Title: Symmetries of Bell correlation scenarios

Date: Jun 06, 2016 02:00 PM

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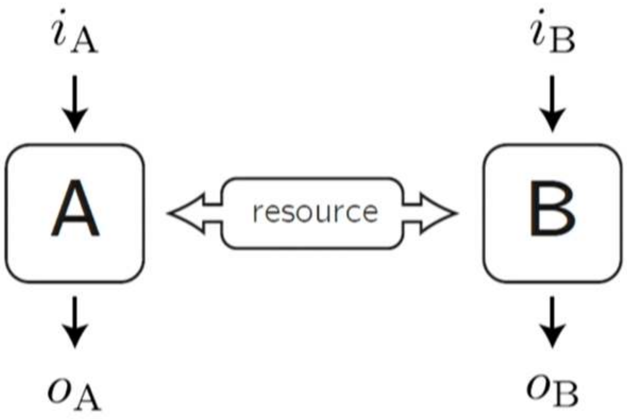
Abstract: <p>In the device-independent paradigm, the labeling of parties/inputs/outputs has no physical meaning and thus the behavior of the system should be studied up to symmetry. We conduct the first formal study of relabelings appearing in Bell scenarios. The talk includes a review of previous works, a definition of Bell relabeling groups illustrated by examples, and applications, including the classification of Bell inequalities, the generalization of binary correlators to d outcomes and the computation of exact bounds using the NPA hierarchy.</p>

This talk = spaghetti on the wall

Spaghetti bench, Pablo Reinoso

Paradigm: "device-independent"

Inputs / measurements settings



Outputs / measurements outcomes

Observed correlations

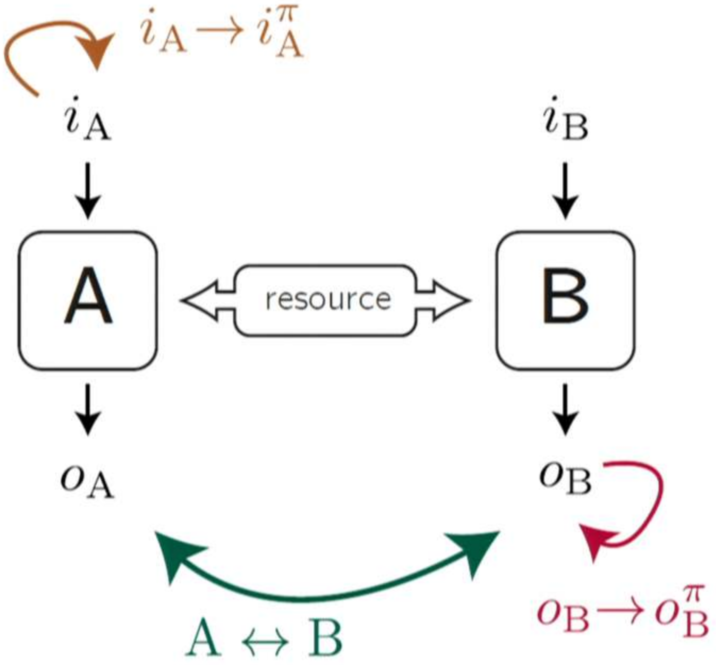
$$P(o_\mathrm{A}, o_\mathrm{B} | i_\mathrm{A}, i_\mathrm{B})$$

Note:
W/o any assumption on the devices,
output, input sets must be finite
in any testable theory

$$\vec{P} \in \mathbb{R}^n = \begin{pmatrix} P(00|00) \\ ... \\ P(11|11) \end{pmatrix}$$

Example: in CHSH scenario, $n = 16$

Paradigm: “device-independent”



Galileo principle:

The laws of quantum information do not depend on a particular labeling of ***parties****,* ***inputs*** *and* ***outputs****.*

Thus:

The relevant physics should be studied “up to symmetry”.

What group structures are involved?

What are their representations?

Any physical meaning associated?

Plan

Definitions

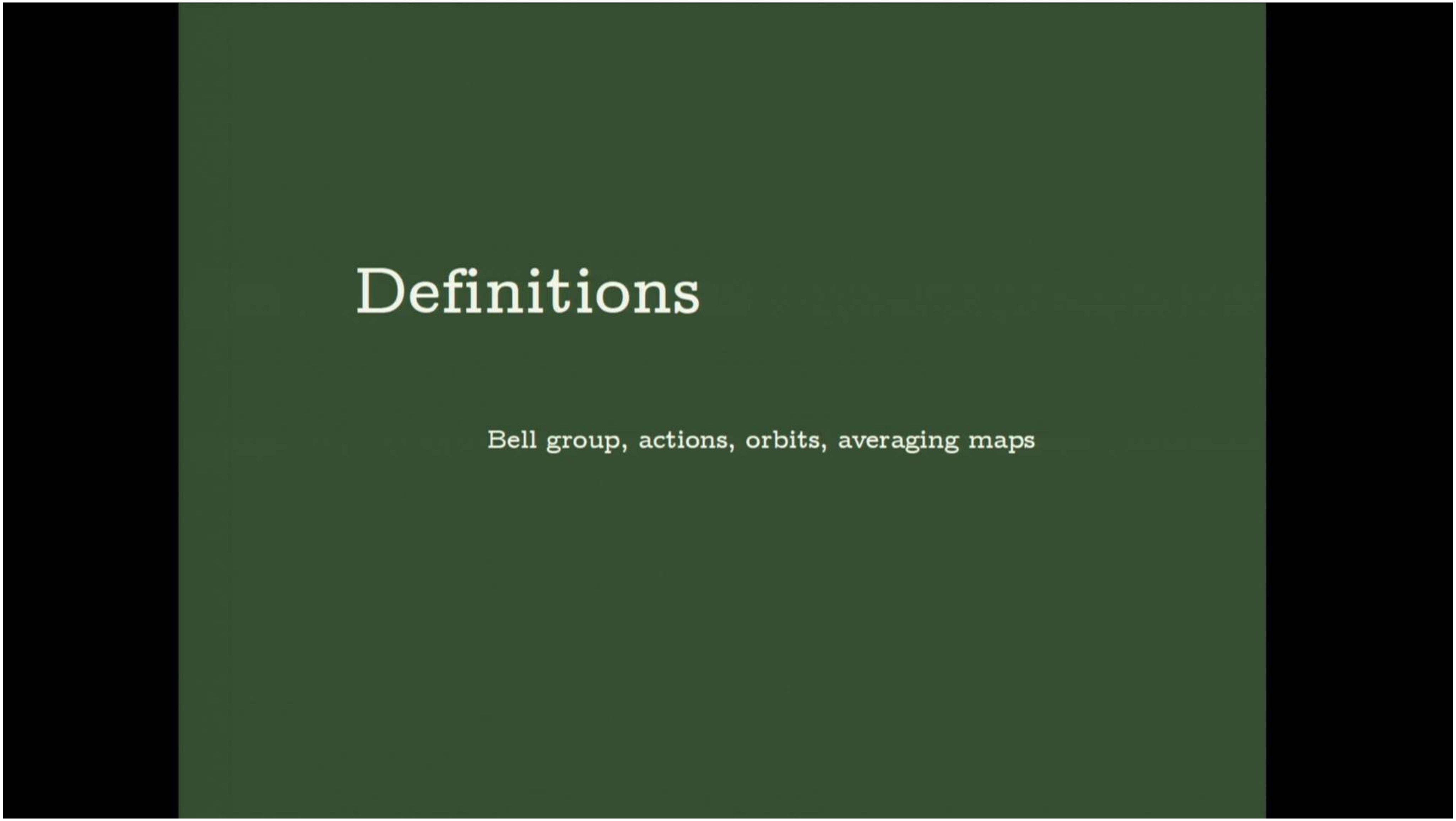
Bell group, actions, orbits, averaging maps

Review of literature

Bell inequality families (Śliwa 2003), Liftings (Pironio 2005),
Invariance under party permutation (Liang 2008, Bancal 2010, Tura 2013)
Creating Bell inequalities from group representations (Güney, 2014)

New applications

Classification of Bell inequalities (Rosset 2014)
Generalized correlators (WIP)
Exact NPA bounds (WIP)

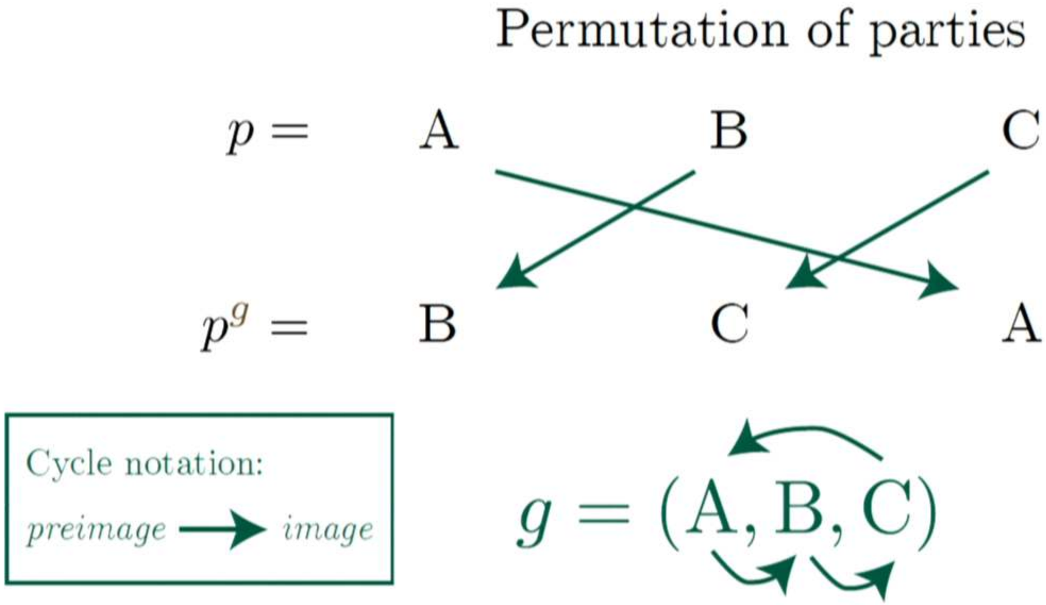


Definitions

Bell group, actions, orbits, averaging maps

Bell group G

What's inside?



