Title: Higher order topological actions

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Abstract: In classical mechanics, an action is defined only modulo additive terms which do not modify the equations of motion; in certain cases, these terms are topological quantities. We construct an infinite sequence of higher order topological actions and argue that they play a role in quantum mechanics, and hence can be accessed experimentally.

Higher order topological actions

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Changing the action without changing the equation of motion

a dynamical system → the action → a stationary point
→ the equation of motion

different actions → the same equation of motion

C: an oriented curve
A: a differential 1-form
Add $S = \int_C A$ to the action

The first variation:

$$\delta S = \int_C \mathscr{L}_{\delta x} A = \int_C \left(d i_{\delta x} + i_{\delta x} d \right) A$$
$$= (A_a \delta x^a)|_{\partial C} + \int_C \left(\partial_a A_b - \partial_b A_a \right) \delta x^a dx^b$$

$\mathscr{L}_{\delta x}$: the Lie derivative

$\delta S = 0,\ \delta x|_{\partial C} = 0 \Rightarrow dA = 0$

Closed 1-forms

A is closed if $dA = 0$
A is exact if $A = d\theta$

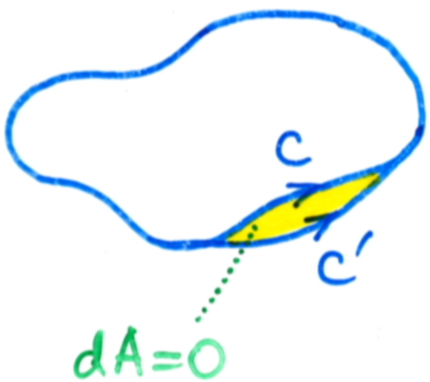
Every exact form is closed
Not every closed form is exact

A is exact $\Rightarrow S = \int_C d\theta = 0$ identically

We are interested in closed 1-forms which are not exact.

Topological terms

A is closed $\Rightarrow$ closed curves C are special



Small deformations of such C do not change the value of $S = \int_C A$.

S depends only on global properties of C
$\Rightarrow$ S is a topological quantity

Topological terms

A^*: the vector space of such 1-forms that $dA = 0$, $A \neq d\theta$ for any θ

We will show that $A^* = \cup_{p \geq 1} A^{(p)}$.
Each space $A^{(p)}$ is recursively constructed from spaces $A^{(q)}$, $q < p$.

The space $A^{(1)}$ is generated by the elements of the first cohomology group $H^1(M)$.

If M is simply connected, then $H^1(M)$ is trivial and the topological term vanishes.

If M is non-simply connected, then $H^1(M)$ is nontrivial and the topological term can be nonzero.

All elements of $A^{(1)}$ are local quantities and all elements of $A^{(p)}$ for $p \geq 2$ are nonlocal quantities; the degree of nonlocality increases with p.

Topological terms

$S^* = \int_C A^*$: a vector space of topological terms for a given C

$S^* = \cup_{p\geq 1} S^{(p)}$ and $S^{(p)} = \int_C A^{(p)}$

A given closed curve C belongs to one of the homotopy classes which are the elements of the fundamental group $G = \pi_1(M)$.

The value of $\int_C A$ is the same for all curves in a class.

Topological terms

In quantum mechanics, the elements of S^* should form abelian representations of the group of allowed curves.

This leads to the set of subgroups $G_* = \{G_p\}_{p \geq 1}$ such that the elements of $S^{(p)}$ form abelian representations of G_p.

This means that a topological term of only one order will occur in any action.

Topological terms

No reason to study topological terms in classical dynamics.

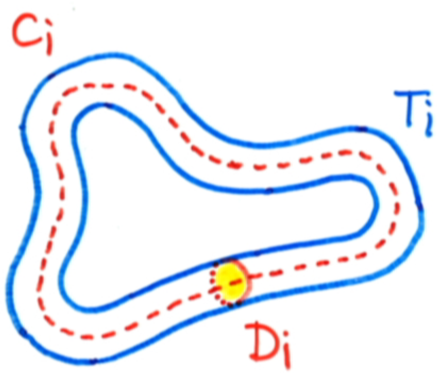
Such terms are important in quantum dynamics.

$S^{(1)}$ is responsible for the Aharonov-Bohm effect.

We argue that the higher order spaces $S^{(p)}$, $p \geq 2$, can also lead to measurable effects in quantum-mechanical systems.

Homology and cohomology groups

A 3-dimensional non-simply connected space: $M = \mathbb{R}^3 \backslash T$



disjoint tubes $T_i = C_i \times D_i$
a closed curve C_i
a disk D_i

$T = \cup_{1 \leq i \leq N} T_i$
$\cap_{1 \leq i \leq N} T_i = \emptyset$

Homology and cohomology groups

Various topological properties of M can be deduced from its homology and cohomology groups.

The first homology group $H_1(M)$ is a group of closed curves modulo those which are boundaries of surfaces.

The first cohomology group $H^1(M)$ is a group of closed 1-forms modulo exact forms.

Singular gauge

$\{\partial D_i\}_{1\leq i\leq N}$ is a basis of $H_1(M)$
$\{A_i\}_{1\leq i\leq N}$ is a basis of $H^1(M)$

De Rham theorem: the 1-forms can be chosen such that the two bases are dual to each other, $\int_{\partial D_i} A_j = \delta^i_j$.

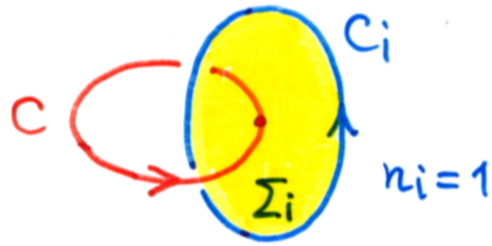
This duality condition cannot be uniquely solved for 1-forms; a convenient particular solution is

$$A_i(x) = \int_{y\in\Sigma_i} \delta(x-y) \sum_{1\leq a\leq 3} dx^a \wedge * dy^a.$$

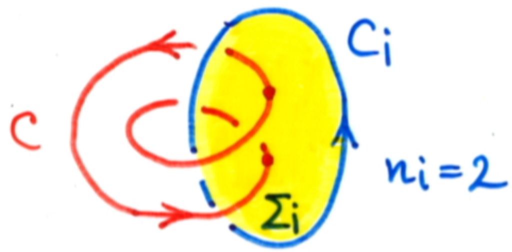
Σ_i is an oriented surface, $\partial\Sigma_i = C_i$
$*$ is the Hodge star operator.

Singular gauge

$$A_i(x) = \int_{y\in\Sigma_i} \delta(x-y) \sum_{1\leq a\leq 3} dx^a \wedge *dy^a$$



$$\operatorname{supp} F_i = C_i$$
$$\operatorname{supp} A_i = \Sigma_i$$
$$\int_C A_i = n_i, \; n_i \in \mathbb{Z}$$



A closed curve C which intersects Σ_i once in the positive direction contributes δ^i_j to the integral $\int_C A_j$.
The duality condition follows.

The first order topological terms

We define $A^{(1)}$ as a vector space with the basis $\{A_i\}_{1\leq i\leq N}$.

