

Title: Reduction of the classification of topological insulators and superconductors by quartic interactions

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Abstract: <p>I will discuss the stability and breakdown of the topological classification of gapped ground states of non-interacting fermions, the tenfold way, in the presence of quartic fermion-fermion interactions. In our approach [1], the effects of interactions on the boundary gapless modes are encoded in terms of boundary dynamical masses. Breakdown of the non-interacting topological classification occurs when the quantum nonlinear sigma models for the boundary dynamical masses favor quantum disordered phases. The non-interacting topological classification \$Z\$ in odd spatial dimensions is unstable and reduces to \$Z_N\$ that can be identified explicitly for any dimension and any defining symmetries.

[1] T. Morimoto, A. Furusaki, and C. Mudry, Phys. Rev. B 91, 235111 (2015).</p>

Reduction of the classification of topological insulators and superconductors by quartic interactions

Akira Furusaki



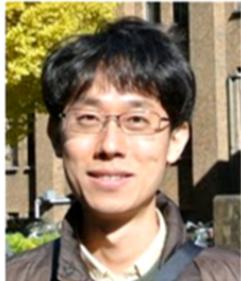
Takahiro Morimoto (UC Berkeley)
Christopher Mudry (Paul Scherrer Inst.)

2016/5/31

@Perimeter Institute

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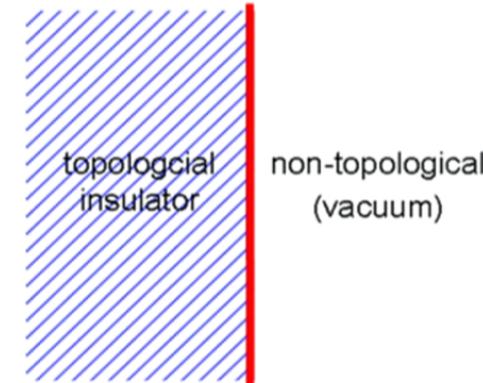
Plan of this talk

- Introduction
 - Classification of topological insulators and superconductors
(gapped phases of free fermions)
- Interactions
 - Stability analysis of boundary Dirac (Majorana) fermions
 - Dynamical Dirac masses
- Examples
 - 1D BDI
 - 3D DIII
 - Higher dimensions
 - 2D DIII + reflection Morimoto, AF, Mudry, Phys. Rev. B 92, 125104 (2015)

Topological insulators

in the broader sense

- band insulators free fermions (ignore e-e int.)
- characterized by topological numbers (Z or Z_2)
- gapless excitations at boundaries
stable



Examples: integer quantum Hall effect,

time reversal symmetry \rightarrow quantum spin Hall insulator, 3D Z_2 topological insulator, ...

2D

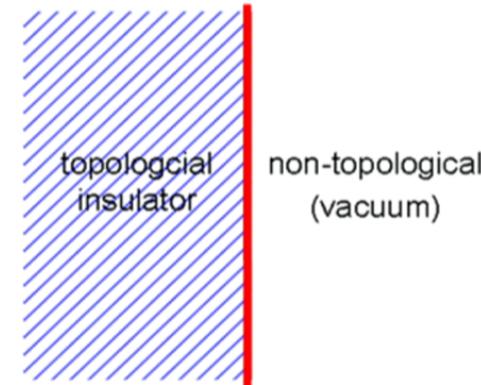
3D

Topological superconductors

- BCS superconductors with a fully gapped Fermi surface
- characterized by topological numbers
- gapless excitations at boundaries (Dirac or Majorana)
stable

Examples: p+ip superconductor, ${}^3\text{He}$, ...

particle-hole symmetry (BdG Hamiltonian)



Classification of free-fermion Hamiltonian in terms of generic discrete symmetries

- Time-reversal symmetry (TRS)

$$THT^{-1} = H$$

T : anti-unitary operator

$$\text{TRS} = \begin{cases} 0 & \text{no TR invariance} \\ +1 & T^2 = +1 \\ -1 & T^2 = -1 \end{cases}$$

spin 0

spin 1/2

- Particle-hole symmetry (PHS)

BdG Hamiltonian

$$CHC^{-1} = -H$$

C : anti-unitary operator

$$\text{PHS} = \begin{cases} 0 & \text{no PH invariance} \\ +1 & C^2 = +1 \\ -1 & C^2 = -1 \end{cases}$$

Singlet SC

- Chiral symmetry (CS)

$$\Gamma H \Gamma^{-1} = -H \quad \Gamma : \text{unitary operator} \quad (\Gamma = TC)$$

Table of topological insulators/superconductors for d=1,2,3

	10 Symmetry Classes	TRS	PHS	CS	d=1	d=2	d=3
Standard (Wigner-Dyson)	A (unitary)	0	0	0	--	\mathbb{Z}	--
	AI (orthogonal)	+1	0	0	--	--	--
	AII (symplectic)	-1	0	0	--	\mathbb{Z}_2	\mathbb{Z}_2
Chiral	AIII (chiral unitary)	0	0	1	\mathbb{Z}	--	\mathbb{Z}
	BDI (chiral orthogonal)	+1	+1	1	\mathbb{Z}	--	--
	CII (chiral symplectic)	-1	-1	1	\mathbb{Z}	--	\mathbb{Z}_2
BdG	D (p-wave SC)	0	+1	0	\mathbb{Z}_2	\mathbb{Z}	--
	C (d-wave SC)	0	-1	0	--	\mathbb{Z}	--
	DIII (p-wave TRS SC)	-1	+1	1	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}
	CI (d-wave TRS SC)	+1	-1	1	--	--	\mathbb{Z}

