Title: Dynamics of Superrotations in 2+1 Dimensions

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Abstract: General relativity is invariant under diffeomorphisms, and

excitations of the metric corresponding to diffeomorphisms

are nonphysical. In the presence of a boundary, though --

including a boundary at infinity -- the Einstein-Hilbert

action with suitable boundary terms is no longer fully

invariant, and certain diffeomorphisms are promoted to

physical degrees of freedom. After briefly describing how

this happens in (2+1)-dimensional AdS gravity, I will

report on work in progress on the asymptotically flat case,

for which the newly dynamical diffeomorphisms are the

superrotations, and the boundary action is related to

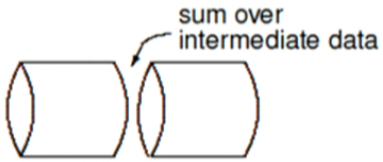
coadjoint orbits of the Virasoro group and Hill's equation.

The idea in a nutshell

- Action on manifold with boundary has two pieces:

$$I = I_{bulk} + I_{bdry} \qquad \text{with} \quad I_{bulk} = \int \mathscr{L} d^n x$$

Boundary piece needed
  – Classically: to allow extrema
  – Quantum mechanically: to ensure proper "sewing" of path integrals



- Gauge symmetries of I_{bulk} will typically be broken by I_{bdry}
  – Formerly nonphysical degrees of freedom become dynamical at boundary
- Action for new degrees of freedom is induced from I_{bdry}
- Results for (2+1)-dimensional gravity:
  – Asymptotically AdS: Liouville action
  – Asymptotically flat: relation to Liouville, coadjoint quantization of Virasoro (work in progress . . .)

$(2+1)$-dimensional gravity

Two tactics

- Start with Chern-Simons formulation
  - simple decomposition $A = g^{-1}dg + g^{-1}\bar{A}g$
  - standard reduction to WZNW model at boundary (plus constraints)
  - boundary term known to be right for "sewing"
  - ***but*** doesn't generalize to higher dimensions, hard to include matter

- Use standard metric or vielbein formulation
  - no simple decomposition into gauge-fixed fields + diffeos
  - boundary theory may not be local
  - but presumably more widely applicable

The AdS case

Interesting because

- AdS/CFT correspondence suggests importance of boundary dynamics
- (2+1)-dimensional AdS gravity includes BTZ black holes
- Asymptotic behavior is well-understood

Fefferman-Graham expansion of metric:

$$ds^2 = -\ell^2 d\rho^2 + g_{ij} dx^i dx^j, \quad \text{with } g_{ij} = e^{2\rho} \overset{(0)}{g}_{ij}(x) + \overset{(2)}{g}_{ij}(x) + \dots$$

Diffeomorphism:

$$\rho \to \rho + \frac{1}{2}\varphi(x) + e^{-2\rho} \overset{(2)}{f}(x) + \dots$$

$$x^i \to x^i + e^{-2\rho} \overset{(2)}{h}{}^i(x) + \dots$$

where form invariance of metric determines $\overset{(2)}{h}{}^i$, $\overset{(2)}{f}$

Action

$$I_{grav} = \frac{1}{16\pi G}\int_M d^3x\sqrt{{}^{(3)}g}\left({}^{(3)}R + \frac{2}{\ell^2}\right)$$
$$+ \frac{1}{8\pi G}\int_{\partial M} d^2x\sqrt{\gamma}K - \frac{1}{8\pi G\ell}\int_{\partial M} d^2x\sqrt{\gamma}$$

Original boundary at $\rho = \bar{\rho}$; new boundary at

$$\rho = \bar{\rho} + \frac{1}{2}\varphi(x) + e^{-2\bar{\rho}}\overset{(2)}{f}(x) = F(x)$$

Compute new normal, extrinsic curvature, induced metric: find

$$I_{grav} = -\frac{\ell}{16\pi G}\int_{\partial M} d^2x\sqrt{\overset{(0)}{g}}\left(\overset{(0)}{g}{}^{ij}\partial_i F\partial_j F - F\overset{(0)}{R}\right)$$