For a given closed curve C, there is an associated vector space of first order topological terms $S^{(1)} = \int_C A^{(1)}$.

The second order topological terms

Let $i \neq j$.
Suppose C_i and C_j are unlinked.

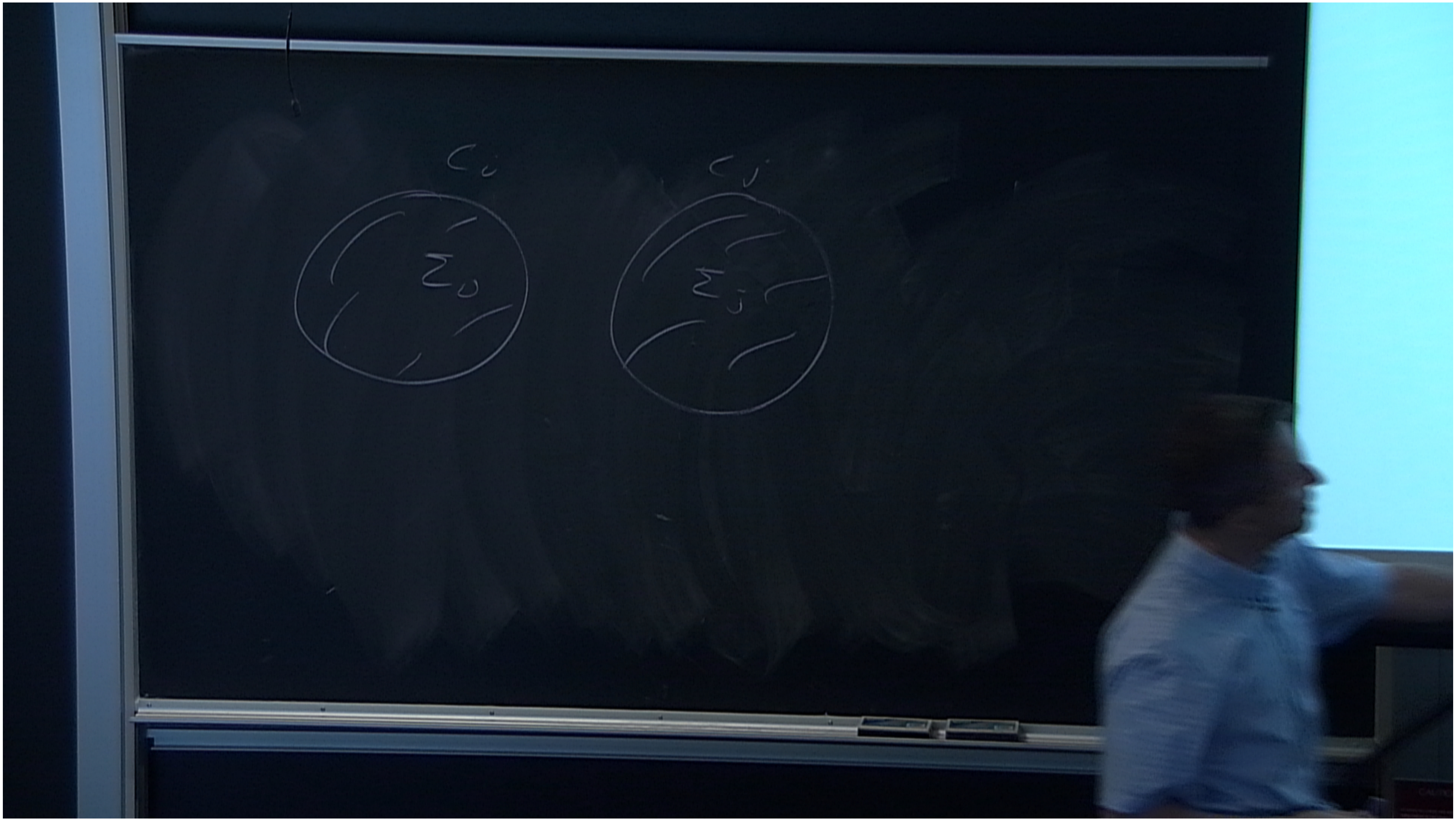
Consider a second order 2-form $F_{ij} = A_i \wedge A_j$.

$dF_{ij} = F_i \wedge A_j - A_i \wedge F_j = 0$ since $F_i|_M = 0$, $F_j|_M = 0$

Modulo a constant factor, F_{ij} is a unique closed 2-form which can be expressed in terms of A_i and A_j.

$dF_{ij} = 0 \Rightarrow$ We can define a second order 1-form A_{ij} by $dA_{ij} = F_{ij}$.

C_i and C_j are unlinked $\Rightarrow \Sigma_i$ and Σ_j can be chosen to be disjoint $\Rightarrow dA_{ij} = 0$.



The second order topological terms

A particular solution of $dA_{ij} = F_{ij}$ is $A_{ij} = \frac{1}{2}\gamma_i A_j - \frac{1}{2}A_i \gamma_j$.

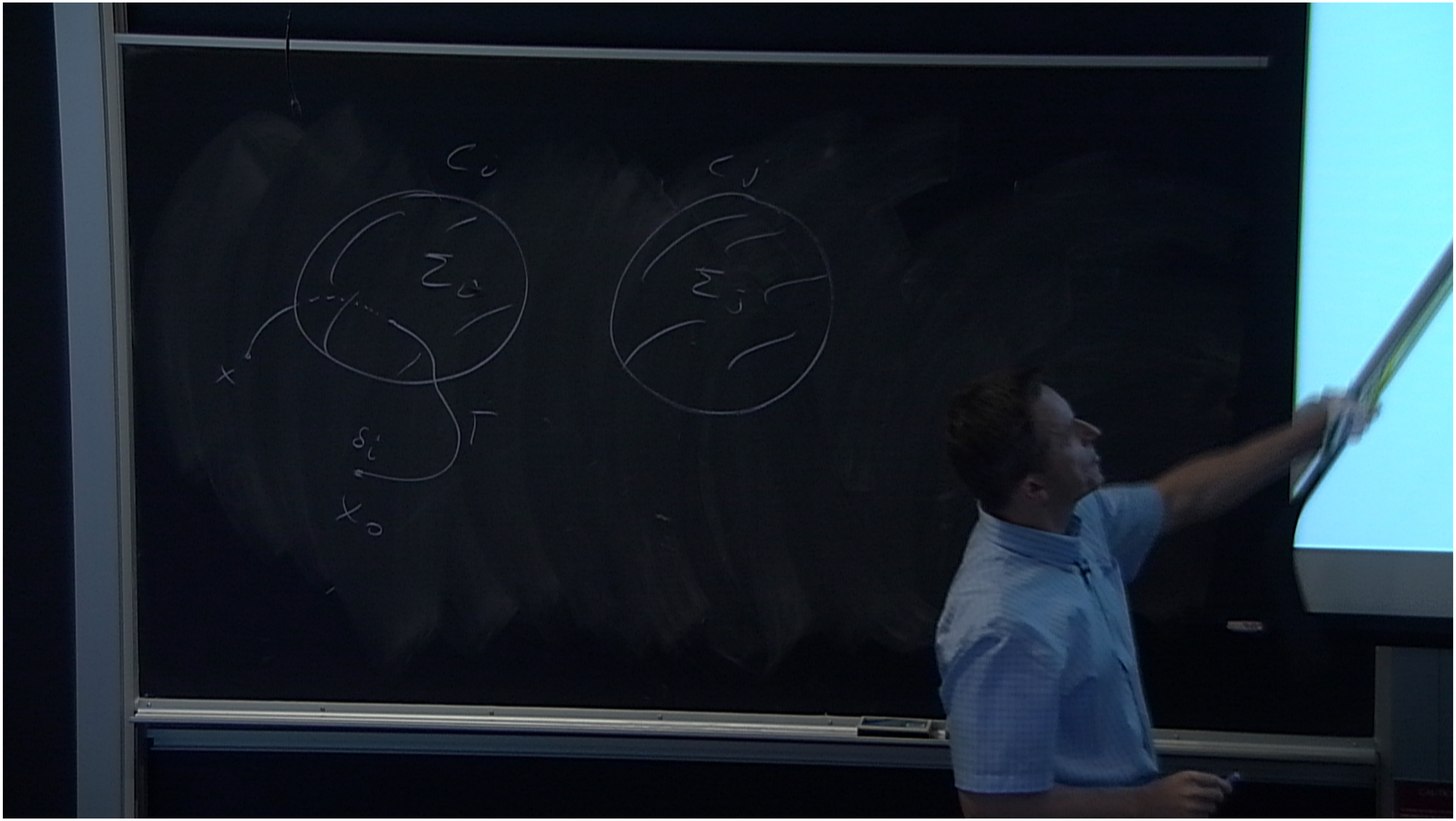
$\gamma_i = \delta_i + \int_\Gamma A_i$, δ_i is a constant.

