Title: Hilbert series for effective field theories

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Abstract: <p>Effective field theories (EFT) are everywhere in particle physics. Given an EFT, the first question we ask is â€œwhat are all the operators consistent with the symmetries and degrees of freedom at a particular expansion order? In this talk I will show how this question can be attacked, and often answered, using an object called a Hilbert series.</p>

Hilbert Series for Effective Field Theory

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based on 1503.07537, 1510.00372 AM, L.Lehman,
also Henning et al 1512.03433, 1507.07240

Perimeter Institute, May 3rd, 2016

Wilsonian picture of field theory

$$\mathcal{L} = \int d^4x \sum_i c_i \, \mathcal{O}_i$$

take all degrees of freedom, form local operators of increasing dimension

all operators consistent with symmetries must be included

lowest mass dimension operators dominate IR physics

SM is a poster child EFT: SMEFT

degrees of freedom are: Q, u^c, d^c, L, e^c, H, gauge fields

symmetry is: Lorentz ⊗ SU(3)c⊗SU(2)w⊗ U(1)Y

low-dimension operators are easy, but quickly gets more complicated

dim ≤4: Standard Model

dim 5: 1 operator (neutrino mass) [Weinberg '79]

dim 6: 63 terms (neglecting flavor) [Büchmuller, Wyler '86, Grzadkowski et al '10]

dim 7: 20 terms [Lehman '14]

dim 8: no complete set known (as of Oct. 2015)

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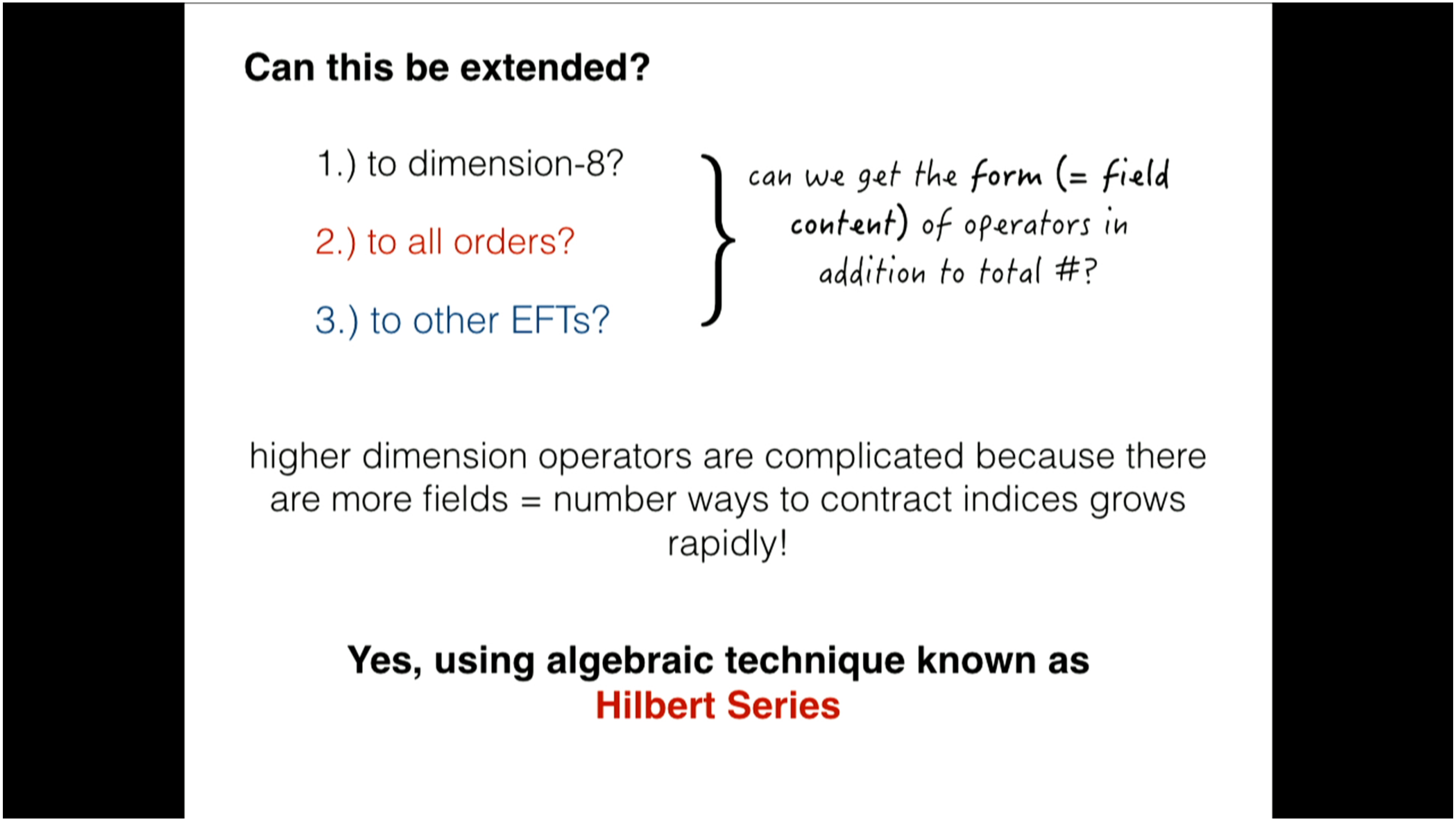


Can this be extended?

1.) to dimension-8?