Permutation of parties

$p =$ A B C

$p^g =$ B C A

Cycle notation:

preimage → *image*

$$g = (\mathrm{A}, \mathrm{B}, \mathrm{C})$$

Bell group G

What's inside?

Permutation of inputs

$$i_{\mathrm{A}} \longrightarrow i_{\mathrm{A}}^{\pi}$$

$$\pi = (0,1,2)$$

$$g = \underset{\text{affected party}}{\mathrm{A}}\ \underset{\text{cycle notation}}{(0,1,2)}$$

Inverse: reverse the cycles

$$g^{-1} = \mathrm{A}(0,2,1)$$

Bell group G

What's inside?

Permutation of outputs

$$o_{\mathrm{A}} \longrightarrow o_{\mathrm{A}}^{\pi} \quad i_{\mathrm{A}} = 0$$

$$\pi = (0, 1, 2)$$

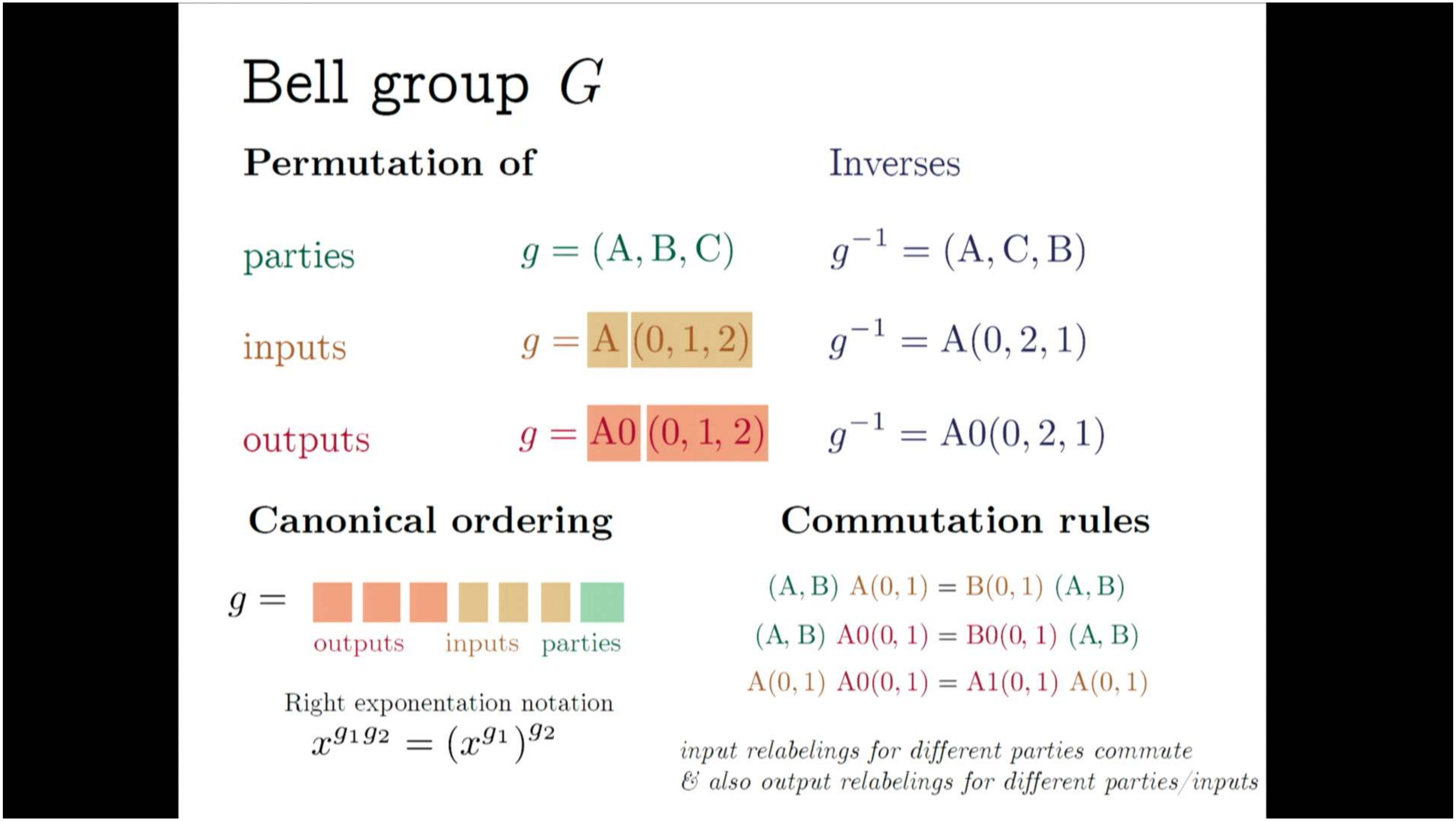
$$g = \mathrm{A0}\ (0, 1, 2)$$

affected party and input

cycle notation

Inverse: reverse the cycles

$$g^{-1} = \mathrm{A0}(0, 2, 1)$$



Bell group G

Permutation of Inverses

parties $g = (\mathrm{A}, \mathrm{B}, \mathrm{C})$ $g^{-1} = (\mathrm{A}, \mathrm{C}, \mathrm{B})$

inputs $g = \boxed{\mathrm{A}}\,\boxed{(0, 1, 2)}$ $g^{-1} = \mathrm{A}(0, 2, 1)$

outputs $g = \boxed{\mathrm{A0}}\,\boxed{(0, 1, 2)}$ $g^{-1} = \mathrm{A0}(0, 2, 1)$

Canonical ordering

$g =$ 🟧🟧🟧🟨🟨🟨🟩

outputs inputs parties

Right exponentation notation

$$x^{g_1 g_2} = (x^{g_1})^{g_2}$$

Commutation rules

$$(\mathrm{A}, \mathrm{B})\ \mathrm{A}(0, 1) = \mathrm{B}(0, 1)\ (\mathrm{A}, \mathrm{B})$$

$$(\mathrm{A}, \mathrm{B})\ \mathrm{A0}(0, 1) = \mathrm{B0}(0, 1)\ (\mathrm{A}, \mathrm{B})$$

$$\mathrm{A}(0, 1)\ \mathrm{A0}(0, 1) = \mathrm{A1}(0, 1)\ \mathrm{A}(0, 1)$$

input relabelings for different parties commute
& also output relabelings for different parties/inputs

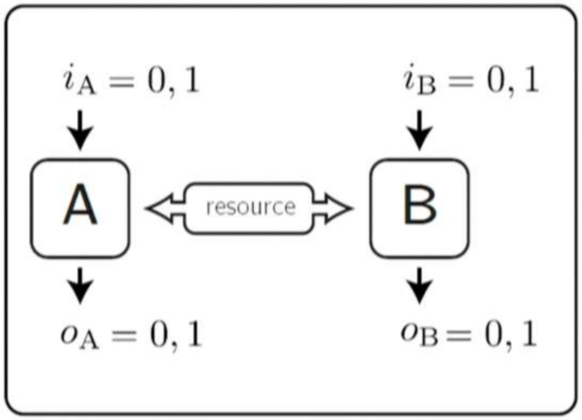
Symmetry subgroup of CHSH game

Simple generators

Game

$$o_{\mathrm{A}} \oplus o_{\mathrm{B}} = i_{\mathrm{A}} i_{\mathrm{B}}$$

Corresponding symmetries



$o_{\mathrm{A}}, i_{\mathrm{A}} \leftrightarrow o_{\mathrm{B}}, i_{\mathrm{B}}$	$g_1 =$ (A, B)
$o_{\mathrm{A}} \to o_{\mathrm{A}} \oplus 1$ $o_{\mathrm{B}} \to o_{\mathrm{B}} \oplus 1$	$g_2 =$ A0(0,1) A1(0,1) B0(0,1) B1(0,1)
$o_{\mathrm{A}} \to o_{\mathrm{A}} \oplus i_{\mathrm{A}}$ $i_{\mathrm{B}} \to i_{\mathrm{B}} \oplus 1$	$g_3 =$ A1(0,1) B(0,1)

Group $G_{\mathrm{CHSH}} = \langle g_1, g_2, g_3 \rangle$ *of order 16*

Actions of Bell groups

(Imprimitive)-imprimitive action (from wreath product)