A path Γ is the part of C which starts at x_0 and ends at x.

The orientations of C and Γ agree.

We define $A^{(2)}$ as a vector space with the basis $\{A_{ij}\}_{1 \le i < j \le N}$.

For a given closed curve C, there is an associated vector space of second order topological terms $S^{(2)} = \int_C A^{(2)}$.


c_i
c_j
Σ_i
Σ_j
x
δ_i
Γ
x_0

The third order topological terms

Let $i \neq j \neq k$.
Suppose the first and second order linkings for (C_i, C_j, C_k) vanish.

Consider a third order 3-form $F_{ijk} = A_{ij} \wedge A_k + A_i \wedge A_{jk}$.

$$\begin{aligned} dF_{ijk} &= F_{ij} \wedge A_k - A_{ij} \wedge F_k + F_i \wedge A_{jk} - A_i \wedge F_{jk} \\ &= A_i \wedge A_j \wedge A_k - A_{ij} \wedge F_k + F_i \wedge A_{jk} - A_i \wedge A_j \wedge A_k \\ &= 0 \end{aligned}$$

Modulo a constant factor, F_{ijk} is the unique closed 3-form which can be expressed in terms of the corresponding elements of $A^{(1)}$ and $A^{(2)}$.

The third order topological terms

$dF_{ijk} = 0 \Rightarrow$ We can define a third order 1-form A_{ijk} by $dA_{ijk} = F_{ijk}$.

The first and second order linkings for (C_i, C_j, C_k) vanish
$\Rightarrow (\Sigma_i, \Sigma_j, \Sigma_k)$ can be chosen to be disjoint
$\Rightarrow dA_{ijk} = 0$.

A particular solutions of $dA_{ijk} = 0$ is $A_{ijk} = \gamma_{ij} A_k - A_i \gamma_{jk}$.

$\gamma_{ij} = \delta_{ij} + \int_\Gamma A_{ij}$, δ_{ij} is a constant.

We define $A^{(3)}$ as a vector space with the basis $\{A_{ijk}\}_{i \neq j \neq k}$.

For a given closed curve C, there is an associated vector space of third order topological terms $S^{(3)} = \int_C A^{(3)}$.

Higher order topological terms

Proceed to iteratively construct topological terms of all orders.

The vector spaces $\{A^{(p)}\}$ are related to what is known in algebraic topology as the Massey products of cohomology groups.

Defining fields inside tubes

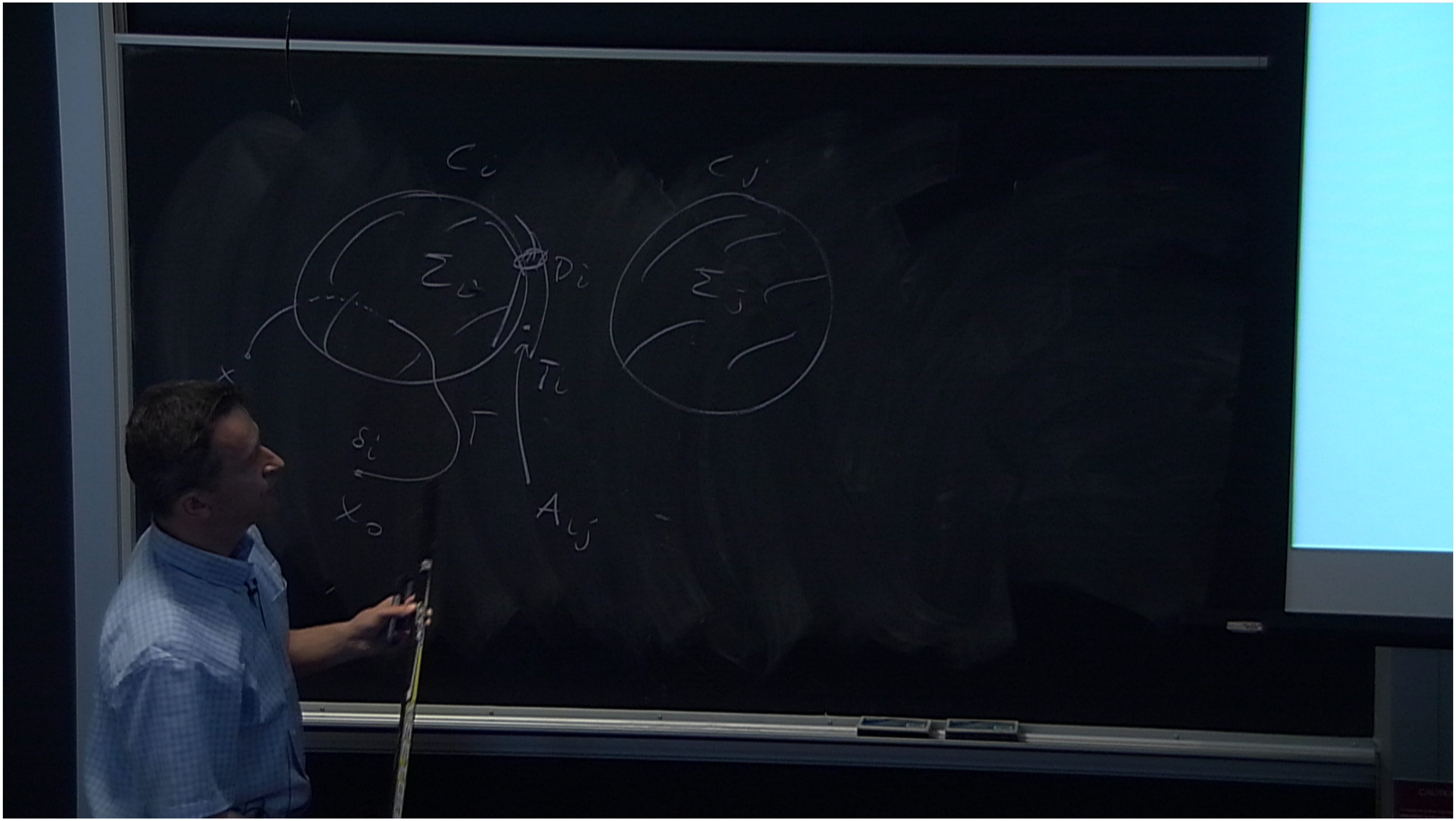
So far the spaces $A^{(p)}$ were defined only on $M = \mathbb{R}^3 \backslash T$.
We now extend these definitions into the interiors of the tubes T.

