

Title: Symmetries of Bell correlation scenarios

Date: Jun 06, 2016 02:00 PM

URL: <http://pirsa.org/16060030>

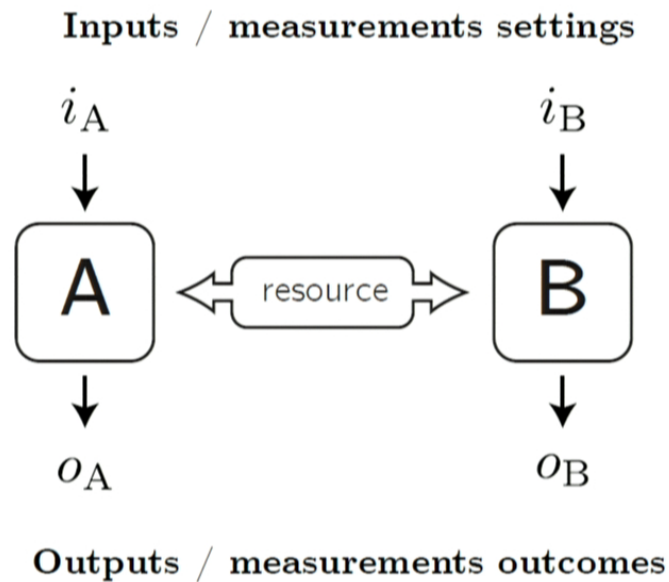
Abstract: <p>In the device-independent paradigm, the labeling of parties/inputs/outputs has no physical meaning and thus the behavior of the system should be studied up to symmetry. We conduct the first formal study of relabelings appearing in Bell scenarios. The talk includes a review of previous works, a definition of Bell relabeling groups illustrated by examples, and applications, including the classification of Bell inequalities, the generalization of binary correlators to  $d$  outcomes and the computation of exact bounds using the NPA hierarchy.</p>

This talk = spaghetti on the wall



*Spaghetti bench, Pablo Reinoso*

# Paradigm: “device-independent”



Observed correlations

$$P(o_A, o_B | i_A, i_B)$$

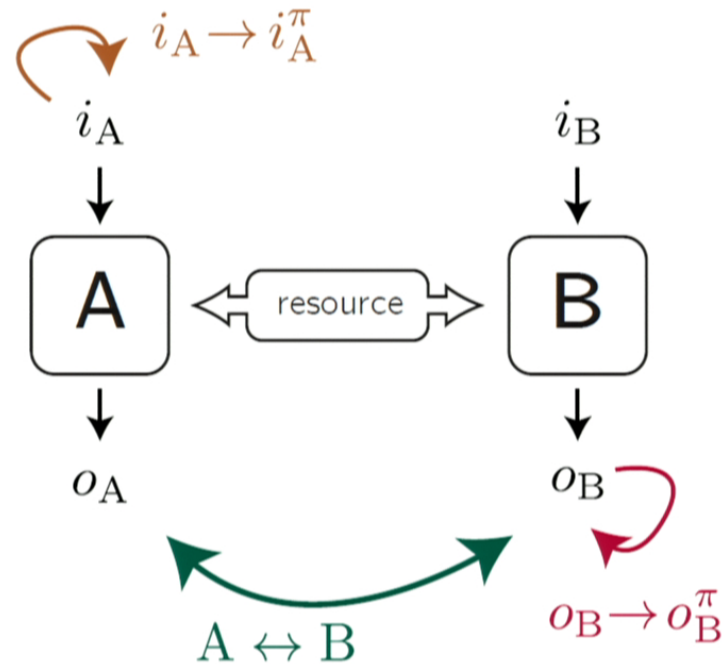
*Note:*

W/o any assumption on the devices,  
output, input sets must be finite  
in any testable theory

$$\vec{P} \in \mathbb{R}^n = \begin{pmatrix} P(00|00) \\ \dots \\ P(11|11) \end{pmatrix}$$

*Example:* in CHSH scenario,  $n = 16$

# Paradigm: “device-independent”



## Galileo principle:

*The laws of quantum information do not depend on a particular labeling of **parties**, **inputs** and **outputs**.*

## Thus:

*The relevant physics should be studied “up to symmetry”.*

*What group structures are involved?*

*What are their representations?*

*Any physical meaning associated?*

# Plan

## Definitions

Bell group, actions, orbits, averaging maps

## Review of literature

Bell inequality families (Śliwa 2003), Liftings (Pironio 2005),  
Invariance under party permutation (Liang 2008, Bancal 2010, Tura 2013)  
Creating Bell inequalities from group representations (Güney, 2014)

## New applications

Classification of Bell inequalities (Rosset 2014)  
Generalized correlators (WIP)  
Exact NPA bounds (WIP)