Triplet

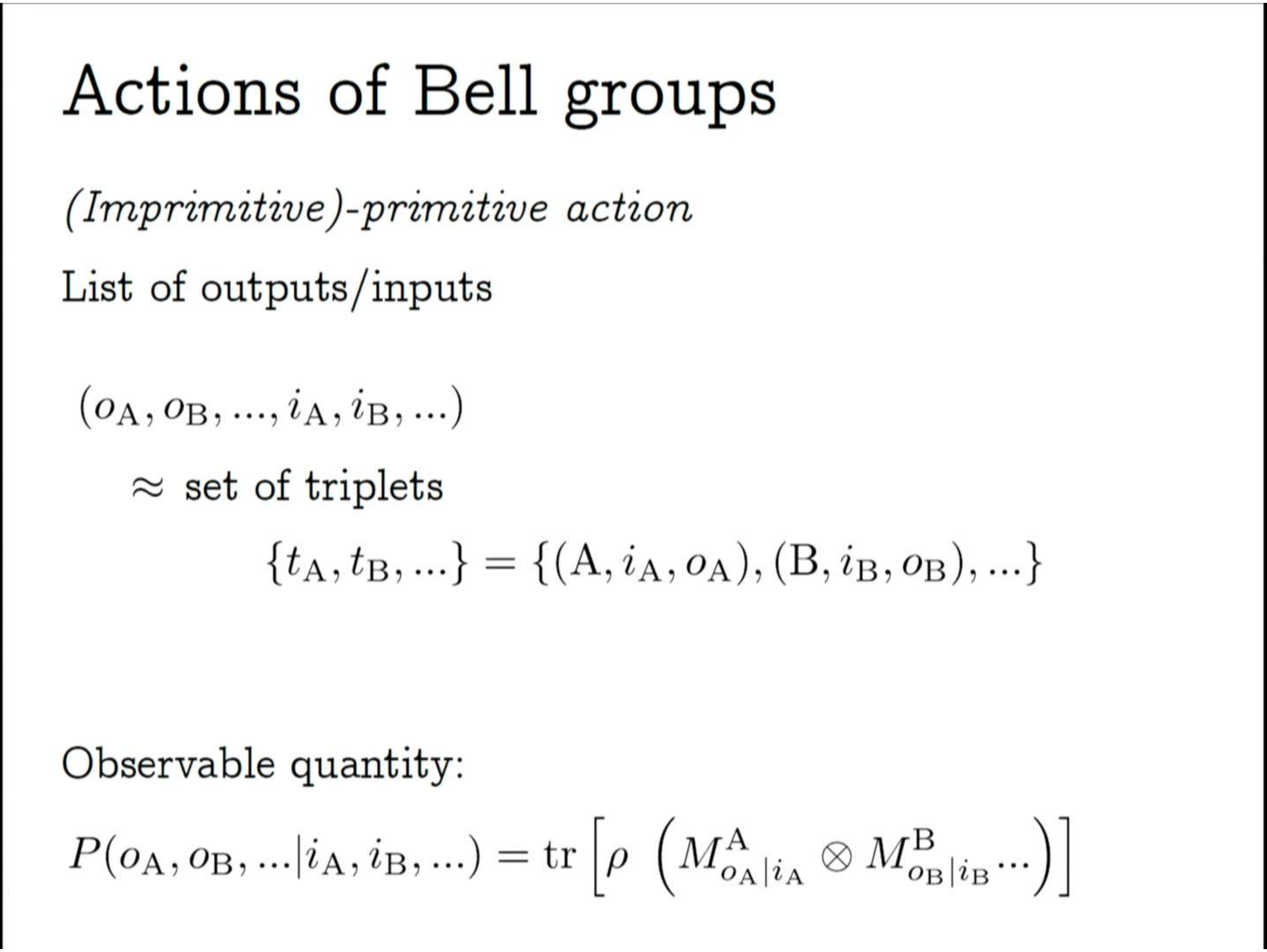
$$t = (p, i_p, o_{p,i}) \xrightarrow{\quad g = \pi \quad} (p^\pi, i, o)$$

$$\xrightarrow{\quad g = P\pi \quad} (p, i^\pi, o) \qquad p = P$$

$$\xrightarrow{\quad g = PI\pi \quad} (p, i, o^\pi) \qquad \begin{array}{c} p = P \text{ and} \\ i = I \end{array}$$

Observable quantity: $P_p(o|i) = \mathrm{tr}\left[\rho_p \; M^p_{o|i}\right]$

Note: use this (faithful) action to retrieve the commutation rules for group elements



Actions of Bell groups

(Imprimitive)-primitive action

List of outputs/inputs

$$(o_{\mathrm{A}}, o_{\mathrm{B}}, ..., i_{\mathrm{A}}, i_{\mathrm{B}}, ...)$$

$\approx$ set of triplets

$$\{t_{\mathrm{A}}, t_{\mathrm{B}}, ...\} = \{(\mathrm{A}, i_{\mathrm{A}}, o_{\mathrm{A}}), (\mathrm{B}, i_{\mathrm{B}}, o_{\mathrm{B}}), ...\}$$

Observable quantity:

$$P(o_{\mathrm{A}}, o_{\mathrm{B}}, ...|i_{\mathrm{A}}, i_{\mathrm{B}}, ...) = \mathrm{tr}\left[\rho\ \left(M^{\mathrm{A}}_{o_{\mathrm{A}}|i_{\mathrm{A}}} \otimes M^{\mathrm{B}}_{o_{\mathrm{B}}|i_{\mathrm{B}}} \cdots\right)\right]$$

Actions of Bell groups

(Imprimitive)-primitive action

List of outputs/inputs

$$(o_{\mathrm{A}}, o_{\mathrm{B}}, ..., i_{\mathrm{A}}, i_{\mathrm{B}}, ...) \xrightarrow{g = \pi} (o_{\mathrm{A}^{\pi^{-1}}}, ..., i_{\mathrm{A}^{\pi^{-1}}}, ...)$$

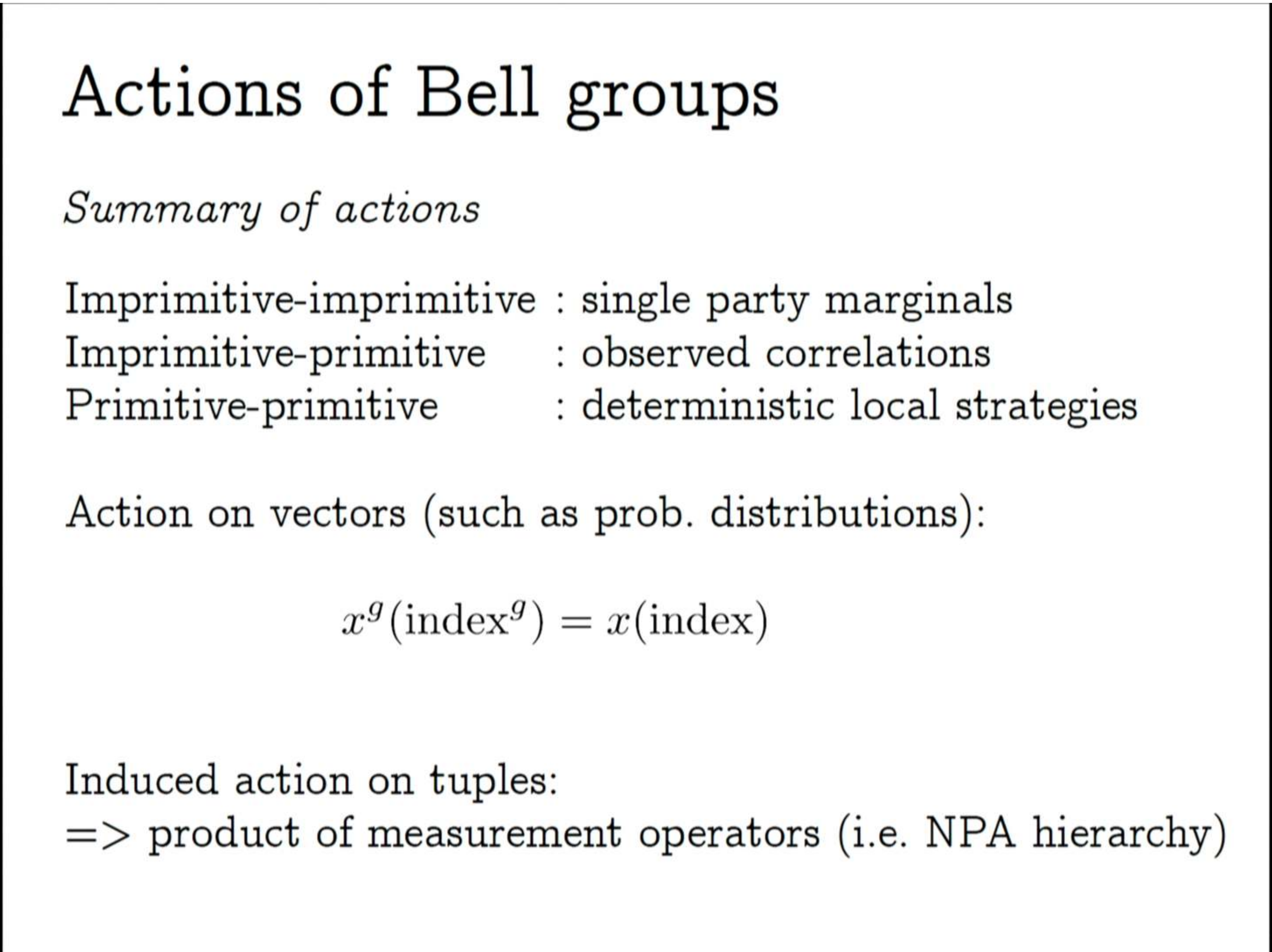
$$(..., o_P, ..., ..., i_P, ...) \xrightarrow{g = P\pi} (..., o_P, ..., ..., {i_P}^{\pi}, ...)$$

$$(..., o_P, ..., ..., i_P, ...) \xrightarrow{g = PI\pi} (..., {o_P}^{\pi}, ..., ..., i_P, ...)$$

$$i_P = I$$

Observable quantity:

$$P(o_{\mathrm{A}}, o_{\mathrm{B}}, ...| i_{\mathrm{A}}, i_{\mathrm{B}}, ...) = \mathrm{tr}\left[\rho \, \left(M^{\mathrm{A}}_{o_{\mathrm{A}}|i_{\mathrm{A}}} \otimes M^{\mathrm{B}}_{o_{\mathrm{B}}|i_{\mathrm{B}}} \cdots\right)\right]$$