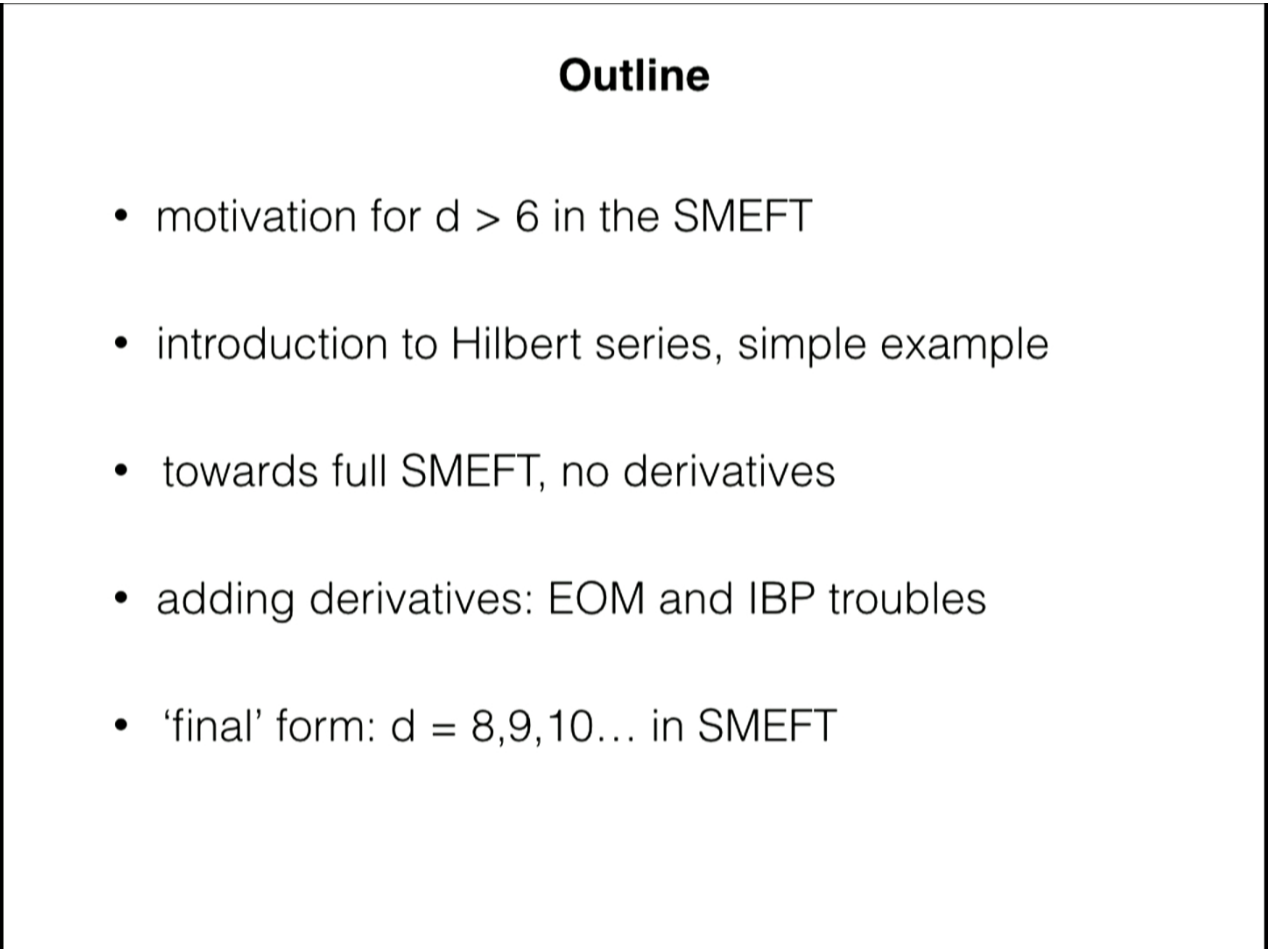
2.) to all orders?

3.) to other EFTs?

} can we get the form (= field content) of operators in addition to total #?

higher dimension operators are complicated because there are more fields = number ways to contract indices grows rapidly!

Yes, using algebraic technique known as
Hilbert Series



Outline

- motivation for d > 6 in the SMEFT
- introduction to Hilbert series, simple example
- towards full SMEFT, no derivatives
- adding derivatives: EOM and IBP troubles
- ‘final’ form: d = 8,9,10… in SMEFT

Why?

precision: LHC, HL-LHC, etc. will soon test SM to unprecedented precision = sensitivity to effects from even higher dimension

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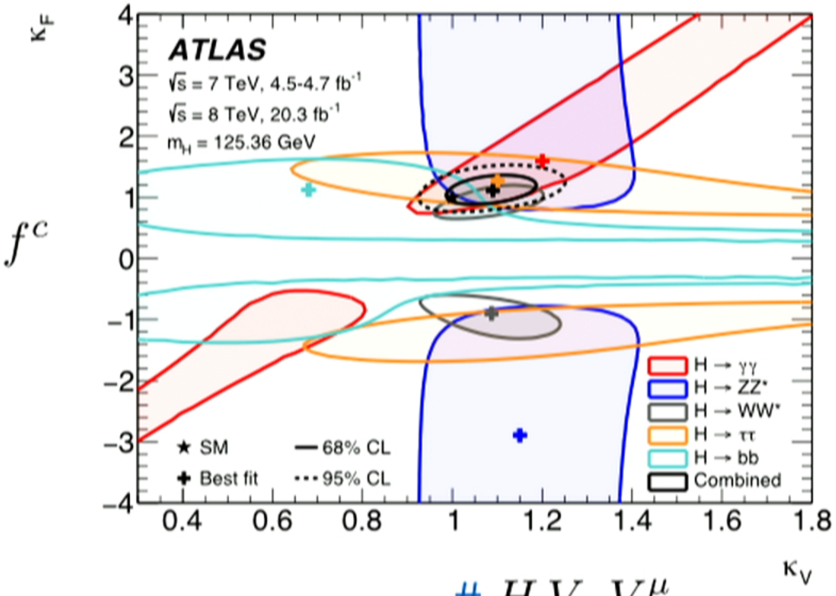

$\# \, H \, f_L \, f^c$

$\# \, H \, V_\mu \, V^\mu$

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precision: LHC, HL-LHC, etc. will soon test SM to unprecedented precision = sensitivity to effects from even higher dimension

new effects: lower dim. operators have accidental symmetries (i.e. baryon #, lepton #). Higher dim. operators are the first place violation of these symmetries occurs

How?

