_p^\dagger \qquad \text{and} \qquad \hat{F}^2\hat{V}_q\hat{F}^2 = \hat{V}_q^\dagger$$

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The infinitesimal versions of the last equations are

$$\hat{F}^{\dagger}\hat{a}\hat{F} = i\hat{a} \qquad \text{and} \qquad \hat{F}^{\dagger}\hat{a}^{\dagger}\hat{F} = -i\hat{a}^{\dagger} \tag{1}$$

Which implies that the Fourier transform operator *commutes* with the number operator

$$[\hat{F}, \hat{N}] = 0$$

So they necessarily share the *same* eigenstates. From the vacuum state fixed point property together with (1), it follows that:

$$\hat{F}|n\rangle = (i)^n|n\rangle \qquad \text{and} \qquad \hat{F} = e^{i\frac{\pi}{2}\hat{N}}$$

which implies that $\hat{N}$ is the hermitian generator of the Fourier transform. It is natural to introduce the *Fractional Fourier Operator* given by[5]:

$$\hat{F}_{\theta} = e^{i\theta\hat{N}}$$

where the usual Fourier transform operator is then recognized as a special

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The Quantum Linear Symplectic Transforms

We can extend the Fractional Fourier transform to the full group of *area preserving* linear maps of classical phase plane $SL(2,\mathbb{R})$. We identify the complex plane with the phase plane through the standard complex-valued coherent states:

$$z = \frac{1}{\sqrt{2}}(q + ip) \qquad |z\rangle = \hat{D}[z]|0\rangle$$

and

$$\hat{D}[z] = \frac{1}{2}\hat{\Delta}(z/2)\hat{F}^2 = e^{(z\hat{a}^\dagger - z\hat{a})}$$

Where $\hat{D}[z]$ is the displacement operator and the "over-completeness" of the coherent states follows from the completeness of the $\{\hat{\Delta}(z)\}$ basis. The *overlap* between two arbitrary coherent states is given by

$$\langle z|z\prime\rangle = \langle p, q|p\prime, q\prime\rangle = e^{-1/4(p-p\prime)^2)+(q-q\prime)^2} e^{i(p\prime q - pq\prime)/2}$$

- Note the symplectic area defined by the vectors (p, q) and $(p\prime, q\prime)$ in the complex phase above.

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One also has

$$\hat{F}_\theta |z\rangle = e^{i\theta\hat{N}}|z\rangle = |e^{i\theta}z\rangle$$

which is a direct manner to represent the Fractional Fourier transform for arbitrary θ.

A complete set of quadratic Hermitian functions of $\hat{Q}$ and $\hat{P}$ are

$$\hat{H}_0 = \frac{1}{2}(\hat{Q}^2 + \hat{P}^2) \qquad \hat{g} = \frac{1}{2}\{\hat{Q}, \hat{P}\} \qquad \text{and} \qquad \hat{k} = \frac{1}{2}(\hat{Q}^2 - \hat{P}^2)$$

where we exclude $i(\hat{Q}\hat{P} - \hat{P}\hat{Q})$ because (due to the Heisenberg commutation relation) it is proportional to the identity operator.

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The three generators $\hat{H}_0$, $\hat{g}$, $\hat{k}$ span the algebra $sl(2,\mathbb{R})$ on the representation space $W^{(\infty)}$. The $\hat{g}$ operator, for instance, is nothing but the *squeezing* generator (known from quantum optics). Indeed, the scale operator

$$\hat{S}_{\xi} = e^{i\hat{g}\ln\xi}$$

generated by $\hat{g}$ act upon the position and momentum basis respectively as

$$\hat{S}_{\xi}|q(x)\rangle = \sqrt{\xi}|q(\xi x)\rangle \qquad \text{and} \qquad \hat{S}_{\xi}|p(x)\rangle = \frac{1}{\sqrt{\xi}}|p(x/\xi)\rangle$$

Analogously, the $\hat{k}$ operator generates *hyperbolic rotations*, that is, linear transformations of the plane that preserve an *indefinite* metric. It takes the hyperbola $x^2 - y^2 = 1$ into itself in an analogous way that the Euclidean rotation takes the circle $x^2 + y^2 = 1$ into itself.

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$SL(2,\mathbb{R})$ is the Lie Group of all area preserving linear transformations of the plane, so we can identify it with the 2×2 real matrices with *unit determinant*. Thus, we can identify the algebra $sl(2,\mathbb{R})$ with all 2×2 real matrices with *null trace*. Thus, it is natural to make the following choice for a basis in this algebra in terms of Pauli matrices:

$$\hat{X}_1 = \hat{\sigma}_1 \quad \hat{X}_2 = i\hat{\sigma}_2 \quad \hat{X}_3 = \hat{\sigma}_3$$

In fact, the isomorphism[6] described by the table below relates these algebra elements directly to the algebra of their representation carried on $W^{(\infty)}$:

generators of $sl(2,\mathbb{R})$	generators of the representation
$\hat{X}_1 = \hat{\sigma}_1$	$-i\hat{k}$
$\hat{X}_2 = i\hat{\sigma}_2$	$-i\hat{H}_0$
$\hat{X}_3 = \hat{\sigma}_3$	$-i\hat{g}$

[6] A. C. Lobo and C. A. Ribeiro, Phys. Rev. A, 80, 012112 (2009)

Weak Values and Quantum Mechanics in Phase Space

The original formulation of weak value concept goes as follows: Let $|\Psi\rangle = |\alpha\rangle \otimes |\phi_i\rangle$ be the initial state of a product space $W = W_S \otimes W_M$ where $|\alpha\rangle$ is the *pre-selected* state of the *system* and $|\phi_i\rangle$ is the initial state of the *apparatus* (a one-dimensional motion of a quantum particle) under the influence of a "weak Hamiltonian":

$$\hat{H}_{int} = \epsilon\delta(t - t_0)\hat{O} \otimes \hat{P} \qquad (\epsilon \rightarrow 0)$$

and $\hat{O}$ is an *arbitrary* observable of the system. After the *ideal* instantaneous interaction that models this von-Neumann (weak) measurement, we post-select a final state $|\beta\rangle$ (*non-orthogonal* to $|\alpha\rangle$) through a strong measurement. The final state of the apparatus is then (up to a normalization constant):

$$|\phi_f\rangle = (\langle\beta| \otimes \hat{I})e^{-i\epsilon\hat{O}\otimes\hat{P}}(|\alpha\rangle \otimes |\phi_i\rangle) \approx \langle\beta|\alpha\rangle(1 - i\epsilon O_w)|\phi_i\rangle$$