Actions of Bell groups

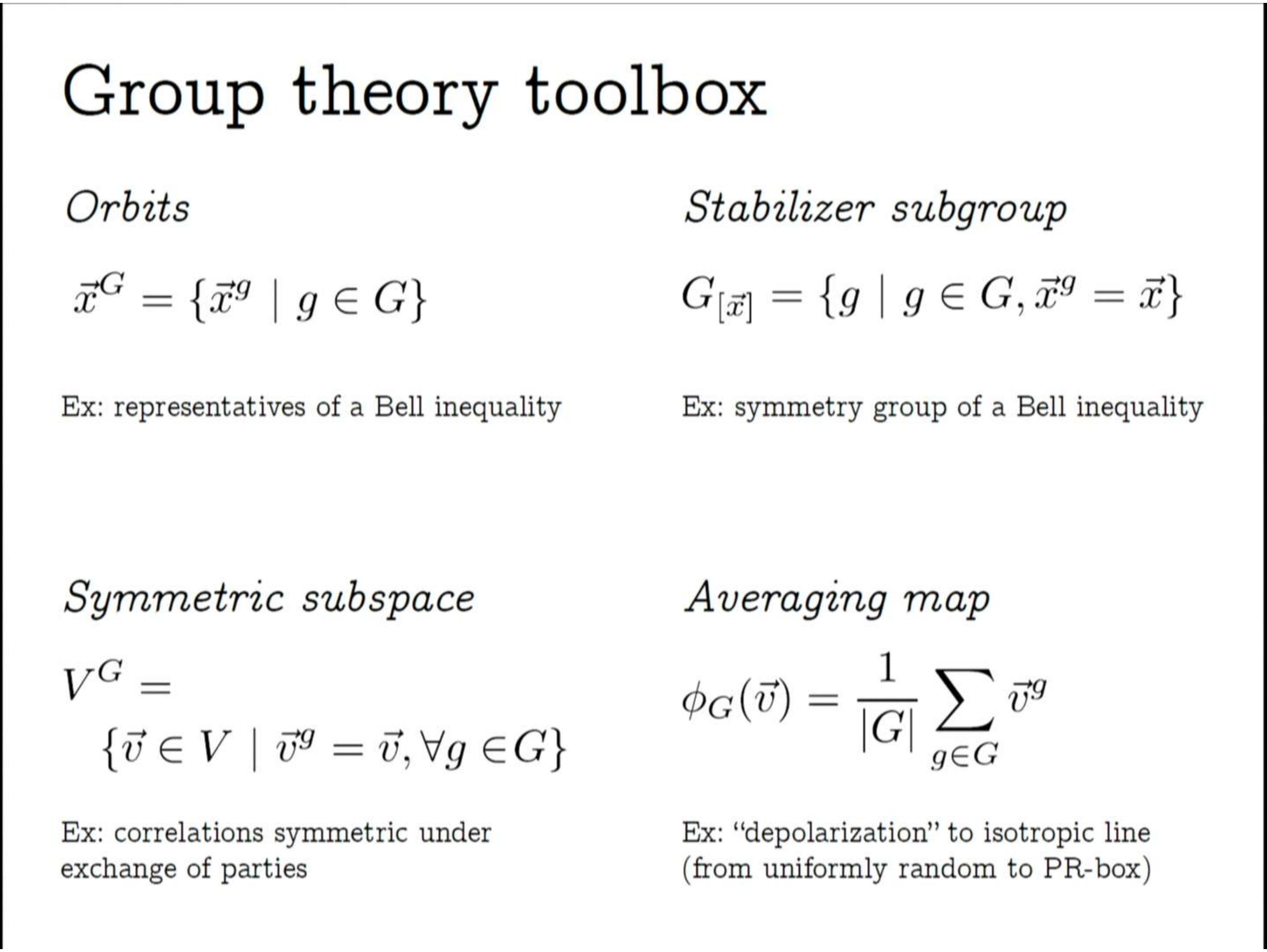
Summary of actions

Imprimitive-imprimitive : single party marginals
Imprimitive-primitive : observed correlations
Primitive-primitive : deterministic local strategies

Action on vectors (such as prob. distributions):

$$x^g(\text{index}^g) = x(\text{index})$$

Induced action on tuples:
=> product of measurement operators (i.e. NPA hierarchy)



Group theory toolbox

Orbits

$$\vec{x}^G = \{\vec{x}^g \mid g \in G\}$$

Ex: representatives of a Bell inequality

Stabilizer subgroup

$$G_{[\vec{x}]} = \{g \mid g \in G, \vec{x}^g = \vec{x}\}$$

Ex: symmetry group of a Bell inequality

Symmetric subspace

$$V^G = \{\vec{v} \in V \mid \vec{v}^g = \vec{v}, \forall g \in G\}$$

Ex: correlations symmetric under exchange of parties

Averaging map

$$\phi_G(\vec{v}) = \frac{1}{|G|} \sum_{g \in G} \vec{v}^g$$

Ex: “depolarization” to isotropic line (from uniformly random to PR-box)

Computational group theory tools

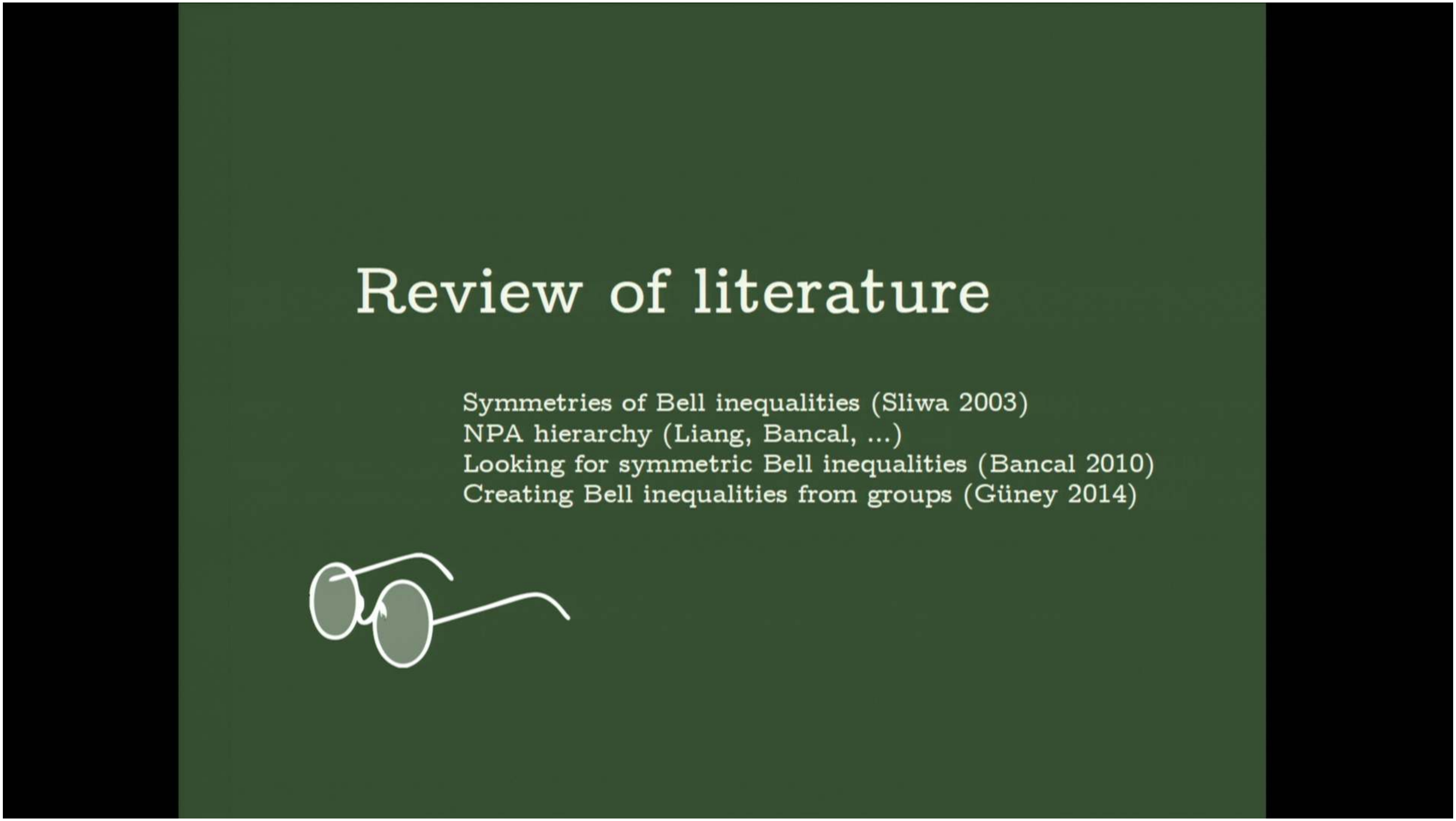
Permutation groups

Efficient algorithms using bases and strong generating sets (Schreier-Sims)

Bible: Derek Holt, Handbook of CGT, 2005
Implemented in Scala (github.com/denisrosset/alasc)

Monomial representations

Using
GAP system / AREP, Markus Püschel (1998-1999)

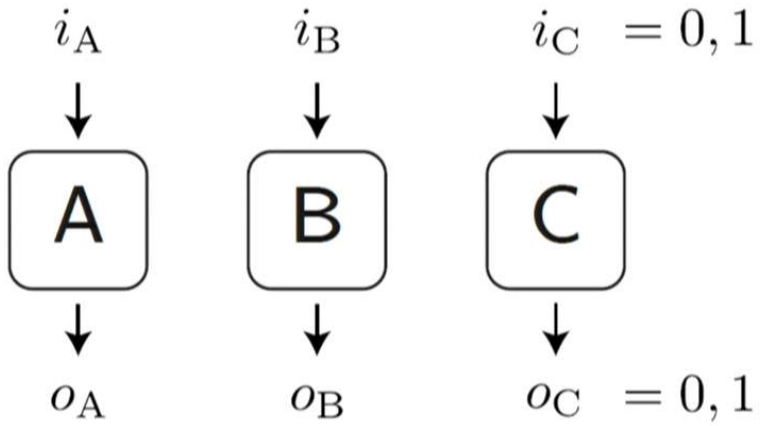


Review of literature

Symmetries of Bell inequalities (Sliwa 2003)
NPA hierarchy (Liang, Bancal, ...)
Looking for symmetric Bell inequalities (Bancal 2010)
Creating Bell inequalities from groups (Güney 2014)

Symmetries of Bell inequalities

Pitowsky & Svozil, 2001:



Local polytope:
53’856 facets

Sliwa, 2003:

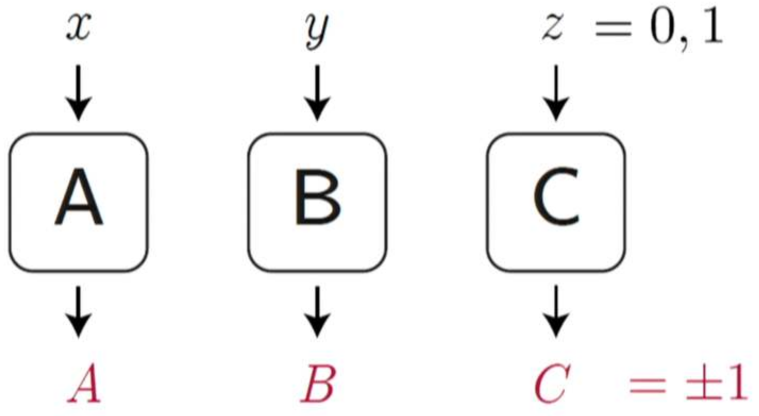
46 equivalence classes under group action