Altland & Zirnbauer, PRB (1997)

Schnyder, Ryu, AF, and Ludwig, PRB (2008)

Table of topological insulators/superconductors for d=1,2,3

	10 Symmetry Classes	TRS	PHS	CS	d=1	d=2	d=3
Standard (Wigner-Dyson)	A (unitary)	0	0	0	--	\mathbb{Z}	IQHE
	AI (orthogonal)	+1	0	0	--		QSHE
	AII (symplectic)	-1	0	0	--	\mathbb{Z}_2	\mathbb{Z}_2 TPI
Chiral	AIII (chiral unitary)	0	0	1	\mathbb{Z}	--	\mathbb{Z}
	BDI (chiral orthogonal)	+1	+1	1	\mathbb{Z}		polyacetylene (SSH)
	CII (chiral symplectic)	-1	-1	1	\mathbb{Z}	--	\mathbb{Z}_2
BdG	D (p-wave SC)	0	+1	0	p SC	\mathbb{Z}_2	\mathbb{Z} p+ip SC
	C (d-wave SC)	0	-1	0	--	\mathbb{Z}	d+id SC
	DIII (p-wave TRS SC)	-1	+1	1	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z} $(p+ip) \times (p-ip)$ SC
	CI (d-wave TRS SC)	+1	-1	1	--	--	\mathbb{Z} ${}^3\text{He-B}$

Altland & Zirnbauer, PRB (1997)

Schnyder, Ryu, AF, and Ludwig, PRB (2008)

Periodic table of topological insulators/superconductors

Class \ d	0	1	2	3	4	5	6	7	8	...
<i>complex case:</i>										
A	\mathbb{Z}	0	\mathbb{Z}	0	\mathbb{Z}	0	\mathbb{Z}	0	\mathbb{Z}	...
AIII	0	\mathbb{Z}	0	\mathbb{Z}	0	\mathbb{Z}	0	\mathbb{Z}	0	...
<i>real case:</i>										
AI	\mathbb{Z}	0	0	0	\mathbb{Z}	0	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}	...
BDI	\mathbb{Z}_2	\mathbb{Z}	0	0	0	\mathbb{Z}	0	\mathbb{Z}_2	\mathbb{Z}_2	...
D	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}	0	0	0	\mathbb{Z}	0	\mathbb{Z}_2	...
DIII	0	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}	0	0	0	\mathbb{Z}	0	...
AII	\mathbb{Z}	0	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}	0	0	0	\mathbb{Z}	...
CII	0	\mathbb{Z}	0	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}	0	0	0	...
C	0	0	\mathbb{Z}	0	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}	0	0	...
CI	0	0	0	\mathbb{Z}	0	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}	0	...

A. Kitaev, AIP Conf. Proc. 1134, 22 (2009); arXiv:0901.2686 K-theory, Bott periodicity

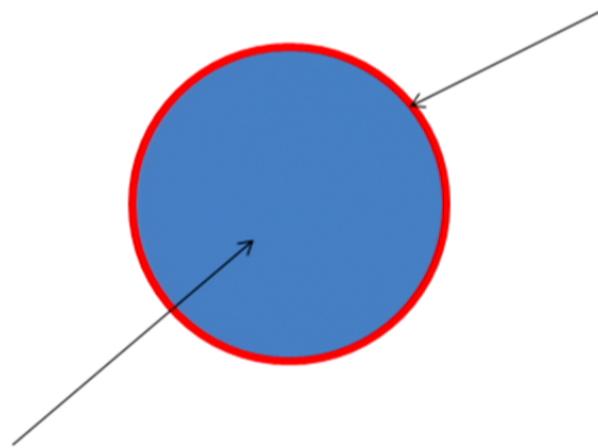
Ryu, Schnyder, AF, Ludwig, NJP 12, 065010 (2010) massive Dirac Hamiltonian

How to achieve classification of TIs and TSCs

- Explicit construction of topological invariants
- Anderson delocalization of boundary fermions
- Classification of Dirac masses in the bulk
- ...

bulk-boundary correspondence

topologically stable, gapless excitations



Topological insulator/superconductor
fully gapped (no excitations)

Anderson delocalization of boundary states

- Gapless boundary modes are topologically protected.
- They are stable against any local perturbation.
(respecting symmetries)
- They should **never** be Anderson localized by **disorder**.