(Liouville action in limit $\lambda \to 0$)

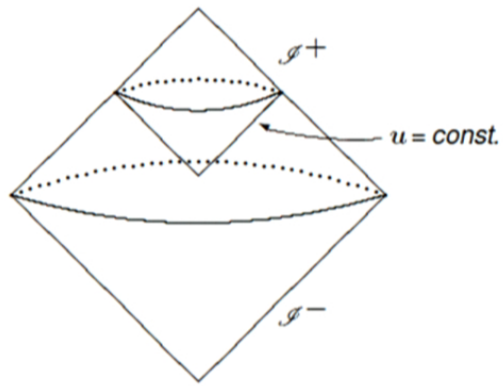
What does this mean?

- CFT with right central charge to match Brown-Henneaux:
  Cardy formula gives correct entropy
- Coupling “classical” source at boundary gives right Hawking radiation (Emparan & Sachs)
- Minimal: “effective description” of black hole states
- Maximal: Liouville theory “really” describes black hole states

The asymptotically flat case (work in progress)

Again partially gauge-fix metric: Bondi coordinates

$$ds^2 = -2dudr + g_{uu}\,du^2 + 2g_{u\phi}\,dud\phi + r^2 e^{2\varphi} d\phi^2$$



$u = const.$: outgoing null surfaces
r is an affine parameter along null geodesics $u = const.$, $\phi = const.$

Behavior of metric near $\mathscr{I}$

Symmetries: supertranslations $\sim u \to u + e^{\varphi} T(\phi)$

superrotations $\sim \phi \to \phi + Y(\phi)$

From field equations (Barnich & Troessaert)

$$g_{uu} = -2r\partial_u\varphi + e^{-2\varphi}\left[-(\partial_\phi\varphi)^2 + 2\partial_\phi^2\varphi + \Theta\right]$$

$$g_{u\phi} = e^{-\varphi}\left[\Xi + \int^u d\tilde{u}\left\{\frac{1}{2}\partial_\phi\Theta - \partial_\phi\varphi[\Theta - (\partial_\phi\varphi)^2 + 3\partial_\phi^2\varphi] + \partial_\phi^3\varphi\right\}\right]$$

with $\partial_u\Theta = \partial_u\Xi = 0$

Θ and Ξ are charges for supertranslations and superrotations

$$\mathcal{Q} \sim \frac{1}{16\pi G}\int d\phi\,(\Theta T + \Xi Y)$$

First problem: need right boundary terms

Approach 1: Compute directly from variation of action

$$\delta I_{grav} = \textit{bulk piece} + \frac{1}{16\pi G}\int_{\partial M} d^2x\,[-\partial_r(re^{\varphi}\delta g_{uu}) - 2g_{uu}\partial_r(re^{\varphi}\delta\varphi)] + \mathcal{O}\left(\frac{1}{r}\right)$$

Use asymptotic expression to integrate: find

$$\delta I = -\frac{1}{8\pi G}\,\delta\int_{\partial M} d^2x\, e^{\tilde{\varphi}}\tilde{g}_{uu} + \frac{1}{16\pi G}\int_{\partial M} d^2x\, e^{\tilde{\varphi}}\delta\tilde{g}_{uu}$$

(where $\tilde{X}$ means the $\mathcal{O}(1)$ part of X)

Fix either $\tilde{g}_{uu}$ (i.e., Θ) or $e^{\tilde{\varphi}}$ (or some combination)

Fix $\tilde{\varphi}$:

$$I_{bdry} = \frac{1}{16\pi G} \int_{\partial M} d^2x \, e^{\tilde{\varphi}} \tilde{g}_{uu}$$

Not quite extrinsic curvature of $r = const.$:

– unit normal n^a to $r = const.$

– induced metric q_{ab} on $r = const.$

– null vector $\ell \propto du$, normalized so $n_a \ell^a = 1$

$$I_{bdry} = \frac{1}{16\pi G} \int_{\partial M} d^2x \, \sqrt{q} \, \nabla_a \ell^a$$