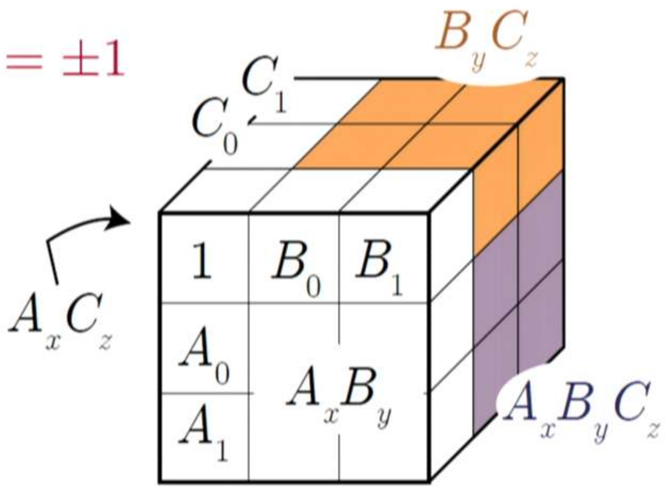
List one representative per class/family
Family representatives = orbit of representative under group

Small group $|G| = 3072$, brute force computation possible

Symmetries of Bell inequalities

Sliwa, 2003:





NPA hierarchy w/ party symmetry

Liang 2008, Bancal (priv. comm.), folklore

$$\chi = \begin{array}{cc} \begin{array}{cc}\text{Alice proj.} & \text{Bob proj.}\end{array} & \\ \left(\begin{array}{c|c} U & V \\ \hline & W \end{array}\right) & \begin{array}{l}\text{Alice proj.}\\ \text{Bob proj.}\end{array} \end{array} \geq 0$$

If inequality symmetric under (A, B),
apply averaging map on χ

$$V = V^{\top}$$
$$U = W$$

$$\longrightarrow \qquad \chi' \equiv \left(\begin{array}{c|c} U+V & \\ \hline & U-V \end{array}\right) \geq 0$$

Looking for symmetric Bell ineqs.

Bancal 2010

Vertices of local polytope

$$\{\vec{v}_i\} \quad \cancel{\xrightarrow[\text{Conversion too difficult}]{}} \quad \{\vec{a}_i^\top \vec{P} \leq b_i\}$$

Looking for symmetric Bell ineqs.

Bancal 2010

Vertices of local polytope

Subgroup of Bell group

$H \subset G$

Apply averaging map

$\phi_H(\vec{x}) = \Pi_H \vec{x}$

projector

$$\begin{tikzcd}\{\vec{v}_i\} \arrow[d] \\ \{\vec{w}_i\}\end{tikzcd}$$

Vertices of polytope
in symmetric subspace

Looking for symmetric Bell ineqs.

Bancal 2010

Vertices of local polytope

A subset of those
are facets of local polytope

$$\begin{tikzcd}
\{\vec{v}_i\} \arrow[d] & \{(\vec{a}_i^\top \Pi_H)\vec{P} \leq b_i\} \\
\{\vec{w}_i\} \arrow[r, "Conversion"'] & \{\vec{a}_i^\top \vec{P} \leq b_i\} \arrow[u]
\end{tikzcd}$$

Subgroup of Bell group

$$H \subset G$$

Apply averaging map

$$\phi_H(\vec{x}) = \Pi_H \vec{x}$$

projector

Vertices of polytope
in symmetric subspace

Looking for symmetric Bell ineqs.

Bancal 2010

Subgroup of Bell group

$H \subset G$

Apply averaging map

$\phi_H(\vec{x}) = \Pi_H \vec{x}$

projector

Vertices of local polytope

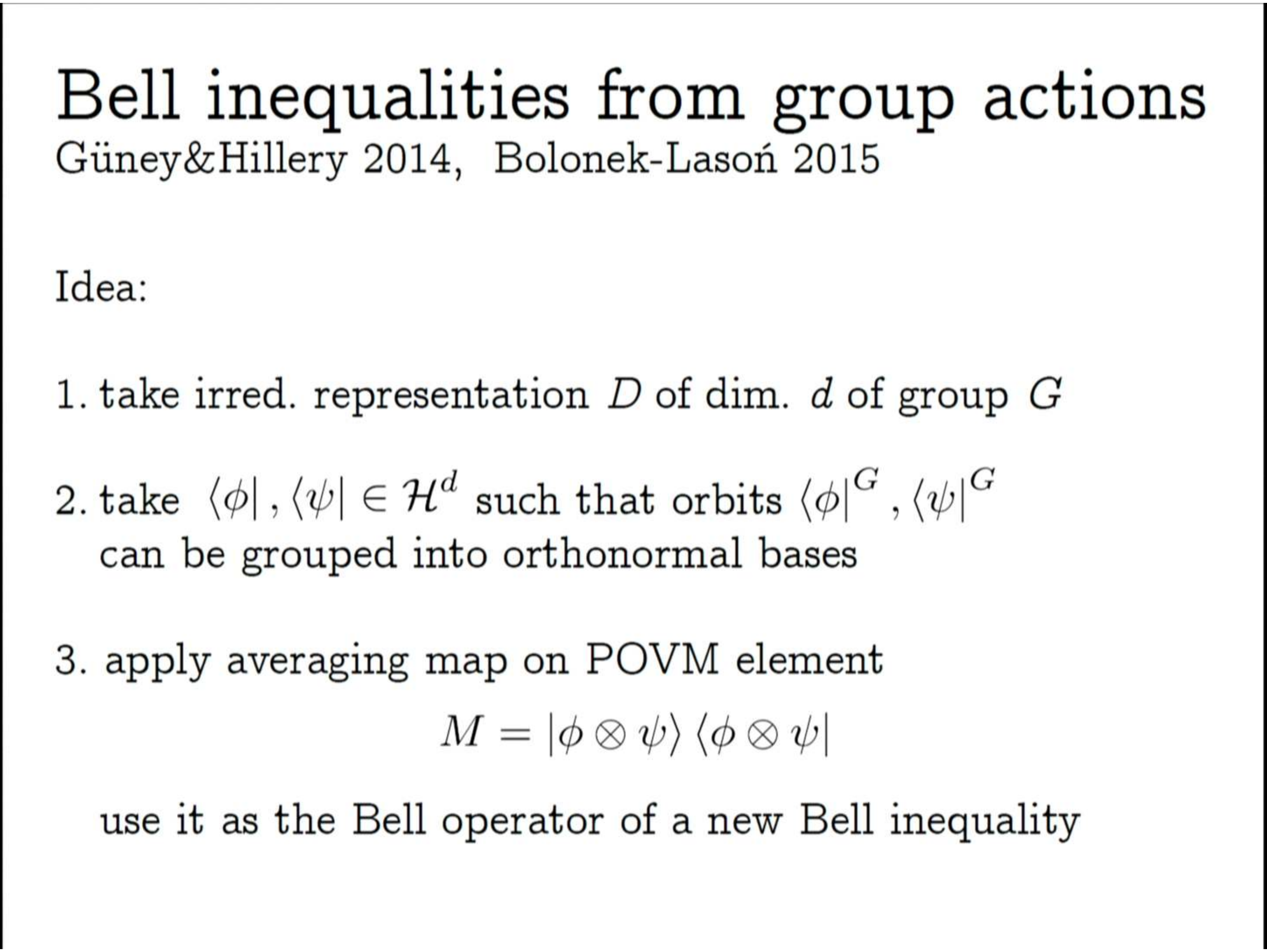
A subset of those
are facets of local polytope

$$\begin{tikzcd}
\{\vec{v}_i\} \arrow[d] & \{(\vec{a}_i^\top \Pi_H)\vec{P} \leq b_i\} \\
\{\vec{w}_i\} \arrow[r, "\text{Conversion}"'] & \{\vec{a}_i^\top \vec{P} \leq b_i\} \arrow[u]
\end{tikzcd}$$

Vertices of polytope
in symmetric subspace

...see also J. Tura 2014-2015

only party permutations considered!



Bell inequalities from group actions

Güney&Hillery 2014, Bolonek-Lasoń 2015

Idea:

1. take irred. representation D of dim. d of group G

2. take $\langle\phi|, \langle\psi| \in \mathcal{H}^d$ such that orbits $\langle\phi|^G, \langle\psi|^G$ can be grouped into orthonormal bases

