Title: A few algebraic surprises in Anti-de Sitter space

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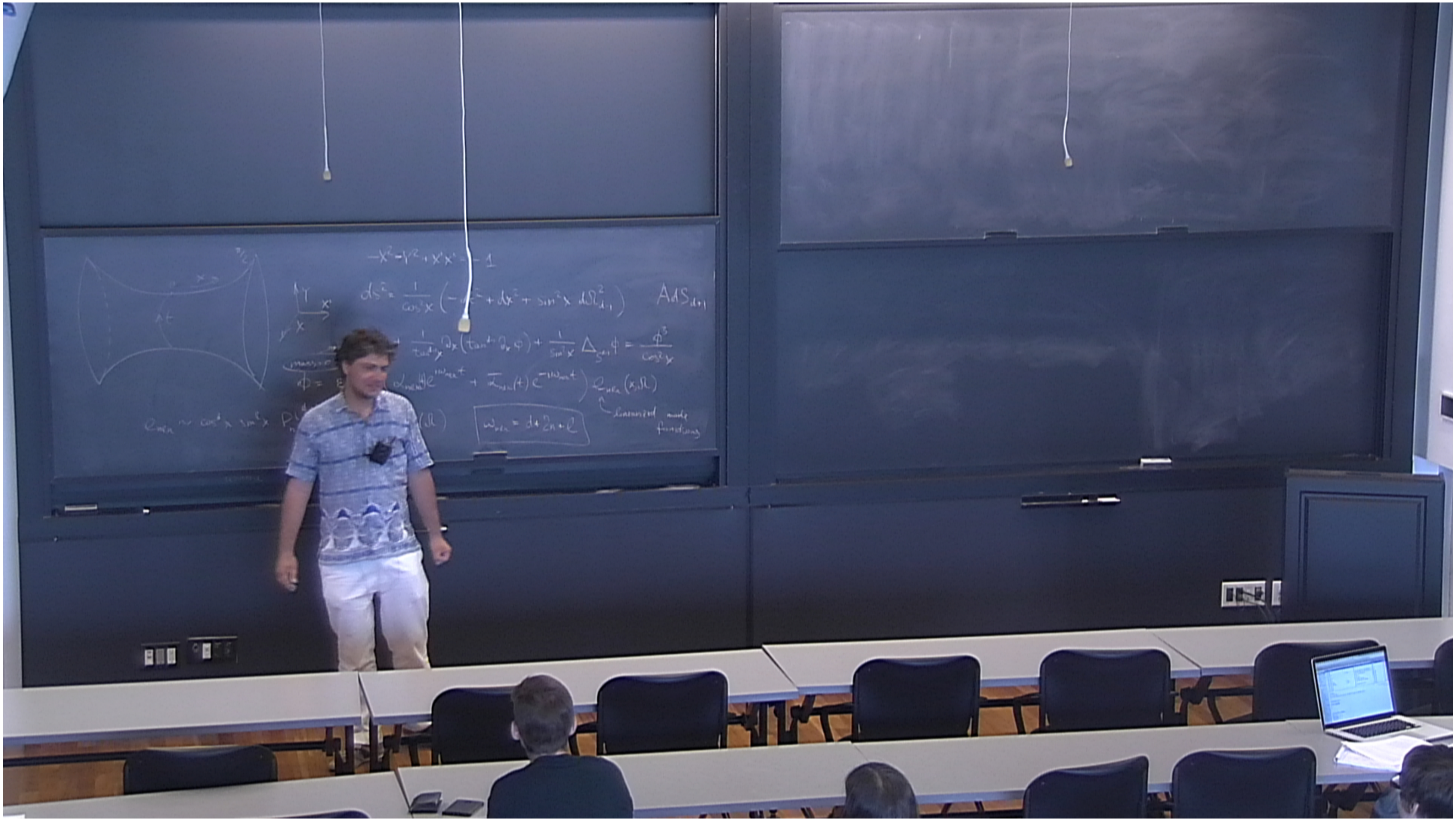
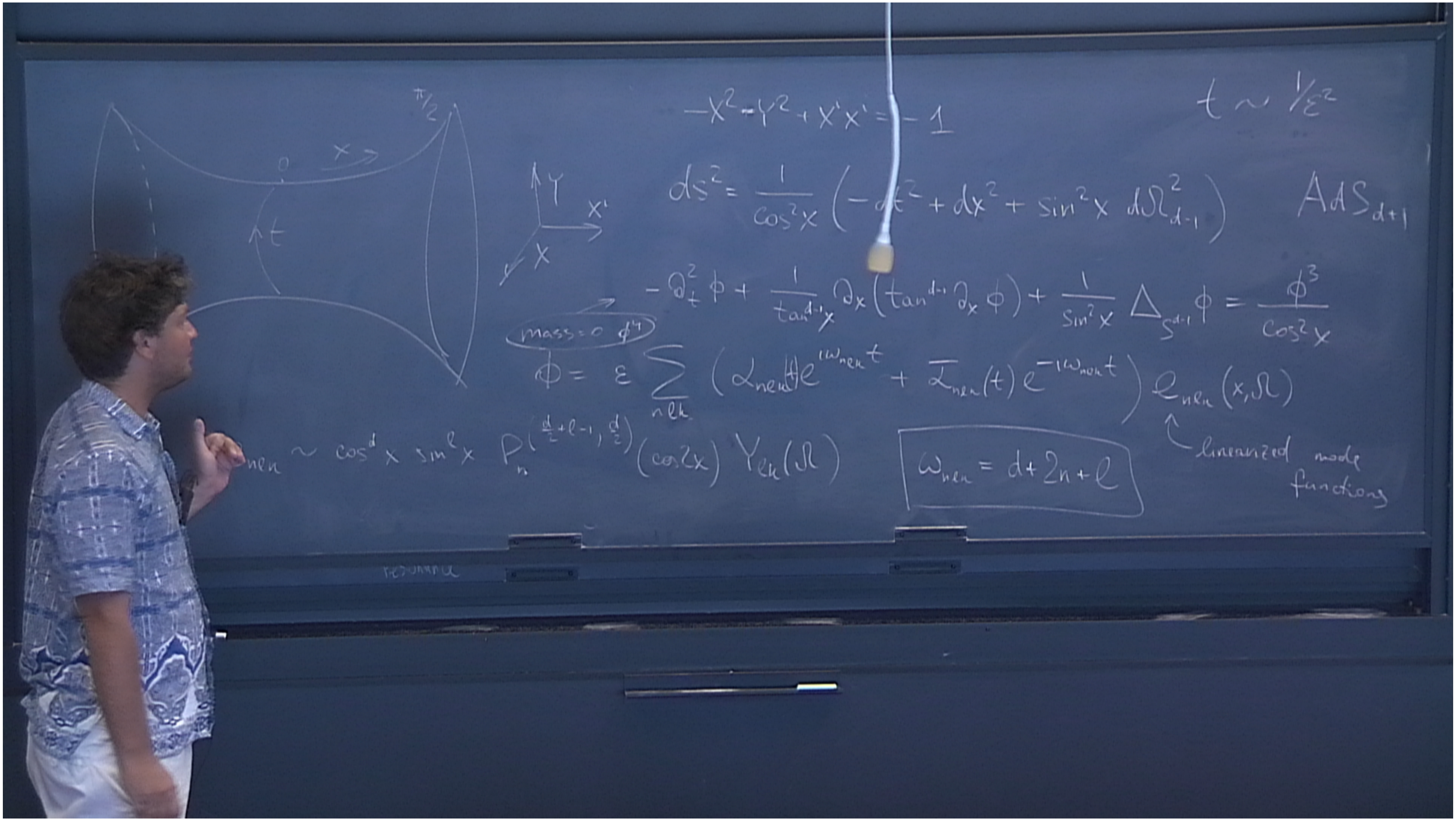
Abstract: <p>Questions of nonlinear stability in global AdS space have recently received a significant amount of attention, both as an interesting problem in mathematical general relativity and nonlinear dynamics, and in relation to thermalization studies within the AdS/CFT paradigm. Working with nonlinear perturbation theory (the main technique available for analytic studies in this area) requires a thorough understanding of the properties of linearized AdS fields'</p>

