Title: Inhomogeneous Anisotropic Cosmology

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Abstract: In homogeneous and isotropic Friedmann-Robertson-Walker cosmology, the topology of the universe determines its ultimate fate. If the Weak Energy Condition is satisfied, open and flat universes must expand forever, while closed cosmologies can recollapse to a Big Crunch. A similar statement holds for homogeneous but anisotropic (Bianchi) universes. Here, we prove that arbitrarily inhomogeneous and anisotropic cosmologies with ``flat'' (including toroidal) and ``open'' (including compact hyperbolic) spatial topology that are initially expanding must continue to expand forever at least in some region at a rate bounded from below by a positive number, despite the presence of arbitrarily large density fluctuations and/or the formation of black holes. Because the set of 3-manifold topologies is countable, a single integer determines the ultimate fate of the universe, and, in a specific sense, most 3-manifolds are ``flat'' or ``open''. Our result has important implications for inflation: if there is a positive cosmological constant (or suitable inflationary potential) and initial conditions for the inflaton, cosmologies with ``flat'' or ``open'' topology must expand forever in some region at least as fast as de Sitter space, and are therefore very likely to begin inflationary expansion eventually, regardless of the scale of the inflationary energy or the spectrum and amplitude of initial inhomogeneities and gravitational waves. Our result is also significant for numerical general relativity, which often makes use of periodic (toroidal) boundary conditions.

Leonardo Senatore (Stanford)

Inhomogeneous Anisotropic Cosmology:
Starting Inflation
in inhomogeneous universe

with East, Linde and Kleban **1511**
with Kleban **1602**

Introduction

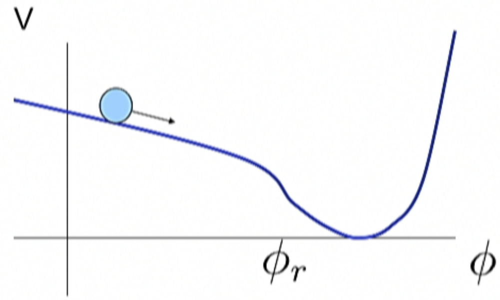
- Inflation is believed to have problems to start.
- This talk will argue (prove) that one of the reasons of concerns is not sustained
  - it will do so by using some interesting mathematics,
    - used to answer a very very physical question

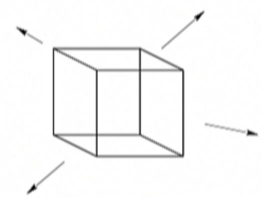
The Problem

– If we have the inflaton on top of his potential

  – and the space is homogeneous on a H_I patch

  – then inflation starts





• Question: how likely is to have an homogenous patch of this size?

  – Here I will first present an apparently compelling argument (at least to me), that if

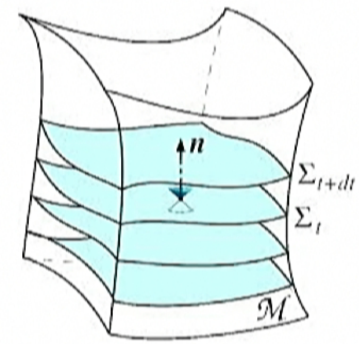
$$H_I \ll M_{\rm Pl} \quad \Rightarrow \quad {\rm prob} \sim e^{-\frac{M_{\rm Pl}}{H_I}}$$

  – and we will show that this argument is not really correct

  – But first, let us open a parenthesis

FRW Cosmology

–Let us review FRW cosmology



$$ds^2 = -dt^2 + a(t)^2 d\Sigma^2$$

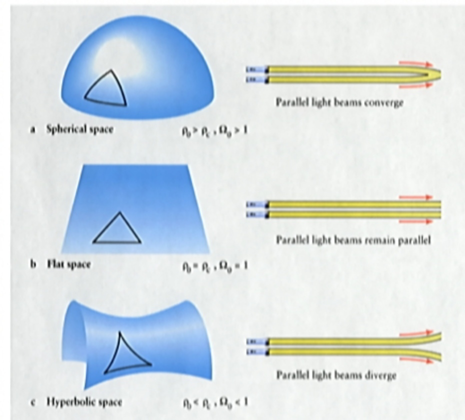
$$d\Sigma^2 = \frac{dr^2}{1-kr^2} + r^2 d\Omega_2^2 \;, \qquad d\Omega_2^2 = d\theta^2 + \sin(\theta)^2 d\phi^2$$

–where $k = +1$, Sphere

$k = 0$, Flat

$k = -1$, Hiperboloid

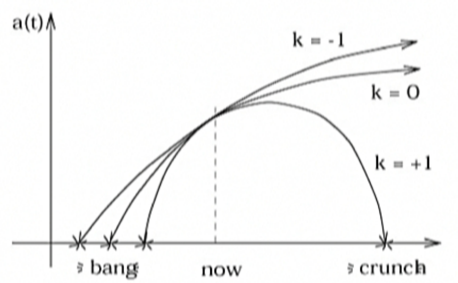
$a(t) = \text{radius of curvature}$



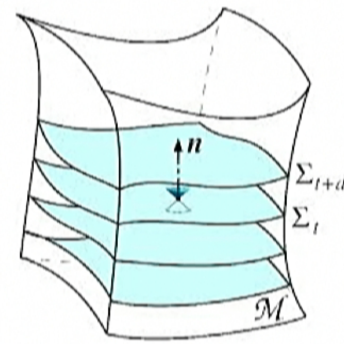
–Typical evolution

–closed universe: recollapses

–open and flat: expand forever



FRW Cosmology



–Let us see that open & flat universes cannot recollapse

- Consider a spatial surface Σ

–normal vector $n_\mu = (1, \vec{0})$

–induced metric $h_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu$

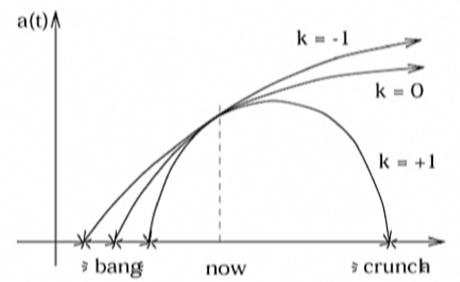
$$\Rightarrow \quad h_{00} = h_{0i} = 0 \ , \quad h_{ij} = g_{ij}$$