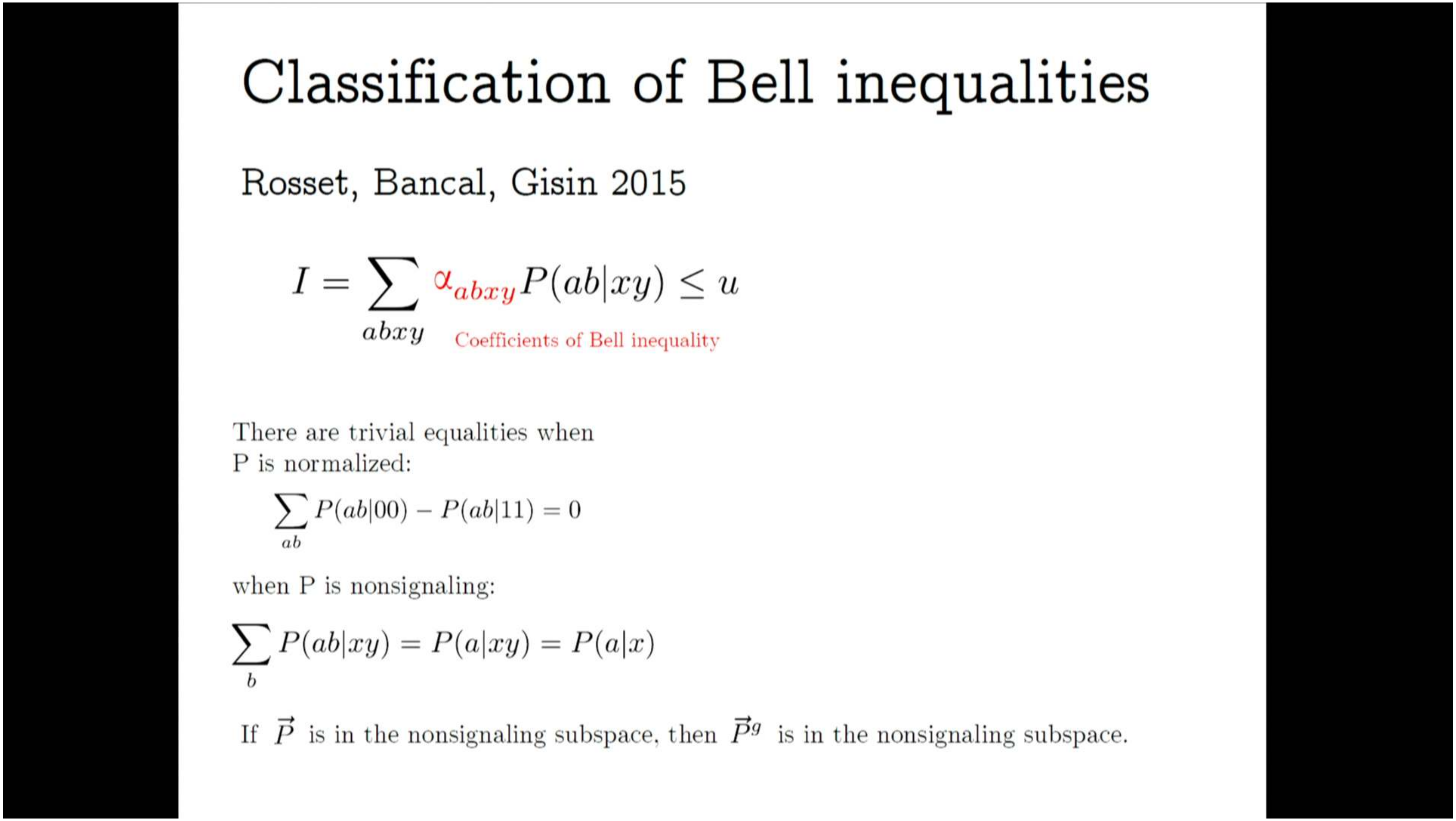
3. apply averaging map on POVM element

$$M = |\phi \otimes \psi\rangle \langle\phi \otimes \psi|$$

use it as the Bell operator of a new Bell inequality

Application & ideas

Classification of Bell inequalities (Rosset, Bancal, Gisin 2014)
Generalized correlators (WIP)
Bell inequalities and statistical noise (w/ M.O. Renou)
Exact solutions to NPA relaxations (WIP)



Classification of Bell inequalities

Rosset, Bancal, Gisin 2015

$$I = \sum_{abxy} \alpha_{abxy} P(ab|xy) \leq u$$

Coefficients of Bell inequality

There are trivial equalities when
P is normalized:

$$\sum_{ab} P(ab|00) - P(ab|11) = 0$$

when P is nonsignaling:

$$\sum_{b} P(ab|xy) = P(a|xy) = P(a|x)$$

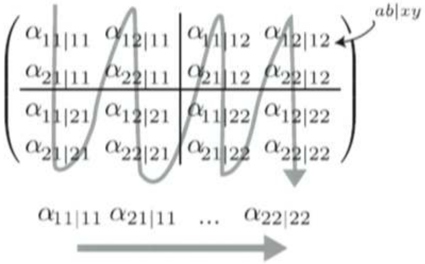
If $\vec{P}$ is in the nonsignaling subspace, then $\vec{P}^g$ is in the nonsignaling subspace.

Classification of Bell inequalities

Rosset, Bancal, Gisin 2015

$$\begin{tikzcd}
\vec{\alpha} \arrow[r, "\Pi"] \arrow[d, "\text{relabeling } g"'] & \Pi\vec{\alpha} \arrow[d, "g"] \\
\vec{\alpha}^g \arrow[r, "\Pi"] & \Pi(\vec{\alpha}^g) = (\Pi\vec{\alpha})^g
\end{tikzcd}$$

projection to
remove "equalities"

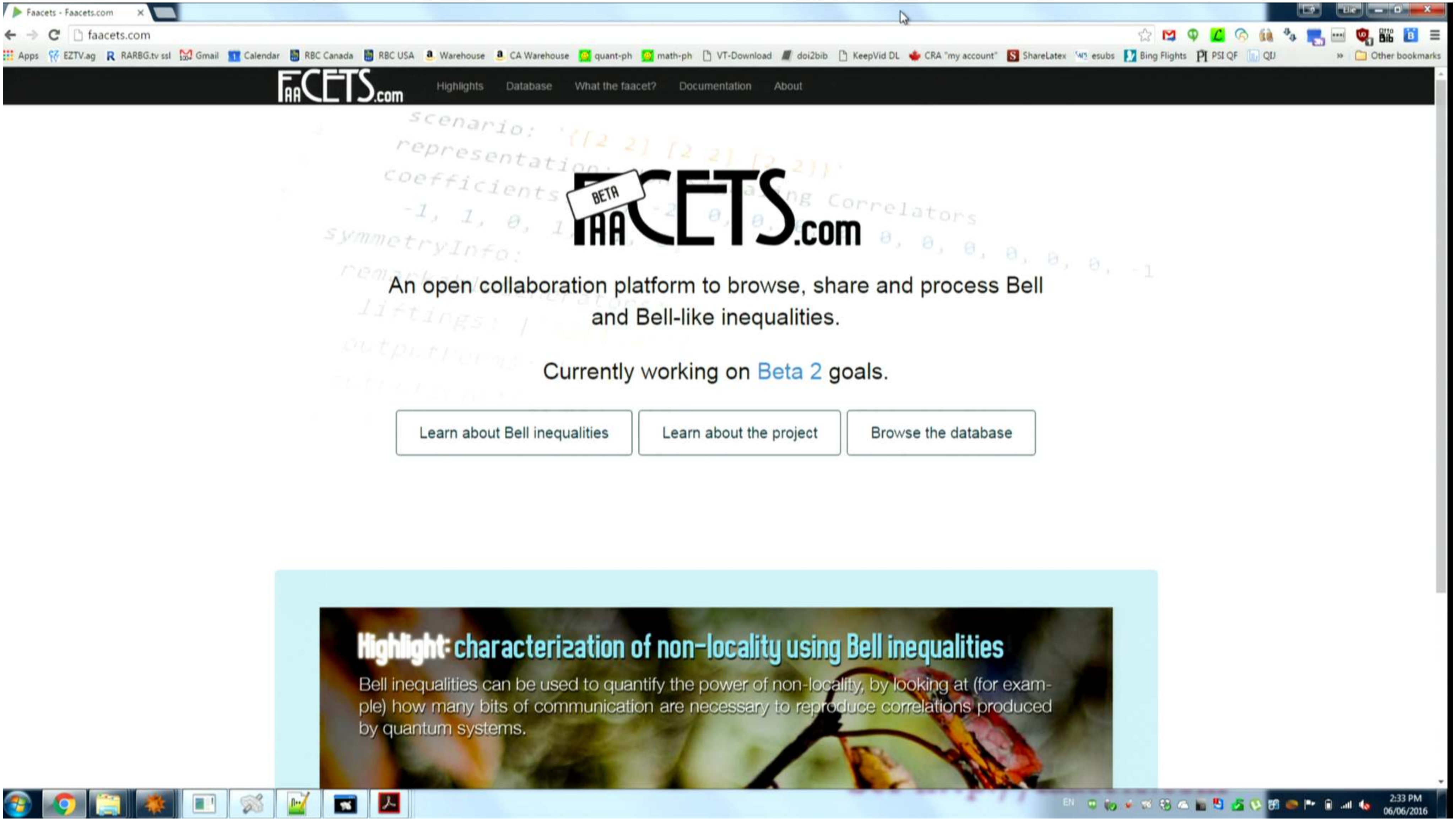


CHSH *representatives in lexicographic order*

minimal

−1	1	−1	1	1	−1	1	−1	−1	1	1	−1	1	−1	−1	1
−1	1	−1	1	1	−1	1	−1	1	−1	−1	1	−1	1	1	−1
−1	1	1	−1	1	−1	−1	1	−1	1	−1	1	1	−1	1	−1
−1	1	1	−1	1	−1	−1	1	1	−1	1	−1	−1	1	−1	1
1	−1	−1	1	−1	1	1	−1	−1	1	−1	1	1	−1	1	−1
1	−1	−1	1	−1	1	1	−1	1	−1	1	−1	−1	1	−1	1
1	−1	1	−1	−1	1	−1	1	−1	1	1	−1	1	−1	−1	1