<p>mode functions, which are the fundamental building blocks in perturbative treatments of nonlinearities. While complicated explicit expressions for these mode functions are available in the literature, they hide a good deal of the elegant underlying structure dictated by the AdS symmetries. Extending the mode functions in the flat embedding space (in which the AdS space can be realized as a hyperboloid) results in families of homogeneous polynomials, on which the AdS isometries act in a straightforward manner. This suggests a simple proof of important selection rules in nonlinear perturbation theory.</p>

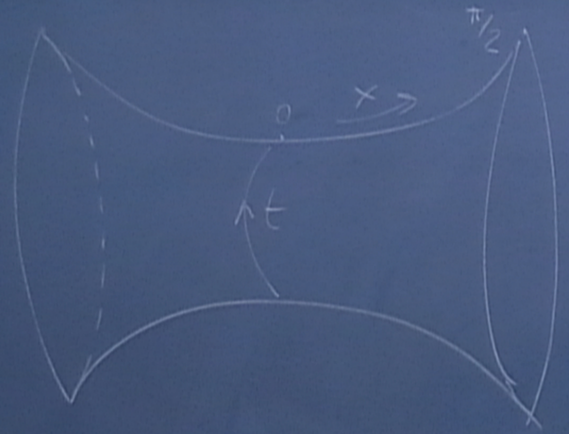
<p>Studies of multiplet structures of the mode functions furthermore reveal a relation to the Higgs oscillator, a well-known quantum-mechanical superintegrable system. This AdS connection leads to an explicit construction of the hidden symmetry generators for the Higgs oscillator, a long-standing problem in mathematical quantum mechanics.</p>

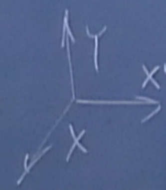

AdS_{d+1}
ω_{nℓ} = d+2n+ℓ
linearized mode functions


$\pi/2$
x
0
t
Y
X^i
X
$-X^2 - Y^2 + X^i X^i = -1$
$t \sim 1/\varepsilon^2$
$ds^2 = \frac{1}{\cos^2 x}\left(-dt^2 + dx^2 + \sin^2 x \, d\Omega^2_{d-1}\right)$
AdS_{d+1}
$-\partial_t^2 \phi + \frac{1}{\tan^{d-1} x}\partial_x\left(\tan^{d-1}\partial_x \phi\right) + \frac{1}{\sin^2 x}\Delta_{S^{d-1}}\phi = \frac{\phi^3}{\cos^2 x}$
mass=0 ϕ^4
$\phi = \varepsilon \sum_{nlk} \left(\alpha_{nlk}(t) e^{i\omega_{nlk}t} + \bar{\alpha}_{nlk}(t) e^{-i\omega_{nlk}t}\right) e_{nlk}(x,\Omega)$
linearized mode functions
$e_{nlk} \sim \cos^d x \sin^l x \, P_n^{\left(\frac{d}{2}+l-1, \frac{d}{2}\right)}(\cos 2x) \, Y_{lk}(\Omega)$
$\boxed{\omega_{nlk} = d + 2n + l}$

$-X^2 - Y^2 + X^i X^i = -1$

$t \sim 1/\varepsilon^2$





$$ds^2 = \frac{1}{\cos^2 x}\left(-dt^2 + dx^2 + \sin^2 x \, d\Omega_{d-1}^2\right) \qquad AdS_{d+1}$$

$$-\partial_t^2 \phi + \frac{1}{\tan^{d-1} x}\partial_x\left(\tan^{d-1}\partial_x \phi\right) + \frac{1}{\sin^2 x}\Delta_{S^{d-1}}\phi = \frac{\phi}{\cos^2 x}$$

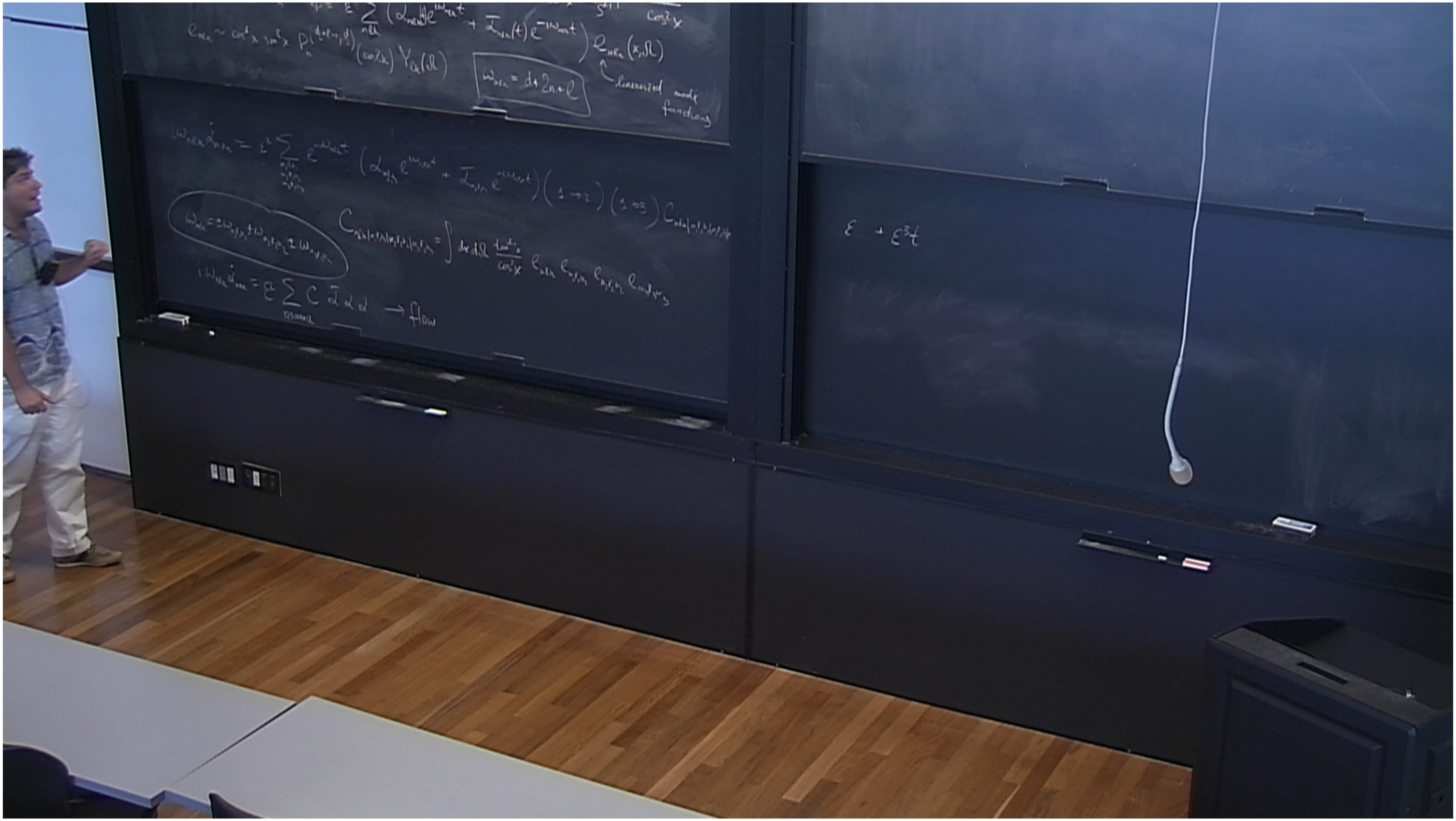
mass = 0 ϕ^4

$$\phi = \varepsilon \sum_{nlk}\left(\alpha_{nlk}(t) e^{i\omega_{nlk} t} + \bar{\alpha}_{nlk}(t) e^{-i\omega_{nlk} t}\right) e_{nlk}(x, \Omega)$$

linearized mode functions

$$e_{nlk} \sim \cos^d x \, \sin^l x \, P_n^{\left(\frac{d}{2}+l-1, \frac{d}{2}\right)}(\cos 2x) \, Y_{lk}(\Omega)$$