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The complex number

$$O_w = \frac{\langle \beta | \hat{O} | \alpha \rangle}{\langle \beta | \alpha \rangle}$$

is the *weak value* of $\hat{O}$.
There is a small shift of the expectation value of the position operator $\hat{Q}$ that can me measured over a *large ensemble* with the same pre and post selected states. Jozsa generalized this procedure by taking an arbitrary operator $\hat{M}$ in the place of $\hat{Q}$ as the observable of $W_{(M)}$ to be measured[7].
The shift between these expectation values of $\hat{M}$ to first order in ϵ is:

$$\begin{aligned} \Delta \hat{M} &= \langle \hat{M} \rangle_f - \langle \hat{M} \rangle_i \\ &= \epsilon \big[Im(O_w) \big(\langle \{ \hat{M}, \hat{P} \} \rangle_{|\phi_i\rangle} - 2 \langle \hat{P} \rangle_{|\phi_i\rangle} \langle \hat{M} \rangle_{|\phi_i\rangle} \big) - \\ &- iRe(O_w) \langle [\hat{M}, \hat{P}] \rangle_{|\phi_i\rangle} \big] \end{aligned}$$

[7] R. Jozsa, Phys. Rev. A, 76, 044103 (2007)

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For $\hat{M} = \hat{Q}$ (respectively $\hat{M} = \hat{P}$) and for the non-relativistic Hamiltonian $\hat{H} = \frac{1}{2m}\hat{P}^2 + V(\hat{Q})$ Jozsa derived

$$\Delta\hat{Q} = \epsilon[Re(O_w) + mIm(O_w)\frac{d}{dt}(\delta^2_{|\phi_i\rangle}\hat{Q})]$$

$$\Delta\hat{P} = 2\epsilon Im(O_w)(\delta^2_{|\phi_i\rangle}\hat{P})$$

where $\delta_{|\psi\rangle}\hat{O}$ is the uncertainty of observable $\hat{O}$ in the initial state $|\psi\rangle$. The former result is clearly *asymmetric* in phase space because of the choice of $\hat{P}$ in the Hamiltonian. One may choose *any* of the symplectic generators of the $sl(2,\mathbb{R})$ algebra with the following interaction Hamiltonian:

$$\hat{H}_{int} = \epsilon\delta(t - t_0)\hat{O}\otimes\hat{R} \qquad (\epsilon \to 0)$$

As an example, we choose the pair $(\hat{N}, \hat{a})$ as an analogous "conjugate pair" to the canonical $(\hat{Q}, \hat{P})$ pair. The shift in this case is:

$$\Delta\hat{a} = \epsilon[-iO_w\langle\hat{a}\rangle_{|\phi_i\rangle} + 2Im(O_w)(\langle\hat{N}\hat{a}\rangle_{|\phi_i\rangle} - \langle\hat{N}\rangle_{|\phi_i\rangle}\langle\hat{a}\rangle_{|\phi_i\rangle})]$$

For quantum optical purposes, we may choose a coherent state $|\phi_i\rangle = |z\rangle$ as the initial state of the system:

$$\Delta\hat{a} = i\epsilon O_w = \epsilon|O_w|e^{i(\theta_z+\theta_w-\pi/2)}$$

where we have $z = |z|e^{i\theta_z}$ and $O_w = |O_w|e^{i\theta_w}$ for the former equation.

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Also by choosing a convenient phase $\theta_z = \pi/2$ and writing the above equation back in terms of the canonical pair $(\hat{Q}, \hat{P})$, we get a very simplified *symmetric* pair of equations

$$\Delta\hat{Q} = \epsilon\sqrt{2}|z|Re(O_w) \qquad \text{and} \qquad \Delta\hat{P} = \epsilon\sqrt{2}|z|Im(O_w)$$

The above equations have some advantages:

- These equations *do not* depend on the *quadratic dispersion* or the *time derivative of the quadratic dispersion* of any observable for the initial state of the measuring system.
- one may "tune" the size of the $\epsilon|z|$ term despite how small ϵ may be by making $|z|$ large enough.
- This may be of practical importance for optical implementations of weak value measurements since $|z|$ for a quantized mode of an electromagnetic field is nothing else but the *mean photon number* in this mode for the coherent state $|z\rangle$.

Discrete measuring device

The measuring device can also be *discrete*. A finite set of position and momentum eigenvectors will be denoted respectively by $|u_j\rangle$ and $|v_j\rangle$ ($j = 0, ..., N-1$) for a N-dimensional space. This notation conforms with Schwinger´s formulation that we shall address at the last section of this presentation.

For a single qubit case, we choose the state $|\alpha\rangle = |u_0\rangle$ for the pre-selected state (the "north pole" of the Bloch sphere), and $|\theta, \varphi\rangle = \cos(\theta/2)\,|u_0\rangle + e^{i\varphi}\sin(\theta/2)\,|u_1\rangle$ for the post-selected states and

$$\hat{O} = \hat{\sigma}_1 = |u_0\rangle\langle u_1| + |u_1\rangle\langle u_0|$$

as the observable to be "weakly" measured.

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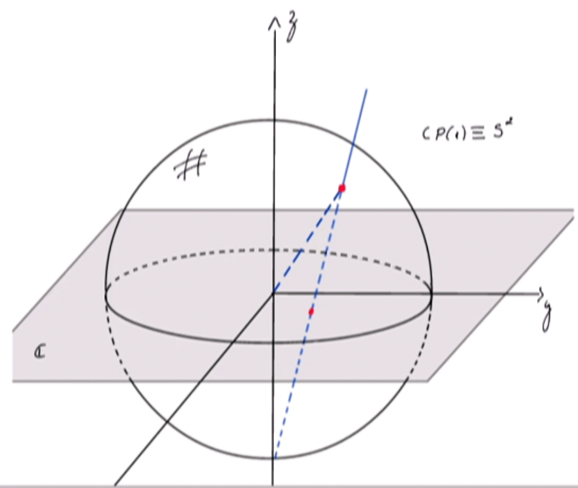
as the observable to be "weakly" measured.

It is straightforward to compute the weak value in this case as

$$O_w = \tan(\theta/2)e^{i\varphi}$$

which is complex-valued in general and remarkably shows that the weak value, in this case, has a direct physical meaning. It measures the complex projective coordinate of the state vector in the Bloch Sphere.