See http://faacets.com

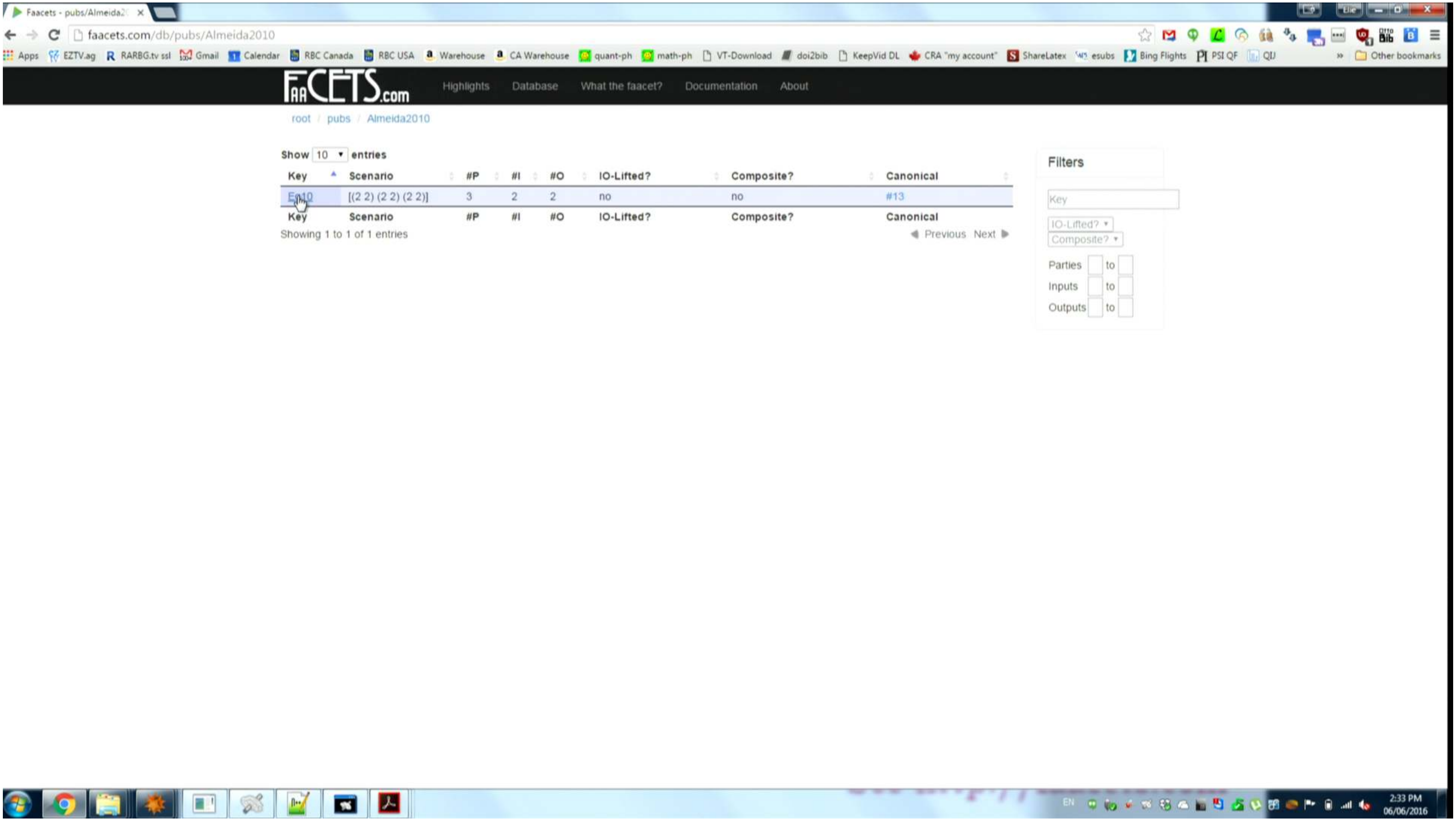


An open collaboration platform to browse, share and process Bell and Bell-like inequalities.

Currently working on Beta 2 goals.

Highlight: characterization of non-locality using Bell inequalities

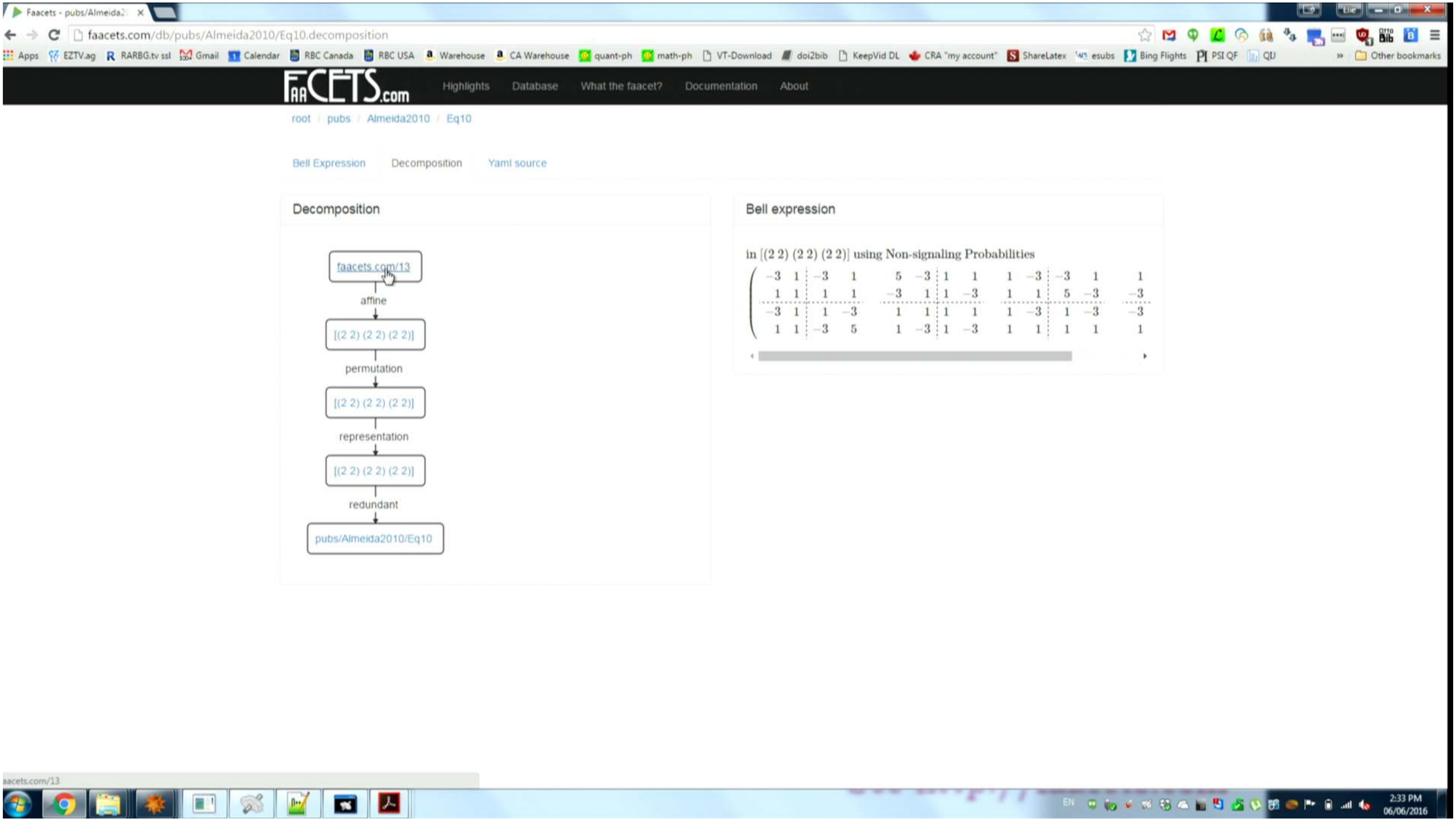
Bell inequalities can be used to quantify the power of non-locality, by looking at (for example) how many bits of communication are necessary to reproduce correlations produced by quantum systems.



Show 10 entries

Key	Scenario	#P	#I	#O	IO-Lifted?	Composite?	Canonical
Eq10	[(2 2) (2 2) (2 2)]	3	2	2	no	no	#13
Key	Scenario	#P	#I	#O	IO-Lifted?	Composite?	Canonical

Showing 1 to 1 of 1 entries



Decomposition

Bell expression

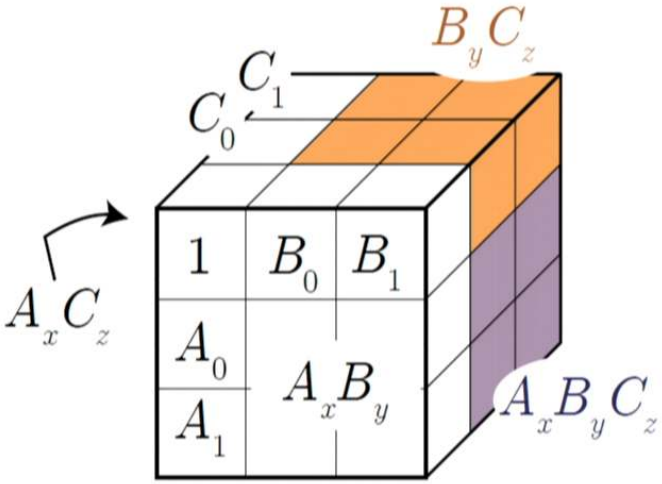
in [(2 2) (2 2) (2 2)] using Non-signaling Probabilities

$$\left(\begin{array}{cc:cc cc:cc cc:cc c}
-3 & 1 & -3 & 1 & 5 & -3 & 1 & 1 & 1 & -3 & -3 & 1 & 1 \\
1 & 1 & 1 & 1 & -3 & 1 & 1 & -3 & 1 & 1 & 5 & -3 & -3 \\
\hdashline
-3 & 1 & 1 & -3 & 1 & 1 & 1 & 1 & 1 & -3 & 1 & -3 & -3 \\
1 & 1 & -3 & 5 & 1 & -3 & 1 & -3 & 1 & 1 & 1 & 1 & 1
\end{array}\right.$$

Irreducible representations

WIP w/ Marc-Olivier Renou

Sliwa 2003: 3 parties, binary I/O

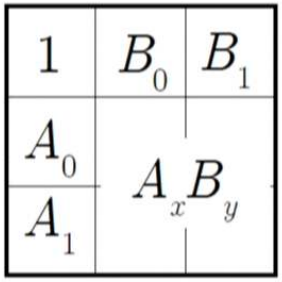


Irreducible representations

WIP w/ Marc-Olivier Renou

CHSH scenario

$\vec{P} \in \mathbb{R}^{16}$



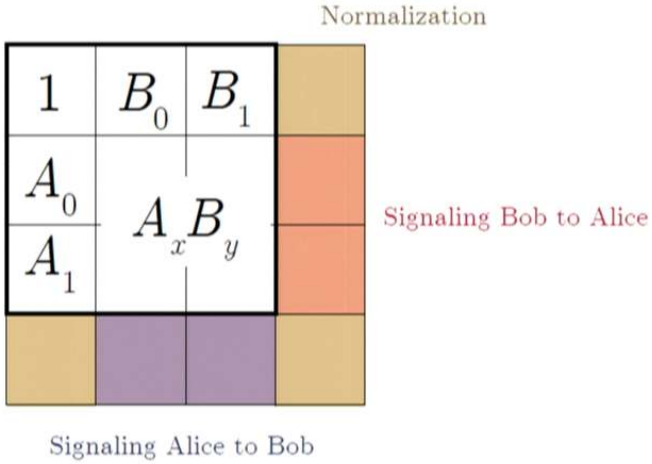
Dimension 16 versus 9 ?