Consider a simple setup: ϕ, ϕ* with charge +1, -1

all invariants are of the form (ϕ ϕ*)^n, and for each n there is only one invariant

Hilbert series is defined as: $$h = \sum_n \kappa_n \, t^n$$

number of invariants of degree n   invariants

for us:

degree = mass dimension, t = symmetry-invariant operators

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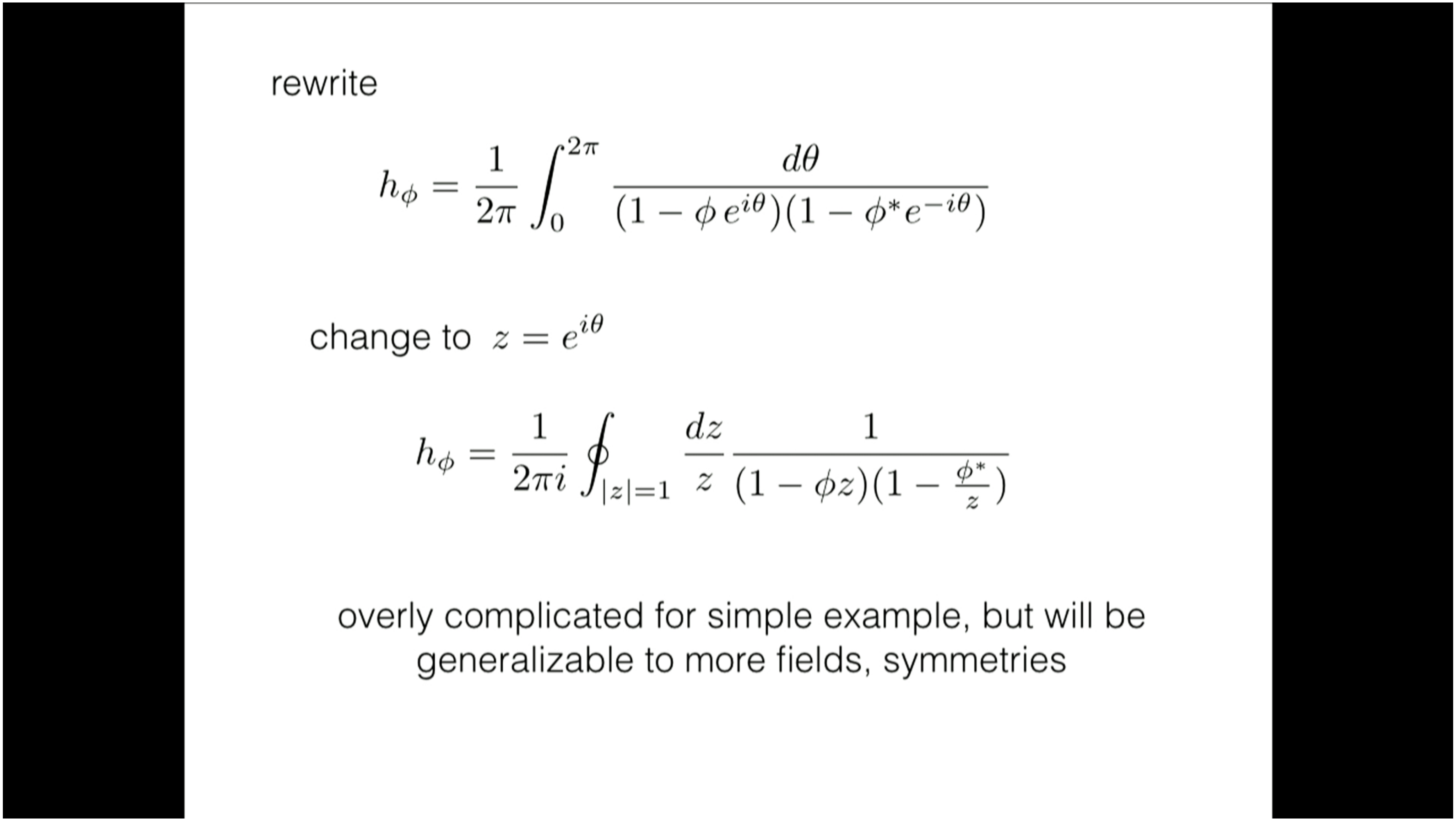

rewrite

$$h_\phi = \frac{1}{2\pi} \int_0^{2\pi} \frac{d\theta}{(1 - \phi\, e^{i\theta})(1 - \phi^* e^{-i\theta})}$$

change to $z = e^{i\theta}$

$$h_\phi = \frac{1}{2\pi i} \oint_{|z|=1} \frac{dz}{z} \frac{1}{(1 - \phi z)(1 - \frac{\phi^*}{z})}$$

overly complicated for simple example, but will be generalizable to more fields, symmetries

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$$\frac{1}{(1-\phi z)(1-\frac{\phi^*}{z})} = 1 + (\phi\phi^*) + (\phi\phi^*)^2 + (\phi\phi^*)^3 + \cdots$$

$$+z(\phi + \phi(\phi\phi^*) + \phi(\phi\phi^*)^2 + \phi(\phi\phi^*)^3 + \cdots)$$

$$+\frac{1}{z}(\phi^* + \phi^*(\phi\phi^*) + \phi^*(\phi\phi^*)^2 + \phi^*(\phi\phi^*)^3 + \cdots)$$

$$+\cdots$$

generates **all** possible combinations of ϕ, ϕ*. Combinations can be grouped according to their charge

only the combinations at O(1) (charge zero) are picked out by the contour integral dz/z

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manipulate further

$$\frac{1}{(1-\phi z)(1-\frac{\phi^*}{z})} = \exp\Big(-\log(1-\phi z) - \log(1-\frac{\phi^*}{z})\Big)$$

$$= \exp\Big(\sum_{r=1}^{\infty}\Big\{\frac{(\phi z)^r}{r} + \frac{1}{r}\Big(\frac{\phi^*}{z}\Big)^r\Big\}\Big)$$

this will be the most useful (= generalizable) form

generating function written as “Plethystic exponential” = PE

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objects `charge'

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Plethystic exponential

Hilbert series

$h = \int d(\text{measure}) \Bigg($  $\Bigg)$

more complicated example:

$$\phi_1,\ \phi_1^*,\ \phi_2,\ \phi_2^*$$

charge: $+1,\ -1,\ +2,\ -2$

now there are four invariants

$$(\phi_1\phi_1^*),\ (\phi_2\phi_2^*),\ (\phi_1^2\phi_2^*),\ (\phi_1^{*2}\phi_2)$$

based on last example, may guess that

$$h_{\phi_1\phi_2} = \frac{1}{(1-(\phi_1\phi_1^*))(1-(\phi_2\phi_2^*))(1-(\phi_1^2\phi_2^*))(1-(\phi_1^{*2}\phi_2))}$$