–Einstein Equations $G_{\mu\nu} = 8\pi G \, T_{\mu\nu} - \Lambda g_{\mu\nu} \ ,$

$$\Rightarrow \quad G_{\mu\nu} n^\mu n^\nu = (8\pi G \, T_{\mu\nu} - \Lambda g_{\mu\nu}) \, n^\mu n^\nu$$

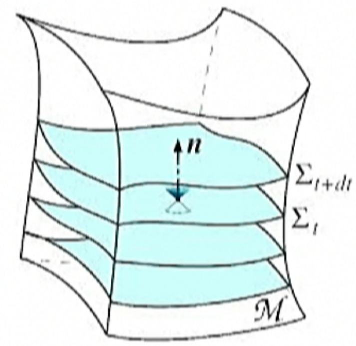
$$\Rightarrow \quad H^2 + \frac{k}{a^2} = \frac{8\pi G}{3}\rho + \frac{\Lambda}{3}$$

–If the universe starts expanding, and it turns back



- $\Rightarrow$ there is a surface with $H = 0$
- $\Rightarrow$ on this surface $\Rightarrow \quad \dfrac{k}{a^2} = \dfrac{8\pi G}{3}\rho + \dfrac{\Lambda}{3}$
- If $(\rho > 0 \, \& \, \Lambda > 0) \, \& \, k \leq 0 \quad \Rightarrow \quad \text{Impossible}$

FRW Cosmology



–Collapse for open and flat universe?

$$\Rightarrow \quad \frac{k}{a^2} = \frac{8\pi G}{3}\rho + \frac{\Lambda}{3}$$

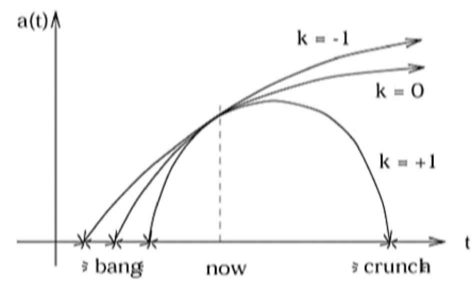
$$(\rho > 0 \,\&\, \Lambda > 0) \,\&\, k \leq 0 \quad \Rightarrow \quad \text{Impossible}$$

–no need of the explicit solution

- Also closed universe can not-recollapse: it needs to become larger than

$$\Rightarrow \frac{k}{a_{\max}^2} \lesssim H_I \sim \Lambda$$

- it needs a large enough CC
- Otherwise, it recollapses in an Hubble time



The Initial Patch Problem

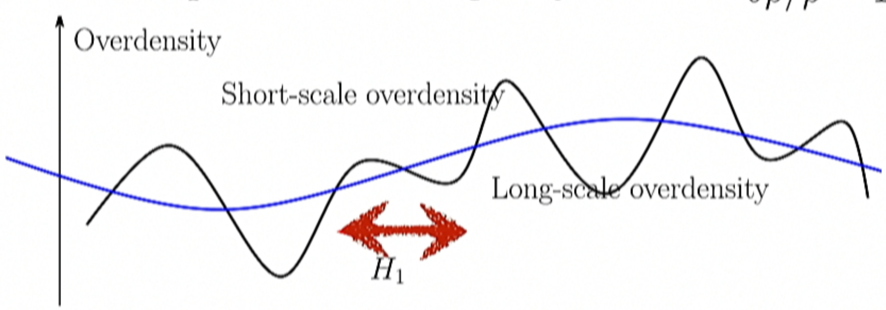
– Imagine we start with a small inhomogeneous universe, not dominated by Inflaton potential. Say it expands in a decelerating way.

– $\Rightarrow$ to start inflation, we need H to decrease to H_I

– During this time, many modes become shorten than H $\rho_{\rm FRW}$

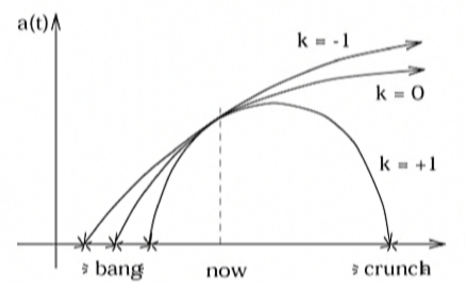
– Modes longer than H count as an effective Homogeneous density

– At each time step, $\rho_{\rm FRW}$ changes by order 1: $\delta\rho/\rho \sim 1 \quad \Rightarrow \quad \delta\rho_{\rm FRW}/\rho_{\rm FRW} \sim 1$



– $\Rightarrow \dfrac{\delta\left(\frac{k}{a^2}\right)}{\frac{k}{a^2}} \sim 1$: the curvature jumps by order one

– at each Hubble time.



The Initial Patch Problem

– Several mechanism to avoid to this problem have been proposed in the years
  – Linde likes starting with a Planckian torus
  – I like starting out of false vacuum eternal inflation.

- but somehow we continue to talk about this problem
    - because they were ad-hoc mechanisms

- The solution we present here is radically different
  – we will show that this problem is non-existent (but for a deep, non-trivial, reason)

Inhomogenous Cosmology

– Already Wald (1983) had shown that if the weak energy condition is preserved, all homogeneous but inosotropic universe (Bianchi universes) that are not `closed' (that is non-Bianchi-Type-IX universes) cannot recollapse.

– WEC: $T_{\mu\nu}t^{\mu}t^{\nu} \geq 0 \;\; (\text{i.e.``}\rho \geq 0, \rho + p > 0'') \;,\;\; \text{for any } t^{\mu} \text{ timelike}$

– But inhomogeneities are more challenging.

  – diff equations become partial diff, and singularities form, geodesic cross, etc. It is a much less symmetric situation.