Such extensions are always possible if certain topological restrictions are satisfied.

To define the space $A^{(p)}$, all spaces $A^{(q)}$ with $q < p$ have to be defined.

If we assume that all $A^{(q)}$ with $q < p$ are defined, then we denote $R^{(p)}$ a set of additional restrictions needed to define the space $A^{(p)}$.

$R^{(p)}$ are constructed iteratively.



C_i
C_j
Σ_i
D_i
Σ_j
T_i
Γ
δ_i
x_0
A_{ij}

Restrictions for the second order fields

Consider extending A_{ij} inside T_i for $i \neq j$.

This extension is possible only if $dF_{ij} = 0$ inside T_i.

$\Rightarrow \int_{\partial T_i} F_{ij} = 0.$

However,
$\int_{\partial T_i} A_i \wedge A_j = \int_{T_i} d(A_i \wedge A_j) = \int_{C_i} A_j.$

There is an obstruction to such a procedure unless $\int_{C_i} A_j = 0$.

No new restriction is needed to extend A_{ij} inside T_j.

Therefore, $A^{(2)}$ can be defined only if a set of restrictions $R^{(2)} = \left\{ \int_{C_i} A_j = 0 \right\}_{i \neq j}$ is satisfied.

$\Rightarrow$ All pairs of distinct loops (C_i, C_j) should be unlinked.

Restrictions for the third order fields

Consider extending A_{ijk} inside T_i, T_j, and T_k for $i \neq j \neq k$.

Reasoning as above, we find that $A^{(3)}$ can be defined only if a set of restrictions $R^{(3)} = \left\{ \int_{C_i} A_{jk} = 0 \right\}_{i \neq j \neq k}$ is satisfied.

$\Rightarrow$ The second order linking between any triple of distinct loops (C_i, C_j, C_k) should vanish.

Restrictions for the higher order fields

Proceed iteratively to construct higher order restrictions $R^{(p)}$, $4 \leq p \leq N$.

To construct all spaces $\{A^{(p)}\}_{1 \leq p \leq N}$, the set of curves $\{C_i\}$ has to satisfy the restrictions $R = \cup_{1 \leq p \leq N} R^{(p)}$.

As a curious observation, note that the simplest topological arrangement of loops (no linking up to the given order) provides the richest structure for the topological term.

Free generators

G is of infinite order and it is freely generated by a set of generators $\{a_i\}_{1\leq i\leq N}$, which are homotopically equivalent to $\{\partial D_i\}_{1\leq i\leq N}$.

A generator a_i is defined as a closed path in M, which starts at the point x_0, intersects Σ_i once in the positive direction, does not intersect any other Σ_j, $j \neq i$, and ends at x_0.

The inverse path a_i^{-1} is the path a_i traversed in the opposite direction.

To multiply paths, we compose them in such a way that the end of the previous path is the beginning of the next path.

Explicit expressions for the topological terms

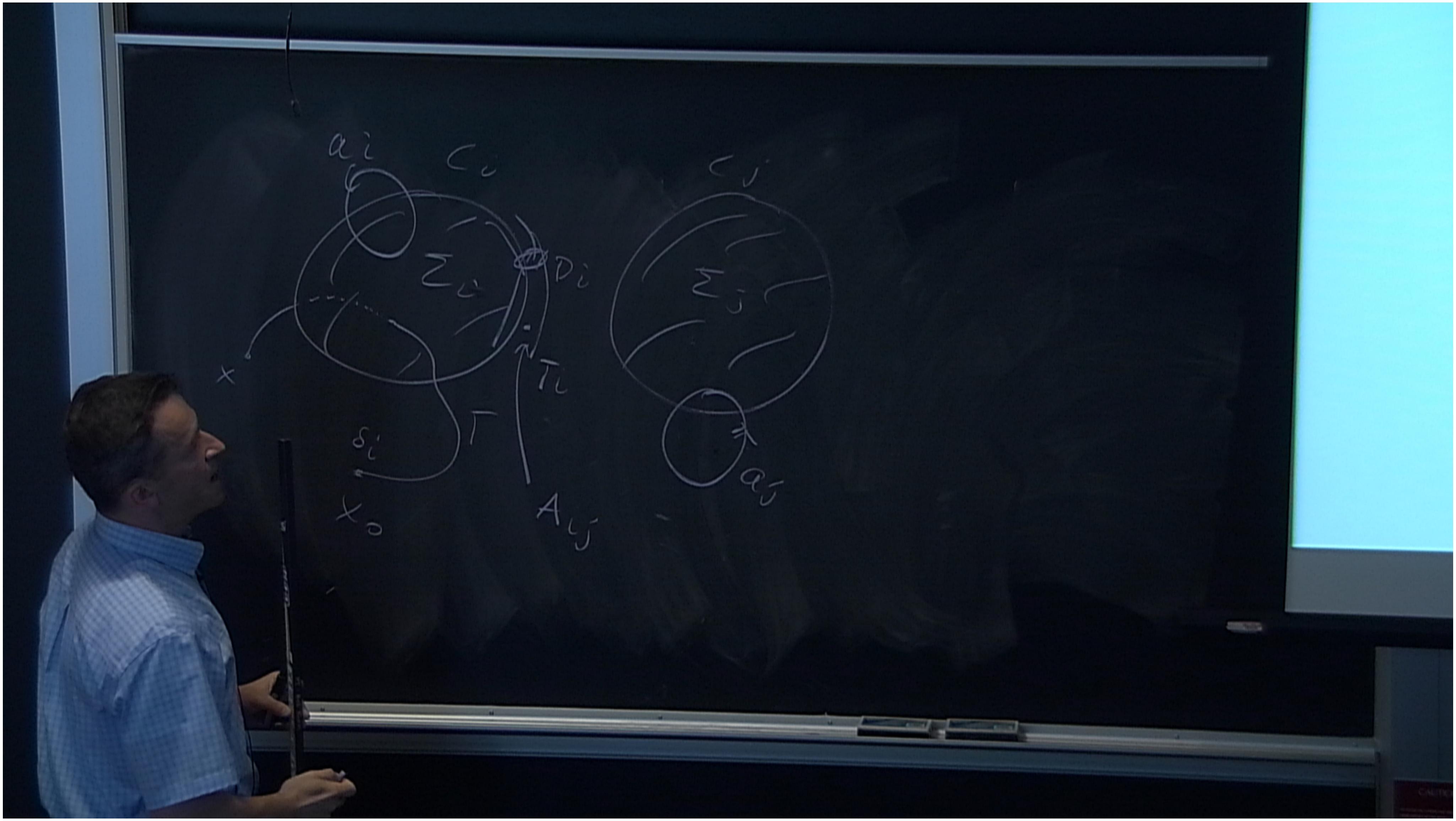
Homotopy classes of paths are labeled by finite sets of integers $(n_{11}, \ldots, n_{N1}, \ldots, n_{1l}, \ldots, n_{Nl})$, and representative paths from such classes are given by $C = a_1^{n_{11}} \cdots a_N^{n_{N1}} \cdots a_1^{n_{1l}} \cdots a_N^{n_{Nl}}$.