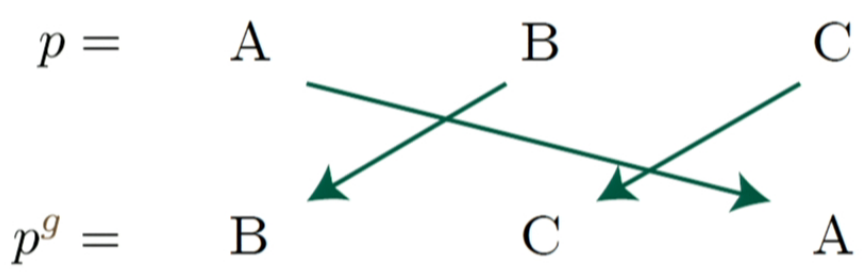
# Definitions

Bell group, actions, orbits, averaging maps

# Bell group $G$

*What's inside?*

Permutation of parties



Cycle notation:  
*preimage*  $\rightarrow$  *image*

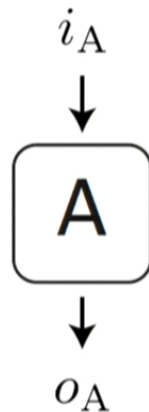
$$g = (A, B, C)$$

The cycle notation  $g = (A, B, C)$  is accompanied by a diagram showing three curved arrows forming a cycle: A to B, B to C, and C to A.

# Bell group $G$

*What's inside?*

Permutation of inputs



$$i_A \longrightarrow i_A^\pi$$

$$\pi = (0, 1, 2)$$

$$g = A(0, 1, 2)$$

affected party      cycle notation

Inverse: reverse the cycles

$$g^{-1} = A(0, 2, 1)$$



# Bell group $G$

*What's inside?*



Permutation of outputs

$$o_A \longrightarrow o_A^\pi \quad i_A = 0$$

$$\pi = (0, 1, 2)$$

$$g = \text{A0} (0, 1, 2)$$

affected party    cycle notation  
and input

Inverse: reverse the cycles

$$g^{-1} = \text{A0}(0, 2, 1)$$

# Bell group $G$

Permutation of

parties

$$g = (A, B, C)$$

Inverses

$$g^{-1} = (A, C, B)$$

inputs

$$g = A(0, 1, 2)$$

$$g^{-1} = A(0, 2, 1)$$

outputs

$$g = A0(0, 1, 2)$$

$$g^{-1} = A0(0, 2, 1)$$

Canonical ordering

$$g = \begin{array}{ccccccc} \color{orange}\blacksquare & \color{orange}\blacksquare & \color{orange}\blacksquare & \color{yellow}\blacksquare & \color{yellow}\blacksquare & \color{yellow}\blacksquare & \color{green}\blacksquare \\ \text{outputs} & & \text{inputs} & & & & \text{parties} \end{array}$$

Right exponentiation notation

$$x^{g_1 g_2} = (x^{g_1})^{g_2}$$

Commutation rules

$$(A, B) A(0, 1) = B(0, 1) (A, B)$$

$$(A, B) A0(0, 1) = B0(0, 1) (A, B)$$

$$A(0, 1) A0(0, 1) = A1(0, 1) A(0, 1)$$

*input relabelings for different parties commute  
& also output relabelings for different parties/inputs*

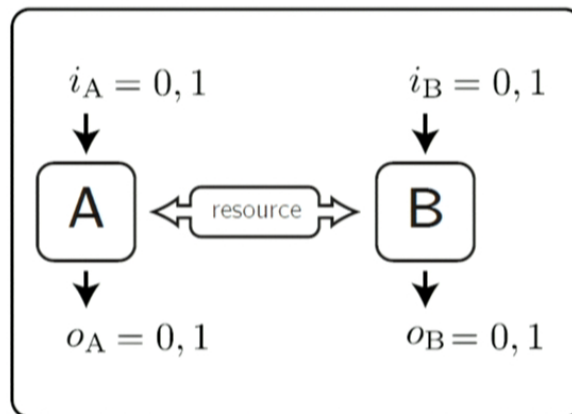
# Symmetry subgroup of CHSH game

*Simple generators*

*Game*

$$o_A \oplus o_B = i_A i_B$$

*Corresponding symmetries*



$$o_A, i_A \leftrightarrow o_B, i_B \quad g_1 = (A, B)$$

$$\begin{aligned} o_A &\rightarrow o_A \oplus 1 \\ o_B &\rightarrow o_B \oplus 1 \end{aligned} \quad g_2 = A0(0,1) \ A1(0,1) \ B0(0,1) \ B1(0,1)$$

$$\begin{aligned} o_A &\rightarrow o_A \oplus i_A \\ i_B &\rightarrow i_B \oplus 1 \end{aligned} \quad g_3 = A1(0,1) \ B(0,1)$$

*Group*  $G_{\text{CHSH}} = \langle g_1, g_2, g_3 \rangle$  of order 16

# Actions of Bell groups

*(Imprimitive)-imprimitive action (from wreath product)*

Triplet

$$\begin{aligned} t = (p, i_p, o_{p,i}) &\xrightarrow{g = \pi} (p^\pi, i, o) \\ &\xrightarrow{g = P\pi} (p, i^\pi, o) \quad p = P \\ &\xrightarrow{g = PI\pi} (p, i, o^\pi) \quad p = P \text{ and } i = I \end{aligned}$$

Observable quantity:  $P_p(o|i) = \text{tr} \left[ \rho_p M_{o|i}^p \right]$

*Note:* use this (faithful) action to retrieve the commutation rules for group elements