  – we will see that a sort of similar conclusion holds

• Let us therefore consider general `cosmologies'.

A Cosmology

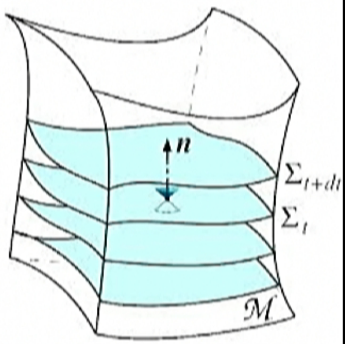
– *First Assumption*: we consider a cosmology:

  – a connected 3+1 dimensional spacetime with a compact Chauchy surface (will comment later on the non-compact case)

- This implies (Geroch 1970):

  – the spacetime is topologically $R \times M$ where M is a 3-manifold

  – it can be foliated by a family of topologically identical Chauchy surfaces M_t



Theorem

with Kleban **1602**

- This implies that in a big bang cosmology, there cannot be a big crunch
  - strongly suggesting that inflation will eventually start, no matter what are the initial inhomogeneities and the scale of inflation

There cannot exist a non-singular spacelike hypersurface with maximum volume: given any time slice, there is another with larger spatial volume. Furthermore, in an initially expanding universe there must be at least one expanding region on every timeslice, and if $\Lambda > 0$ the expansion rate in that region is bounded from below by that of de Sitter spacetime in the flat slicing.

For the first sentence, see also Barrow and Tippler **1985**

Introduction

- Inflation is believed to have problems to start.
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Proof

–The proof is very simple. A regular surface of maximum volume has extrinsic curvature

$$K = 0 \quad \text{everywhere}$$

–To satisfy Einstein equations, this implies

$$R^{(3)} \geq 0 \quad \text{everywhere}$$

–But topologically some manifolds require

$$R^{(3)} \leq 0 \quad \text{at least at one point}$$

–Therefore regular maximal surfaces cannot exist

Hypothesis

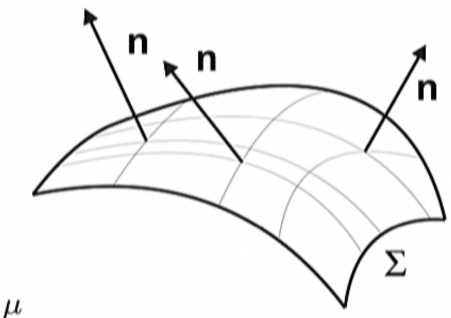
– A cosmology

– The spatial topology of M_t must not be `closed', i.e. it must not be of type (i) that we define below (roughly, M_t must not be a sphere)

– The weak energy condition holds (satisfied by matter, radiation and V>0)

Notation

– n_μ is the orthonormal vector to M_t: $n_\mu n^\mu = -1$



– Spatial metric $h_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu$

– Extrinsic curvature $K_{\mu\nu} = h_\mu{}^\rho \nabla_\rho n_\nu$, $K = \nabla_\mu n^\mu$

–how much the family of geodesics induced by n_μ deviates

–In particular $K = \nabla_\mu n^\mu$, $\sigma_{\mu\nu} = K_{\mu\nu} - \frac{1}{3} K h_{\mu\nu}$

–Notice $\mathcal{L}_n \log \sqrt{h} = K,$: rate of growth of volume

$$\Rightarrow \quad \sqrt{h} \sim \sqrt{h_0}\, e^{Kt}$$

Proof

–Similarly to FRW case, consider (cc reabsorbed in stress tensor)

$$n^\mu n^\nu G_{\mu\nu} = 8\pi G_N \, T_{\mu\nu} \, n^\mu n^\nu$$

–From Gauss-Codazzi

$$n^\mu n^\nu G_{\mu\nu} = \frac{1}{2}\left\{R^{(3)} + \left(K^\mu_\mu\right)^2 - K_{\mu\nu}K^{\mu\nu}\right\} = \; 3 - \text{surface quantities}$$

–$\Longrightarrow$ we have

$$16\pi G_N T_{\mu\nu} n^\mu n^\nu = R^{(3)} + \frac{2}{3}K^2 - \sigma_{\mu\nu}\sigma^{\mu\nu}$$

–If a surface has extremal volume, the volume is stationary wrt any variations. Since

$$\mathcal{L}_n \log\sqrt{h} = K, \quad \Rightarrow \quad K = 0 \quad \text{everywhere}$$

–Then we have, on an extremal surface

$$16\pi G_N \, T_{\mu\nu} n^\mu n^\nu = R^{(3)} - \sigma^{\mu\nu}\sigma_{\mu\nu}$$

Proof

– One Einstein eq. is

$$\underbrace{16\pi G_N\, T_{\mu\nu} n^\mu n^\nu}_{\geq 0 \text{ by WEC}} = R^{(3)} \underbrace{-\sigma^{\mu\nu}\sigma_{\mu\nu}}_{\leq 0}$$

– If $R^{(3)} \leq 0$ at least at one point , $\Longrightarrow$ this equation cannot be satisfied

– $\Longrightarrow$ an extremal surface cannot exist

– It turns out that for order-one fraction of topologies $R^{(3)} \leq 0$ at least at one point