Nonlinear sigma models for Anderson localization

of gapless boundary modes

$$S = \int d^{d-1}r \operatorname{tr} (\partial Q)^2 + \text{topological term} \text{ (with no adjustable parameter)}$$

$$Q \in M$$

$$Z_2 \text{ top. term} \quad \pi_{\underline{d-1}}(M) = Z_2$$

bulk: d dimensions
boundary: $d-1$ dimensions

$$\text{WZW term} \quad \pi_{\underline{d}}(M) = Z$$

NLSM topological terms

$$\pi_d(G/H)$$

complex case:

	$G/H \setminus d$	$d = 0$	$d = 1$	$d = 2$	$d = 3$
A	$U(N+M)/U(N) \times U(M)$	\mathbb{Z}	0	\mathbb{Z}	0
AIII	$U(N)$	0	\mathbb{Z}	0	\mathbb{Z}

real case:

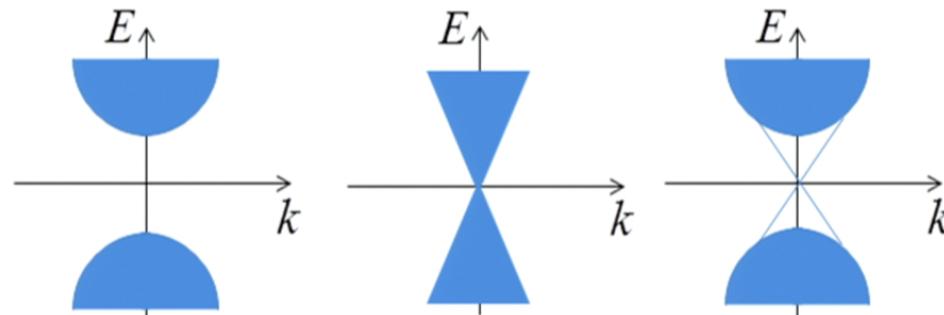
	$G/H \setminus d$	$d = 0$	$d = 1$	$d = 2$	$d = 3$
AI	$Sp(N+M)/Sp(N) \times Sp(M)$	\mathbb{Z}	0	0	0
BDI	$U(2N)/Sp(N)$	0	\mathbb{Z}	0	0
D	$O(2N)/U(N)$	\mathbb{Z}_2	0	\mathbb{Z}	0
DIII	$O(N)$	\mathbb{Z}_2	\mathbb{Z}_2	0	\mathbb{Z}
All	$O(N+M)/O(N) \times O(M)$	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_2	0
CII	$U(N)/O(N)$	0	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_2
C	$Sp(N)/U(N)$	0	0	\mathbb{Z}	\mathbb{Z}_2
CI	$Sp(N)$	0	0	0	\mathbb{Z}

a \mathbb{Z}_2 topological term can appear if $\pi_{d-1}(M) = \mathbb{Z}_2$

a WZW term can appear if $\pi_d = \mathbb{Z}$

Classification of TIs and TSCs using Clifford algebras

A. Kitaev (2009); T. Morimoto and AF, Phys. Rev. B 88, 125129 (2013)



Dirac Hamiltonian

$$H = \sum_{\mu=1}^d k_\mu \gamma_\mu + m \gamma_0$$

gamma matrices

$$\{\gamma_a, \gamma_b\} = 2\delta_{a,b}$$

Minimal representative models for TIs and TSCs

effective theory for topological phase transitions (closing of a band gap)

classification of TIs and TSCs \longleftrightarrow classification of Dirac mass $m\gamma_0$

Set of possible mass terms: classifying space

Example: $d = 2$ class A (IQHE)

$$H = k_x \underbrace{\sigma_x \otimes 1_N}_{\gamma_1} + k_y \underbrace{\sigma_y \otimes 1_N}_{\gamma_2} + \gamma_0 \quad \{ \gamma_a, \gamma_b \} = 2\delta_{a,b}$$

$$\gamma_0 = \sigma_z \otimes A \quad A = U \begin{pmatrix} 1_n & 0 \\ 0 & -1_m \end{pmatrix} U^\dagger \quad (N = n+m)$$

$$\gamma_0 \iff U \in U(n+m)/[U(n) \times U(m)]$$

Classifying space \mathcal{C}_0

$$\pi_0 \left[\bigoplus_n U(N)/U(N-n) \times U(n) \right] = \mathbb{Z} \quad \begin{matrix} n=3 & n=4 & n=5 & n=6 \\ \cdots & \bullet & \bullet & \bullet & \cdots \end{matrix}$$

There are topologically distinct gapped phases labelled by an integer index.

The parameter n corresponds to Chern number.

$$H = k_x \sigma_x + k_y \sigma_y + (\varepsilon - k^2) \sigma_z \quad \text{Chern \#} = \begin{cases} 1 & (\varepsilon > 0) \\ 0 & (\varepsilon < 0) \end{cases}$$

Example: $d = 1$ class A (no symmetry constraint)

$$H = k_x \sigma_z \otimes 1_N + \gamma_0$$

$$\gamma_0 = \begin{pmatrix} 0 & U \\ U^\dagger & 0 \end{pmatrix} \quad U \in U(N) \quad \text{Classifying space } C_1$$

$$\pi_0(U(N)) = 0$$

There is only a single gapped phase.



Example: $d = 3$ class AII (time-reversal symmetry $T = i\sigma_y K$)

$$H = (k_x \sigma_x + k_y \sigma_y + k_z \sigma_z) \tau_z \otimes 1_N + \gamma_0$$

$$\gamma_0 = \sigma_0 \otimes (\tau_x \otimes S + i\tau_y \otimes A) = \sigma_0 \otimes \begin{pmatrix} 0 & S+A \\ S-A & 0 \end{pmatrix} = \sigma_0 \otimes \begin{pmatrix} 0 & O \\ O^T & 0 \end{pmatrix}$$

$$S^T = S, \quad A^T = -A$$

$$\gamma_0^2 = 1_{4N} \quad \xrightarrow{\hspace{1cm}} \quad O \in O(N) \quad \text{Classifying space } R_1$$

$$\pi_0(O(N)) = \mathbb{Z}_2$$

There are two gapped phases.



