Title: Diagonalising states: an operational route towards a resource theory of purity

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Abstract: <p>In quantum theory every state can be diagonalised, i.e. decomposed as a convex combination of perfectly distinguishable pure states. This fact is crucial in quantum statistical mechanics, as it provides the foundation for the notions of majorisation and entropy. A natural question then arises: can we give an operational characterisation of them? We address this question in the framework of general probabilistic theories, presenting a set of axioms that guarantee that every state can be diagonalised: Causality, Purity Preservation, Purification, and Pure Sharpness. If we add the Permutability and Strong Symmetry axioms, which are in fact completely equivalent in theories satisfying the other axioms, the diagonalisation result allows us to define a well-behaved majorisation preorder on states. Indeed this majorisation criterion fully captures the convertibility of states in the operational resource theory of purity where random reversible transformations are regarded as free operations. One can also put forward two alternative notions of purity as a resource: one where free operations are unital channels, and another where free operations are generated by reversible interactions with an environment in the invariant state. Under the validity of the above axioms, all these definitions are in fact equivalent, i.e. they all lead to the same preorder on states, which is given by majorisation, in the very same way as in quantum theory.</p>

Diagonalising states

An operational route towards a resource theory of purity

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Introduction

- Joint work with G. Chiribella, final goal: foundations of thermodynamics.

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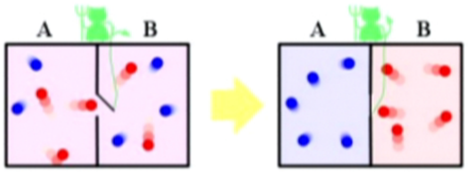
- Joint work with G. Chiribella, final goal: foundations of thermodynamics.
- Some tension between microscopic and macroscopic dynamics in classical physics
- Quantum physics is a solution!
- We wish to study thermodynamics even with only a few particles (important for nanotechnology)

Foundations of thermodynamics and GPTs

- Thermodynamics related to information theory: Maxwell's demon and Landauer's principle



- We look for an approach where the role of "information" is evident

Why not address the issue using *GPTs*?

New insights on quantum foundations from a thermodynamic angle!

Contents

OPTs

We use a specific variant of GPTs, known as OPTs.
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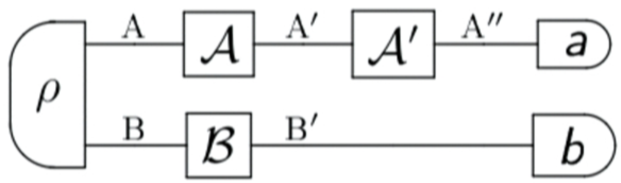

- A, B, etc. are systems
- $\mathcal{A}$, $\mathcal{B}$, etc. are transformations: they can be composed in sequence (e.g. $\mathcal{A}$ and $\mathcal{A}'$) or in parallel (e.g. $\mathcal{A}$ and $\mathcal{B}$)
- ρ is a state (a transformation with *no* input)
- a and b are effects (transformations with *no* output)

Probabilistic structure & purity

- Circuits with no external wires represent probabilities

$$\boxed{\rho_j} \xrightarrow{\text{A}} \boxed{a_i} = p_{ij} \in [0, 1] \, .$$

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- and defines *real* vector spaces spanned by states and effects. We assume they are finite-dimensional.
- We can define coarse-graining and purity.

Purity (mathematical)

A transformation $\mathcal{T}$ is pure if $\mathcal{T} = \sum_i \mathcal{T}_i$ implies $\mathcal{T}_i = p_i \mathcal{T}$, where $\{p_i\}$ is a probability distribution.

Resource theories

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Resource theories

We have a set of resources, and transformations between them.
Some resources are abundant: they come for free; other are valuable.
Define a set of free operations (i.e. easy to implement).

Resource theory of purity in QM

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Resource theory of purity

Free states: only the maximally mixed state $\chi = \frac{1}{d}\mathbf{1}$.
Free operations of the form (noisy operations)

$$\mathcal{N}\left(\rho_{\mathrm{A}}\right) = \mathrm{tr}_{\mathrm{E}}\left[U_{\mathrm{AE}}\left(\rho_{\mathrm{A}} \otimes \chi_{\mathrm{E}}\right) U_{\mathrm{AE}}^{\dagger}\right]$$

and its topological closure [Gour et al.].