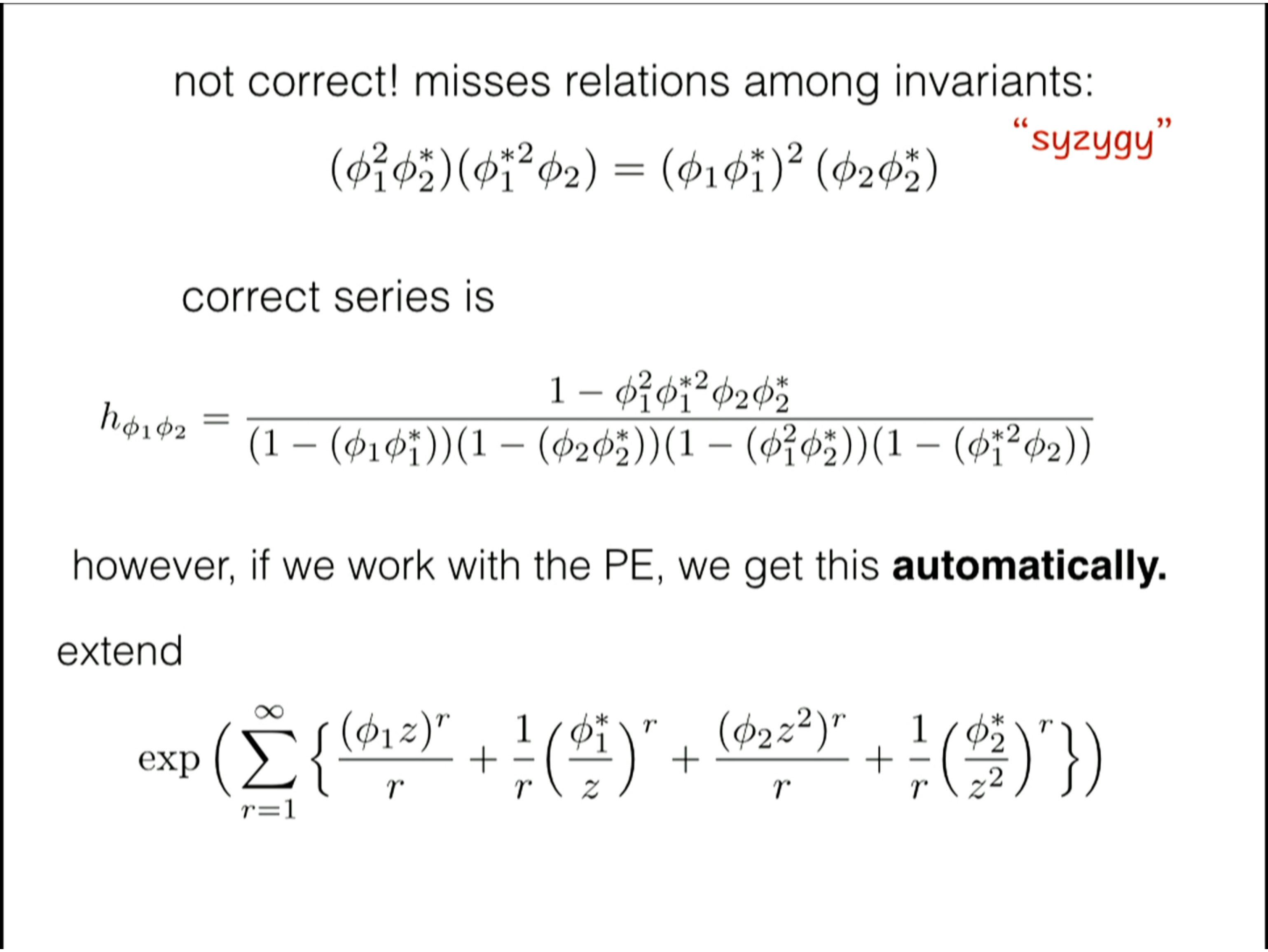
generates all invariants

not correct! misses relations among invariants:

$$(\phi_1^2\phi_2^*)(\phi_1^{*2}\phi_2) = (\phi_1\phi_1^*)^2\,(\phi_2\phi_2^*)$$

correct series is

$$h_{\phi_1\phi_2} = \frac{1-\phi_1^2\phi_1^{*2}\phi_2\phi_2^*}{(1-(\phi_1\phi_1^*))(1-(\phi_2\phi_2^*))(1-(\phi_1^2\phi_2^*))(1-(\phi_1^{*2}\phi_2))}$$



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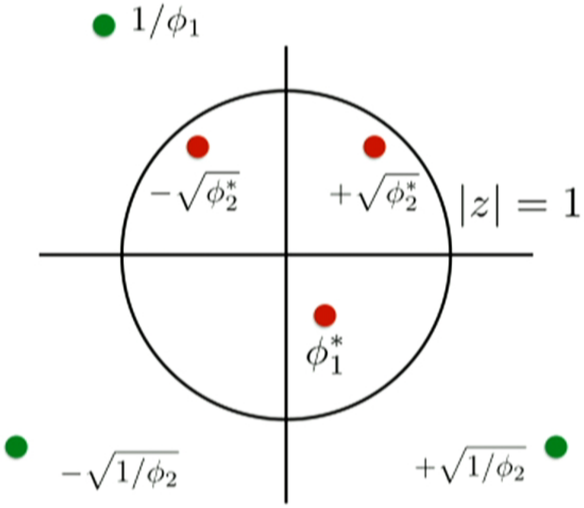
however, if we work with the PE, we get this **automatically.**

extend

$$\exp\Big(\sum_{r=1}^{\infty}\Big\{\frac{(\phi_1 z)^r}{r}+\frac{1}{r}\Big(\frac{\phi_1^*}{z}\Big)^r+\frac{(\phi_2 z^2)^r}{r}+\frac{1}{r}\Big(\frac{\phi_2^*}{z^2}\Big)^r\Big\}\Big)$$

$$\frac{1}{2\pi i}\oint_{|z|=1}\frac{dz}{z}\frac{1}{(1-\phi_1 z)(1-\frac{\phi_1^*}{z})(1-\phi_2 z^2)(1-\frac{\phi_2^*}{z^2})}$$

multiple poles, but not all reside in |z| < 1 (ϕ_1, ϕ_2 are also mod <1)



Molien form = PE

developed to capture invariants correctly

[Melia]

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Molien/Hilbert

developed these tools to study invariant polynomials related to mathematics of **rings**, **ideals**

Use in formal field theory (counting BPS states, moduli space)