$$\boxed{\omega_{nlk} = d + 2n + l}$$



$$e_{n\ell k} \sim \cos^d x \, \sin^\ell x \, P_n^{(\ldots)}(\cos 2x)\, Y_{\ell k}(\Omega)$$

$$\boxed{\omega_{n\ell k} = d + 2n + \ell}$$

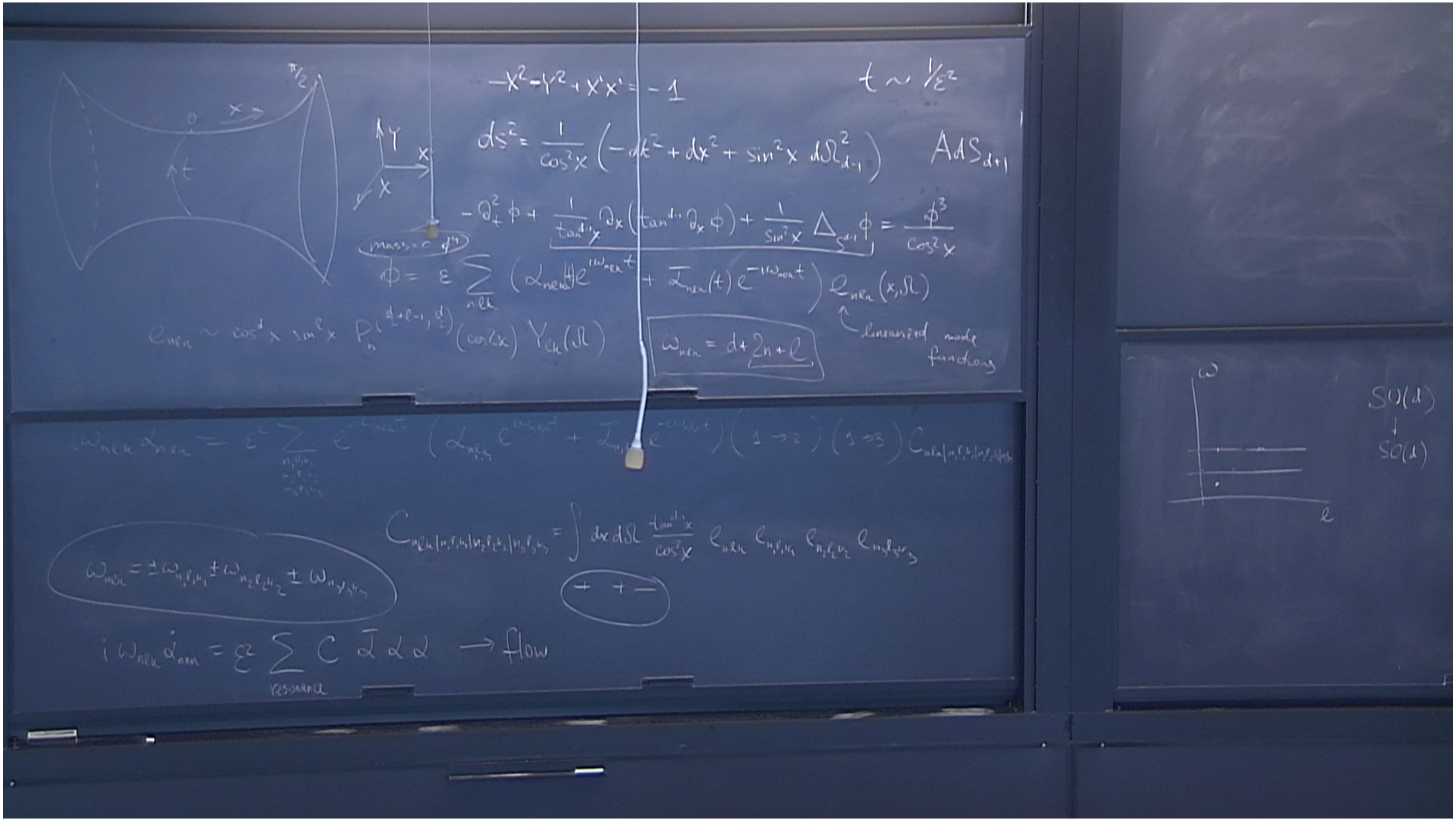
linearized mode functions

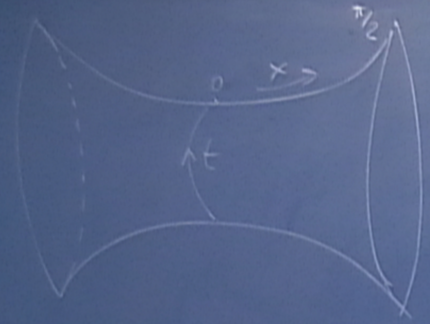
$$i\omega_{n\ell k}\dot{\alpha}_{n\ell k} = \varepsilon^2 \sum_{\substack{n_1\ell_1k_1\\ n_2\ell_2k_2\\ n_3\ell_3k_3}} e^{-i\omega_{n\ell k}t}\left(\alpha_{n_1\ell_1k_1} e^{i\omega_{n_1\ell_1k_1}t} + \bar{\alpha}_{n_1\ell_1k_1} e^{-i\omega_{n_1\ell_1k_1}t}\right)(1\to 2)(1\to 3)\, C_{n\ell k|n_1\ell_1k_1|n_2\ell_2k_2|n_3\ell_3k_3}$$

$$C_{n\ell k|n_1\ell_1k_1|n_2\ell_2k_2|n_3\ell_3k_3} = \int dx\, d\Omega\, \frac{\tan^{d-1}x}{\cos^2 x}\, e_{n\ell k}\, e_{n_1\ell_1k_1}\, e_{n_2\ell_2k_2}\, e_{n_3\ell_3k_3}$$

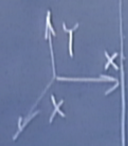
$$\omega_{n\ell k} = \pm\omega_{n_1\ell_1k_1} \pm \omega_{n_2\ell_2k_2} \pm \omega_{n_3\ell_3k_3}$$

$$i\omega_{n\ell k}\dot{\alpha}_{n\ell k} = \varepsilon^2 \sum_{\text{resonance}} C\,\bar{\alpha}\,\alpha\,\alpha \longrightarrow \text{flow}$$


$-X^2 - Y^2 + X^i X^i = -1$
$t \sim 1/\varepsilon^2$
$ds^2 = \frac{1}{\cos^2 x}\left(-dt^2 + dx^2 + \sin^2 x \, d\Omega^2_{d-1}\right)$
AdS_{d+1}
$-\partial_t^2 \phi + \frac{1}{\tan^{d-1} x}\partial_x\left(\tan^{d-1}\partial_x \phi\right) + \frac{1}{\sin^2 x}\Delta_{S^{d-1}}\phi = \frac{\phi^3}{\cos^2 x}$
mass = 0 ϕ^4
$\phi = \varepsilon \sum_{nlk}\left(\alpha_{nlk}(t) e^{i\omega_{nlk}t} + \bar{\alpha}_{nlk}(t) e^{-i\omega_{nlk}t}\right) e_{nlk}(x,\Omega)$
linearized mode functions
$e_{nlk} \sim \cos^d x \sin^l x \, P_n^{(\frac{d}{2}+l-1, \frac{d}{2})}(\cos 2x)\, Y_{lk}(\Omega)$
$\omega_{nlk} = d + 2n + l$
$C_{nlk|n_1l_1k_1|n_2l_2k_2|n_3l_3k_3} = \int dx\, d\Omega \frac{\tan^{d-1} x}{\cos^2 x} e_{nlk} e_{n_1l_1k_1} e_{n_2l_2k_2} e_{n_3l_3k_3}$
$\omega_{nlk} = \pm\omega_{n_1l_1k_1} \pm \omega_{n_2l_2k_2} \pm \omega_{n_3l_3k_3}$
$+ \; + \; -$
$i\omega_{nlk}\dot{\alpha}_{nlk} = \varepsilon^2 \sum_{\text{resonance}} C \bar{\alpha}\alpha\alpha \rightarrow$ flow
$SO(d)$
$SO(d)$
ω
l



$$-X^2 - Y^2 + X^i X^i = -1 \qquad t \sim 1/\varepsilon^2$$



$$ds^2 = \frac{1}{\cos^2 x}\left(-dt^2 + dx^2 + \sin^2 x \, d\Omega^2_{d-1}\right) \qquad AdS_{d+1}$$

$$-\partial_t^2 \phi + \underbrace{\frac{1}{\tan^{d-1} x}\partial_x\left(\tan^{d-1} x\, \partial_x \phi\right) + \frac{1}{\sin^2 x}\Delta_{S^{d-1}}}\phi = \frac{\phi^3}{\cos^2 x}$$

mass = 0 ϕ^4

$$\phi = \varepsilon \sum_{n\ell k}\left(\alpha_{n\ell k}(t)\, e^{i\omega_{n\ell} t} + \bar{\alpha}_{n\ell k}(t)\, e^{-i\omega_{n\ell} t}\right) e_{n\ell k}(x,\Omega)$$

linearized mode functions

$$e_{n\ell k} \sim \cos^d x \sin^\ell x \; P_n^{\left(\frac{d}{2}+\ell-1,\, \frac{d}{2}\right)}(\cos 2x)\, Y_{\ell k}(\Omega) \qquad \boxed{\omega_{n\ell} = d + \underline{2n+\ell}}$$