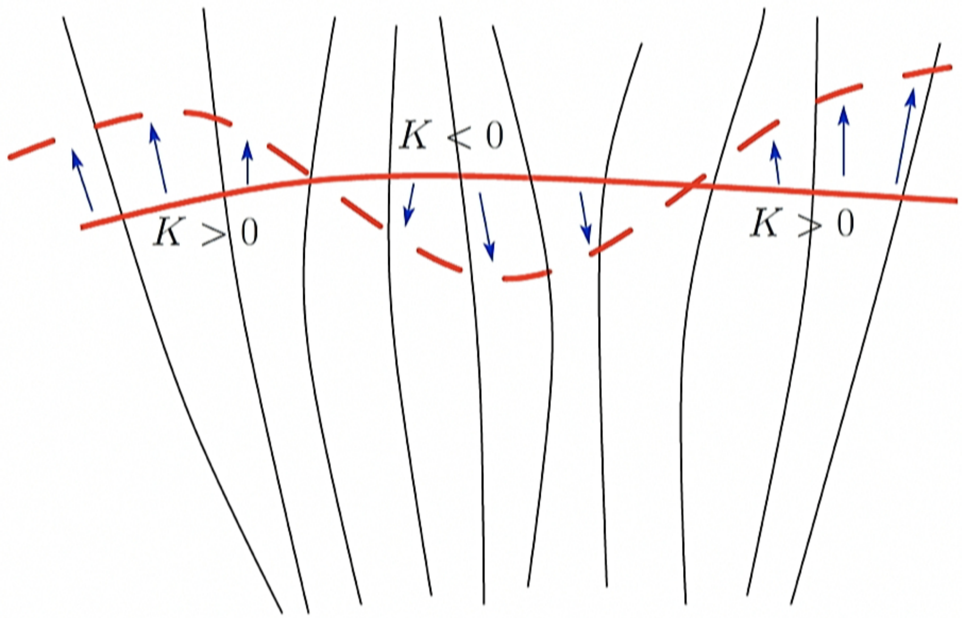
– $\Longrightarrow$ an extremal surface cannot exist

– This means that in a spacetime with this topology, and with a big-bang or big-crunch, given a surface, we can always find a surface with larger volume (either in the future or in the past)

– This surface can be found with the following procedure

Mean Curvature Flow

– Take a surface, and deform it forward or backward according to sign of K



– The change of volume: $\dfrac{\partial V}{\partial \lambda} = \int d^3x K^2 \sqrt{h} \equiv \langle K^2 \rangle \geq 0$

– So this procedure either converges to an extremal surface, if it can exists, $K = 0$ everywhere

– or it gives a surface of larger volume indefinitely

Thorston Classification Conjecture

Thorston, Hamilton, Perelman

– To determine which manifolds must have $R^{(3)} \leq 0$ at least at one point , consider that all compact oriented 3-manifold fall into one of these three classes

– (i) ``Closed'': any function on M_t can be the $R^{(3)}$ of a smooth metric on M_t

- ex: $S^3,\ S^2 \times S^1,\ S^3/\Gamma(\text{with } \Gamma \in SO(4)), RP^3$

– (ii) ``Flat'': any function on M_t can be the of a smooth metric on if it is negative somewhere or zero everywhere

- ex: $R^3/\Gamma(\text{ with } \Gamma \text{ an isometryof } R^3)$

– (iii) ``Open'': any function on M_t can be the $R^{(3)}$ of a smooth metric on M_t if it is negative somewhere

- ex: $H^3/\Gamma,\ H^2 \times R, nil, sol, \widetilde{SL}(2, R)$

– Any connected sum of (i) and (ii) with a factor of (iii) is of kind (iii)

Thorston Classification ~~Conjecture~~ Classification

Thorston, Hamilton, Perelman

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Thorston Classification Conjecture

Thorston, Hamilton, Perelman

– The set of topologies in countable! Can be represented by an integer number (~FRW).

– For type (iii), the theorem is established.

– For type (ii), it could be that $R^{(3)} = 0$ everywhere

– if extremal surface exists, we have:

$$\Rightarrow \sigma_{\mu\nu} = 0, \quad T_{\mu\nu} = 0 \text{ (with dominant energy condition)}$$

$$\text{DEC} : T_{\mu\nu} t^{\nu} \text{ is past directed for all future directed timelike } t^{\mu}, \text{ ``}\rho > p\text{''}$$

– empty universe

Open universes

–In all cases where $R^{(3)} \leq 0$ at least at one point , at that point we have

$$16\pi G_N T_{\mu\nu} n^\mu n^\nu = R^{(3)} + \frac{2}{3} K^2 - \sigma_{\mu\nu}\sigma^{\mu\nu}$$

$$\Rightarrow \quad |K| \geq K_\star \equiv \sqrt{24\pi G_N T_{\mu\nu} n^\mu n^\nu}.$$

–If we add that the universe is initially expanding, than we have that at that point

$$K \geq K_\star$$

Open universes

– We have at one point on any surface $|K| \geq K_\star \equiv \sqrt{24\pi G_N T_{\mu\nu} n^\mu n^\nu}$.

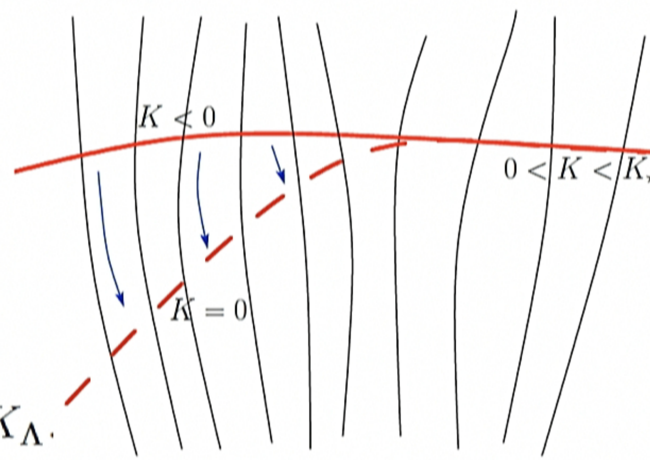
– the universe is initial expansing

$$\Rightarrow \; K \geq K_\star \text{ at that point}$$

– Assume there is a surface where $K < K_\star$ everywhere

– can pull back the surface to obtain a surface where $|K| < K_\star$ everywhere

- contradiction



- Therefore, in any slice there must be a point that
- expands with $K \geq K_\star$

– Suppose now there is a positive CC

– (as in inflation), then $K \geq \sqrt{24\pi G_N \Lambda} \equiv K_\Lambda$.