Straightforward computation gives

$$S_i = \int_C A_i = \sum_{i'} n_{ii'},$$

$$S_{ij} = \int_C A_{ij} = \frac{1}{2}\Big[\delta_i S_j - S_i \delta_j + \sum_{i',j'} \sigma_{i'j'} n_{ii'} n_{jj'}\Big],$$

$$S_{ijk} = \int_C A_{ijk} = \frac{1}{4}\Big[\delta_i S_{jk} - S_i \delta_j S_k - S_{ij}\delta_k - \delta_i S_j \delta_k + 2\delta_{ij} S_k - 2 S_i \delta_{jk} + \sum_{i',j',k'} \sigma_{i'j'k'} n_{ii'} n_{jj'} n_{kk'}\Big],$$



ai
Ci
Cj
Σi
Di
Σj
x
Ti
Γ
δi
x0
Aij
aj

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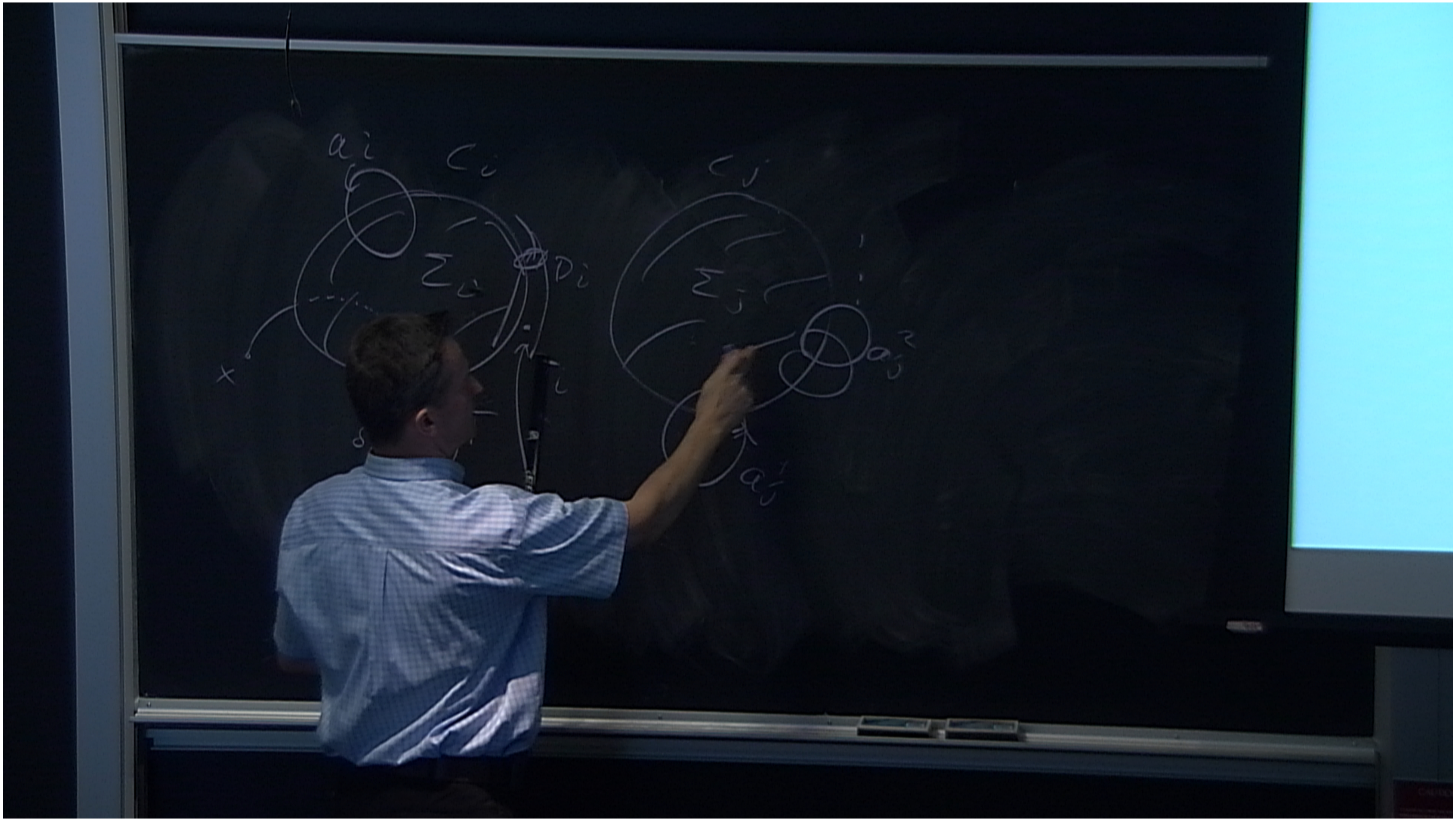