The geometry of von Neumann's pre-measurement and weak values

In the last section we discussed weak measurements in terms of an analysis of the *measuring apparatus*. In this section we do the *opposite* approach: We discuss certain geometric structures of the *measured system* based on previous work of Tamate *et al*[8].
Let $W = W_{(S)} \otimes W_{(M)}$ the composition of a *measured* subsystem $W_{(S)}$ with the *measuring* apparatus $W_{(M)}$. We suppose initially that both are *discrete* and defined respectively by:

$$\hat{O} = |o_k\rangle o_k \langle o^k| \qquad \text{(sum convention implied)}$$

where $\hat{O}$ is an arbitrary observable in $W_{(S)}$ and

$$\hat{P} = |v_\sigma\rangle p_\sigma \langle v^\sigma|$$

$$\hat{P} = |v_\sigma\rangle p_\sigma \langle v^\sigma|$$

is a discrete momentum observable in $W_{(M)}$.

[8] S Tamate, H Kobayashi, T Nakanishi, K Sugiyama and M Kitano, New Journal of Physics, Vol. 11, September 2009

After performing an *ideal* von-Neumann pre-measurement defined by the unitary evolution given by the Hamiltonian:

$$\hat{H}_{int} = \lambda\delta(t - t_0)\hat{O} \otimes \hat{P}$$

of an initially unentangled product state:

$$|\Psi_i\rangle = |\alpha\rangle \otimes |\phi_i\rangle$$

the total state becomes entangled:

$$|\Psi_f\rangle = |A_\sigma\rangle \otimes |v_\sigma\rangle\phi^\sigma \qquad \text{with} \qquad |\phi_i\rangle = |v_\sigma\rangle\phi^\sigma$$

where we have defined a set of states (in general non-orthogonal) $\{|A_\sigma\rangle\}$ in $W_{(M)}$ as:

$$|A_\sigma\rangle = e^{-i\lambda p_\sigma \hat{O}}|\alpha\rangle$$

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[8] S Tamate, H Kobayashi, T Nakanishi, K Sugiyama and M Kitano, New Journal of Physics, Vol. 11, September 2009

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$$|A_\sigma\rangle = e^{-i\lambda p_\sigma \hat{O}}|\alpha\rangle$$

By tracing out the measured subsystem in $|\Psi_f\rangle\langle\Psi_f|$ we get the density matrix:

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unitary evolution given by the Hamiltonian:

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of an initially unentangled product state:

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By tracing out the measured subsystem in $|\Psi_f\rangle\langle\Psi_f|$ we get the density matrix:

$$\hat{\rho}_{|\Psi_f\rangle} = tr_1\left(|\Psi_f\rangle\langle\Psi_f|\right) = |v_\sigma\rangle\phi^\sigma\langle A^\sigma|A_\tau\rangle\bar{\phi}_\tau\langle v^\tau|$$

Consider now the important example of a measuring apparatus defined by

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Consider now the important example of a measuring apparatus defined by a *single qubit* described within the usual Bloch sphere notation:

$$|\phi_i\rangle = \cos(\theta/2)\,|v_0\rangle + e^{i\varphi}\sin(\theta/2)\,|v_1\rangle$$

with

$$\langle A^0|A_1\rangle = |\langle A^0|A_1\rangle|e^{-i\eta}$$

A protocol for measuring the *intrinsic phase* between $|A_0\rangle$ and $|A_1\rangle$ can be executed in the following way:
Performing *a* von Neumann *pre-measurement and* subsequently implementing a *strong* measurement in the second subsystem. The probability $P(\eta)$ of finding the measuring apparatus in the following *reference state*

$$|ref\rangle = |\theta = \pi/2, \varphi = 0\rangle = \frac{1}{\sqrt{2}}\left(|v_0\rangle + |v_1\rangle\right)$$

is then:

$$P(\eta) = tr(\hat{\rho}_{|\Psi_f\rangle}|ref\rangle\langle ref|) = \frac{1}{2} + \frac{1}{4}|\langle A^0|A_1\rangle|\sin\theta\cos(\varphi - \eta)$$

$$P(\eta) = tr(\hat{\rho}_{|\Psi_f\rangle}|ref\rangle\langle ref|) = \frac{1}{2} + \frac{1}{4}|\langle A^0|A_1\rangle| \sin\theta\cos(\varphi - \eta)$$

For a fixed θ, this probability is *maximized* when $\varphi = \eta$. This allows one to measure the so called *geometric phase*

$$\arg\langle A^0|A_1\rangle = \eta$$

between the two states $|A_0\rangle$ and $|A_1\rangle \in W_{(S)}$.

This *geometric phase* was originally proposed in 1956 by Pancharatnam[9] for optical states and rediscovered by Berry in 1984[10] for adiabatic cyclic evolutions. In 1987, Anandan and Aharonov[11] provided a more general description of the geometric phase in terms of natural structures of the $U(1)$ fiber-bundle over $CP(N)$.

Given $|\Psi_f\rangle$, one may "post-select" a chosen state $|\beta\rangle \in W_{(S)}$. The resulting state of the *total system* is now again a *non-entangled* state:

$$|\Psi_f\rangle = |\beta\rangle \otimes C|v_\sigma\rangle\langle\beta|A_\sigma\rangle\phi^\sigma$$

11 Y. Aharonov and J. Anandan, Phys. Rev. Lett, vol.50 N.16 (April 1987) 1593-1596

The state of the measuring system is then:

$$|\phi_f\rangle = C\left(|v_0\rangle\langle\beta|A_0\rangle\cos\theta + |v_1\rangle\langle\beta|A_1\rangle e^{i\varphi}\sin\theta\right)$$

By writing the following complex phases as

$$\langle\beta|A_0\rangle = |\langle\beta|A_0\rangle|e^{i\beta_0} \qquad \text{and} \qquad \langle\beta|A_1\rangle = \langle\beta|A_1\rangle e^{-i\beta_1}$$

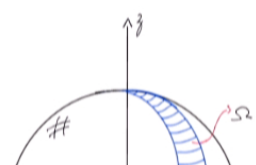
We can establish a *protocol* for computing the probability of finding the second subsystem in state $|ref\rangle = |\theta = \pi/2, \varphi = 0\rangle$. For a *fixed* angle θ, the *maximum* probability occurs for

$$\varphi = \beta_0 + \beta_1 = arg\left[(\langle A^1|\beta\rangle\langle\beta|A_0\rangle\right]$$

This implies that there is an *overall* phase change Θ given by

$$\Theta = \arg(\langle A^1|\beta\rangle\langle\beta|A^0\rangle\langle A^0|A^1\rangle)$$

which is a well-known *geometric invariant* in the sense that it depends only on the projection of the state-vectors $|A_0\rangle$, $|A_1\rangle$ and $|\beta\rangle$ on $CP(N)$. It is the geometric phase picked by a state-vector when it is *parallel transported* through the closed geodesic triangle defined by the projection of these states on $CP(N)$. For a single qubit, the geometric invariant is proportional to the *symplectic area* of the geodesic triangle on Bloch sphere.