Sets of symmetry-allowed Dirac masses (classifying spaces V)

Class	T	C	Γ	Extension	V_d	$\pi_0(V_{d=0})$	$\pi_0(V_{d=1})$	$\pi_0(V_{d=2})$	$\pi_0(V_{d=3})$
A	0	0	0	$Cl_d \rightarrow Cl_{d+1}$	C_{0+d}	\mathbb{Z}	0	\mathbb{Z}	0
AIII	0	0	1	$Cl_{d+1} \rightarrow Cl_{d+2}$	C_{1+d}	0	\mathbb{Z}	0	\mathbb{Z}
AI	+1	0	0	$Cl_{0,d+2} \rightarrow Cl_{1,d+2}$	R_{0-d}	\mathbb{Z}	0	0	0
BDI	+1	+1	1	$Cl_{d+1,2} \rightarrow Cl_{d+1,3}$	R_{1-d}	\mathbb{Z}_2	\mathbb{Z}	0	0
D	0	+1	0	$Cl_{d,2} \rightarrow Cl_{d,3}$	R_{2-d}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}	0
DIII	-1	+1	1	$Cl_{d,3} \rightarrow Cl_{d,4}$	R_{3-d}	0	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}
AII	-1	0	0	$Cl_{2,d} \rightarrow Cl_{3,d}$	R_{4-d}	\mathbb{Z}	0	\mathbb{Z}_2	\mathbb{Z}_2
CII	-1	-1	1	$Cl_{d+3,0} \rightarrow Cl_{d+3,1}$	R_{5-d}	0	\mathbb{Z}	0	\mathbb{Z}_2
C	0	-1	0	$Cl_{d+2,0} \rightarrow Cl_{d+2,1}$	R_{6-d}	0	0	\mathbb{Z}	0
CI	+1	-1	1	$Cl_{d+2,1} \rightarrow Cl_{d+2,2}$	R_{7-d}	0	0	0	\mathbb{Z}

Label	Classifying space V
C_0	$\cup_{n=0}^N \{\mathrm{U}(N)/[\mathrm{U}(n) \times \mathrm{U}(N-n)]\}$
C_1	$\mathrm{U}(N)$
R_0	$\cup_{n=0}^N \{\mathrm{O}(N)/[\mathrm{O}(n) \times \mathrm{O}(N-n)]\}$
R_1	$\mathrm{O}(N)$
R_2	$\mathrm{O}(2N)/\mathrm{U}(N)$
R_3	$\mathrm{U}(2N)/\mathrm{Sp}(N)$
R_4	$\cup_{n=0}^N \{\mathrm{Sp}(N)/[\mathrm{Sp}(n) \times \mathrm{Sp}(N-n)]\}$
R_5	$\mathrm{Sp}(N)$
R_6	$\mathrm{Sp}(N)/\mathrm{U}(N)$
R_7	$\mathrm{U}(N)/\mathrm{O}(N)$

$$C_{q+2} = C_q \quad R_{q+8} = R_q$$

$\pi_0(V)$ counts the # of path-connected parts in the set V

$\pi_0(V) = 0$ trivial insulators

$\pi_0(V) = \mathbb{Z}, \mathbb{Z}_2$ topologically nontrivial insulators

$$\pi_n(V)$$

Label	Classifying space V	$\pi_0(V)$	$\pi_1(V)$	$\pi_2(V)$	$\pi_3(V)$	$\pi_4(V)$
C_0	$\cup_{n=0}^N \{\mathrm{U}(N)/[\mathrm{U}(n) \times \mathrm{U}(N-n)]\}$	\mathbb{Z}	0	\mathbb{Z}	0	\mathbb{Z}
C_1	$\mathrm{U}(N)$	0	\mathbb{Z}	0	\mathbb{Z}	0
R_0	$\cup_{n=0}^N \{\mathrm{O}(N)/[\mathrm{O}(n) \times \mathrm{O}(N-n)]\}$	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_2	0	\mathbb{Z}
R_1	$\mathrm{O}(N)$	\mathbb{Z}_2	\mathbb{Z}_2	0	\mathbb{Z}	0
R_2	$\mathrm{O}(2N)/\mathrm{U}(N)$	\mathbb{Z}_2	0	\mathbb{Z}	0	0
R_3	$\mathrm{U}(2N)/\mathrm{Sp}(N)$	0	\mathbb{Z}	0	0	0
R_4	$\cup_{n=0}^N \{\mathrm{Sp}(N)/[\mathrm{Sp}(n) \times \mathrm{Sp}(N-n)]\}$	\mathbb{Z}	0	0	0	\mathbb{Z}
R_5	$\mathrm{Sp}(N)$	0	0	0	\mathbb{Z}	\mathbb{Z}_2
R_6	$\mathrm{Sp}(N)/\mathrm{U}(N)$	0	0	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_2
R_7	$\mathrm{U}(N)/\mathrm{O}(N)$	0	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_2	0