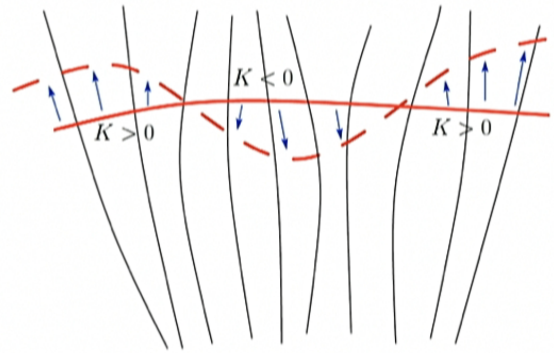
– a region expands always faster than in inflation

Subtleties

- We need to assume the absence of finite volume singularities
  - the universe could stop expanding all of a sudden
  - these singularities are conjectured to be removed with some mild assumptions of regularity of stress tensor (Barrow and Tippler 1985)
  - these singularities sound quite unphysical to us

Does the universe reach infinite volume?

– It is tempting to conclude that the slices obtained with mean curvature flow reach infinite volume



– But it could be that the overall volume keeps

– growing reaching an asymptote

– This is very unlikely:

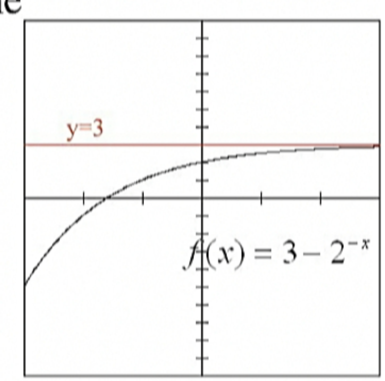
– there is always a region that expands fast

– therefore this region should shrink, becoming singular

– this can only happen if surface reaches embedding infinite time

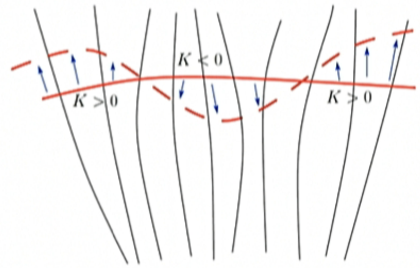
• which it must as otherwise it would reach extremality

– Then, unless the surface is null, it has infinite volume



– (this is not proven yet: we are trying to prove this)

Implications for inflation



– Modulo these concerns, these surfaces will reach

– infinite volume, so that the vacuum energy will dominate

– The probability to start inflation is therefore reduced to estimating the probability for these topologies (clearly order one), as well as for the inflaton of being on top of the potential (see later, but I will not make a judgement on this)

– The problem of initial homogeneity seems to have been resolved (but showing it does not hold for topological reasons)

  – the random walk of $\delta\rho/\rho$ did not hold because topology induced a correlation among modes.

More `practical' considerations with East, Linde and Kleban **1511**

–This is as far as pure Mathematics has brought us.

- There is a strong suggestion that inflation will always start for these topologies.
- We can check this by performing numerical simulations (actually historically we have done in reverse order)
- Approach: start with highly inhomogenous universe, and simulate its evolution.

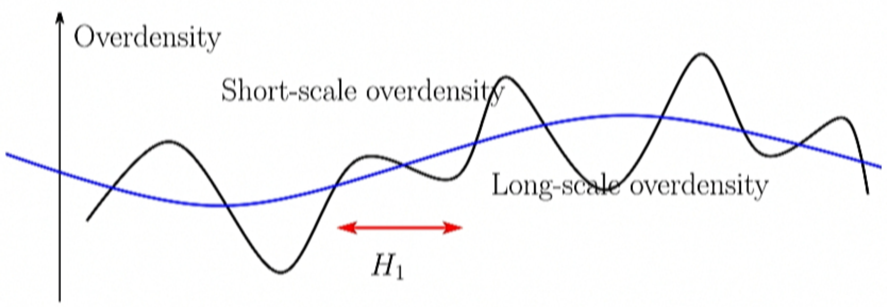
  –need a code that can handle singularities, batch holes and horizons.

  –Solution: ask William East! and the problem is solved.

Strategy of simulations

with East, Linde and Kleban **1511**

- Consider an initially expanding highly inhomogenous universe.
- Define a local hubble rate by the extrinsic curvature: H_0
  - There will be modes longer and shorter than H_0

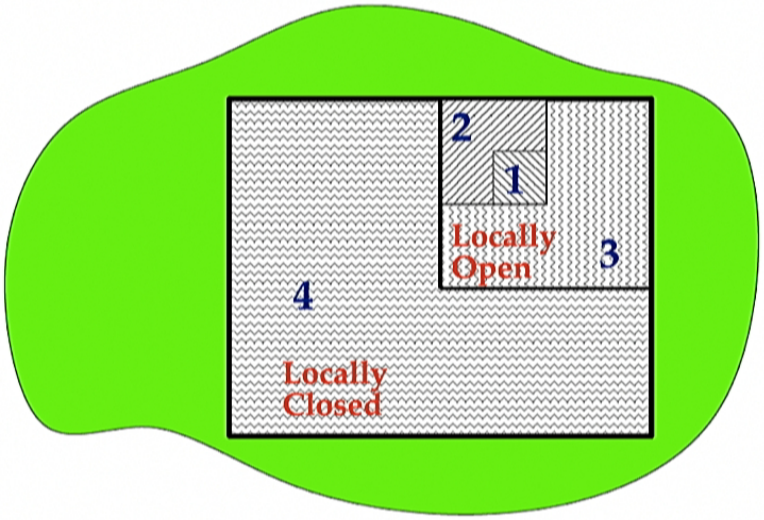


Strategy of simulations

with East, Linde and Kleban **1511**

- The effect of long modes $k/H \ll 1$ can be understood analytically.
  - As long as a mode in longer than Hubble, its effect is simply to renormalize the energy density: it induces a local homogenous anisotropic (Bianchi) universe
  - Their evolution is well understood (Wald 1983): only closed universes can recollapse
  - If a universe is open on average (meaning that the zero mode of the energy is such that it makes is open-Bianchi), there must be a region where the effect of long modes is to keep it open at all times