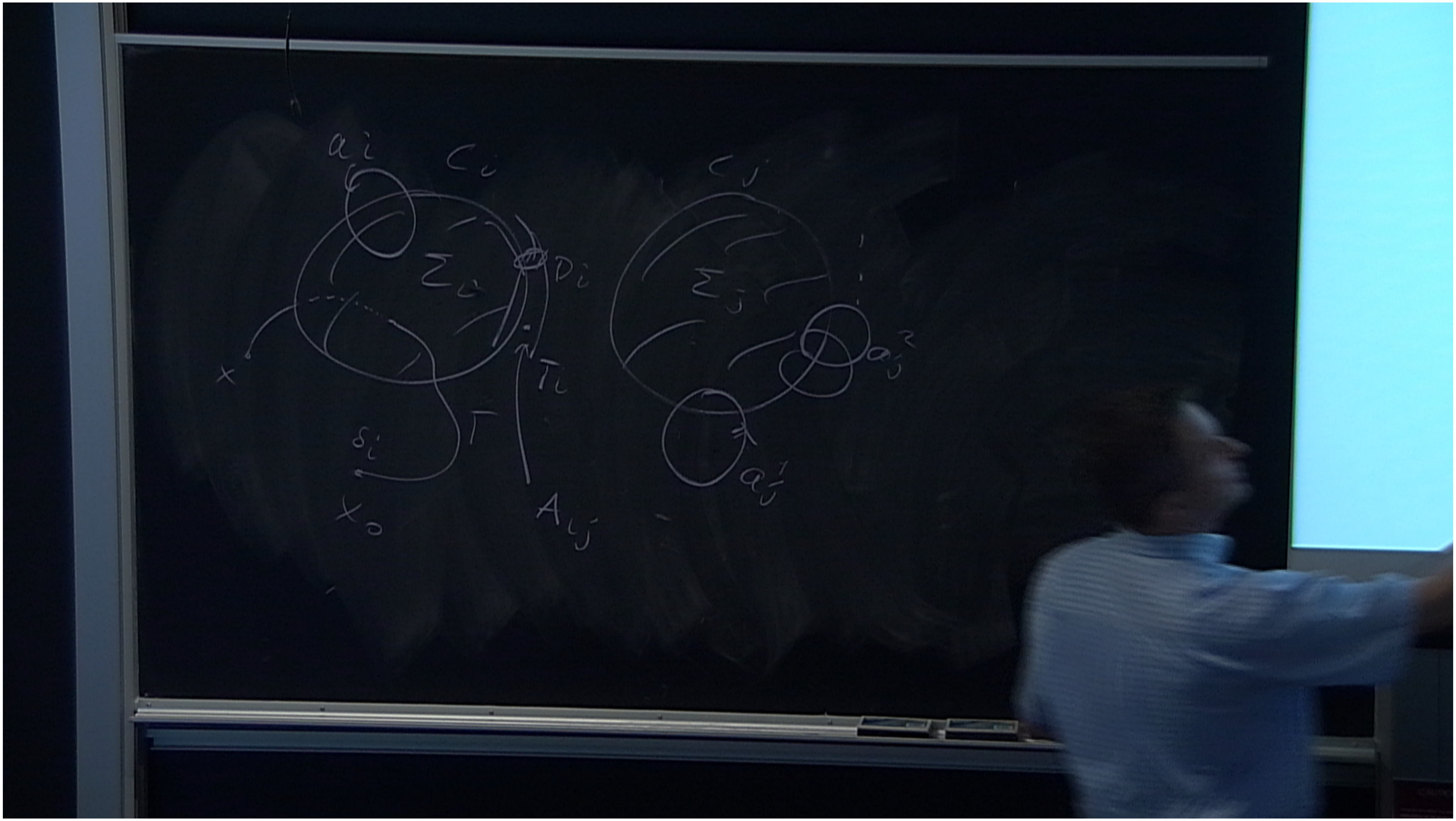
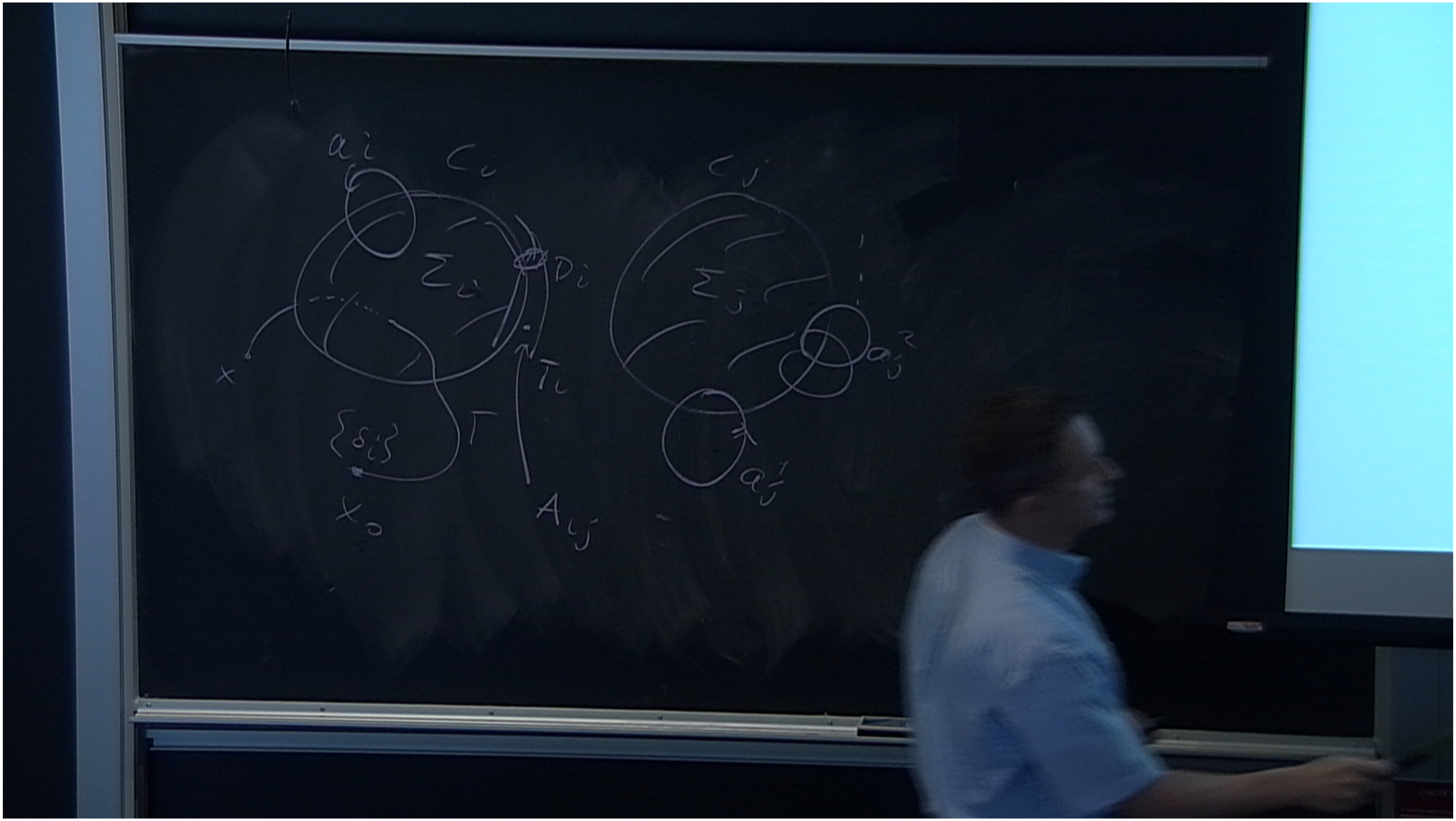

a_i
c_i
Σ_a
D_i
c_j
Σ_j
a_j^2
a_j^1



a_i
c_i
Σ_i
D_i
c_j
Σ_j
a_j^2
a_j^1
T_i
Γ
δ_i
x_0
A_{ij}



a_i
c_i
Σ_i
D_i
T_i
$\{\delta_i\}$
Γ
x_0
A_{ij}
c_j
Σ_j
a_j^2
a_j^1

Explicit expressions for the topological terms

$$\sigma_{ij} = \begin{cases} 1, & i \leq j \\ -1, & i > j \end{cases}$$

$$\sigma_{ijk} = \begin{cases} 1, & i \leq j \leq k \text{ or } \; k+2 \leq j+1 \leq i \\ -1, & \text{otherwise} \end{cases}$$

Expressions for higher order topological terms are similarly found.