Interacting fermions

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Classification of (fermionic) SPT phases

Symmetry Protected Topological (SPT) Phases \ni TIs & TSCs

= topological phases of (interacting) particles
with short-range entanglement
(no topological order / no fractionalization)

- group (super-)cohomology X. Chen, Z.-C. Gu, Z.-X. Liu, X.-G. Wen
- (1+1)-dimension: matrix product states $H^2(G, U(1))$
Fidkowski-Kitaev, Pollmann et al., ...
- (2+1)-dimension: Chern-Simons approach
 $Ka\partial a \rightarrow (\partial\varphi)^2 + \lambda \cos l\varphi$ Lu & Vishwanath,
- Braiding statistics Levin & Gu, Wang & Levin, ..
- Cobordism Kapustin, ...
- Anomaly Ryu, Witten, ...
- Breakdown of free-fermion classification Z of TIs/TSCs

Breakdown of non-interacting topological phases labeled by \mathbb{Z} with interactions

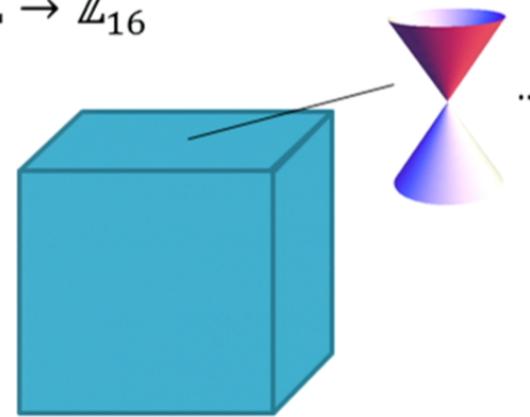
Time-reversal symmetric
Majorana chain (1D class BDI)
 $\mathbb{Z} \rightarrow \mathbb{Z}_8$



8 Majorana zero modes at the boundary can be gapped without breaking TRS.

Fidkowski and Kitaev, PRB (2010), PRB (2011)

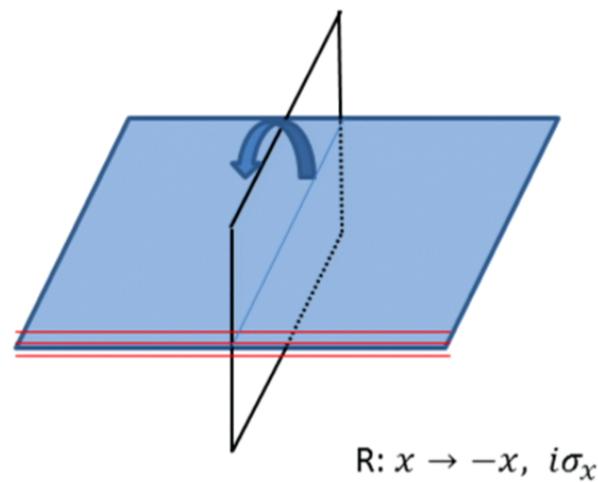
Time-reversal symmetric 3D topological SC (3D class DIII)
 $\mathbb{Z} \rightarrow \mathbb{Z}_{16}$



$\nu \neq 16n$ flavors of Dirac surface fermions lead to nontrivial topological order with TRS.

Kitaev (2011), Fidkowski et al. PRX (2013), Metlitski, Kane & Fisher. (2014), ...

Time-reversal and reflection
symmetric 2D superconductors
(2D class DIII+R) $\mathbb{Z} \rightarrow \mathbb{Z}_8$



$$R: x \rightarrow -x, i\sigma_x$$

8 pairs of Majorana helical modes
Can be gapped out by interactions
Yao and Ryu, PRB (2013); Qi, NJP (2013).

Aim:

Systematic study of the breakdown of Z classification
for any spatial dimension and all symmetry classes

Can boundary states be gapped out without breaking symmetries?

Analyze the stability of boundary gapless states against interactions
using the topology of the space of dynamical boundary Dirac masses

Kitaev's talk @ UCLA (2015)

We only consider the contact interactions obtained from taking squares of
the bilinears built from Dirac mass matrices $(\psi^\dagger \beta \psi)^2$

Result:

Class	T	C	Γ_5	V_d	$d = 1$	$d = 2$	$d = 3$	$d = 4$	$d = 5$	$d = 6$	$d = 7$	$d = 8$
A	0	0	0	C_{0+d}	0	\mathbb{Z}	0	\mathbb{Z}	0	\mathbb{Z}	0	\mathbb{Z}
AIII	0	0	1	C_{1+d}	\mathbb{Z}_4	0	\mathbb{Z}_8	0	\mathbb{Z}_{16}	0	\mathbb{Z}_{32}	0
AI	+1	0	0	R_{0-d}	0	0	0	\mathbb{Z}	0	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}
BDI	+1	+1	1	R_{1-d}	\mathbb{Z}_8	0	0	0	\mathbb{Z}_{16}	0	\mathbb{Z}_2	\mathbb{Z}_2
D	0	+1	0	R_{2-d}	\mathbb{Z}_2	\mathbb{Z}	0	0	0	\mathbb{Z}	0	\mathbb{Z}_2
DIII	-1	+1	1	R_{3-d}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_{16}	0	0	0	\mathbb{Z}_{32}	0
AII	-1	0	0	R_{4-d}	0	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}	0	0	0	\mathbb{Z}
CII	-1	-1	1	R_{5-d}	\mathbb{Z}_2	0	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_{16}	0	0	0
C	0	-1	0	R_{6-d}	0	\mathbb{Z}	0	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}	0	0
CI	+1	-1	1	R_{7-d}	0	0	\mathbb{Z}_4	0	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_{32}	0