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Thermodynamic interpretation

χ is a thermal state (of a trivial Hamiltonian)!

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Thermodynamic interpretation

χ is a thermal state (of a trivial Hamiltonian)!

Free operations are:

- preparing thermal states
- putting a system into contact with a thermal environment
- removing the thermal environment

Problems in GPTs

- The maximally mixed state (invariant state) χ may not exist.
- If it exists, it may not be unique.
- If it exists, and it's unique, there may be another problem...

Free states must be stable under tensor product [Coecke et al.], but in general $\chi_{\mathrm{A}} \otimes \chi_{\mathrm{B}} \neq \chi_{\mathrm{AB}}$!

We have to follow another approach! In QM there are indeed several ways of defining a theory of purity...

RaRe channels

- Consider an agent controlling a *single* closed system
- Allowed evolutions are reversible
- He hasn't got perfect control on the reversible transformation he applies
- We have a mixture of reversible channels [Uhlmann]

RaRe channels

A channel $\mathcal{R}$ is RaRe if it's of the form

$$\mathcal{R} = \sum_i p_i \mathcal{U}_i,$$

where $\{p_i\}$ is a probability distribution, and the $\mathcal{U}_i$'s are *reversible* channels.

Purity via RaRe channels [Chiribella & Scandolo '15b]

- RaRe channels are free operations
- There are *no* free states
- ρ is purer than σ if $\mathcal{R}\rho = \sigma$, for some RaRe channel $\mathcal{R}$ (cf. also [Müller & Masanes] in a different context).

This is the most general definition of resource theory of purity we can give (no assumptions!).

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To be very precise, this preorder coincides with purity iff the action of reversible channels is transitive on pure states [Chiribella & Scandolo '15b].

Section 3

Diagonalisation and majorisation

Majorisation

Important in quantum thermodynamics.

Majorization

Let $\mathbf{p}, \mathbf{p}' \in \mathbb{R}^n$ be two probability distributions. We say that $\mathbf{p}$ is *majorized* by $\mathbf{p}'$ ($\mathbf{p} \preceq \mathbf{p}'$) if

$$\sum_{i=1}^{k} p_{[i]} \leq \sum_{i=1}^{k} p'_{[i]} \quad \text{for } k = 1, \ldots, n-1,$$

where $p_{[i]}$ is the i-th entry of the decreasing rearrangement of $\mathbf{p}$.

It describes the purity preorder on *quantum* states, based on their eigenvalues [Gour et al.].

Working plan

Does majorisation characterise purity even in GPTs?

- From majorization we get entropies as Schur-concave functions.

We need to define the "eigenvalues" of states even in GPTs.

Cf. also [Barnum et al. '15] (from a different angle) and [Chiribella et al. '11] (with stronger axioms).

Purity Preservation

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The sequential and parallel composition of pure transformations is a pure transformation.

- The product of two pure states is pure.
- Without Purity Preservation, we may have a "*non-local*" loss of information when composing transformations.

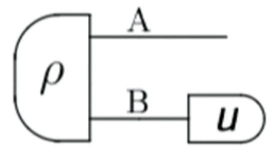
Causality

Causality [Chiribella et al. '10]

The outcome probabilities of present experiments aren't affected by the choice of future measurements.

- Equivalently, for every system A there is a *unique* deterministic effect u_{A}.
- We can use u to define the marginals of bipartite states:

$$\rho_{\mathrm{A}} := \mathrm{tr}_{\mathrm{B}} \rho_{\mathrm{AB}} =$$



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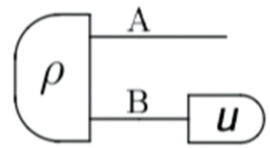
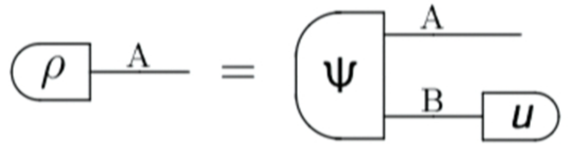