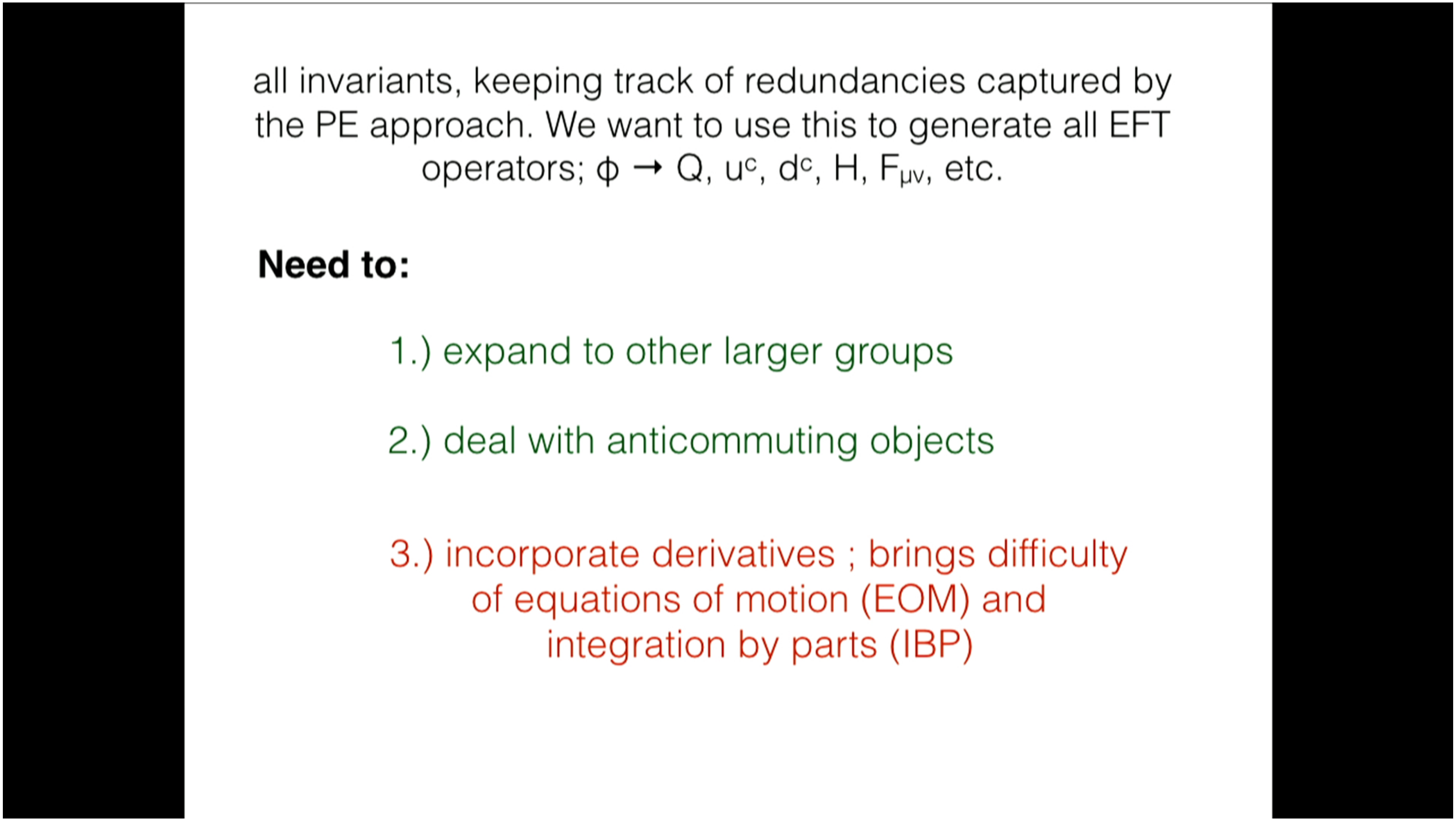
[Benvenuti, Feng, Hanany, He '06] [Feng, Hanany, He '07]

[Dolan '07] [Gray, Hanany, He, Jejjala, Mekareeya '08]

[Hanany, Mekareeya, Torri '08] …many more

Some use in counting flavor invariants

[Jenkins, Manohar '09], [Hanany, Jenkins, Manohar, Torri '10], [Merle, Zwicky '11]



all invariants, keeping track of redundancies captured by the PE approach. We want to use this to generate all EFT operators; $\phi \rightarrow Q$, u^c, d^c, H, $F_{\mu\nu}$, etc.

Need to:

1.) expand to other larger groups

2.) deal with anticommuting objects

3.) incorporate derivatives ; brings difficulty of equations of motion (EOM) and integration by parts (IBP) all invariants, keeping track of redundancies captured by the PE approach. We want to use this to generate all EFT operators; $\phi \rightarrow Q, u^c, d^c, H, F_{\mu\nu}$, etc.

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for a `field’ in a representation R of a group G,

z → $\chi_R(z_i)$, the **character** of the representation R

character?

if, under G $\phi_i \to D_{R,ij}\phi_j$ then $\chi_R = tr(D_R)$

χ_R are functions of **j** complex numbers, **j = rank of G**

(1 for SU(2), 2 for SU(3), etc..)