Reproduces:

1D BDI ($\mathbb{Z} \rightarrow \mathbb{Z}_8$), Fidkowski & Kitaev (2010)

3D DIII ($\mathbb{Z} \rightarrow \mathbb{Z}_{16}$), Kitaev (2011), Fidkowski-Chen-Vishwanath (2013),
Metlitski-Kane-Fisher (2014), ...

...

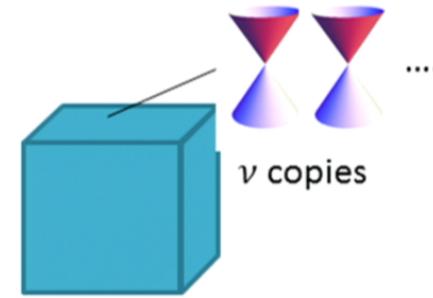
However, we do not have the sectors of bosonic SPT .

e.g., 3D AII: \mathbb{Z}_2^3 , 3D AIII: $\mathbb{Z}_8 \times \mathbb{Z}_2$, 3D CII: \mathbb{Z}_2^5 , Wang & Senthil, PRB (2014)

Our approach

ν copies of gapless boundary states

$$\mathcal{H}_0 = \sum_{j=1}^{d-1} (-i) \partial_j \alpha_j \otimes 1_\nu$$



Boundary massless Dirac fermions + quartic interactions

$$\mathcal{L}_{\text{bd}} = \Psi^\dagger (\partial_\tau + \mathcal{H}_0) \Psi + \lambda \sum_{\{\beta\}} (\Psi^\dagger \beta_n \Psi)^2$$

$\{\beta_1, \beta_2, \dots, \beta_N\}$

marginal at $d - 1 = 1$
irrelevant for $d - 1 > 1$

We assume strong enough
interactions when $d > 2$
(smaller than the bulk gap).

$\alpha \otimes 1$, β : mutually anti-commuting gamma matrices

$\alpha_j \otimes 1$ respect symmetries (such as TRS).

β_n are odd ($\beta_n \rightarrow -\beta_n$) under some symmetry transformation.

$$\mathcal{L}_{\text{bd}} = \Psi^\dagger (\partial_\tau + \mathcal{H}_0) \Psi + \lambda \sum_{\{\beta\}} (\Psi^\dagger \beta_n \Psi)^2$$

Hubbard-Stratonovich transformtation

$$\begin{aligned} \mathcal{L}'_{\text{bd}} &= \Psi^\dagger \left(\partial_\tau + \mathcal{H}_{\text{bd}}^{(\text{dyn})} \right) \Psi + \frac{1}{\lambda} \sum_{n=1}^N \phi_n^2 \\ \mathcal{H}_{\text{bd}}^{(\text{dyn})}(\tau, \mathbf{x}) &= \mathcal{H}_0(\mathbf{x}) + \sum_{\{\beta\}} \frac{2i\beta_n \phi_n(\tau, \mathbf{x})}{\text{dynamical Dirac masses}} \\ \mathcal{H}_0 &= \sum_{j=1}^{d-1} (-i) \partial_j \alpha_j \otimes \mathbf{1}_\nu \end{aligned}$$

cf. You and Xu, PRB (2014), Kitaev's talk (2015)

Integrating out fermions

$$S_{\text{eff}}[\boldsymbol{\phi}] = -\text{Tr} \log \left[\partial_\tau + \sum_{j=1}^{d-1} (-i\partial_j) \alpha_j \otimes 1_\nu + \sum_{\{\beta\}} 2i\boldsymbol{\beta} \cdot \boldsymbol{\phi} \right] + \frac{1}{\lambda} \boldsymbol{\phi}^2$$



Saddle point approximation

+ including fluctuations about the direction in which ϕ freezes

Nonlinear sigma model + topological term

$$Z_{\text{bd}} \approx \int D[\phi] \delta(\boldsymbol{\phi}^2 - 1) e^{-S_{\text{QNLSM}} - S_{\text{top}}}$$

$$S_{\text{QNLSM}} = \frac{1}{2g} \int d\tau \int d^{d-1}x (\partial_j \boldsymbol{\phi})^2$$

Abanov, Wiegmann
Nucl. Phys. B (2000)

$$\boldsymbol{\phi} \in S^{N(\nu)-1}$$

Target space of NLSM is a sphere generated by $N(\nu)$ anticommuting dynamical masses.

$$\mathcal{L}_{\text{bd}} = \Psi^\dagger (\partial_\tau + \mathcal{H}_0) \Psi + \lambda \sum_{\{\beta\}} (\Psi^\dagger \beta_n \Psi)^2$$