Non-additivity for multiplicative paths

Elements of $S^{(1)}$ depend only on a path
$\Rightarrow$ they are additive for multiplicative paths,
$S_i(CC') = S_i(C) + S_i(C')$.
Elements of $S^{(1)}$ form abelian representations of the group G.

The situation is different for elements of $S^{(p)}$ for $p \geq 2$.
They depend on both the path and the location of the point x_0 through constants $\{\delta_i\}$, $\{\delta_{ij}\}, \ldots$.

Since the constants can be different for different loops in a product of loops, these topological terms are not in general additive for multiplicative paths.

However, there is a particular set of terms that are additive.

The abelian property for the second order fields

$$S_{ij} = \int_C A_{ij} = \frac{1}{2}\Big[\delta_i S_j - S_i \delta_j + \sum_{i',j'} \sigma_{i'j'} n_{ii'} n_{jj'}\Big]$$

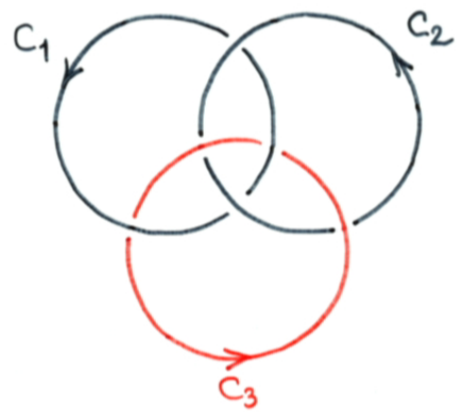
$\{\delta_i\}$ depend on x_0
$S^{(2)}$ is independent of x_0 only if $S^{(1)}$ is the zero vector space.

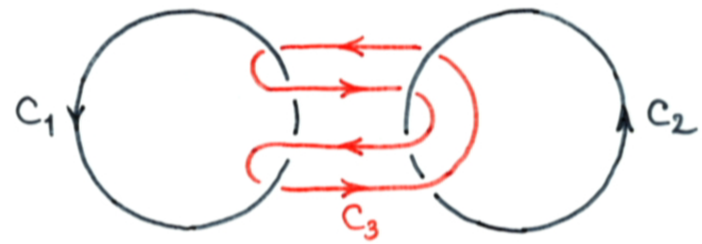
It can be shown that in this case a closed curve C is a product of commutator loops.

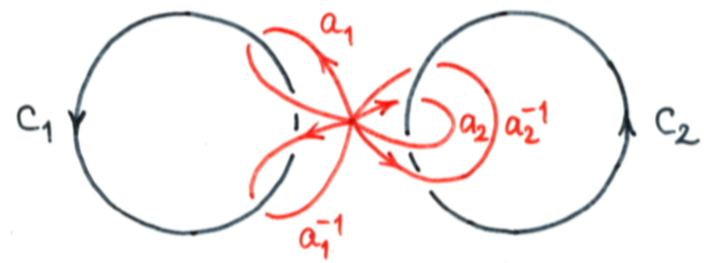
A commutator loop is a path $[g_1, g_2] = g_1 g_2 g_1^{-1} g_2^{-1}$, where $g_i \in G$.

For the product of commutator loops, an element of $S^{(2)}$ is the sum of the corresponding terms for each component,
$S_{ij}(CC') = S_{ij}(C) + S_{ij}(C')$.

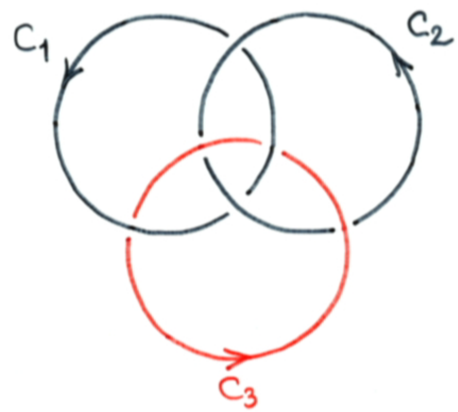
The Borromean rings

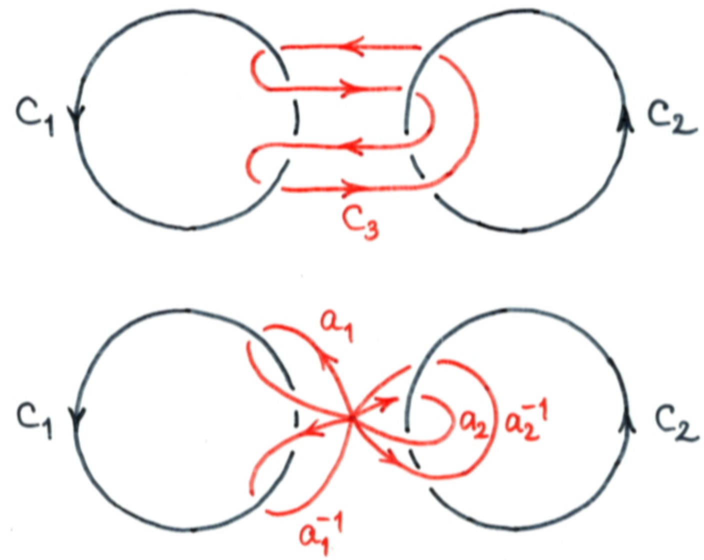






The Borromean rings





Higher order quantum phases

Computations via the topological terms.

$p = 1$
$C = a_i^{n_i},\ S_i = n_i,\ \phi_i = \xi S_i \Phi_i$

$p = 2$
$C = [a_i^{n_i}, a_j^{n_j}],\ S_{ij} = n_i n_j,\ \phi_{ij} = K_2 \xi^2 S_{ij} \Phi_i \Phi_j,\ K_2 = \mathrm{const}$

General order p
$\phi_{i_1 \cdots i_p} = K_p \xi^p S_{i_1 \cdots i_p} \Phi_{i_1} \cdots \Phi_{i_p},\ K_p = \mathrm{const}$

Except for $K_1 = 1$, constants K_p are undetermined.

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Higher order quantum phases

For path integrals in non-simply connected spaces, the phase of the wave function in quantum mechanics has to form an abelian representations of the fundamental group.

$\Rightarrow$ Higher order boundary terms can be included and the phase of order p is $\int_C A$, where $A \in A^{(p)}$ and $C \in G_p$.

Quantum mechanics imposes restrictions on what elements of $S^{(p)}$ are allowed to contribute to the phase.

If a charged particle is transported along a closed curve C outside a solenoid, then its action changes by $\int_C A$, where A is the gauge potential of the magnetic field in the solenoid.

The Aharonov-Bohm effect: the wave function acquires a phase $\phi = \xi n \Phi$, where $\xi = e(\hbar c)^{-1}$, n is the number of times the curve wraps around the solenoid, and Φ is the flux of the magnetic field.

Higher order quantum phases

Computations via the topological terms.