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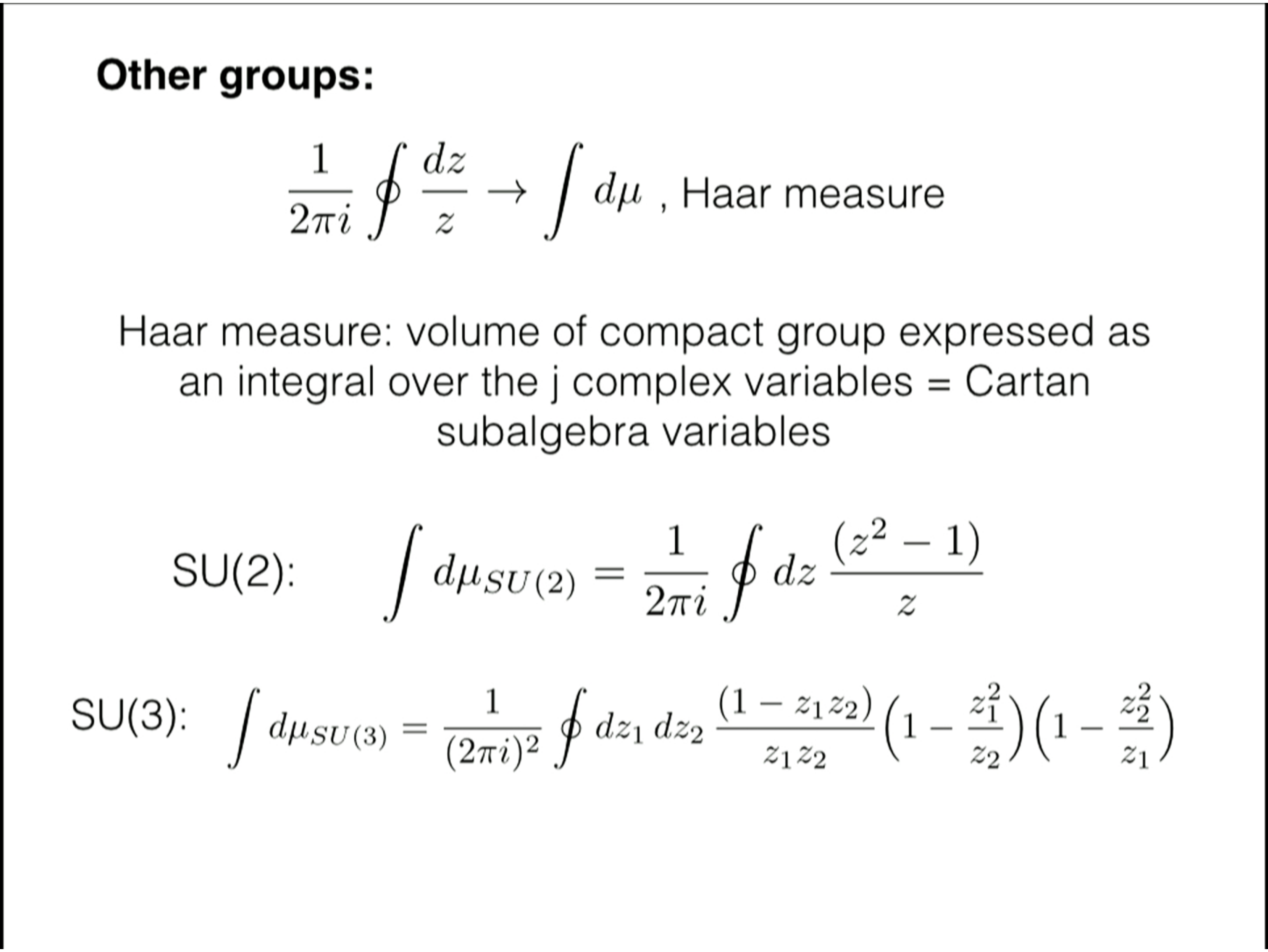


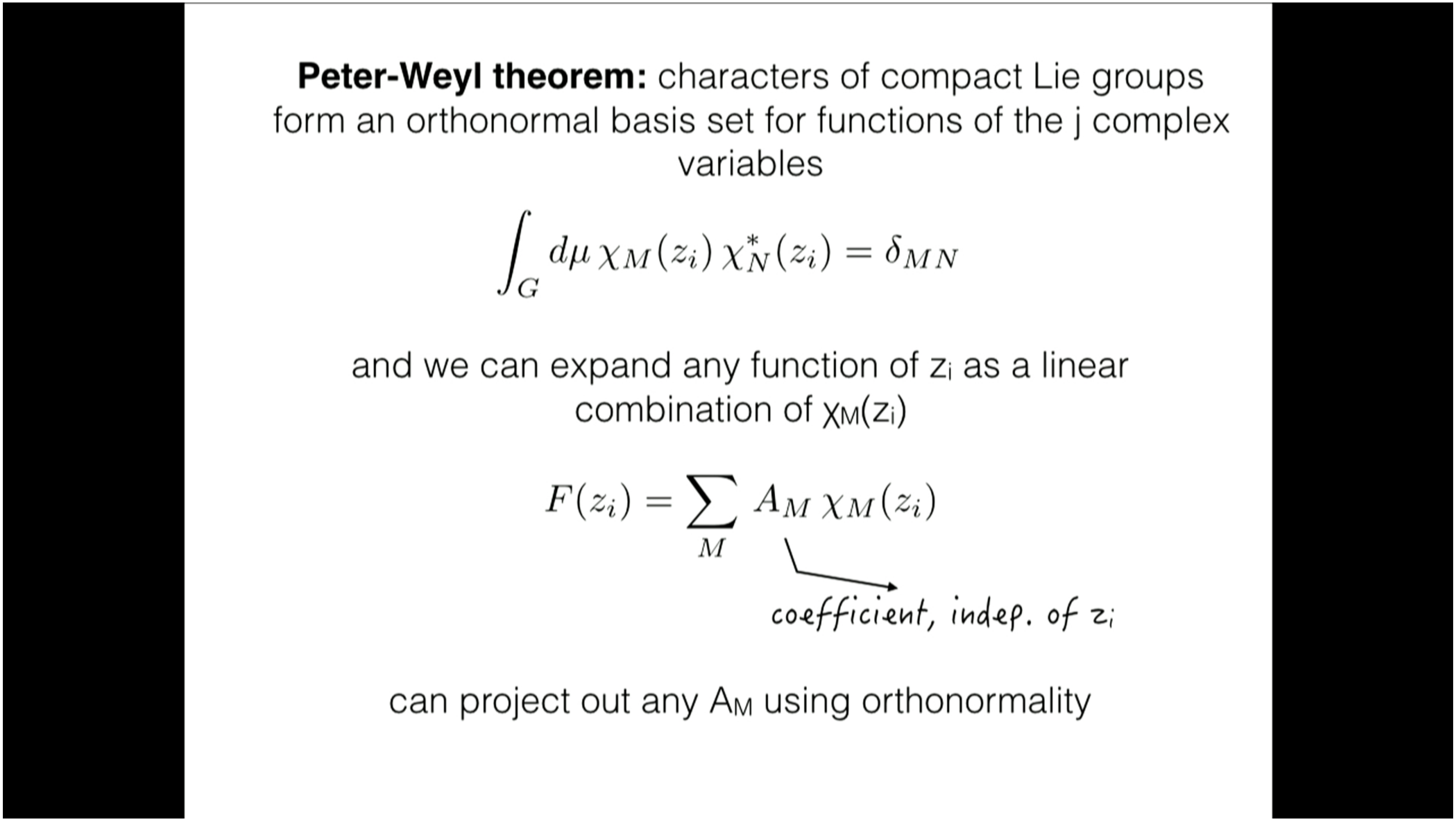
Other groups:

$$\frac{1}{2\pi i}\oint \frac{dz}{z} \to \int d\mu \text{ , Haar measure}$$

Haar measure: volume of compact group expressed as an integral over the j complex variables = Cartan subalgebra variables