Approach 2: Look at symplectic structure at $\mathscr{I}$

"Radial quantization":

– surface of constant r

– induced metric q_{ab}

– extrinsic curvature K_{ab}

– conjugate momentum $\pi^{ab} = \sqrt{q}(K^{ab} - q^{ab}K)$

Find

$$\pi^{ab}\delta q_{ab} \sim \delta(2\pi^{a}{}_{a} - g_{uu}e^{\varphi}) - g_{uu}\delta e^{\varphi}$$

$\Rightarrow$ e^{φ} and g_{uu} are conjugate variables

Relating this to diffeomorphisms

Start with flat base metric

$$ds^2 = -2d\bar{u}d\bar{r} + d\bar{u}^2 + \bar{r}^2 d\bar{\phi}^2$$

Diffeomorphism

$$\bar{u} = u_0 + \frac{u_1}{r} + \ldots, \qquad \bar{\phi} = \phi_0 + \frac{\phi_1}{r} + \ldots, \qquad \bar{r} = ar + b_0 + \frac{b_1}{r} + \ldots$$

Form invariance of metric $\Rightarrow$

$$\partial_u u_0 = \frac{1}{a}$$

$$\partial_\phi \phi_0 = \beta \qquad \text{with } \partial_u \beta = 0$$

$$\beta^2 b_0 = -\frac{\partial_\phi \beta}{\beta}\partial_\phi u_0 + \partial_\phi^2 u_0 + \alpha\beta^2 a$$

where

$$g_{\phi\phi} = a^2\beta^2(r^2 + 2\alpha r) + \mathcal{O}(1)$$

What can we say about this action?

- No u derivatives (why?)
- No Ξ (why?)
- Vary φ: Hill's equation

Let $e^{-\varphi/2} = \sqrt{2\pi G}\chi$

$$T = \frac{c}{24}(\partial_\phi \phi_0)^2 + \frac{c}{12}\{\phi_0; \phi\}$$

$$\Rightarrow \partial_\phi^2 \chi + \frac{6}{c} T \chi = 0$$

- Closely related to action for coadjoint orbit quantization of Virasoro group

  Virasoro = central extension of $Diff\,(S^1)$

  Hill's equation $\partial_\phi^2 \chi + \frac{6}{c} T \chi = 0$

  Solutions χ_1, χ_2

  Let $\varepsilon = \chi_1^2,\ \chi_1\chi_2,\ \chi_2^2$

  Then $\delta_\varepsilon T = -\frac{c}{12}\varepsilon''' - 2\varepsilon' T - \varepsilon T' = 0$

  (orbit of Virasoro group)

Action for ϕ_0 closely related to Alexe'ev-Shatashvili action for coadjoint quantization:

$$I = \int dt dx \left(-b_0 F' \dot{F} + \frac{c}{48\pi} \frac{\dot{F}}{F'} \{F; x\} \right)$$

(matches if $F' \leftrightarrow \partial_\phi \phi_0,\ \dot{F} \leftrightarrow \partial_u u_0$)

- Related to quantization of Liouville theory a la Polyakov:

Consider the auxiliary two-dimensional metric

$$d\tilde{s}^2 = \beta \left(2du d\phi + e^{-\varphi} du^2 \right)$$

If $\partial_\phi \partial_u \varphi = 0$, then

$$I_{bdry} = \frac{1}{16\pi G} \int d^2x \sqrt{-\tilde{g}} \left(\tilde{R} \Box^{-1} \tilde{R} + \Lambda \right) \quad \text{with } \Lambda = -\partial_u u_0$$

(Polyakov action with $c = 6/G$)

Note that $d\tilde{s}^2 = \dfrac{1}{(\partial_u u_0)} \left(d\tau^2 - (\partial_u u_0)^2 d\phi^2 \right)$ with $\tau = u_0 + \phi_0$

$\Rightarrow$ connections with CFT, but to be worked out ...