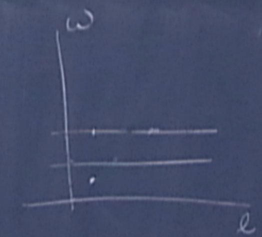
$$\omega_{n\ell k}\dot{\alpha}_{n\ell k} = \varepsilon^2 \sum_{\substack{n_1\ell_1 k_1 \\ n_2\ell_2 k_2 \\ n_3\ell_3 k_3}} e^{-i\omega_{n\ell} t}\left(\alpha_{n_1\ell_1 k_1} e^{i\omega_{n_1\ell_1} t} + \bar{\alpha}_{n_1\ell_1 k_1} e^{-i\omega_{n_1\ell_1} t}\right)(1 \to 2)(1 \to 3)\, C_{n\ell k|n_1\ell_1 k_1|n_2\ell_2 k_2|n_3\ell_3 k_3}$$

$$C_{n\ell k|n_1\ell_1 k_1|n_2\ell_2 k_2|n_3\ell_3 k_3} = \int dx\, d\Omega \, \frac{\tan^{d-1} x}{\cos^2 x}\, e_{n\ell k}\, e_{n_1\ell_1 k_1}\, e_{n_2\ell_2 k_2}\, e_{n_3\ell_3 k_3}$$