$p = 1$
$C = a_i^{n_i}, \; S_i = n_i, \; \phi_i = \xi S_i \Phi_i$

$p = 2$
$C = [a_i^{n_i}, a_j^{n_j}], \; S_{ij} = n_i n_j, \; \phi_{ij} = K_2 \xi^2 S_{ij} \Phi_i \Phi_j, \; K_2 = \text{const}$

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$\phi_{i_1 \cdots i_p} = K_p \xi^p S_{i_1 \cdots i_p} \Phi_{i_1} \cdots \Phi_{i_p}, \; K_p = \text{const}$

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Higher order quantum phases

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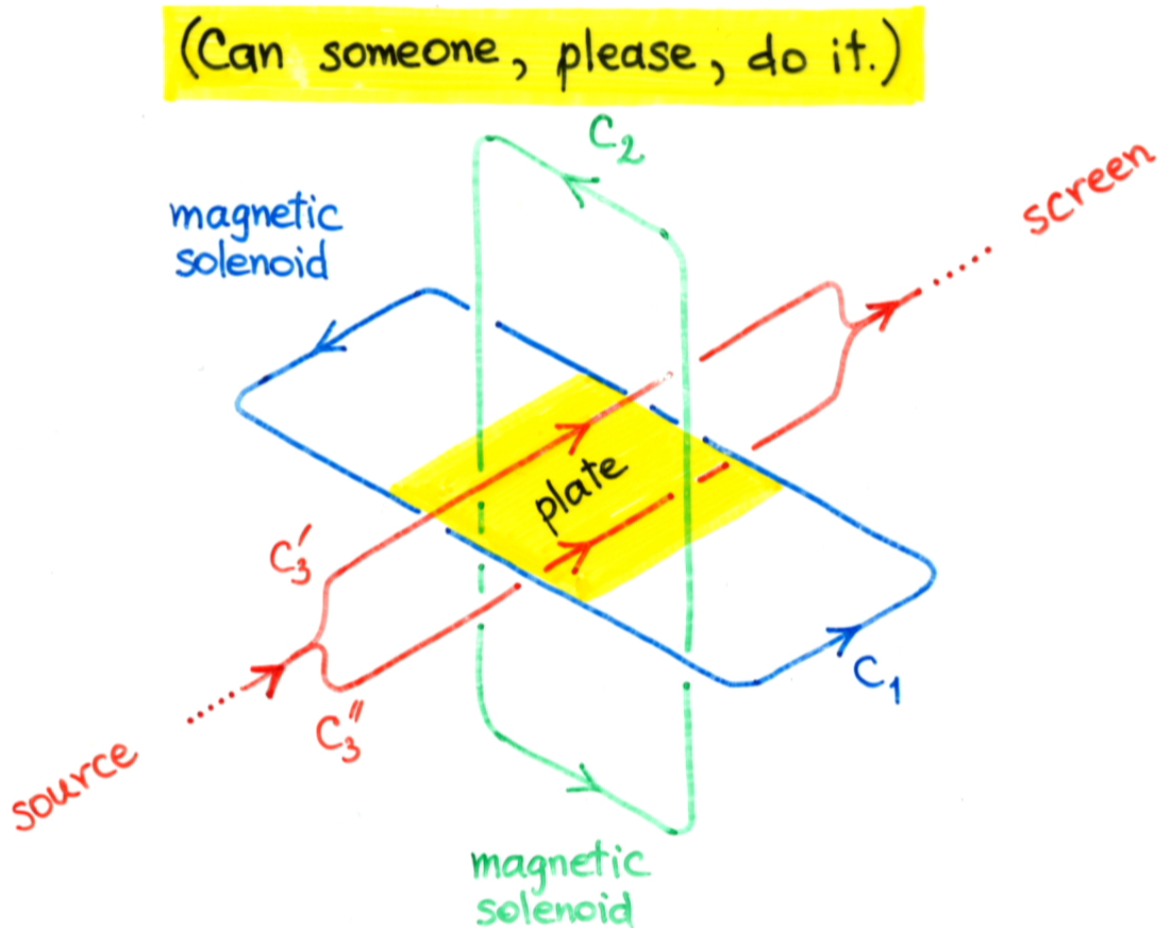
Experiment



(Can someone, please, do it.)
C2
magnetic solenoid
screen
plate
C3′
C3″
C1
source
magnetic solenoid

Experiment

$C_3 = C_3' \cup C_3''^{-1}$
Observe the phase $\phi_{1,2}(C_3) = K_2(e/\hbar c)^2 \Phi_1 \Phi_2$

Note:

$\phi_1(C_3) = 0$
$\phi_2(C_3) = 0$
$\phi_{1,2}(C_3) \neq 0$

Generalized Dirac quantization condition: $K_2 = (2\pi)^{-1}$

Needs to be checked experimentally

Higher order quantum phases

Computations via the topological terms.

$p = 1$
$C = a_i^{n_i}, \ S_i = n_i, \ \phi_i = \xi S_i \Phi_i$

$p = 2$
$C = [a_i^{n_i}, a_j^{n_j}], \ S_{ij} = n_i n_j, \ \phi_{ij} = K_2 \xi^2 S_{ij} \Phi_i \Phi_j, \ K_2 = \text{const}$

General order p
$\phi_{i_1 \cdots i_p} = K_p \xi^p S_{i_1 \cdots i_p} \Phi_{i_1} \cdots \Phi_{i_p}, \ K_p = \text{const}$

Except for $K_1 = 1$, constants K_p are undetermined.

Higher order quantum phases

$$\phi_{i_1 \cdots i_p} = K_p \xi^p S_{i_1 \cdots i_p} \Phi_{i_1} \cdots \Phi_{i_p}, \;\; K_p = \text{const}$$

We are not aware of any fundamental quantum-mechanical principle forbidding the presence of terms with $p \geq 2$ and therefore suggest this be tested experimentally.

An argument allowing to calculate the constants K_p for $p \geq 2$.

From the Aharonov-Bohm result, if $(2\pi)^{-1}\xi\Phi_i \in \mathbb{Z}$, then the phase ϕ_i is unobservable. If this is also the case for the higher order phases, then we find $K_p = (2\pi)^{-p+1}$.

Higher order quantum phases

A possible objection: all elements of $A^{(p)}$ for $p \geq 2$ are nonlocal quantities.

After addition of these terms, the coordinate and momentum operators are still local, but the hamiltonian operator becomes nonlocal.

This nonlocality, however, has no local consequences. (In the magnetic field analogy, the only measurable effect is the force acting on the particle and it is absent outside the tubes.)

This is analogous to the first order term having no local consequences despite being the nonlocal operator itself.

Conclusions

- The action of a system is not uniquely defined since arbitrary topological terms can be added to the action without changing the equation of motion.
- Although classical dynamics is immune to such terms, they affect the quantum dynamics.
- These terms can be classified according to their topological properties.
- Each term contributes a phase to the wave function, the functional form of which is easily distinguishable from the phases due to terms of other orders.
- In particular, the phase of order p is proportional to the product of p fluxes.
- The usual Aharonov-Bohm phase corresponds to $p = 1$, and its simplest generalization is the Borromean ring phase which corresponds to $p = 2$.

Conclusions

- Higher order quantum phases can clearly exist in any physical situation that supports higher order linking of a wave function with fluxes, e.g., superconducting loops containing Josephson junctions that have higher order linking with solenoids.
- It should not be difficult to conduct an experiment capable of answering the question whether higher order topological phases play a role in quantum mechanics.
- Can someone, please, do the Borromean ring experiment!