# Actions of Bell groups

*(Imprimitive)-primitive action*

List of outputs/inputs

$$(o_A, o_B, \dots, i_A, i_B, \dots)$$

$\approx$  set of triplets

$$\{t_A, t_B, \dots\} = \{(A, i_A, o_A), (B, i_B, o_B), \dots\}$$

Observable quantity:

$$P(o_A, o_B, \dots | i_A, i_B, \dots) = \text{tr} \left[ \rho \left( M_{o_A|i_A}^A \otimes M_{o_B|i_B}^B \dots \right) \right]$$

# Actions of Bell groups

*(Imprimitive)-primitive action*

List of outputs/inputs

$$(o_A, o_B, \dots, i_A, i_B, \dots) \xrightarrow{g = \pi} (o_{A^{\pi^{-1}}}, \dots, i_{A^{\pi^{-1}}}, \dots)$$

$$(\dots, o_P, \dots, \dots, i_P, \dots) \xrightarrow{g = P\pi} (\dots, o_P, \dots, \dots, i_P^\pi, \dots)$$

$$(\dots, o_P, \dots, \dots, i_P, \dots) \xrightarrow{g = PI\pi} (\dots, o_P^\pi, \dots, \dots, i_P, \dots)$$

$$i_P = I$$

Observable quantity:

$$P(o_A, o_B, \dots | i_A, i_B, \dots) = \text{tr} \left[ \rho \left( M_{o_A | i_A}^A \otimes M_{o_B | i_B}^B \dots \right) \right]$$

# Actions of Bell groups

## *Summary of actions*

Imprimitive-imprimitive : single party marginals

Imprimitive-primitive : observed correlations

Primitive-primitive : deterministic local strategies

Action on vectors (such as prob. distributions):

$$x^g(\text{index}^g) = x(\text{index})$$

Induced action on tuples:

=> product of measurement operators (i.e. NPA hierarchy)

# Group theory toolbox

## *Orbits*

$$\vec{x}^G = \{\vec{x}^g \mid g \in G\}$$

Ex: representatives of a Bell inequality

## *Stabilizer subgroup*

$$G_{[\vec{x}]} = \{g \mid g \in G, \vec{x}^g = \vec{x}\}$$

Ex: symmetry group of a Bell inequality

## *Symmetric subspace*

$$V^G = \{\vec{v} \in V \mid \vec{v}^g = \vec{v}, \forall g \in G\}$$

Ex: correlations symmetric under exchange of parties

## *Averaging map*

$$\phi_G(\vec{v}) = \frac{1}{|G|} \sum_{g \in G} \vec{v}^g$$

Ex: “depolarization” to isotropic line (from uniformly random to PR-box)



# *Computational* group theory tools

## *Permutation groups*

Efficient algorithms using bases and strong generating sets  
(Schreier-Sims)

Bible: Derek Holt, Handbook of CGT, 2005  
Implemented in Scala ([github.com/denisrosset/alasc](https://github.com/denisrosset/alasc))

## *Monomial representations*

Using  
GAP system / AREP, Markus Püschel (1998-1999)

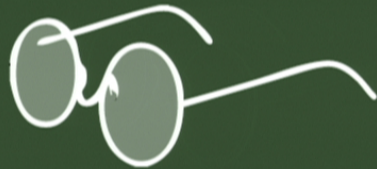
# Review of literature

Symmetries of Bell inequalities (Sliwa 2003)

NPA hierarchy (Liang, Bancal, ...)

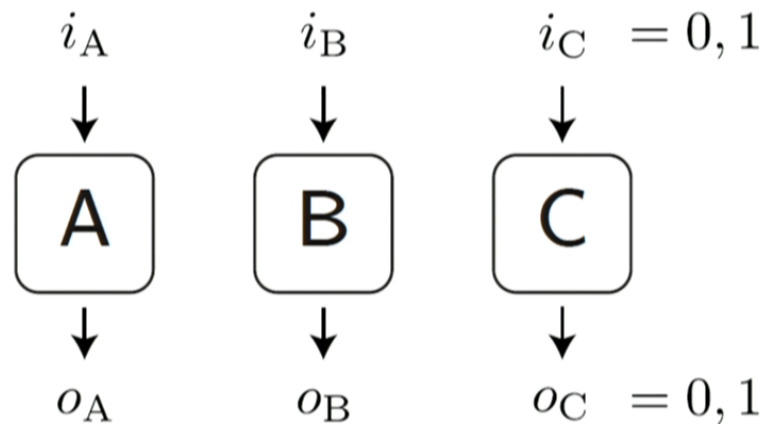
Looking for symmetric Bell inequalities (Bancal 2010)

Creating Bell inequalities from groups (Güney 2014)



# Symmetries of Bell inequalities

Pitowsky & Svozil, 2001:



Local polytope:  
53'856 facets

Sliwa, 2003:

46 equivalence classes under group action

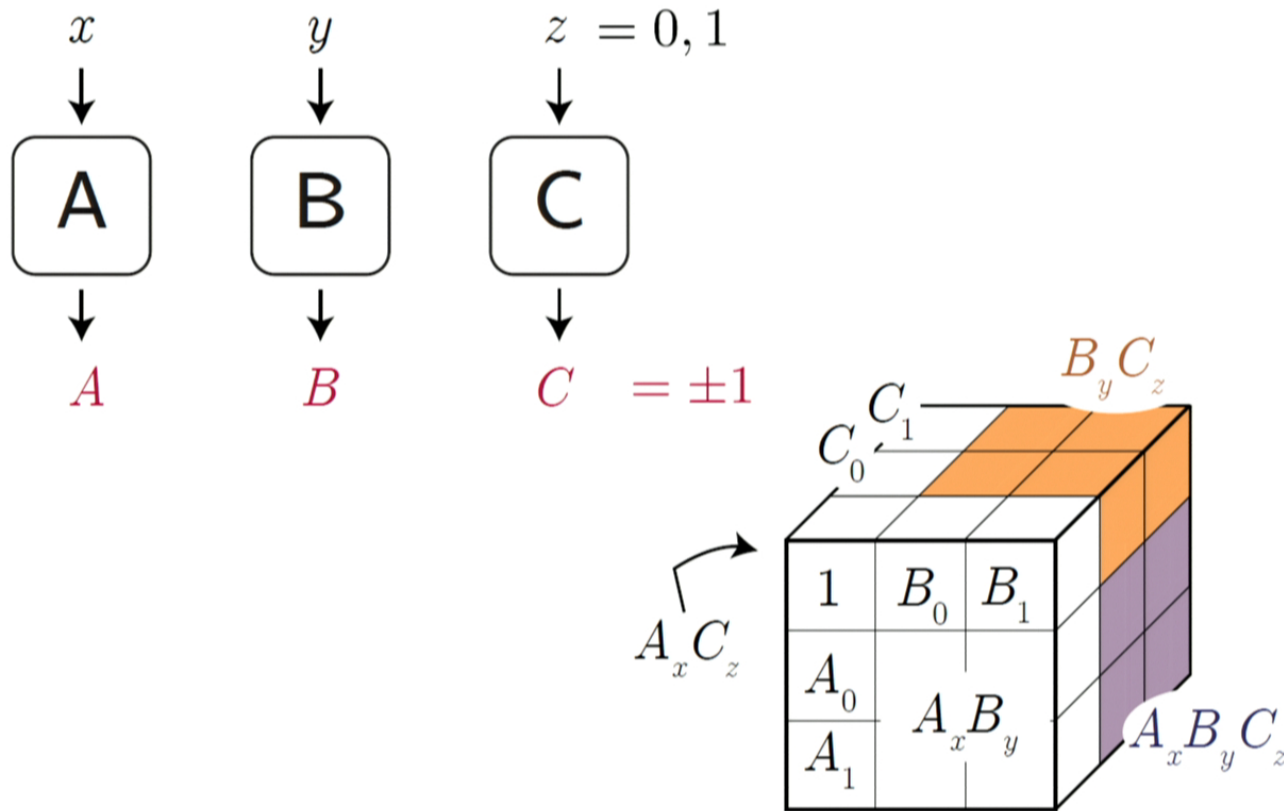
List one representative per class/family

Family representatives = orbit of representative under group

Small group  $|G| = 3072$ , brute force computation possible

# Symmetries of Bell inequalities

Sliwa, 2003:



# NPA hierarchy w/ party symmetry

Liang 2008, Bancal (priv. comm.), folklore

$$\chi = \begin{pmatrix} \text{Alice proj.} & \text{Bob proj.} \\ U & V \\ \hline & W \end{pmatrix} \geq 0$$

■ Alice proj.     ■ Bob proj.

If inequality symmetric under (A, B),  
apply averaging map on  $\chi$

$$\begin{aligned} V &= V^\top \\ U &= W \end{aligned}$$

$$\longrightarrow \chi' \equiv \begin{pmatrix} U+V & \\ \hline & U-V \end{pmatrix} \geq 0$$