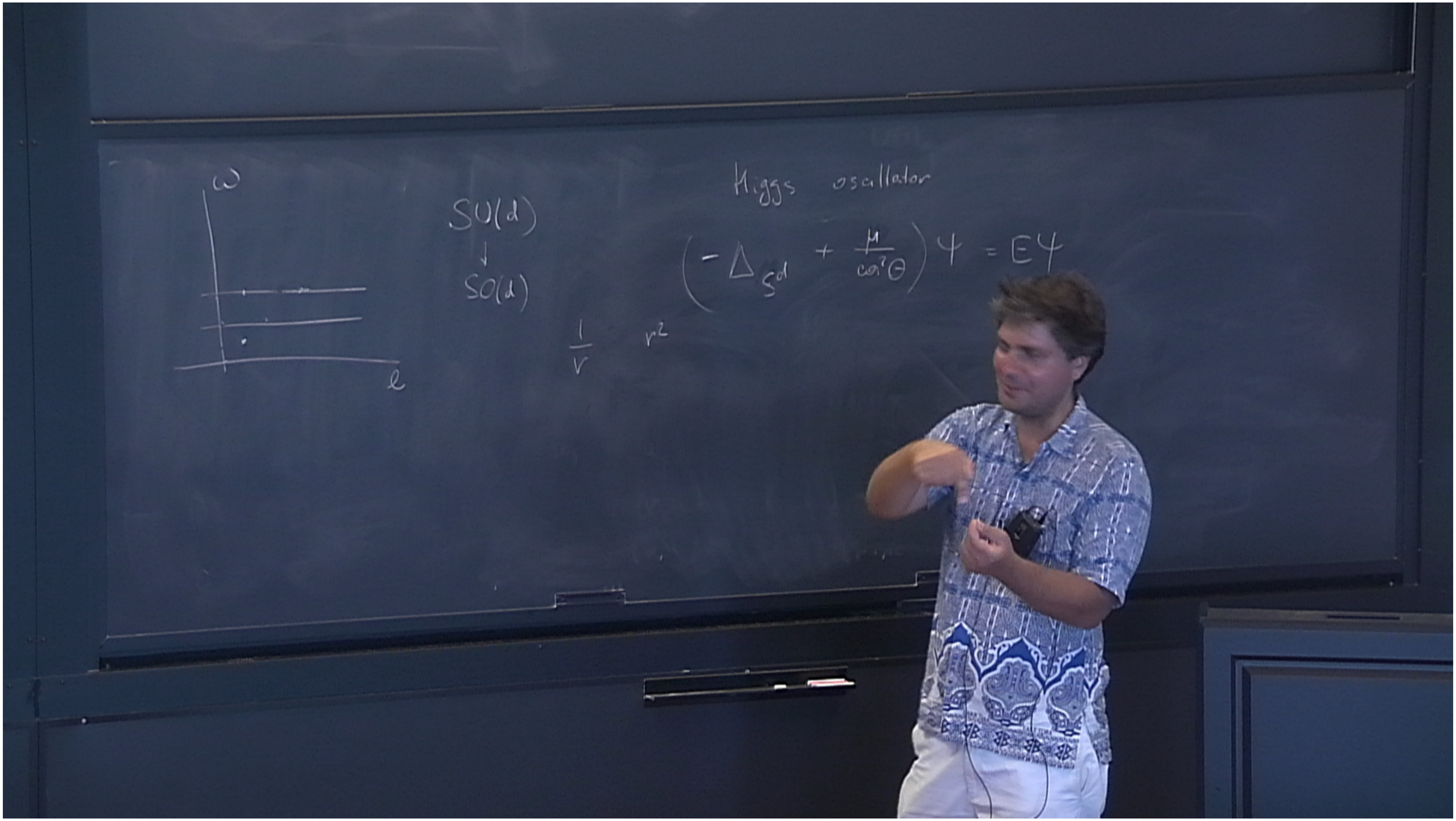
$+ \quad + \quad -$

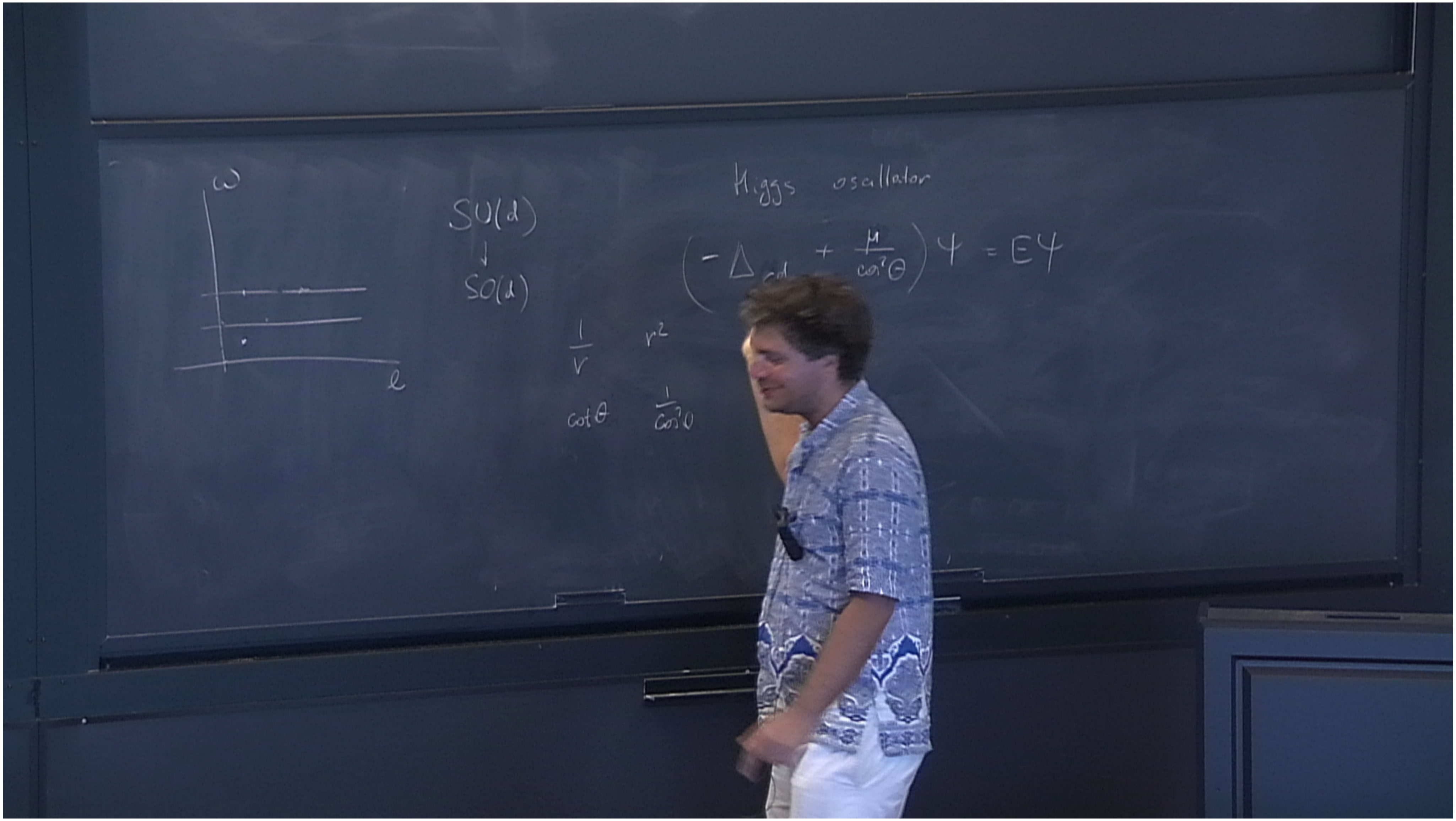
$$\omega_{n\ell} = \pm\omega_{n_1\ell_1} \pm \omega_{n_2\ell_2} \pm \omega_{n_3\ell_3}$$

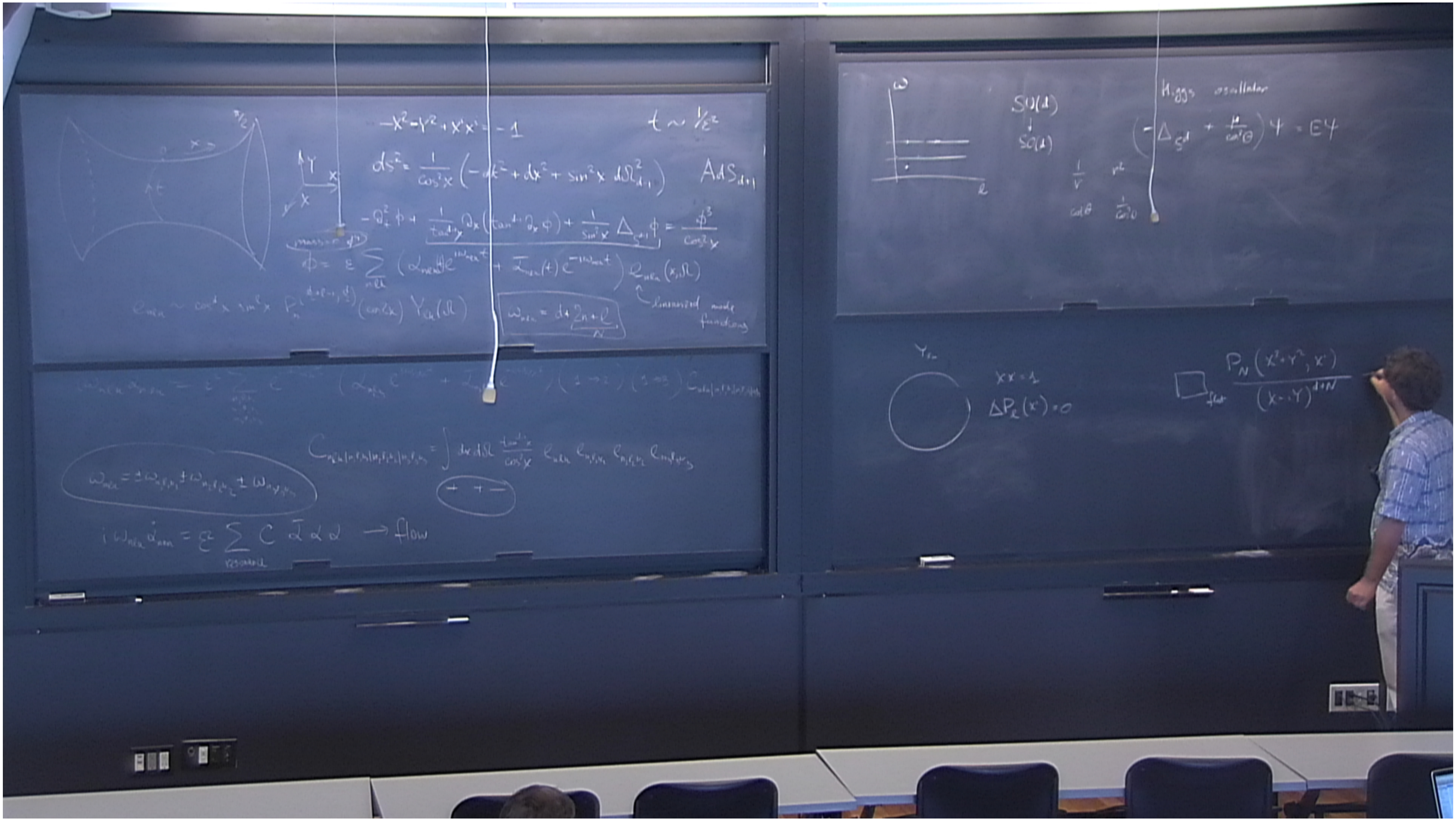
$$i\omega_{n\ell}\dot{\alpha}_{n\ell k} = \varepsilon^2 \sum_{\text{resonance}} C\, \bar{\alpha}\,\alpha\,\alpha \longrightarrow \text{flow}$$