$$\sim \int_V (\rho - M_{\rm Pl}^2 H^2) \leq 0$$



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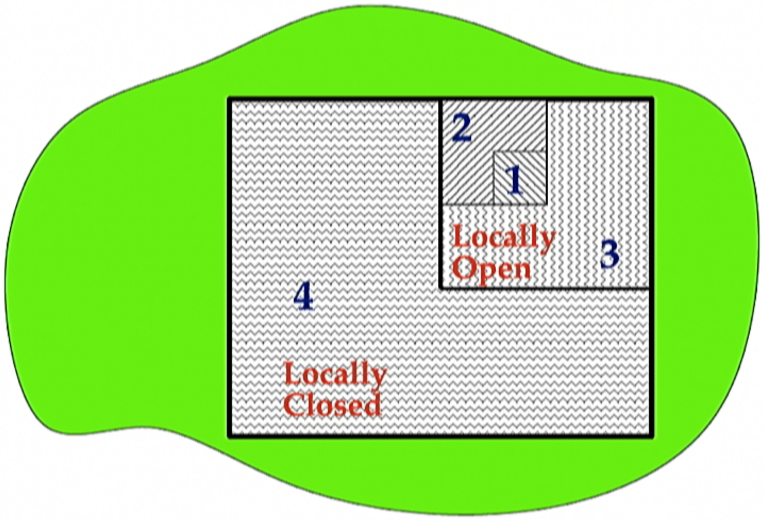


Strategy of simulations

with East, Linde and Kleban **1511**

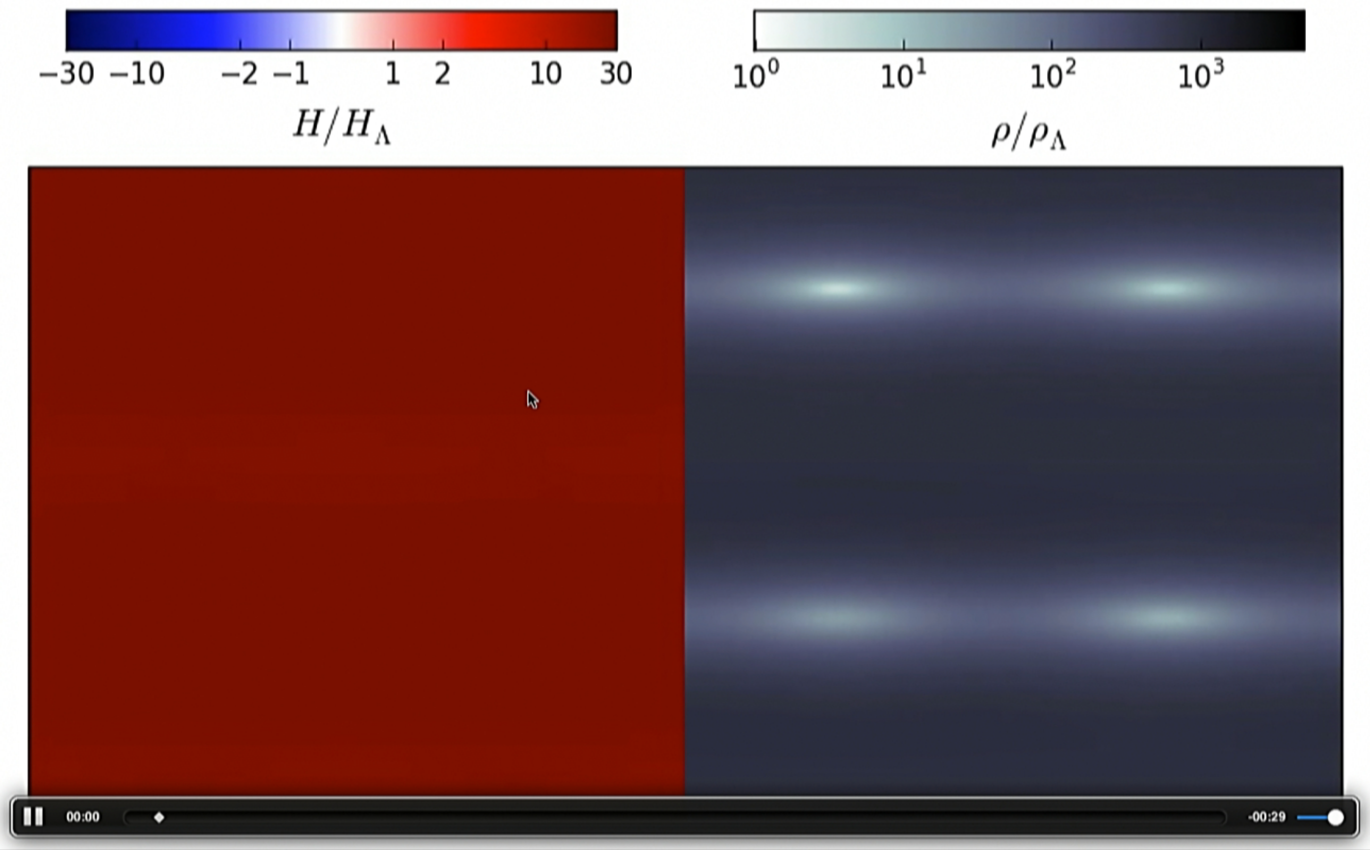
- The effect of short modes $k/H \gtrsim 1$ have strong dynamical effect
  - they will form overdensity that end in Black Holes
    - but, by conservation of energy, empty regions will form, and in those locations inflation will start
- We start with simulation with domination of gradient energy:

$$\phi(t=0,\mathbf{x}) = \phi_0 + \delta\phi \left[\sum_{1 \leq |\mathbf{k}L/2\pi|^2 \leq N} \cos(\mathbf{k}\cdot\mathbf{x} + \theta_{\mathbf{k}}) \right],$$