# Looking for symmetric Bell ineqs.

Bancal 2010

*Vertices of local polytope*

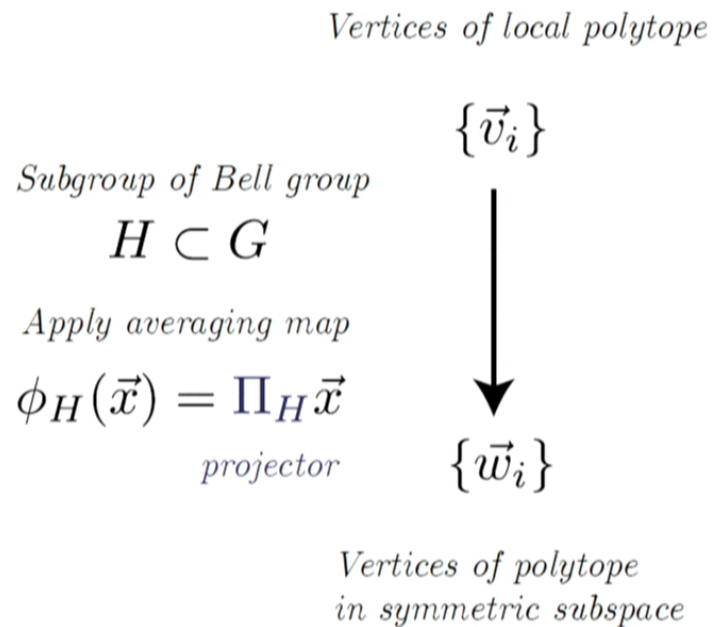
$$\{\vec{v}_i\}$$



$$\{\vec{a}_i^\top \vec{P} \leq b_i\}$$

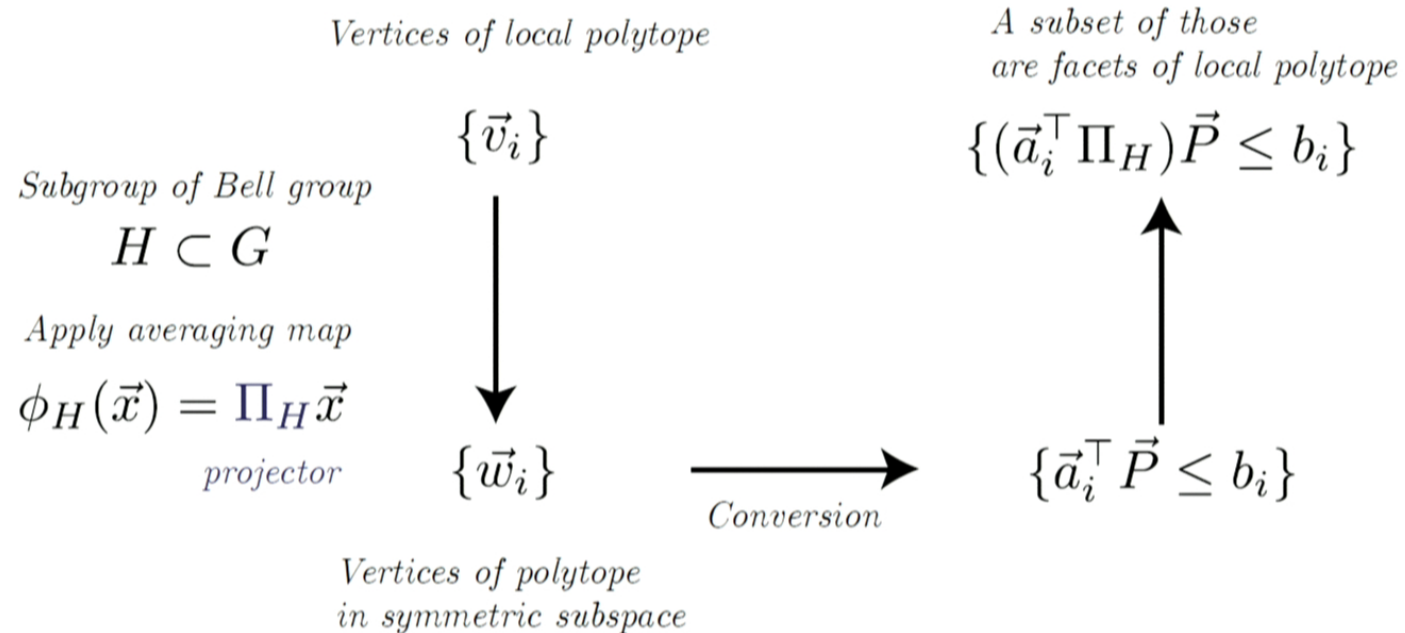
# Looking for symmetric Bell ineqs.