Hubbard-Stratonovich transformtation

$$\begin{aligned} \mathcal{L}'_{\text{bd}} &= \Psi^\dagger \left(\partial_\tau + \mathcal{H}_{\text{bd}}^{(\text{dyn})} \right) \Psi + \frac{1}{\lambda} \sum_{n=1}^N \phi_n^2 \\ \mathcal{H}_{\text{bd}}^{(\text{dyn})}(\tau, \mathbf{x}) &= \mathcal{H}_0(\mathbf{x}) + \sum_{\{\beta\}} \frac{2i\beta_n \phi_n(\tau, \mathbf{x})}{\text{dynamical Dirac masses}} \\ \mathcal{H}_0 &= \sum_{j=1}^{d-1} (-i) \partial_j \alpha_j \otimes \mathbf{1}_\nu \end{aligned}$$

cf. You and Xu, PRB (2014), Kitaev's talk (2015)

Topological term in NLSM

The presence or absence of a topological term is determined by the homotopy group of the target space

$$\pi_0(S^{N(v)-1}) \neq 0 \quad \text{domain wall}$$

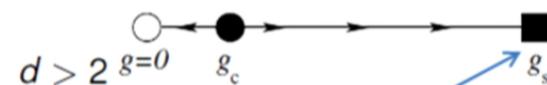
$$\pi_1(S^{N(v)-1}) \neq 0 \quad \text{vortex}$$

Topological defects in the dynamical mass bind fermion zero-energy states.

$$\pi_d(S^{N(v)-1}) \neq 0$$

$$\pi_{d+1}(S^{N(v)-1}) \neq 0 \quad \text{Wess-Zumino term}$$

Without topological term



quantum disorderd (gapped) phase

Nontrivial homotopy group



Topological term in NLSM



Boundary states
remain gapless

With topological term



gapless phase

Example 1: 1D class BDI

$v = 1$

Bulk: $\mathcal{H}^{(0)}(x) := -i\partial_x \tau_3 + m(x) \tau_2.$ TRS $\mathcal{T} := \tau_1 K,$
 PHS $\mathcal{C} := \tau_0 K.$

Boundary: $\mathcal{H}_{bd}^{(0)} = 0.$ $\mathcal{T}_{bd} := K, \mathcal{C}_{bd} := K.$

v copies Quartic interaction + HS trasnf. \rightarrow Dynamical mass at the boundary

$$\mathcal{H}_{bd \nu}^{(dyn)}(\tau) = iM(\tau), \quad M : v \times v \text{ Real anti symmetric matrix}$$

iM breaks TRS, but preserves PHS.

	Dynamical masses	Target space	Topological obstruction
$v=1$	No mass term	-	-
$v=2$	$iM = \pm \sigma_y$	pt.+pt. $\pi_0 \neq 0$	Domain wall
$v=4$	$iM = X_{21}, X_{02}, X_{23}$	$S^2 \quad \pi_2 \neq 0$	WZ term
$v=8$	$iM = X_{213}, X_{023}, X_{233}, X_{002}$	$S^3 \quad (S^6)$	None
	$X_{ijk\dots} = \sigma_i \otimes \sigma_j \otimes \sigma_k \otimes \dots$		$\mathbb{Z} \rightarrow \mathbb{Z}_8$

Example 1: 1D class BDI (ctn'd)

Space of dynamical Dirac masses at boundary
= space Dirac masses for d=0 class D,
 $R_2 = O(2N)/U(N)$

D	$\pi_D(R_2)$	v	Topological obstruction
$iM = \pm\sigma_y$	0	\mathbb{Z}_2	2
	1	0	Domain wall
$iM = X_{21}, X_{02}, X_{23}$	2	\mathbb{Z}	4
	3	0	WZ term
	4	0	
	5	0	
	6	\mathbb{Z}	8
$iM = X_{213}, X_{023}, X_{233}, X_{002}$	7	\mathbb{Z}_2	None

Example 2: 3D class DIII

$$X_{jk} = \tau_j \otimes \sigma_k$$

$\nu = 1$

Bulk: $\mathcal{H}^{(0)}(\mathbf{x}) := -i\partial_1 X_{31} - i\partial_2 X_{02} - i\partial_3 X_{11} + m(\mathbf{x}) X_{03}$. $\mathcal{T} := iX_{20} \mathbb{K}$, $\mathcal{C} := X_{01} \mathbb{K}$.

Boundary: $\mathcal{H}_{\text{bd}}^{(0)}(x, z) = -i\partial_x \tau_3 - i\partial_z \tau_1$, $\mathcal{T}_{\text{bd } \nu} := i\tau_2 \otimes \mathbb{1} \mathbb{K}$
 $\mathcal{C}_{\text{bd } \nu} := \tau_0 \otimes \mathbb{1} \mathbb{K}$

ν copies

Dynamical mass: $\tau_2 \otimes \underbrace{M(\tau, x, z)}_{\nu \times \nu \text{ Real symmetric matrix}}$

Dynamical masses break T, but preserves C.

v=1: $M = \pm 1$ pt.+pt. $\pi_0 \neq 0$ domain wall

v=2: $M = X_1, X_3$ $\pi_1(S^1) = \mathbb{Z}$ vortex

v=4: $M = X_{13}, X_{33}, X_{01}$ $\pi_2(S^2) = \mathbb{Z}$ monopole

v=8: $M = X_{133}, X_{333}, X_{013}, X_{001}, X_{212}$ $\pi_4(S^4) = \mathbb{Z}$ WZ term

$$X_{ijk\dots} = \sigma_i \otimes \sigma_j \otimes \sigma_k \otimes \dots$$

Higher dimensions

- \mathbb{Z}_2 in any dimension and any symmetry class is **stable**.
 - $\exists D \leq d + 1, \pi_D(S^{N(\nu)}) \neq 0$ for $\nu = 1$
- \mathbb{Z} in even dimensions is **stable**.
 - Either (a) no dynamical mass exists, or (b) $\pi_1 \neq 0$.
- \mathbb{Z} in odd dimensions is **unstable**.