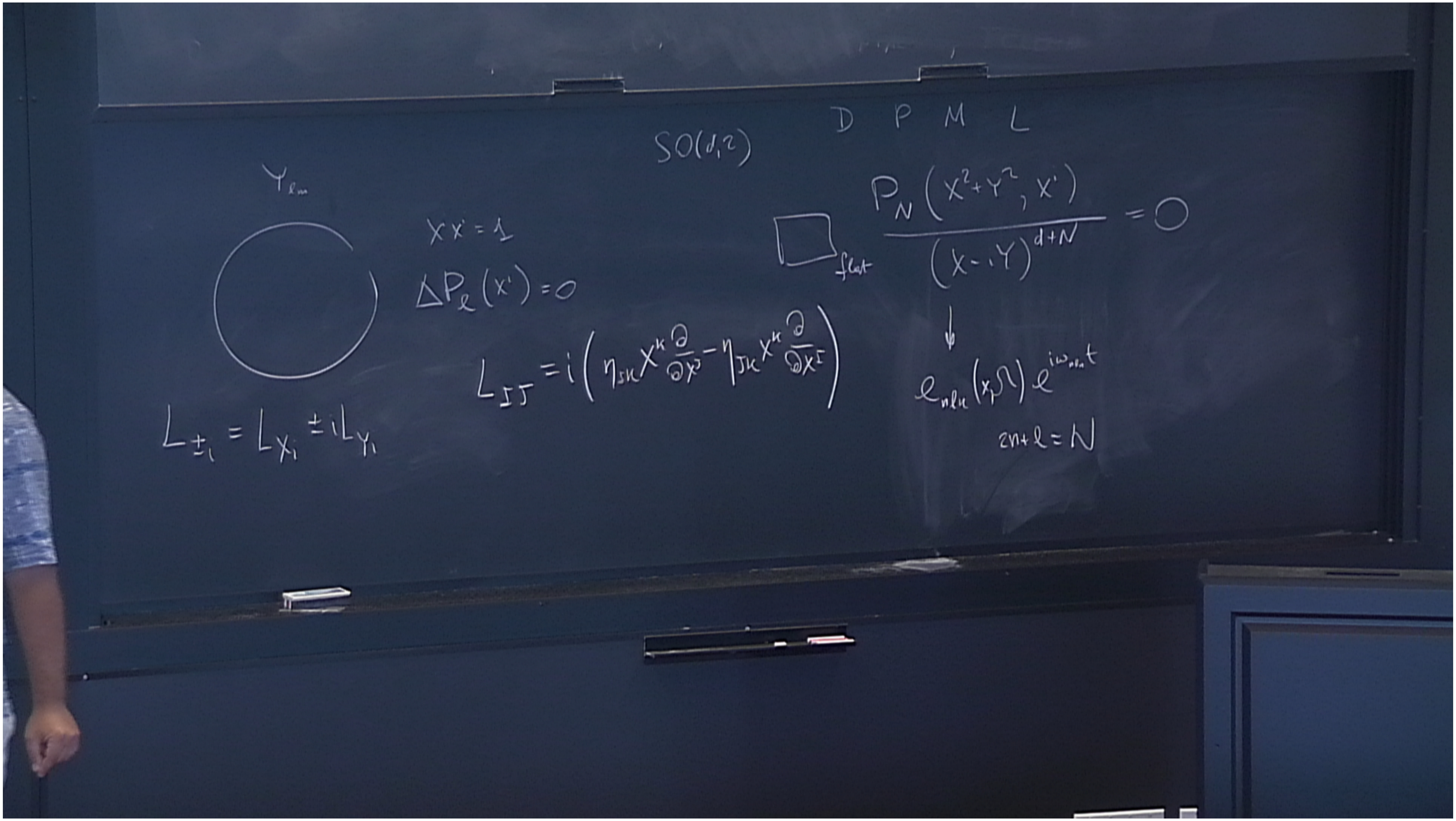


SU(d)
↓
SO(d)