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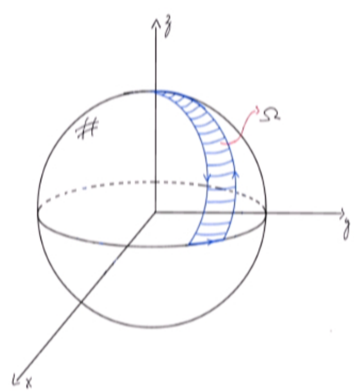

Suppose now that the *measuring* system is *continuous*, given by the motion of a quantum particle. The initial state is again the product state $|\Psi_i\rangle = |\alpha\rangle \otimes |\phi_i\rangle$ submitted to an instantaneous interaction coupling an observable $\hat{O}$ of the system of interest with the momentum $\hat{P}$ of the "quantum-particle apparatus". The final entangled state of the *system + apparatus* is then:

$$|\Psi_f\rangle = \int_{\mathbb{R}} dx |A(x)\rangle \otimes |p(x)\rangle \tilde{\phi}_i(x) \qquad \text{with} \qquad |A(x)\rangle = e^{-i\lambda x \hat{O}}|\alpha\rangle$$

and $\tilde{\phi}_i(x) = \langle p(x)|\tilde{\phi}_i\rangle$ is the wave-function of the measuring particle in the *momentum* basis. The phase shift between $|A(x)\rangle$ and $|A(x+dx)\rangle$ (to first order in dx) is then:

$$\arg(\langle A(x)|A(x+dx)\rangle) \simeq -\lambda dx \langle \hat{O}\rangle_{|\alpha\rangle}$$

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A geometric interpretation of this von Neumann's pre-measurement can be presented through the following analogy: Let $|\psi(t)\rangle$ be the curve in Hilbert space generated by the Hamiltonian $\hat{H}(t)$. The Schrödinger equation implies a relation between $|\psi(t)\rangle$ and $|\psi(t+dt)\rangle$ given by:

$$|d\psi(t)\rangle = |\psi(t+dt)\rangle - |\psi(t)\rangle = -i\hat{H}|\psi(t)\rangle dt$$

The squared distance between two infinitesimally nearby projection of state vectors in an time interval dt over $CP(N)$ is

$$ds^2(CP(N)) = [\langle\psi(t)|\hat{H}^2|\psi(t)\rangle - \langle\psi(t)|\hat{H}|\psi(t)\rangle^2]dt^2 = \left(\delta^2_{|\psi(t)\rangle}E\right) dt$$

The expression $|A(x)\rangle = e^{-i\lambda x\hat{O}}|\alpha\rangle$ is formally equivalent to the unitary time evolution equation $|\psi(t)\rangle = e^{-i\hat{H}t}|\psi(0)\rangle$. The analogy between these two distinct physical processes is shown in the following table:

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$\vert\psi(t)\rangle$	$\vert A(x)\rangle$
$\vert\psi(0)\rangle$	$\vert\alpha\rangle$
t	x
$\hat{H}$	$\lambda\hat{O}$

Regarding x as an *external parameter* (just as the time variable t) an analog for the *infinitesimal* distance in $CP(N)$ is:

$$\begin{aligned} ds^2 &= [\langle A(x)|\hat{O}^2|A(x)\rangle - \langle A(x)|\hat{O}|A(x)\rangle^2]\lambda^2 dx^2 = \\ &= [\langle\alpha|\hat{O}^2|\alpha\rangle - \langle\alpha|\hat{O}|\alpha\rangle^2]\lambda^2 dx^2 \end{aligned}$$

For the *weak measurement*, the *global* geometric phase related to $|A(x)\rangle$, $|A(x+dx)\rangle$ and $|\beta\rangle$ on $CP(N)$ is:

$$\Theta = \arg(\langle A(x)|\beta\rangle\langle\beta|A(x+dx)\rangle\langle A(x+dx)|A(x)\rangle)$$

$$\begin{array}{cc} |\psi(t)\rangle & |A(x)\rangle \\ |\psi(0)\rangle & |\alpha\rangle \\ t & x \\ \hat{H} & \lambda\hat{O} \end{array}$$

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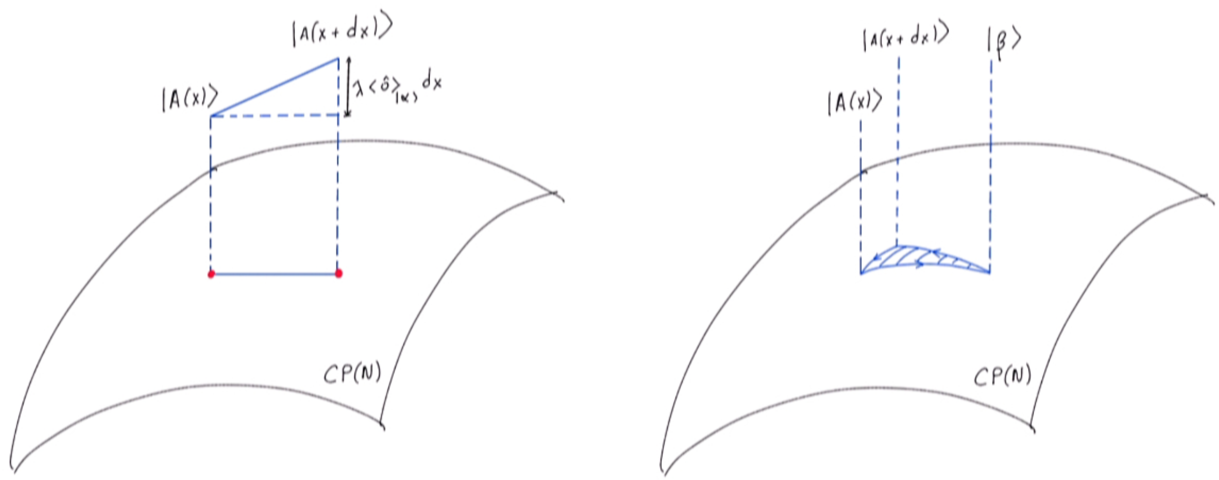
Expanding to first order in ϵ, we have:

$$\Theta = -\epsilon[Re(O_w) - \langle\hat{O}\rangle_{|\alpha\rangle}]dx$$


|A(x + dx)⟩
|A(x)⟩
CP(N)
|A(x + dx)⟩
|β⟩
|A(x)⟩
CP(N)

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The Finite Phase Space and Schwinger's Formalism[12]