Bancal 2010



# Looking for symmetric Bell ineqs.

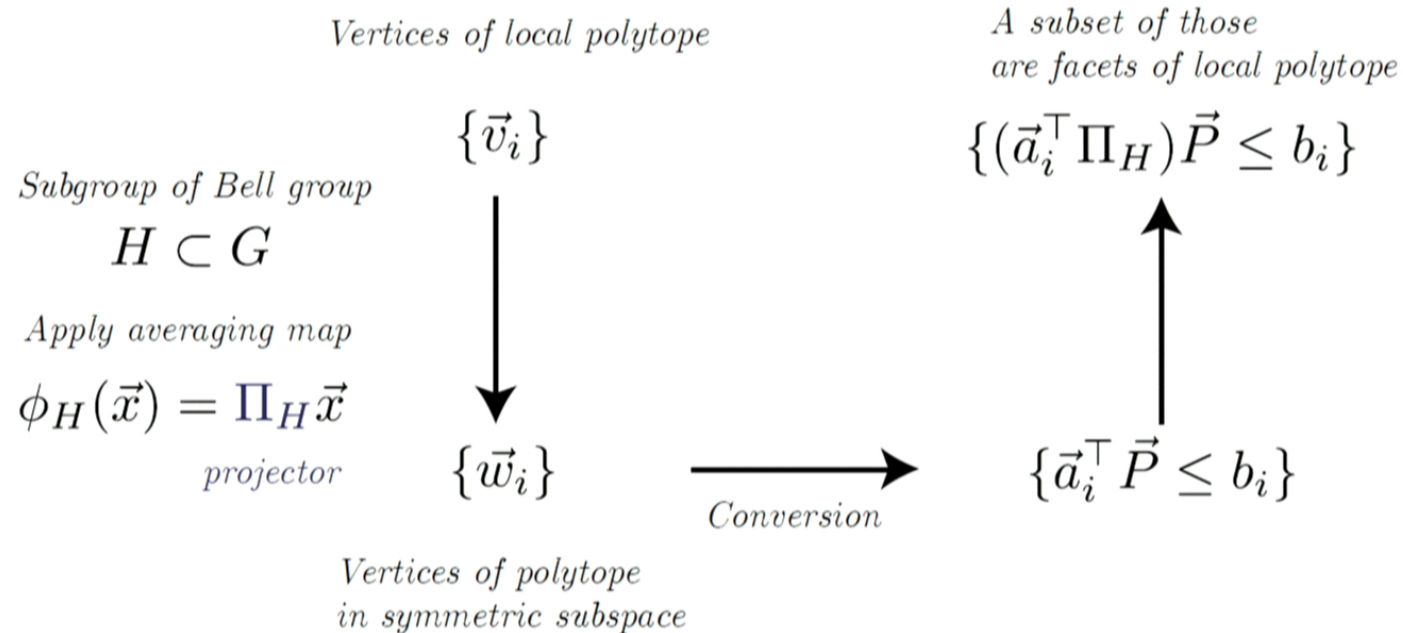
Bancal 2010





# Looking for symmetric Bell ineqs.

Bancal 2010



...see also J. Tura 2014-2015

only party permutations considered!

# Bell inequalities from group actions

Güney&Hillery 2014, Bolonek-Lasoń 2015

Idea:

1. take irred. representation  $D$  of dim.  $d$  of group  $G$
2. take  $\langle\phi|, \langle\psi| \in \mathcal{H}^d$  such that orbits  $\langle\phi|^G, \langle\psi|^G$  can be grouped into orthonormal bases
3. apply averaging map on POVM element

$$M = |\phi \otimes \psi\rangle \langle\phi \otimes \psi|$$

use it as the Bell operator of a new Bell inequality



# Application & ideas

Classification of Bell inequalities (Rosset, Bancal, Gisin 2014)  
Generalized correlators (WIP)  
Bell inequalities and statistical noise (w/ M.O. Renou)  
Exact solutions to NPA relaxations (WIP)

# Classification of Bell inequalities

Rosset, Bancal, Gisin 2015

$$I = \sum_{abxy} \alpha_{abxy} P(ab|xy) \leq u$$

$\alpha_{abxy}$  Coefficients of Bell inequality

There are trivial equalities when P is normalized:

$$\sum_{ab} P(ab|00) - P(ab|11) = 0$$

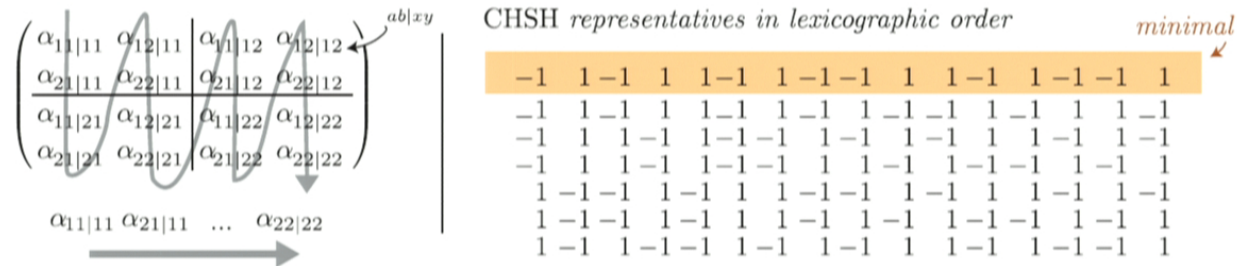
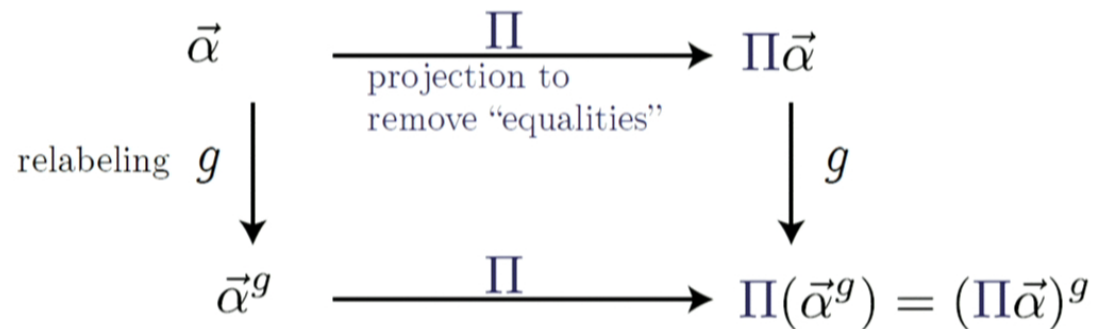
when P is nonsignaling:

$$\sum_b P(ab|xy) = P(a|xy) = P(a|x)$$

If  $\vec{P}$  is in the nonsignaling subspace, then  $\vec{P}^g$  is in the nonsignaling subspace.

# Classification of Bell inequalities

Rosset, Bancal, Gisin 2015



See <http://faacets.com>



# BETA faacets.com

An open collaboration platform to browse, share and process Bell and Bell-like inequalities.