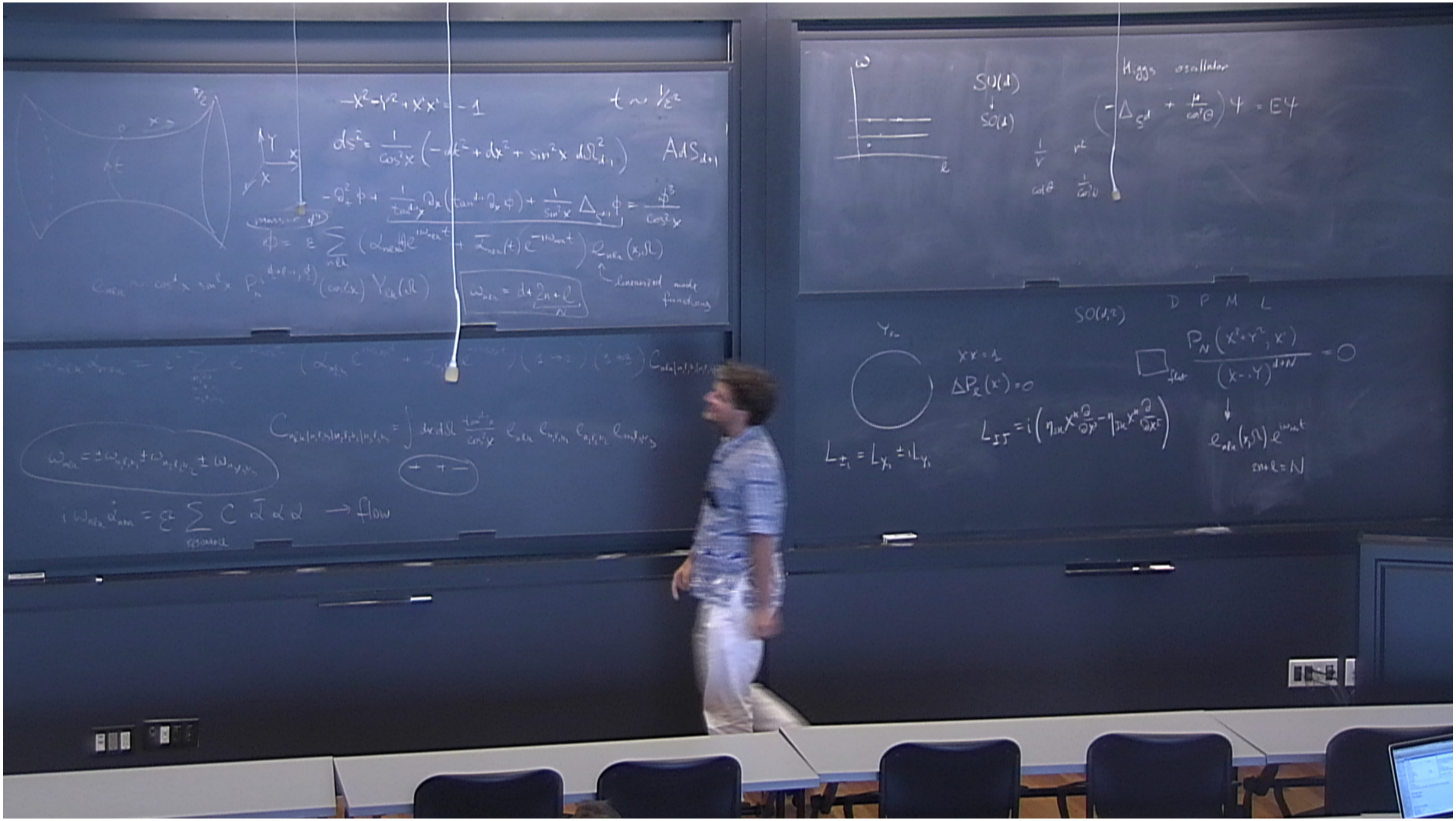

Higgs oscillator
SU(d)
SO(d)
ω
l
1/r
r²
(−Δ_{S^d} + μ/cos²θ) ψ = Eψ

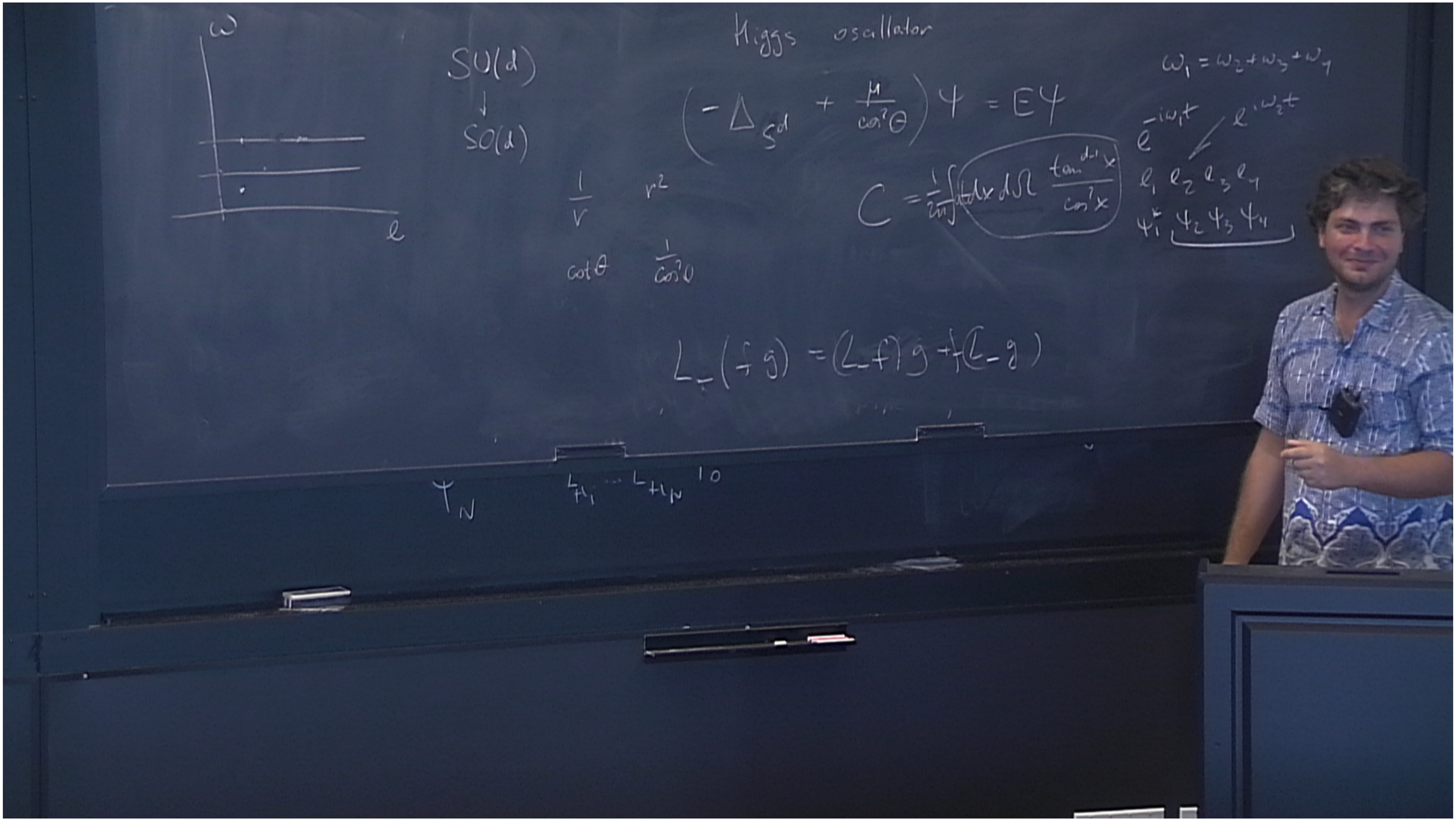

Higgs oscillator
SU(d)
SO(d)
ω
l
1/r
r²
cot θ
1/cos²θ
= EΨ

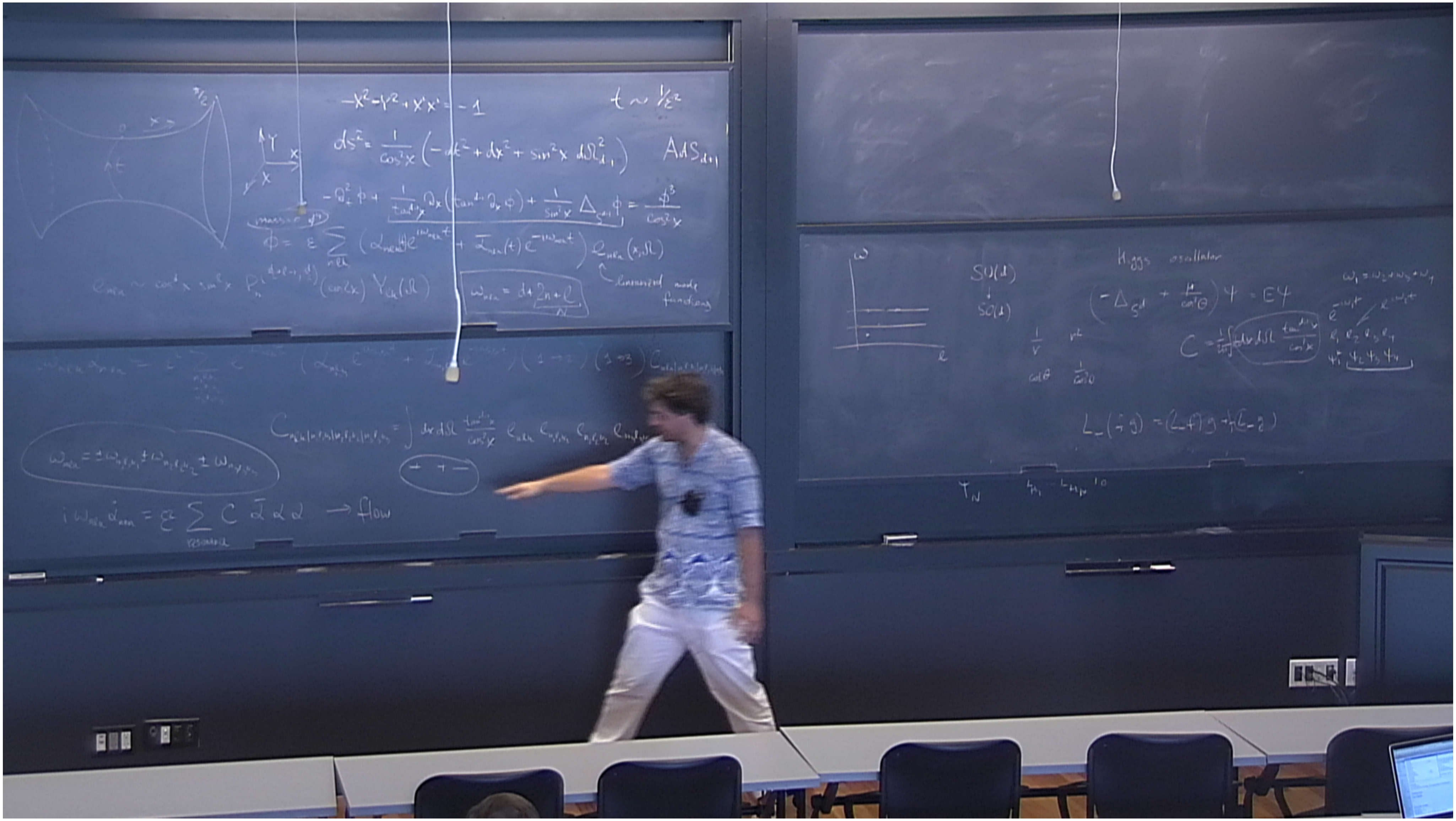

-X²-Y²+X'X' = -1
AdS_{d+1}
linearized mode functions
flow
resonant
Higgs oscillator
xx = 1
ΔP_ℓ(x) = 0