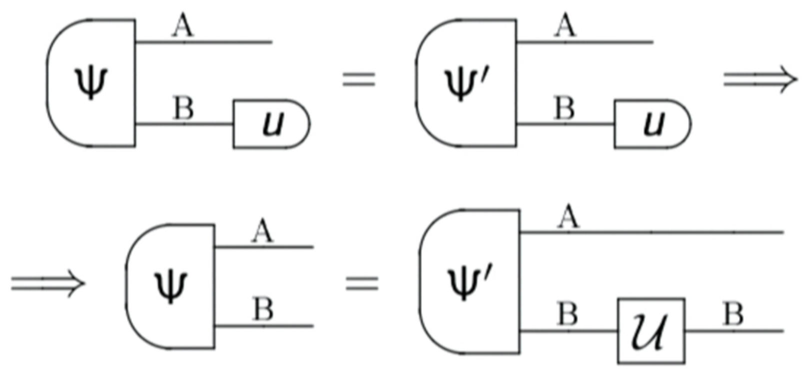
Important in thermodynamics: we need to restrict ourselves to subsystems!

Purification [Chiribella et al. '10]

1. Every state ρ_{A} can be purified: there exists a pure state Ψ_{AB} such that



2. Different purifications of the same state differ by a reversible transformation $\mathcal{U}$ on the purifying system:



Why Purification?

- It reconciles partial information and irreversibility with a picture where everything is pure and reversible.
- Dilation and extension theorems can be reconstructed from it [Chiribella et al. '10].
- It provides a formal justification of the thermodynamic procedure of enlarging a system to deal with an isolated system.

Purification is a good starting point for a theory of thermodynamics.
This narrows down the class of theories.

Pure Sharpness

Pure Sharpness [Chiribella & Scandolo '15c]

For every system, there exists at least one pure effect a occurring with probability 1 on some state ρ.

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- We can think of a as part of a yes/no test to check an *elementary property* of the system.
- Pure Sharpness guarantees that every system has an elementary property.

Consequences of the axioms [Chiribella & Scandolo '15c]

1. Duality pure states-pure effects: for every *pure* state α there is a unique *pure* effect $\alpha^\dagger$ such that $(\alpha^\dagger|\alpha) = 1$.
2. Existence of perfectly distinguishable (pure) states.
3. Every state can be diagonalised, i.e. decomposed as a mixture of perfectly distinguishable pure states.

$$\rho = \sum_i p_i \alpha_i$$

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Permutability and Strong Symmetry

Strong Symmetry [Barnum et al. '14]

If $\{\alpha_i\}_{i=1}^n$ and $\{\beta_i\}_{i=1}^n$ are two maximal sets of perfectly distinguishable *pure* states, there is a reversible channel $\mathcal{U}$ such that $\alpha_i = \mathcal{U}\beta_i$ for $i = 1, \ldots, n$.

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Permutability [Hardy]

Every permutation of a maximal set of perfectly distinguishable *pure* states is implemented by a reversible channel.

Consequences

1. The eigenvalues of a state are *uniquely* defined.

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2. The product of two invariant states is invariant: $\chi_{\mathrm{A}} \otimes \chi_{\mathrm{B}} = \chi_{\mathrm{AB}}$.

Now we can rightfully take χ to be a free state: other definitions of purity become possible! (QPL 2016).

Again, strong evidence that Permutability/Strong Symmetry isn't necessary (work in progress)...

Majorisation and purity

Theorem ([Chiribella & Scandolo '15c])

If ρ and ρ' are states with eigenvalues $\mathbf{p}$ and $\mathbf{p}'$ respectively, under the present axioms the following are equivalent:

1. $\mathcal{R}\rho = \rho'$ *for a RaRe channel* $\mathcal{R}$*;*
2. $\mathbf{p}' \preceq \mathbf{p}$.

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- These axioms open the way for other definitions of purity in GPTs (see QPL 2016).
- Majorisation plays a major role in characterising and describing purity.

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- Majorisation plays a major role in characterising and describing purity.

Even with the other definitions of purity (QPL 2016)!

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