SU(2): $$\int d\mu_{SU(2)} = \frac{1}{2\pi i}\oint dz\, \frac{(z^2-1)}{z}$$

SU(3): $$\int d\mu_{SU(3)} = \frac{1}{(2\pi i)^2}\oint dz_1\, dz_2\, \frac{(1-z_1 z_2)}{z_1 z_2}\Big(1-\frac{z_1^2}{z_2}\Big)\Big(1-\frac{z_2^2}{z_1}\Big)$$



Peter-Weyl theorem: characters of compact Lie groups form an orthonormal basis set for functions of the j complex variables

$$\int_G d\mu\, \chi_M(z_i)\, \chi_N^*(z_i) = \delta_{MN}$$

and we can expand any function of z_i as a linear combination of $\chi_M(z_i)$

$$F(z_i) = \sum_M A_M\, \chi_M(z_i)$$

coefficient, indep. of z_i

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So, for some field content in irreps under a SU(2):

1.) form the PE: PE[$\phi_1(\chi(z))$ +$\phi_2(\chi'(z))$ + …]

2.) PE is a function of the complex variables parameterizing the groups, z. Can be expanded in terms of characters

$$PE = \sum_M A_M \, \chi_M(z)$$

(singlet, doublet, triplet, etc..)

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Generalizes to multiple symmetry groups

1.) form the PE: $PE[\phi_1(\chi_1(z_1), \chi_2(z_2)\ldots) + \phi_2(\chi_1'(z_1), \chi'_2(z_2)) + \ldots]$

2.) PE is a function of the complex variables parameterizing the groups, z. can be expanded in terms of characters

$$PE = \prod_G \Big(\sum_M A^G_M \, \chi^G_M(z_i) \Big)$$ (combo of all reps of all groups)

3.) Integrate over Haar measure

$$\int \prod_G d\mu_G \prod_G \Big(\sum_M A^G_M \, \chi^G_M(z_i) \Big) = \prod_G A^G_0$$

only piece that survives is A_0, coefficient of **overall** singlet/ invariant irrep

Ex: doublet scalar with Higgs charges under SU(2)$_W$⊗U(1)$_Y$

$$PE[H(0,\frac{1}{2},-\frac{1}{2})+H^\dagger(0,\frac{1}{2},\frac{1}{2})]$$

$$\Downarrow$$

$$PE[H\left(z+\frac{1}{z}\right)u^{-1/2}+H^\dagger\left(z+\frac{1}{z}\right)u^{1/2}]$$

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asymmetric, plus they transform under Lorentz group

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Plethystic Exponential (PE)

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[Hanany '14]

→ Fermionic Plethystic Exponential (PEF)

$$PEF[\psi]=\exp\left\{\sum_{r=1}^{\infty}\frac{(-1)^{r+1}}{r}(\psi\,\chi(z_i))^r\right\}$$

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character

Lorentz group:

LH, RH fermions are in 2D reps of the Lorentz group

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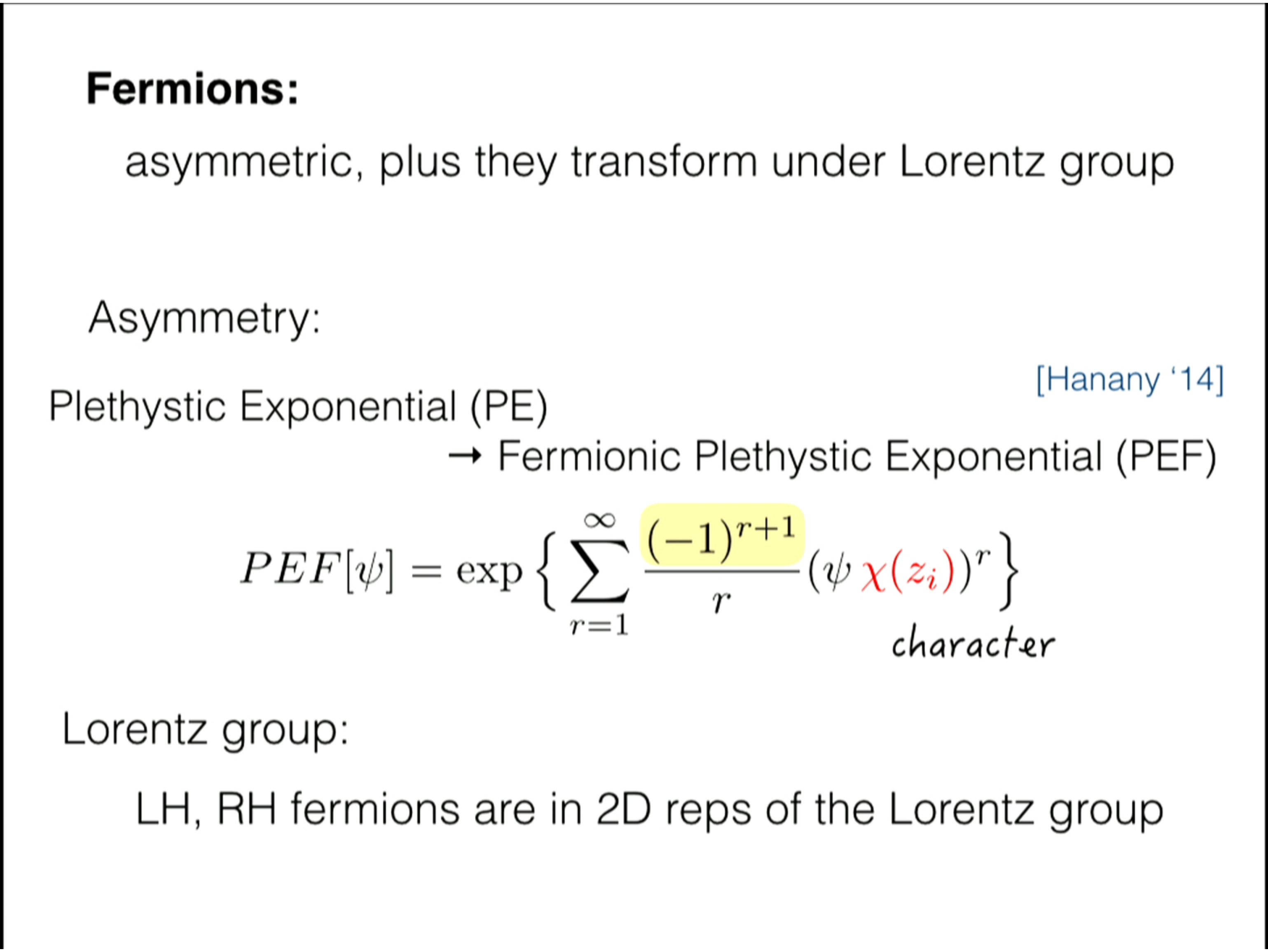