Irreducible representations

WIP w/ Marc-Olivier Renou

CHSH scenario

$\vec{P} \in \mathbb{R}^{16}$



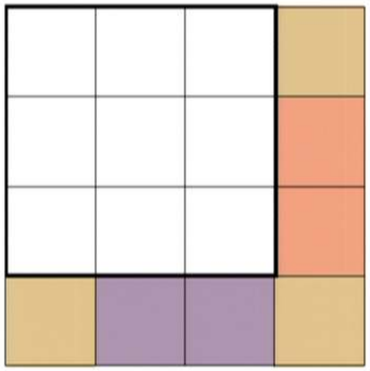
Irreducible representations

WIP w/ Marc-Olivier Renou

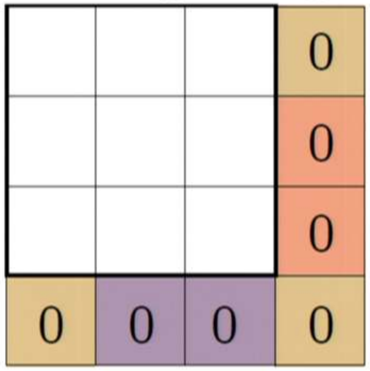
Bell inequality:

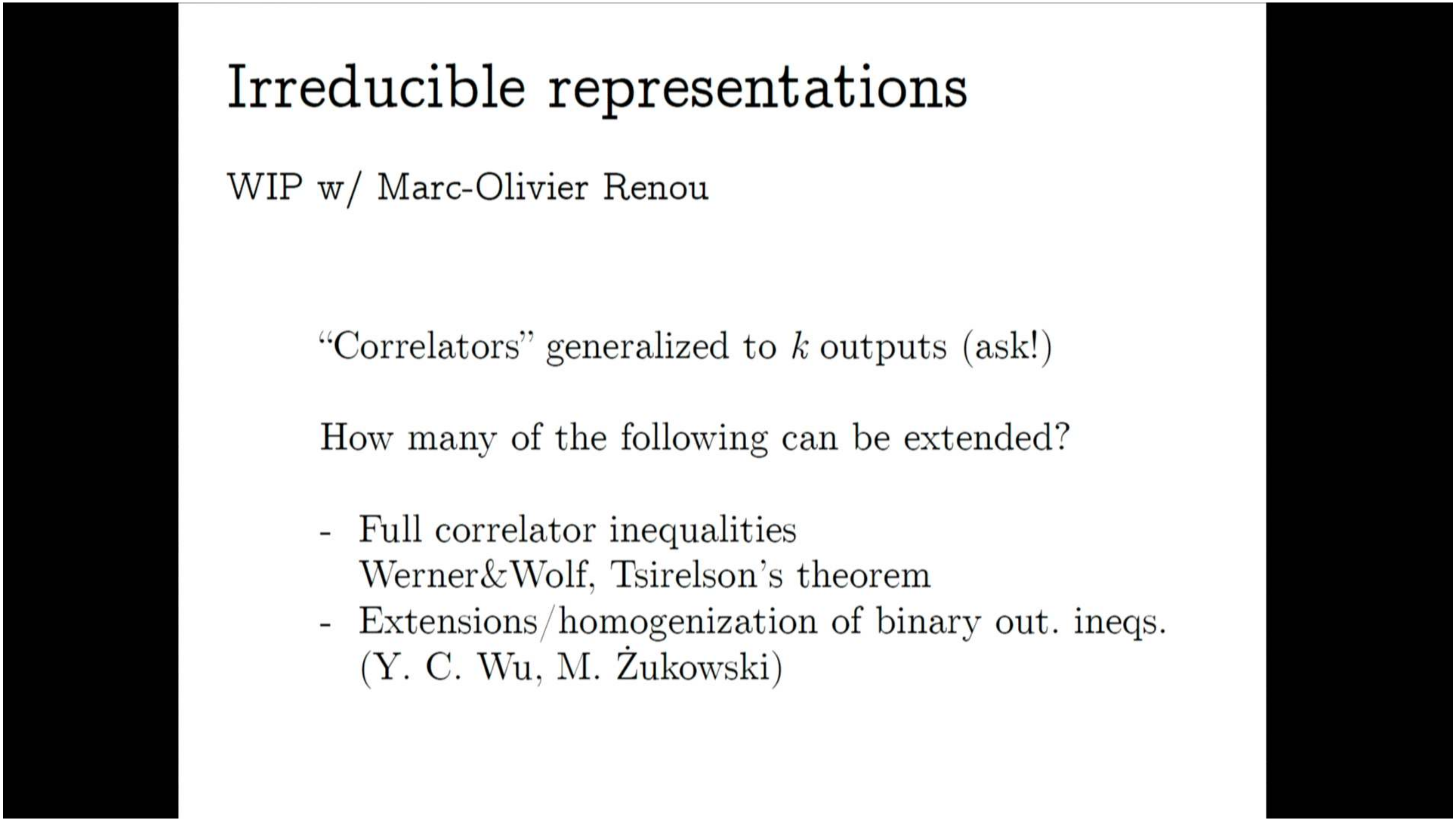
$$\vec{\alpha} \cdot \vec{P} \leq u$$

$\vec{\alpha} \in \mathbb{R}^{16}$



$\vec{P} \in \mathbb{R}^{16}$
nonsignaling





Irreducible representations

WIP w/ Marc-Olivier Renou

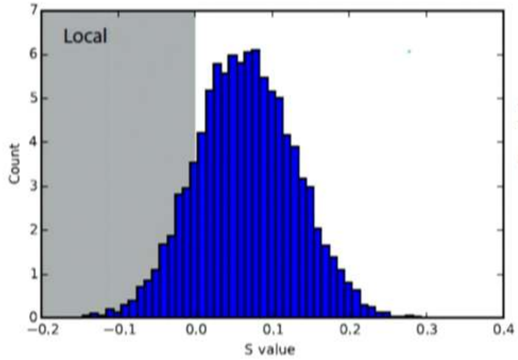
"Correlators" generalized to k outputs (ask!)

How many of the following can be extended?

- Full correlator inequalities
  Werner&Wolf, Tsirelson's theorem
- Extensions/homogenization of binary out. ineqs.
  (Y. C. Wu, M. Żukowski)

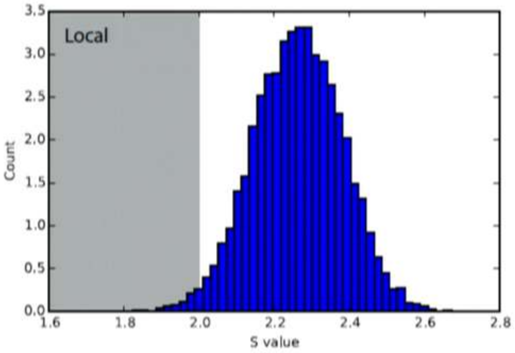
Bell ineqs. and finite statistics

WIP w/ Marc-Olivier Renou, Anthony Martin



S_CH = 0.06 ± 0.06
(local ≤ 0)

$$S^{\mathrm{CHSH}} = \langle A_0 B_0 \rangle + \langle A_0 B_1 \rangle + \langle A_1 B_0 \rangle - \langle A_1 B_1 \rangle$$



S_CHSH = 2.3 ± 0.1
(local ≤ 2)

$$S^{\mathrm{CH}} = -P_{\mathrm{A}}(0|0) - P_{\mathrm{B}}(0|0) + P_{\mathrm{AB}}(00|00) + P_{\mathrm{AB}}(00|10) + P_{\mathrm{AB}}(00|01) - P_{\mathrm{AB}}(00|11)$$

Exact bounds for NPA relaxations

Operators in NPA

$(\mathbb{1}, A_0, A_1, B_0, B_1)$

with ±1 eigenvalues

$$I^* = \max \sum_{xy} (-1)^{xy} \langle A_x B_y \rangle \text{ such that } \Gamma \geq 0$$

$$\Gamma = \begin{pmatrix} 1 & \langle A_0 \rangle & \langle A_1 \rangle & \langle B_0 \rangle & \langle B_1 \rangle \\ \langle A_0 \rangle & 1 & \langle A_0 A_1 \rangle & \langle A_0 B_0 \rangle & \langle A_0 B_1 \rangle \\ \langle A_1 \rangle & \langle A_0 A_1 \rangle & 1 & \langle A_1 B_0 \rangle & \langle A_1 B_1 \rangle \\ \langle B_0 \rangle & \langle A_0 B_0 \rangle & \langle A_1 B_0 \rangle & 1 & \langle B_0 B_1 \rangle \\ \langle B_1 \rangle & \langle A_0 B_1 \rangle & \langle A_1 B_1 \rangle & \langle B_0 B_1 \rangle & 1 \end{pmatrix}$$

Apply averaging map:

$$I^* = \max 4a \quad \text{such that } \Gamma \geq 0$$

$$\Gamma = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & a & a \\ 0 & 0 & 1 & a & -a \\ 0 & a & a & 1 & 0 \\ 0 & a & -a & 0 & 1 \end{pmatrix}$$

$$\rho = \theta_1 \oplus \theta_2 \oplus \theta_3$$

trivial representation

faithful representation of degree 2

another faithful representation of degree 2

3 blocks of size 1 (all multiplicities = 1)

$$I^* = \max 4a \quad \text{such that } \Gamma' \geq 0$$

$$\Gamma' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 + 2\sqrt{2}a & 0 \\ 0 & 0 & 2 - 2\sqrt{2}a \end{pmatrix} \geq 0$$

Gain: blocks (1x1,4x4) → (1x1,1x1,1x1)

Exact bounds for NPA relaxations

CGMLP k=3 Q_1

1) Operators

$\mathbb{1}, A_{0|0}, A_{1|0}, A_{2|0}, A_{0|1}, A_{1|1}, A_{2|1}, B_{0|0}, B_{1|0}, B$

Generators:	*Perm. of operators*
A0(0,2,1) A1(0,2,1) B0(0,1,2) B1(0,1,2)	(1,3,2)(4,6,5)(7,8,9)(10,11,12)
A0(0,2) A1(0,2) B0(1,2) B1(0,2) A(0,1)	(1,6)(2,5)(3,4)(8,9)(10,12)
(A,B)	(1,7)(2,8)(3,9)(4,10)(5,11)(6,12)

2) 6 variables in reduced basis
+ 3 equality constraints

3) SDP with 1x1 blocks

$$I^* = \frac{2\sqrt{3}+6}{3} \approx 3.1547$$

Conclusion

Full understanding of symmetries opens avenues of research

Useful for **classification**

Extension of known tricks/results?

Robustness (theory & experiments)

Deeper **theoretical understanding?**