The reduction pattern is
determined by the topology of
the space of dynamical masses

	$d = 8n + 1$	$d = 8n + 3$	$d = 8n + 5$	$d = 8n + 7$
BDI	$\mathbb{Z} \rightarrow \mathbb{Z}_{2^{4n+3}}$	—	$\mathbb{Z} \rightarrow \mathbb{Z}_{2^{4n+4}}$	—
DIII	—	$\mathbb{Z} \rightarrow \mathbb{Z}_{2^{4n+4}}$	—	$\mathbb{Z} \rightarrow \mathbb{Z}_{2^{4n+5}}$
CII	$\mathbb{Z} \rightarrow \mathbb{Z}_{2^{4n+1}}$	—	$\mathbb{Z} \rightarrow \mathbb{Z}_{2^{4n+4}}$	—
CI	—	$\mathbb{Z} \rightarrow \mathbb{Z}_{2^{4n+2}}$	—	$\mathbb{Z} \rightarrow \mathbb{Z}_{2^{4n+5}}$

Class	T	C	Γ_5	V_d	$d = 1$	$d = 2$	$d = 3$	$d = 4$	$d = 5$	$d = 6$	$d = 7$	$d = 8$
A	0	0	0	C_{0+d}	0	\mathbb{Z}	0	\mathbb{Z}	0	\mathbb{Z}	0	\mathbb{Z}
AIII	0	0	1	C_{1+d}	\mathbb{Z}_4	0	\mathbb{Z}_8	0	\mathbb{Z}_{16}	0	\mathbb{Z}_{32}	0
AI	+1	0	0	R_{0-d}	0	0	0	\mathbb{Z}	0	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}
BDI	+1	+1	1	R_{1-d}	\mathbb{Z}_8	0	0	0	\mathbb{Z}_{16}	0	\mathbb{Z}_2	\mathbb{Z}_2
D	0	+1	0	R_{2-d}	\mathbb{Z}_2	\mathbb{Z}	0	0	0	\mathbb{Z}	0	\mathbb{Z}_2
DIII	-1	+1	1	R_{3-d}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_{16}	0	0	0	\mathbb{Z}_{32}	0
AII	-1	0	0	R_{4-d}	0	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}	0	0	0	\mathbb{Z}
CII	-1	-1	1	R_{5-d}	\mathbb{Z}_2	0	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_{16}	0	0	0
C	0	-1	0	R_{6-d}	0	\mathbb{Z}	0	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}	0	0
CI	+1	-1	1	R_{7-d}	0	0	\mathbb{Z}_4	0	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_{32}	0

2D DIII + R

(superconductor w/time-reversal & reflection)

$$\nu = 1 \text{ Bulk} \quad \mathcal{H}^{(0)} = -i\partial_x X_{31} - i\partial_y X_{02} + m(x, y)X_{03} \quad X_{ij} = \sigma_i \otimes \sigma_j$$

$$T = iX_{20}\mathbf{K}, \quad C = X_{01}\mathbf{K}, \quad R = iX_{20}$$

ν copies Boundary

$$\mathcal{H}_{\text{bd}}^{\text{(dyn)}} = -i\partial_x \sigma_3 \otimes \mathbf{1}_\nu + M(\tau, x) \quad M^* = -M$$

$$M = \begin{pmatrix} 0 & -iA \\ iA^T & 0 \end{pmatrix} \quad A \in O(\nu) = R_1$$

ν	$\pm \sigma_y$	$\pi_0 \neq 0$	D	$\pi_D(R_1)$	ν	Topological obstruction
$\nu = 1$			0	\mathbb{Z}_2	1	Domain wall
			1	\mathbb{Z}_2	2	Vortex
$\nu = 2$	X_{21}, X_{23}	$\pi_1(S^1) = \mathbb{Z}$	2	0		
$\nu = 4$	$X_{210}, X_{230}, X_{102}, X_{222}$	$\pi_3(\text{?})$	3	\mathbb{Z}	4	WZ term
$\nu = 8$	$X_{2100}, X_{2310}, X_{2331}, X_{2333}$		4	0		
			5	0		
			6	0		
	\mathbb{Z}_8		7	\mathbb{Z}	8	None

Summary

- Classification of TIs \leftrightarrow classification of Dirac masses
- Reduction of the topological classification \mathbb{Z}
 - Interactions represented in terms of dynamical Dirac masses (on the boundary)
 - Stability/breakdown = nontrivial/trivial topology of the space of dynamical Dirac masses

T. Morimoto, A. Furusaki, C. Mudry, Phys. Rev. B 92, 125104 (2015)

cf: R. Queiroz, E. Khalaf, A. Stern, 1601.01596