Currently working on [Beta 2](#) goals.

- [Learn about Bell inequalities](#)
- [Learn about the project](#)
- [Browse the database](#)

### Highlight: characterization of non-locality using Bell inequalities

Bell inequalities can be used to quantify the power of non-locality, by looking at (for example) how many bits of communication are necessary to reproduce correlations produced by quantum systems.

root / pubs / Almeida2010

Show 10 entries

Key	Scenario	#P	#I	#O	IO-Lifted?	Composite?	Canonical
Eq10	[(2 2) (2 2) (2 2)]	3	2	2	no	no	#13

Showing 1 to 1 of 1 entries ◀ Previous Next ▶

**Filters**

Key

IO-Lifted?

Composite?

Parties  to

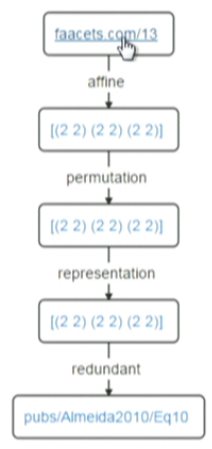
Inputs  to

Outputs  to

root / pubs / Almeida2010 / Eq10

Bell Expression Decomposition Yaml source

Decomposition



Bell expression

in [(2 2) (2 2) (2 2)] using Non-signaling Probabilities

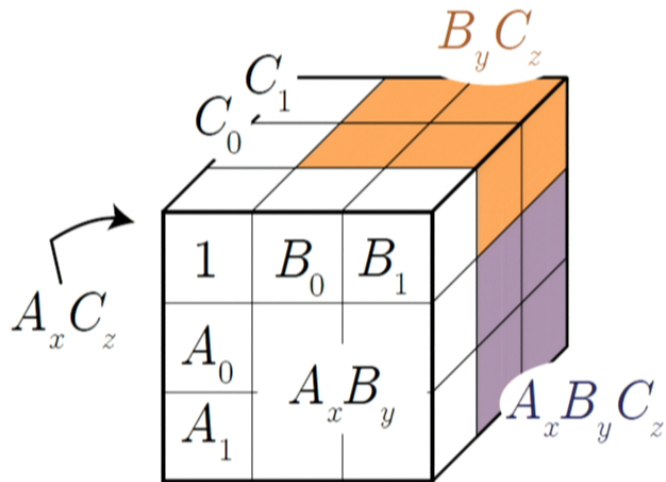
$$\left( \begin{array}{ccc|ccc|ccc|ccc} -3 & 1 & -3 & 1 & 5 & -3 & 1 & 1 & 1 & -3 & -3 & 1 & 1 \\ 1 & 1 & 1 & 1 & -3 & 1 & 1 & -3 & 1 & 1 & 5 & -3 & -3 \\ -3 & 1 & 1 & -3 & 1 & 1 & 1 & 1 & 1 & -3 & 1 & -3 & -3 \\ 1 & 1 & -3 & 5 & 1 & -3 & 1 & -3 & 1 & 1 & 1 & 1 & 1 \end{array} \right)$$



# Irreducible representations

WIP w/ Marc-Olivier Renou

Sliwa 2003: 3 parties, binary I/O



# Irreducible representations

WIP w/ Marc-Olivier Renou

CHSH scenario

$$\vec{P} \in \mathbb{R}^{16}$$

1	$B_0$	$B_1$
$A_0$	$A_x B_y$	
$A_1$		

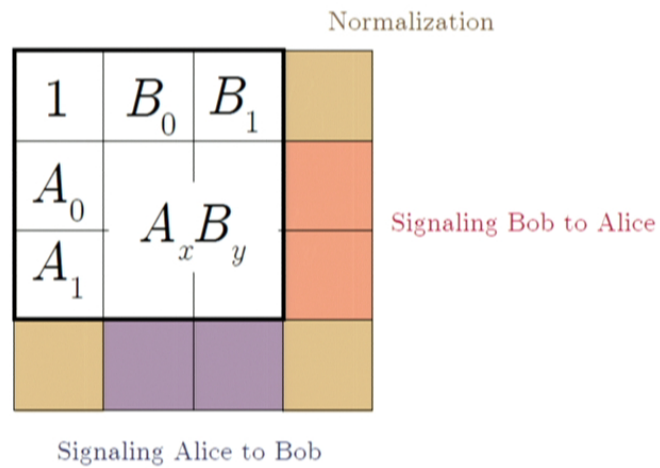
Dimension 16 versus 9 ?

# Irreducible representations

WIP w/ Marc-Olivier Renou

CHSH scenario

$$\vec{P} \in \mathbb{R}^{16}$$





# Irreducible representations

WIP w/ Marc-Olivier Renou

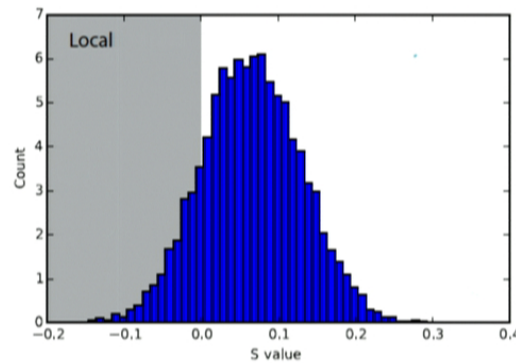
“Correlators” generalized to  $k$  outputs (ask!)