Let $W^{(N)}$ be an N-dimensional space generated by *finite position and momentum* $\{|u_k\rangle\}$ or $\{|v_k\rangle\}$ $(k = 0, 1, 2, \ldots N-1)$:

$$|u_k\rangle\langle u^k| = |v_k\rangle\langle v^k| = \hat{1} \qquad \text{and} \qquad \langle u^j|u_k\rangle = \langle v^j|v_k\rangle = \delta^j_k$$

We also define unitary translations operators $\hat{V}$ and $\hat{U}$ such that:

$$\hat{V}|u_k\rangle = |u_{k-1}\rangle \qquad \hat{U}|v_k\rangle = |v_{k+1}\rangle \qquad |u_{k+N}\rangle = |u_k\rangle \qquad |v_{k+N}\rangle = |v_k\rangle$$

$$\hat{V}^N = \hat{U}^N = \hat{1} \qquad \text{which eigenvalues} \qquad \omega^k = e^{\frac{2\pi i}{N}k}$$

where $\{|u_k\rangle\}$ and $\{|v_k\rangle\}$ are *eigenvectors* $\hat{U}$ and $\hat{V}$

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Schwinger's Finite Quantum Kinematics and Modular Variables

The heuristic continuum limit as a re-scaling

The implementation of a "continuum heuristic limit" (when $N \longrightarrow \infty$) can be performed in two ways: one *symmetric* and the other *non-symmetric* between the position and momentum states.

- **The symmetric continuum limit**

We re-scale in equal manner the position and momentum basis:

$$|q(x_j)\rangle = \left(\frac{N}{2\pi}\right)^{1/4} |u_j\rangle \quad \text{and} \quad |p(y_k)\rangle = \left(\frac{N}{2\pi}\right)^{1/4} |v_k\rangle$$

with

$$x_j = \left(\frac{2\pi}{N}\right)^{1/4} j \quad \text{and} \quad y_k = \left(\frac{2\pi}{N}\right)^{1/4} k$$

The discrete indices are disposed *symmetrically* in relation to zero. We take N as an odd number without loss of generality.

$$j, k = -\frac{(N-1)}{2}, \ldots, +\frac{(N-1)}{2} \quad \text{and} \quad \Delta x_j = \Delta y_k = \left(\frac{2\pi}{N}\right)^{1/2}$$

We recover the continuous case in a very intuitive sense. The completeness relations for both basis are:

$$\hat{1} = \sum_{j=-\frac{(N-1)}{2}}^{+\frac{(N-1)}{2}} \left(\frac{2\pi}{N}\right)^{1/2} |q(x_j)\rangle\langle q(x_j)| = \sum_{k=-\frac{(N-1)}{2}}^{+\frac{(N-1)}{2}} \left(\frac{2\pi}{N}\right)^{1/2} |p(y_k)\rangle\langle p(y_k)|$$

So that for $N \longrightarrow \infty$ we have the usual completeness relation for the continuum:

$$\hat{1} = \int_{\mathbb{R}} dx |q(x)\rangle\langle q(x)| = \int_{\mathbb{R}} dy |p(y)\rangle\langle p(y))|$$

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Which can be seen as an heuristic definition of the *Dirac delta function* as a "continuous limit" of the Kronecker delta.

The overlap of elements of the two different bases is:

$$\langle q(x_j)|p(y_k)\rangle = \frac{1}{\sqrt{2\pi}}e^{ix_jy_k} \qquad (\hbar = 1)$$

- **The non-symmetric continuum limit**

We introduce a different scaling for the variables of position and momentum with a given $\xi \in \mathbb{R}$:

$$|q(x_j)\rangle = \left(\frac{N}{\xi}\right)^{1/2}|u_j\rangle \qquad \text{and} \qquad |p(y_k)\rangle = \left(\frac{\xi}{2\pi}\right)^{1/2}|v_k\rangle$$

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with

$$x_j = \frac{\xi}{N} j \qquad \text{and} \qquad y_k = \frac{2\pi}{\xi} k$$

so that

$$\Delta x_j = \frac{\xi}{N} \to 0 \qquad \text{for} \qquad N \to \infty$$

Only the position states become *singular* and the x_j variable takes value in a *bounded quasi-continuum* set so that the resolution of identity Is:

$$\hat{I}_\xi = \int^{+\xi/2} dx |q(x)\rangle\langle q(x)| = \frac{2\pi}{\xi} \sum^{+\infty} |p(y_k)\rangle\langle p(y_k)|$$

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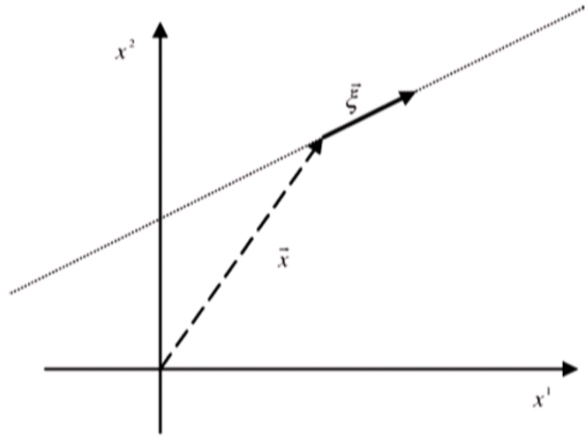
Modular Variables

Pseudo Degrees of Freedom

The translations in the $x^1 - x^2$ plane as a result of the repeated action of $\hat{V}_{\vec{\xi}} = \hat{V}_{\xi^1} \otimes \hat{V}_{\xi^2}$ upon $|q(\vec{x})\rangle = |q(x^1)\rangle \otimes |q(x^2)\rangle$ forms a straight line that contains the point $\vec{x}$ but with direction given by $\vec{\xi}$:



Consider the six-dimensional space $W^{(6)} = W^{(2)} \otimes W^{(3)}$ (the product of a *qubit* and a *qutrit*).The action of $\hat{V} \otimes \hat{V}$ on the basis $\{|u_j\rangle \otimes |u_\sigma\rangle\}$, $(j = 0, 1$ and $\sigma = 0, 1, 2)$ is identical to the action of $\hat{V}^{(6)} = \hat{V}^{(2)} \otimes \hat{V}^{(3)}$ on the same basis relabeled as $\{|u_0\rangle, |u_1\rangle, |u_2\rangle, |u_3\rangle, |u_4\rangle, |u_5\rangle\}$. One starts with $|u_0\rangle \otimes |u_0\rangle$ and covers