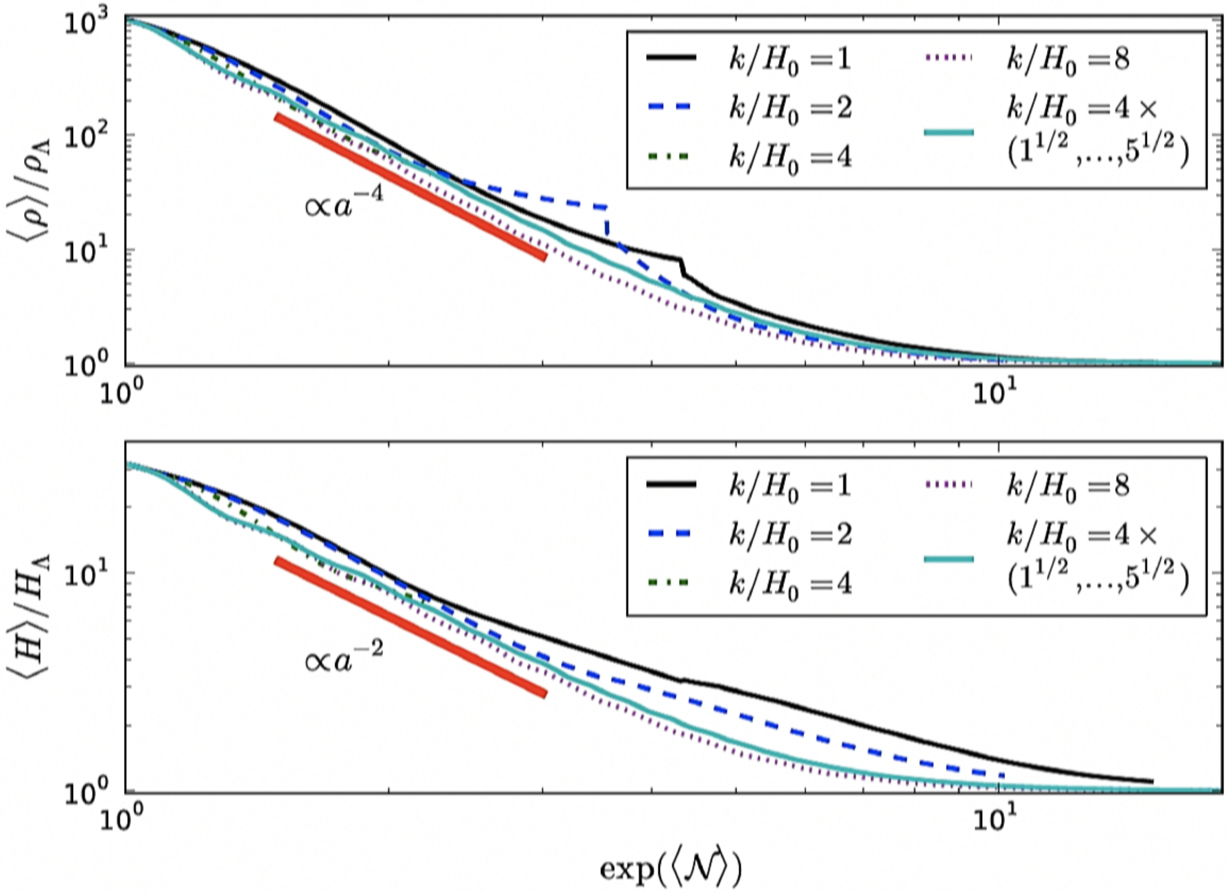
Simulation

– Inflation starts

– BH emit GW

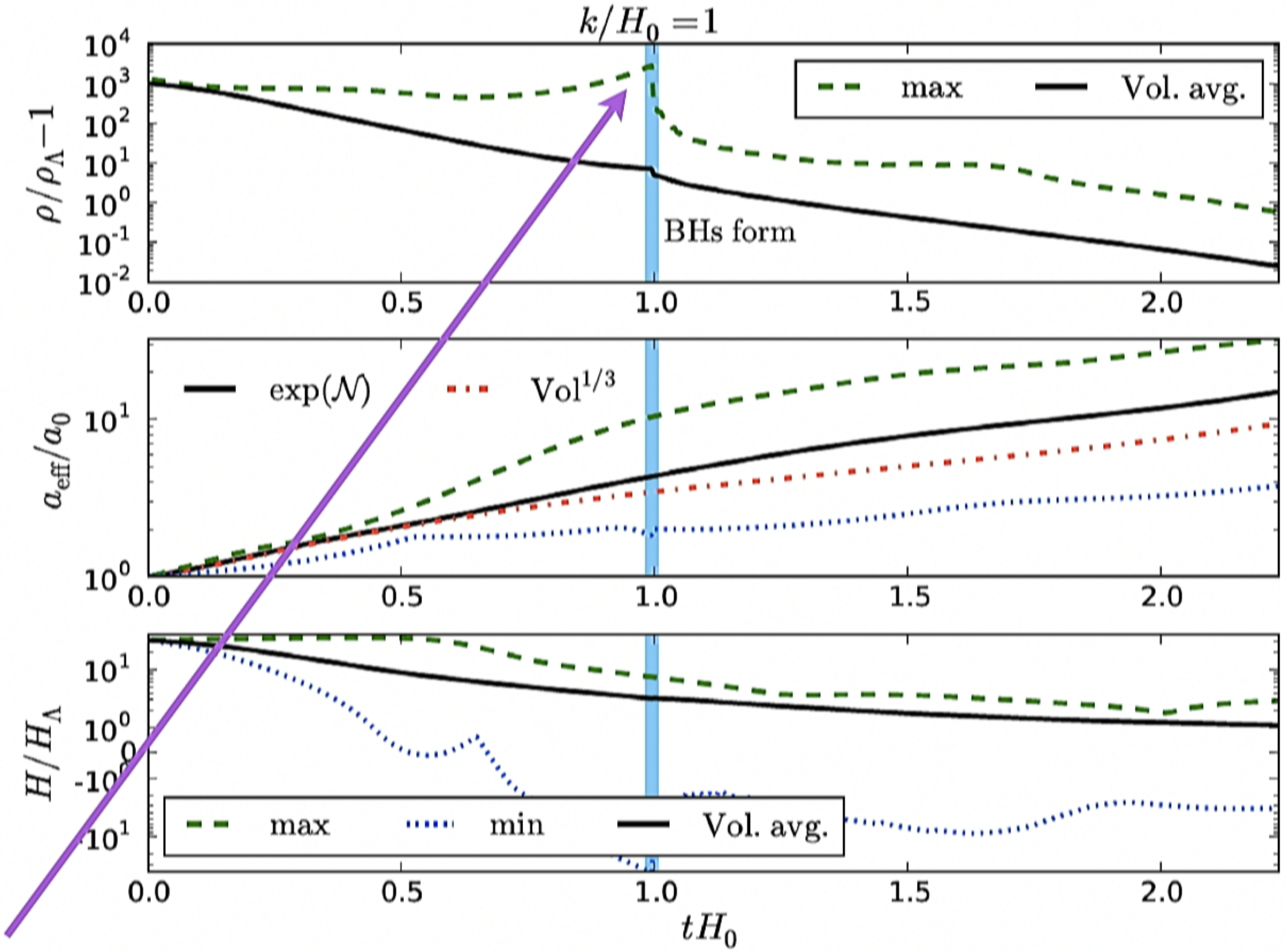


Approaching Inflation



– Universe on average expands like radiation and then as CC

Approaching Inflation



–BH form but dilute, universe homogenizes, inflation starts

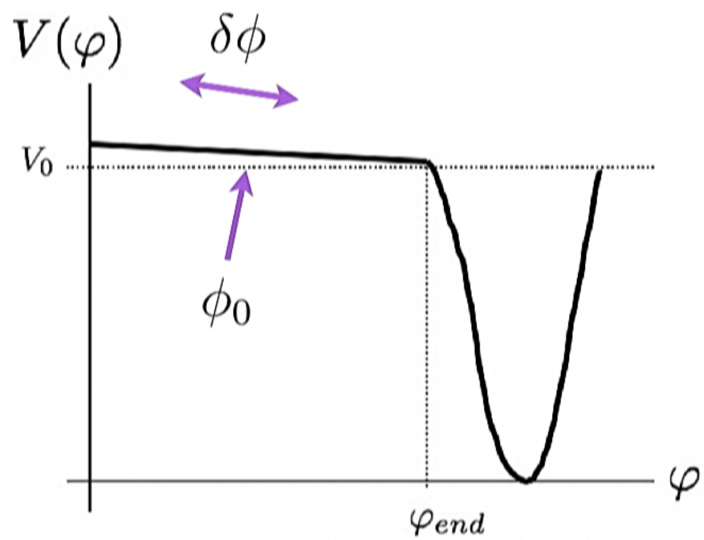
Slight Issue for Inflation

– It looks like there is no need to impose with initial homogeneity to have inflation somewhere.
    – just estimate probability of topology (or of zero mode)

- But we require the field to be on top of the potential.
    – Why?

Slight Issue for Inflation

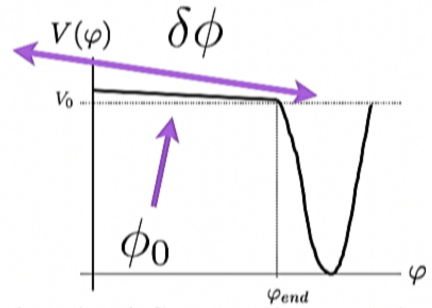
–Consider an inflaton potential with a very perturbed initial field configuration.



–The fluctuations can be very large, going beyond the plateau

Slight Issue for Inflation

– The dynamics of the average field is given by



– the average of the eq. of motion

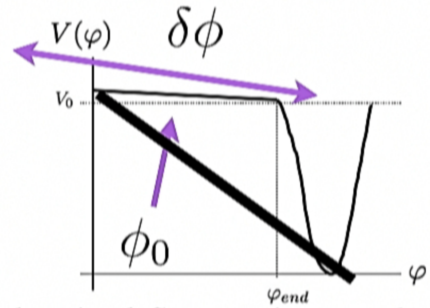
$$\langle V'(\phi)\rangle \gg V'(\langle\phi\rangle)$$

– From an EFT point of view: integrate out the fast moving classical fluctuations to keep only the homogeneous mode. Since the classical fluctuations probe the high gradient region, the effective potential for the zero mode looks more something like this:

Slight Issue for Inflation

–The dynamics of the average field is given by

–the average of the eq. of motion



$$\langle V'(\phi)\rangle \gg V'(\langle\phi\rangle)$$