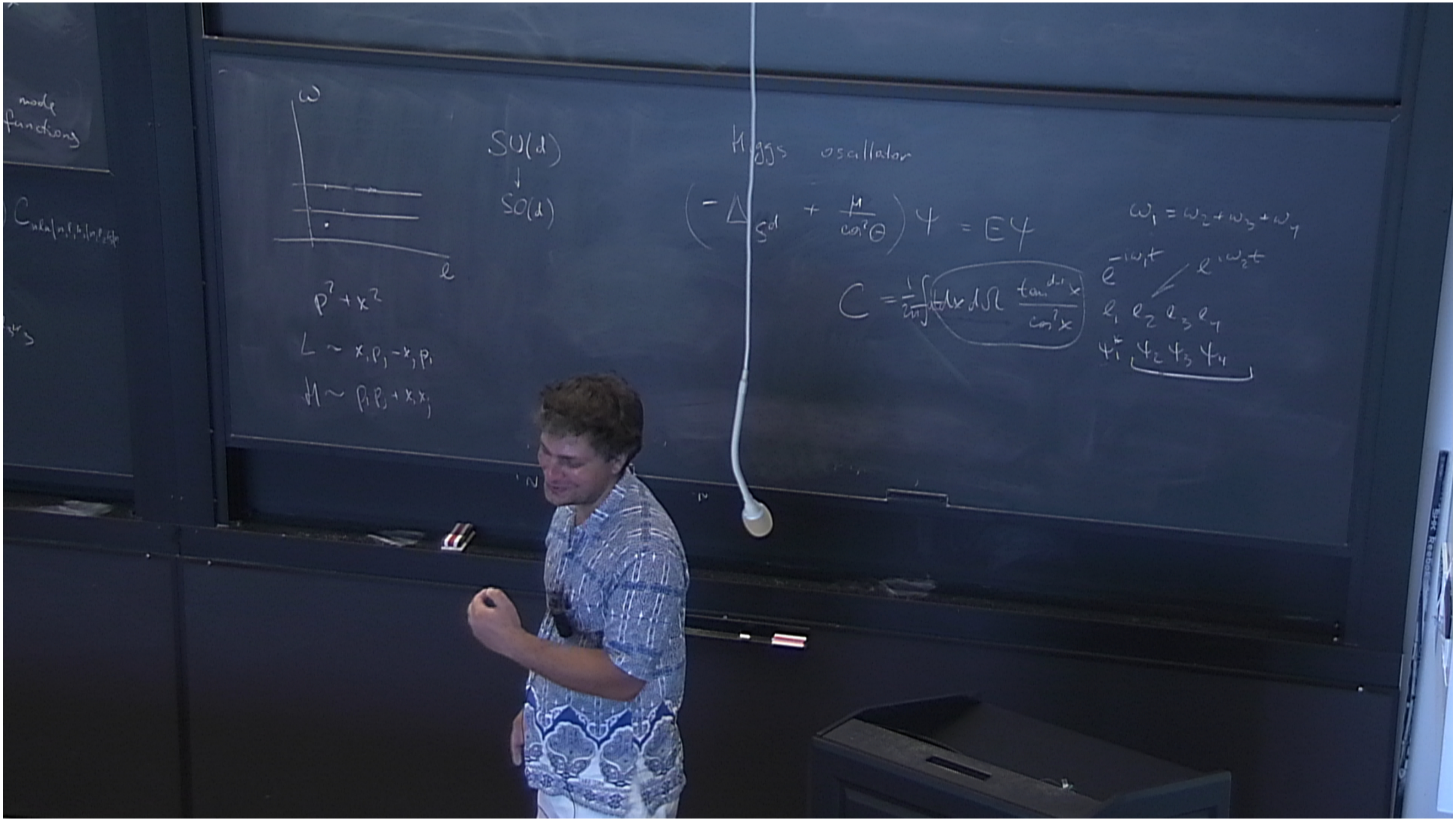
D P M L
SO(d,2)
Y_lm
XX' = 1
ΔP_l(X') = 0
P_N(X^2+Y^2, X') / (X-iY)^(d+N) = 0
flat
L_IJ = i(η_JK X^K ∂/∂X^J − η_JK X^K ∂/∂X^I)
e_nlm(x,Ω) e^(iω_nlm t)
2n+l = N
L_±i = L_Xi ± i L_Yi

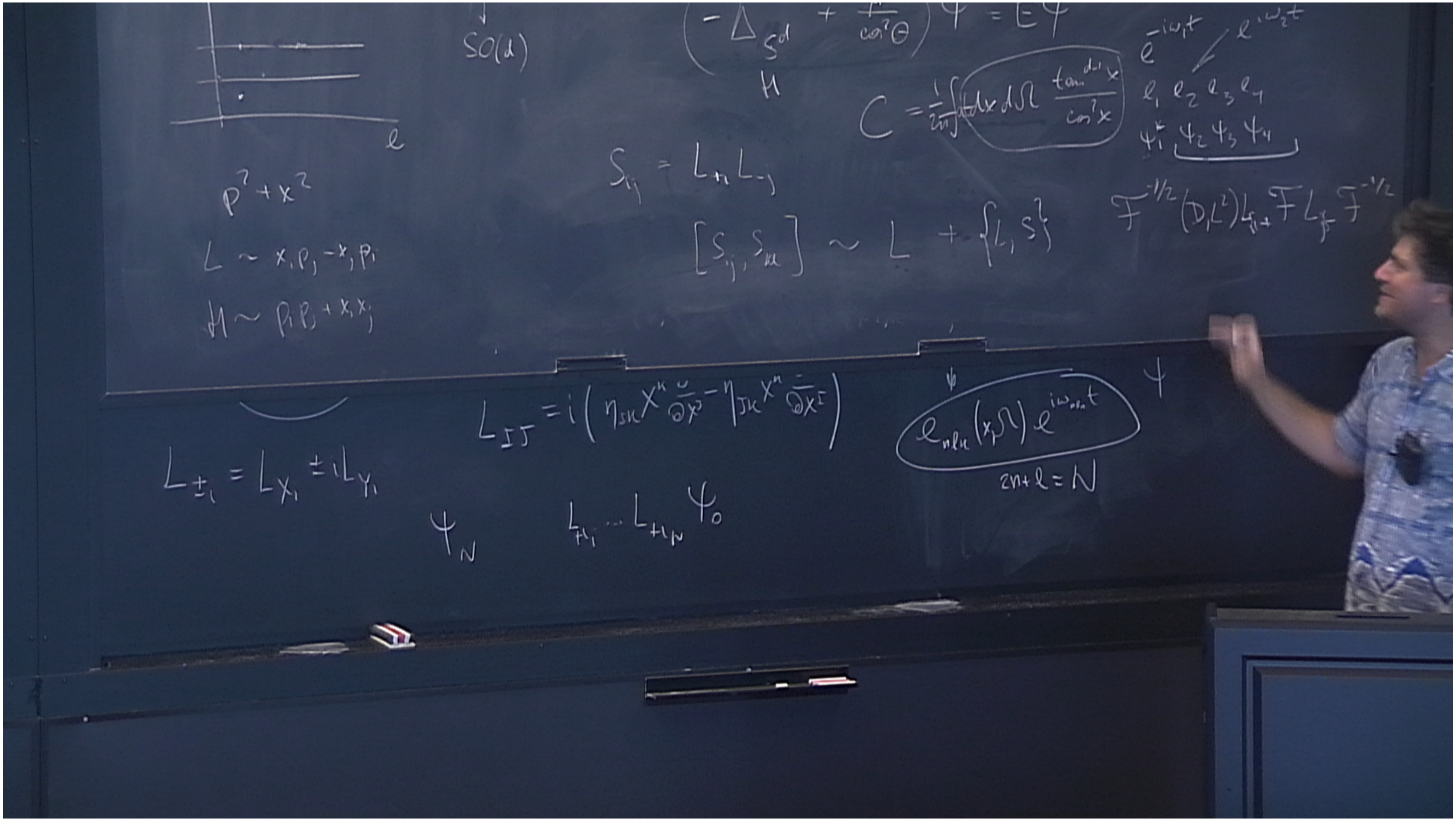



Higgs oscillator
SU(d)
SO(d)
ω
l
1/r
r^2
cot θ
1/cos^2 θ
(−Δ_{S^d} + μ/cos^2 θ) ψ = E ψ
C = 1/(2π) ∫ dt dx dΩ tan^{d−1} x / cos^2 x
ω_1 = ω_2 + ω_3 + ω_4
e^{−iω_1 t}
e^{iω_2 t}
e_1 e_2 e_3 e_4
ψ_1^* ψ_2 ψ_3 ψ_4
L_−(f g) = (L_− f) g + f (L_− g)
ψ_N
L_{+l_1} ... L_{+l_N} |0



AdS_{d+1}
linearized mode functions
flow
resonance
Higgs oscillator


mode functions
SU(d)
SO(d)
Higgs oscillator
$p^2 + x^2$
$L \sim x_i p_j - x_j p_i$
$H \sim p_i p_j + x_i x_j$
$\left(-\Delta_{S^d} + \frac{\mu}{\cos^2\theta}\right)\psi = E\psi$
$\omega_1 = \omega_2 + \omega_3 + \omega_4$
$e^{-i\omega_1 t}$
$e^{i\omega_2 t}$
$l_1\ l_2\ l_3\ l_4$
$\psi_1^*\ \psi_2\ \psi_3\ \psi_4$


SO(d)
$p^2 + x^2$
$L \sim x_i p_j - x_j p_i$
$H \sim p_i p_j + x_i x_j$
$S_{ij} = L_{+i} L_{-j}$
$[S_{ij}, S_{kl}] \sim L + \{L, S\}$
$L_{IJ} = i\left(\eta_{JK} X^K \frac{\partial}{\partial X^I} - \eta_{IK} X^K \frac{\partial}{\partial X^J}\right)$
$L_{\pm 1} = L_{x_1} \pm i L_{y_1}$
$\Psi_N \quad L_{+1_1} \cdots L_{+1_N} \Psi_0$
$e_{nl_n}(x, \Omega)\, e^{i\omega_{nl} t}$
$2n + l = N$