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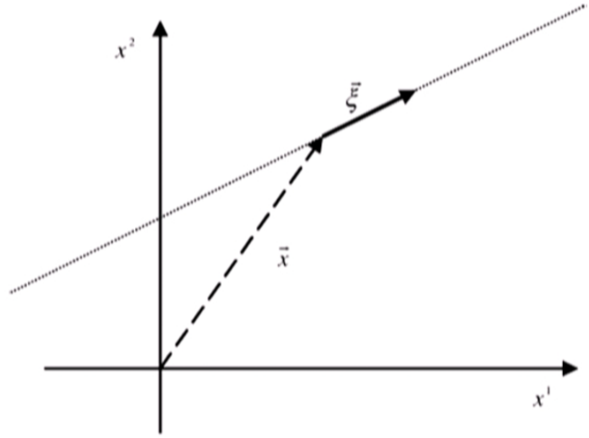

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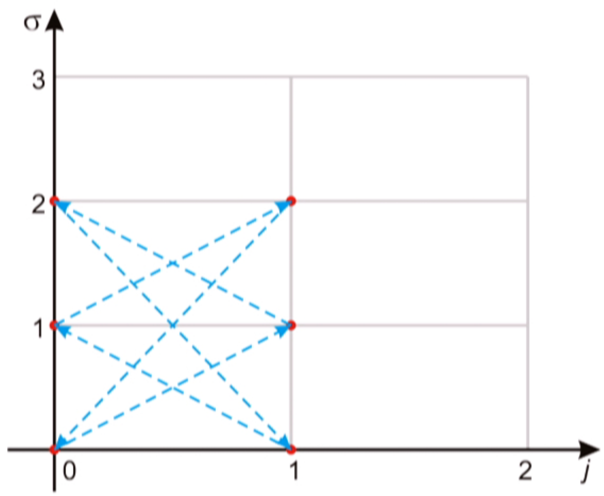
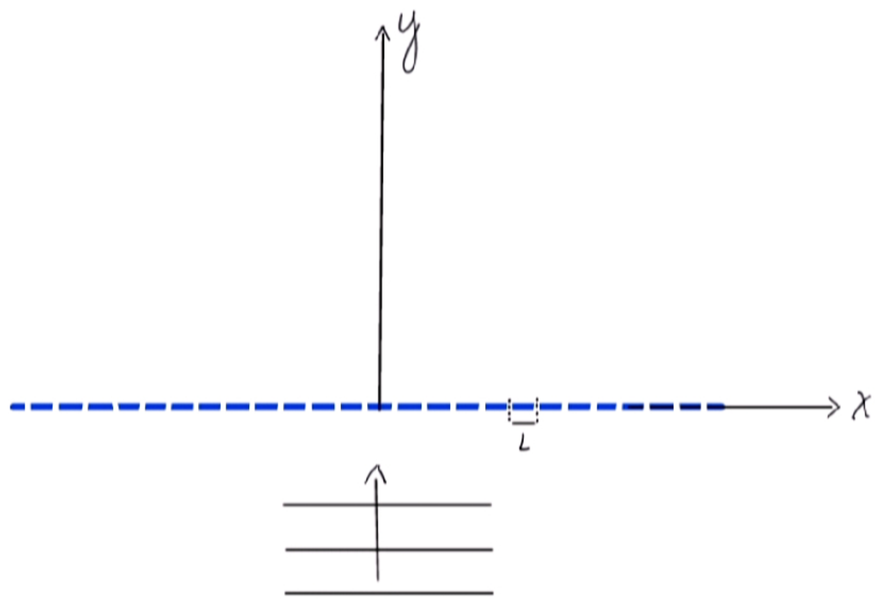


This follows from elementary number theory for all spaces $W^{(N_a)}$ and $W^{(N_b)}$ when $\gcd(N_a, N_b) = 1$. These two "degrees of freedom" are actually *pseudo-degrees* of freedom. One dimension is the periodic factor of the total dimension. For instance, the six-dimensional space is a single "true" degree of freedom comprised as three periods of two or two periods of three.

The N-Slit Experiment

Consider the paradigmatic experiment of *diffraction* of a quantum particle through an apparatus with a large set of equally distant slits. (Distance L)



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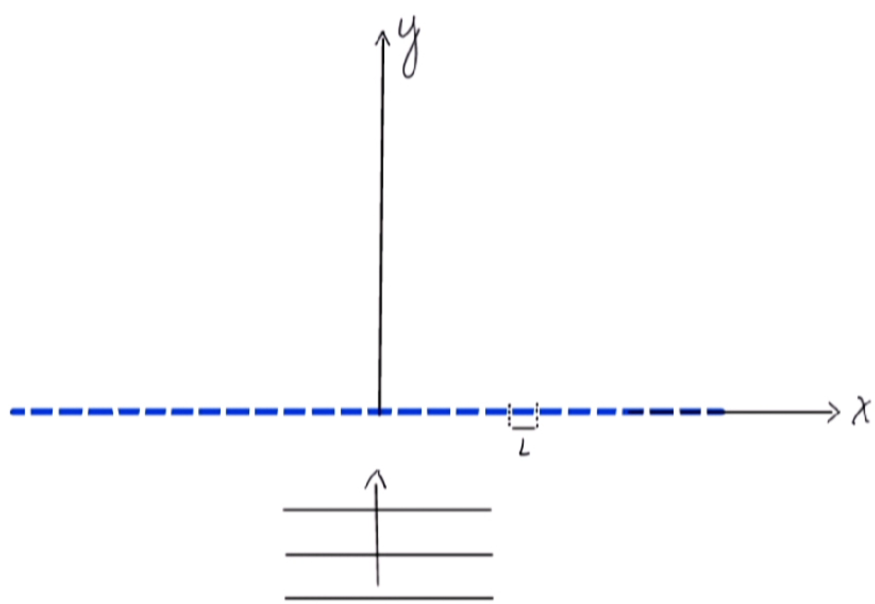

The N-slit experiment

The interaction can be modeled by the following Hamiltonian in the x direction:

$$\hat{H}(t) = \frac{1}{2m}\hat{P}^2 + V(\hat{Q})\delta(t) \qquad \text{with} \qquad V(\hat{Q}+L) = V(\hat{Q})$$