How many of the following can be extended?

- Full correlator inequalities  
Werner&Wolf, Tsirelson’s theorem
- Extensions/homogenization of binary out. ineqs.  
(Y. C. Wu, M. Żukowski)

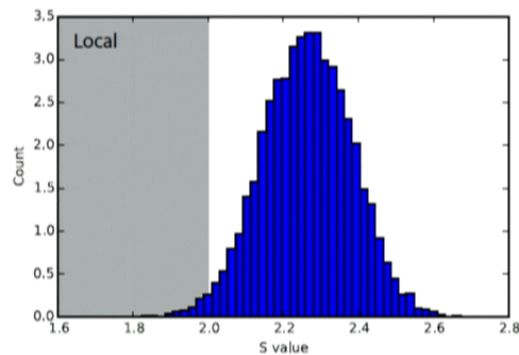
# Bell ineqs. and finite statistics

WIP w/ Marc-Olivier Renou, Anthony Martin



$S_{\text{CH}} = 0.06 \pm 0.06$   
(local  $\leq 0$ )

$$S^{\text{CHSH}} = \langle A_0 B_0 \rangle + \langle A_0 B_1 \rangle + \langle A_1 B_0 \rangle - \langle A_1 B_1 \rangle$$



$S_{\text{CHSH}} = 2.3 \pm 0.1$   
(local  $\leq 2$ )

$$S^{\text{CH}} = -P_A(0|0) - P_B(0|0) + P_{\text{AB}}(00|00) + P_{\text{AB}}(00|10) + P_{\text{AB}}(00|01) - P_{\text{AB}}(00|11)$$

# Exact bounds for NPA relaxations

Operators in NPA

$$(\mathbb{1}, A_0, A_1, B_0, B_1)$$

with  $\pm 1$  eigenvalues

$$I^* = \max \sum_{xy} (-1)^{xy} \langle A_x B_y \rangle \text{ such that } \Gamma \geq 0$$

$$\Gamma = \begin{pmatrix} 1 & \langle A_0 \rangle & \langle A_1 \rangle & \langle B_0 \rangle & \langle B_1 \rangle \\ \langle A_0 \rangle & 1 & \langle A_0 A_1 \rangle & \langle A_0 B_0 \rangle & \langle A_0 B_1 \rangle \\ \langle A_1 \rangle & \langle A_0 A_1 \rangle & 1 & \langle A_1 B_0 \rangle & \langle A_1 B_1 \rangle \\ \langle B_0 \rangle & \langle A_0 B_0 \rangle & \langle A_1 B_0 \rangle & 1 & \langle B_0 B_1 \rangle \\ \langle B_1 \rangle & \langle A_0 B_1 \rangle & \langle A_1 B_1 \rangle & \langle B_0 B_1 \rangle & 1 \end{pmatrix}$$

Apply averaging map:

$$I^* = \max 4a \text{ such that } \Gamma \geq 0$$

$$\Gamma = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & a & a \\ 0 & 0 & 1 & a & -a \\ 0 & a & a & 1 & 0 \\ 0 & a & -a & 0 & 1 \end{pmatrix}$$

$$\rho = \theta_1 \oplus \theta_2 \oplus \theta_3$$

*trivial representation*  $\nearrow$   $\uparrow$   $\nwarrow$  *another faithful representation of degree 2*  
*faithful representation of degree 2*

3 blocks of size 1 (all multiplicities = 1)

$$I^* = \max 4a \text{ such that } \Gamma' \geq 0$$

$$\Gamma' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 + 2\sqrt{2}a & 0 \\ 0 & 0 & 2 - 2\sqrt{2}a \end{pmatrix} \geq 0$$

Gain: blocks  $(1 \times 1, 4 \times 4) \rightarrow (1 \times 1, 1 \times 1, 1 \times 1)$

# Exact bounds for NPA relaxations

CGMLP  $k=3$   $Q_1$

## 1) Operators

$\mathbb{1}, A_{0|0}, A_{1|0}, A_{2|0}, A_{0|1}, A_{1|1}, A_{2|1}, B_{0|0}, B_{1|0}, B$

*Generators:*

*Perm. of operators*

$A_{0(0.2.1)} A_{1(0.2.1)} B_{0(0.1.2)} B_{1(0.1.2)} (1.3.2)(4.6.5)(7.8.9)(10.11.12)$

$A_{0(0.2)} A_{1(0.2)} B_{0(1.2)} B_{1(0.2)} A(0.1) (1.6)(2.5)(3.4)(8.9)(10.12)$

$(A.B) (1.7)(2.8)(3.9)(4.10)(5.11)(6.12)$

2) 6 variables in reduced basis  
+ 3 equality constraints

3) SDP with 1x1 blocks

$$I^* = \frac{2\sqrt{3} + 6}{3} \approx 3.1547$$



# Conclusion

Full understanding of symmetries opens avenues of research

Useful for classification

Extension of known tricks/results?

Robustness (theory & experiments)

Deeper theoretical understanding?