–From an EFT point of view: integrate out the fast moving classical fluctuations to keep only the homogeneous mode. Since the classical fluctuations probe the high gradient region, the effective potential for the zero mode looks more something like this:

–a non-inflating potential: in fact, we find no inflation in simulations

–In particular: $\langle V'(\phi)\rangle \sim \frac{V(\langle\phi\rangle)}{\langle\phi\rangle} = \left(\frac{M_P}{\langle\phi\rangle}\right)\left(\frac{V'(\langle\phi\rangle)}{\sqrt{\epsilon}}\right) > V'(\langle\phi\rangle)$

unless $\langle\phi\rangle > M_P/\sqrt{\epsilon}$

- Disclaimer: this is different than the former homogeneity problem
    - and quantitatively is highly model dependent

Conclusions

- It is widely believed that in order to start inflation, we need an homogenous patch of order H_I
- By using numerical GR simulations that are able to handle singularities and horizons
- & less-usual mathematical techniques such as mean curvature flow and topology
- we have shown that this does not seem to be necessary:
  - even by starting with an highly inhomogeneous universe,
  - if we condition on the topology of the manifold (or on the zero mode of the fluctuations)
  - there will be a (potentially small region in terms of initial coordinates) region were inflation will start with certainty.

• Both the numerical approach and the analytical approach are relatively new and unfamiliar. It looks like we just scratched the surface of a new attack to the study of the initial conditions of the universe.