The "instantaneous" interaction happens for $t = 0$. We model this interaction as an *ideal* strong measurement of position (up to a period L) resulting in the following state (in the x direction):

$$|\psi_f\rangle \sim \sum_{m\in\mathbb{Z}} |q_x[(m+1/2)L]\rangle$$

which gives us the probability proportional to:

$$P(\theta) = |\langle p_x(\eta)|\psi_f\rangle|^2 = \left[1 + 2\sum_{m=1}^{+\infty}\cos\left(\frac{2\pi L}{\lambda} m \sin\theta\right)\right]^2$$

where

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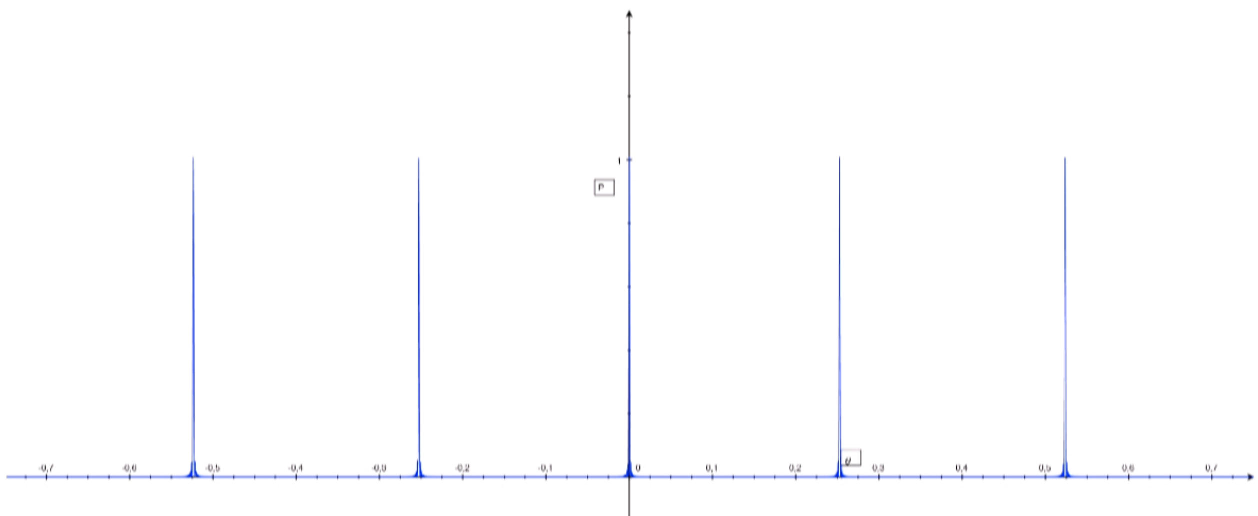
where

$$p_x = \eta = |\vec{p}| \sin\theta = \frac{2\pi}{\lambda} \sin\theta$$

Augusto César Lobo (UFOP) Phase Space Methods in QM & Weak Values June 2016 43 / 55

The probability $p(\theta)$ gives us the well-known interference pattern $(L = 4\lambda)$:



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One can also characterize the state in the following way: The interaction is so *fast* that the time evolution is given by $\hat{U}(t) \approx e^{-iV(\hat{Q})}$ which can be expanded in a Fourier series:

$$e^{-iV(\hat{Q})} = \sum_{n \in Z} c_n e^{\frac{2\pi i n}{L}\hat{Q}} = \sum_{n \in Z} c_n \hat{U}_{\frac{2\pi n}{L}}$$

The *initial state* (in the x axis) of the particle is an eigenstate of zero momentum $|\psi_i\rangle = |p_x(0)\rangle$ so that $|\psi_f\rangle = \hat{U}(t)|p_x(0)\rangle$. The *final state* is

$$|\psi_f\rangle = \hat{U}(t)|p(0)\rangle = \sum_{n \in Z} c_n \hat{U}_{\frac{2\pi n}{L}}|p(0)\rangle = \sum_{n \in Z} c_n |p(\frac{2\pi n}{L})\rangle$$

The resulting state has the remarkable property of being an eigenstate *both* of $\hat{U}_{2\pi/L}$ and $\hat{V}_L$ and since they *commute*, their eigenvectors are mutual and their eigenvalues are complex phases. These phases are Aharonov´s *modular variables*.

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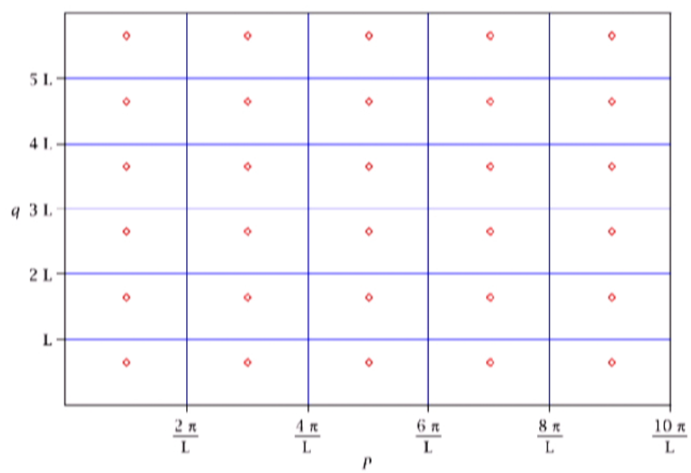
A phase space description of such a state is pictured as follows:



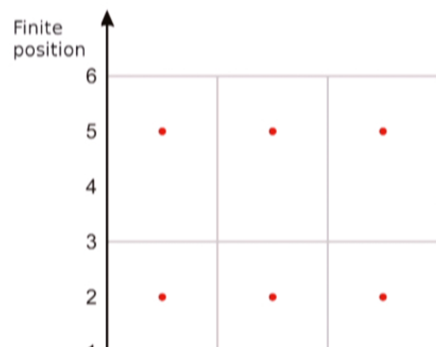
[13] Y. Aharonov, D. Rohrlich, "*Quantum paradoxes: quantum theory for the perplexed*", Weinheim: Wiley-VCH (2005)

[14] A. C. Lobo, Y. Aharonov, J. Tollaksen, E. M. Berrigan and C. A. Ribeiro, Quantum Stud.: Math. Found. (May 2014)

A finite analog of this state in $W^{(N)}$ is

$$|j, \sigma^{(N)}\rangle = |v_j{}^{(N_a)}\rangle \otimes |u_\sigma{}^{(N_b)}\rangle$$

with $N = N_a \cdot N_b$ and $\gcd(N_a, N_b) = 1$. This state is *simultaneously* an eigenstate of finite momentum in $W^{(N_a)}$ and finite position in $W^{(N_b)}$ and are also simultaneous eigenstates of $\hat{U}^{(N_a)} \otimes \hat{I}^{(N_b)}$ and $\hat{I}^{(N_a)} \otimes \hat{U}^{(N_b)}$ which clearly commute. For instance, consider the finite phase space of state $|1, 2^{(6)}\rangle = |v_1{}^{(2)}\rangle \otimes |u_2{}^{(3)}\rangle$ defined in $W^{(6)} = W^{(2)} \otimes W^{(3)}$:



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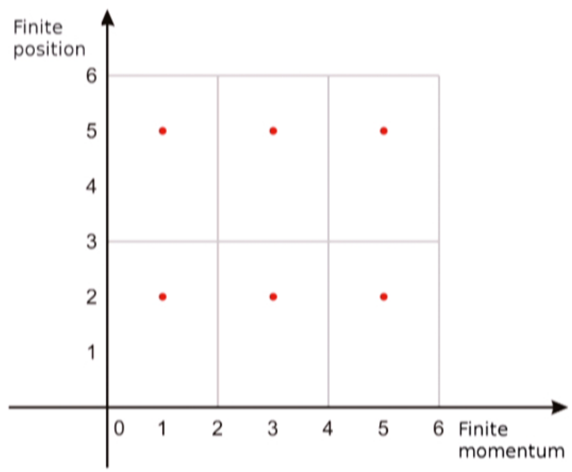


The interaction of the particle with the N-slit apparatus ($N \gg 1$) can be described as a *finite space measurement* performed on the *second* modular variable subspace of the particle:

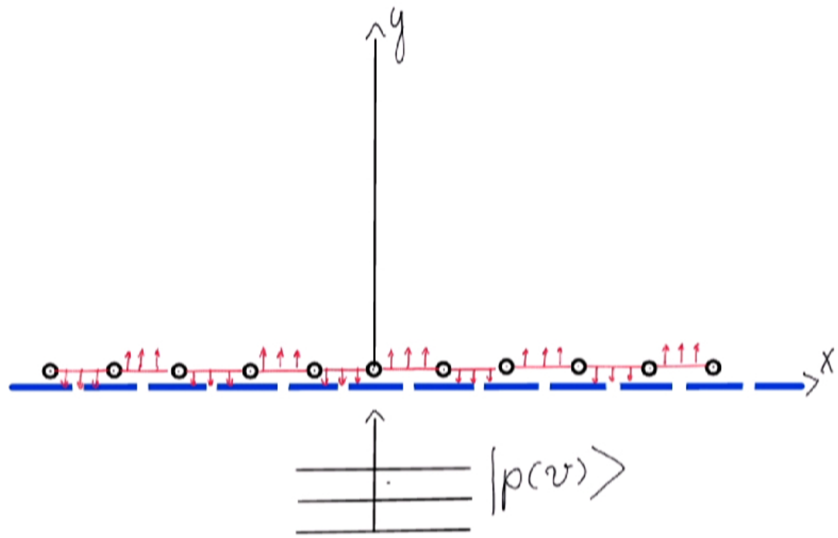
$$|v_0{}^{(N_a)}\rangle \otimes |v_0{}^{(N_b)}\rangle \xrightarrow[\textit{in the second subspace}]{\text{"space" measurement}} |v_0{}^{(N_a)}\rangle \otimes |u_0{}^{(N_b)}\rangle =$$

$$= \frac{1}{\sqrt{N_a}} \sum_{j=0}^{N_a-1} |u_j{}^{(N_a)}\rangle \otimes |u_0^{(N_b)}\rangle = \frac{1}{\sqrt{N_b}} \sum_{\sigma=0}^{N_b-1} |v_0{}^{(N_a)}\rangle \otimes |v_\sigma^{(N_b)}\rangle$$

The Continuum limit of this state can be constructed through the *non-symmetric* limit discussed previously. Given the two subspaces, the *opposite* limit must be taken for each one. That is, if one chooses the *momentum* basis of the *first* subspace as a bounded quasi-continuum, then for the *second* subspace, it is the *position* basis that is a bounded quasi-continuum and vice-versa. The discrete degrees of freedom designates the phase space cells.

The AB effect

Consider an AB setup where a quantum particle with charge q is fired against a N-slit grid with N equal solenoids, each one with magnetic flux Φ:



We choose the following *singular gauge* for the vector potential for a reference system where the electron is at rest[15]:

$$\vec{A}\,(x,y,t) = \frac{\Phi}{2}\Theta(x)\delta(y+vt)\hat{\jmath}$$

with

$$\Theta(x) = \sum_{m\in\mathbb{Z}} \left[\theta(x+mL) - \theta(-x-mL)\right]$$

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$$|\psi_i\rangle = \left(\sum_{n\in\mathbb{Z}} |q_x\left[(2n+1)\,L/2\right]\rangle\right) \otimes |p_y(v)\rangle$$

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So that

$$|\Psi_f\rangle \sim \left(\sum_{n\in\mathbb{Z}} e^{i\frac{\Phi}{2}(-1)^n}\left|q_x\left[(2n+1)\frac{L}{2}\right]\right\rangle\right) \otimes |p_y(v)\rangle =$$

$$= e^{i\frac{\Phi}{2}}\left(\ldots + \left|q_x\left(-\frac{5L}{2}\right)\right\rangle + \left|q_x\left(-\frac{L}{2}\right)\right\rangle + \left|q_x\left(\frac{L}{2}\right)\right\rangle + \left|q_x\left(\frac{5L}{2}\right)\right\rangle \ldots\right)$$

$$+e^{-i\frac{\Phi}{2}}\left(\ldots + \left|q_x\left(-\frac{7L}{2}\right)\right\rangle + \left|q_x\left(-\frac{3L}{2}\right)\right\rangle + \left|q_x\left(\frac{3L}{2}\right)\right\rangle + \left|q_x\left(\frac{7L}{2}\right)\right\rangle \ldots$$

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So that

$$|\Psi_f\rangle \sim \left(\sum_{n\in\mathbb{Z}} e^{i\frac{\Phi}{2}(-1)^n}\left|q_x\left[(2n+1)\frac{L}{2}\right]\right\rangle\right)\otimes|p_y(v)\rangle =$$

$$= e^{i\frac{\Phi}{2}}\left(\ldots+\left|q_x\left(-\frac{5L}{2}\right)\right\rangle+\left|q_x\left(-\frac{L}{2}\right)\right\rangle+\left|q_x\left(\frac{L}{2}\right)\right\rangle+\left|q_x\left(\frac{5L}{2}\right)\right\rangle\ldots\right)$$

$$+e^{-i\frac{\Phi}{2}}\left(\ldots+\left|q_x\left(-\frac{7L}{2}\right)\right\rangle+\left|q_x\left(-\frac{3L}{2}\right)\right\rangle+\left|q_x\left(\frac{3L}{2}\right)\right\rangle+\left|q_x\left(\frac{7L}{2}\right)\right\rangle.$$

which represents a qubit state on the "equator" of its Bloch sphere parametrized by the flux Φ.

- With all this, we can again proclaim or mantra: **Look at phase space!**

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Some open questions

- By modulating the fluxes of the solenoids, how can we use the resulting qudits in order to implement quantum informational tasks?
- How can we describe *modular time* and *modular energy* variables within the Schwinger formulation?

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