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BUT: orthonormality of characters only applies to compact Lie groups; SO(3,1) not compact

However: we don't care about dynamics, just counting invariants.
Work in Euclidean space!

$$SO(3,1) \rightarrow SO(4) \cong SU(2)_R \otimes SU(2)_L$$

Lorentz group just looks like two more symmetry groups

use LH fermions only for simplicity: Q, u^c, d^c, etc $\sim (0, 1/2)$

$Q^\dagger, u^{\dagger c}, d^{\dagger c}$, etc $\sim (1/2, 0)$

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$Q^\dagger$, $u^{\dagger c}$, $d^{\dagger c}$, etc ~ (1/2, 0)

Field strengths:

$$X^{\pm}_{\mu\nu} = X_{\mu\nu} \pm i\tilde{X}_{\mu\nu} \qquad \text{in (1,0) or (0,1) irrep.}$$

put the pieces together:

$$\mathcal{H}_{0,SM} = \int \prod_G d\mu_{Gi}\, PE[H, F^+, W^+, G^+ + c.c.]\times$$
$$PEF[Q, u^c, d^c, L, e^c + c.c]$$

$$[SU(2)_L \times SU(2)_R] \times SU(3)_c \times SU(2)_W \times U(1)_Y$$

generates **all** invariants (with one flavor of QUDLE) with no derivatives

comments:

1.) $N_F > 1$ easy:

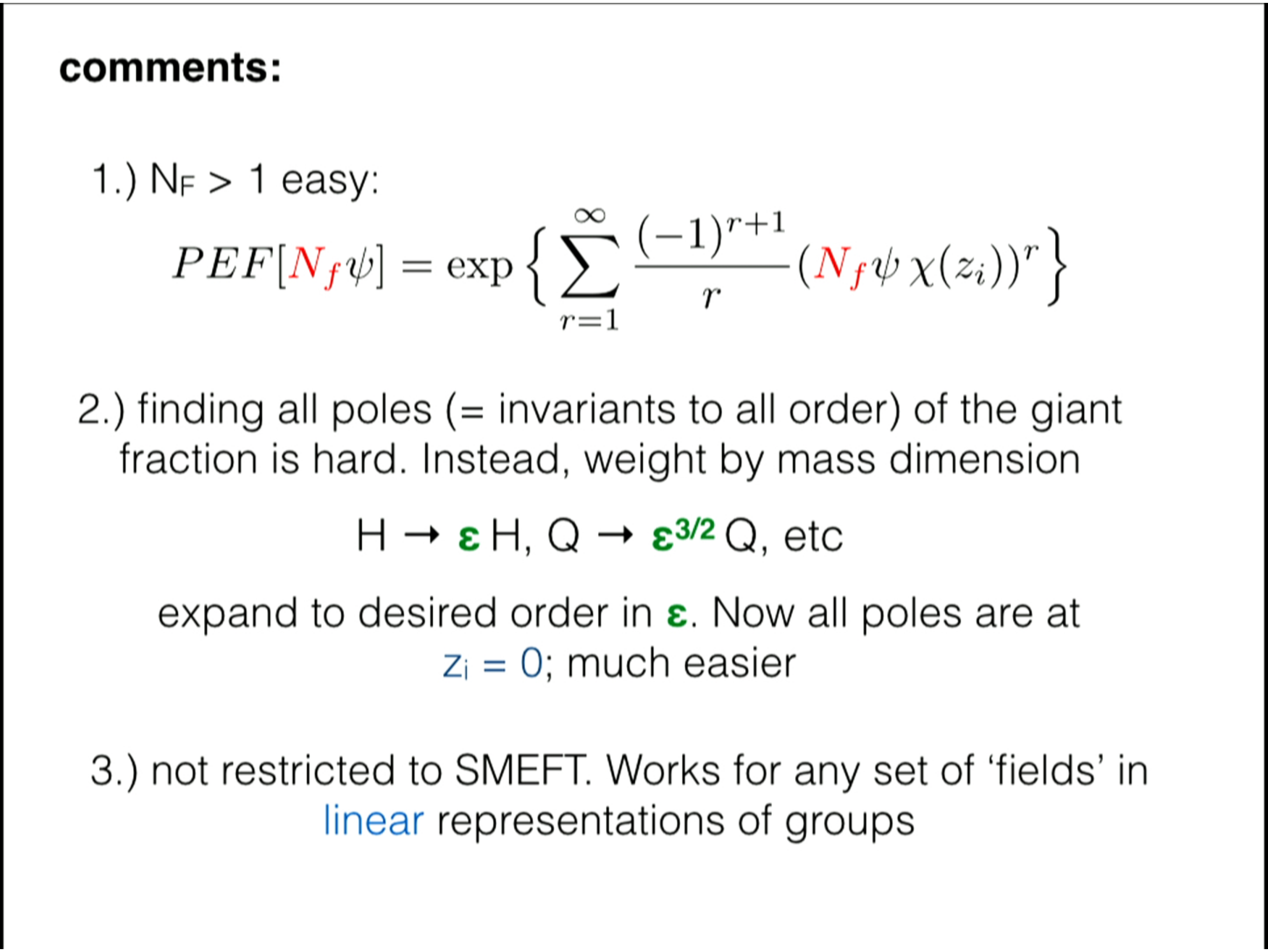
$$PEF[N_f\psi] = \exp\left\{\sum_{r=1}^{\infty}\frac{(-1)^{r+1}}{r}(N_f\psi\,\chi(z_i))^r\right\}$$

2.) finding all poles (= invariants to all order) of the giant fraction is hard. Instead, weight by mass dimension

H → $\boldsymbol{\varepsilon}$ H, Q → $\boldsymbol{\varepsilon}^{3/2}$ Q, etc

expand to desired order in $\boldsymbol{\varepsilon}$. Now all poles are at $z_i = 0$; much easier

3.) not restricted to SMEFT. Works for any set of ‘fields’ in linear representations of groups



comments:

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example: QQQL operators, $N_f = 3$

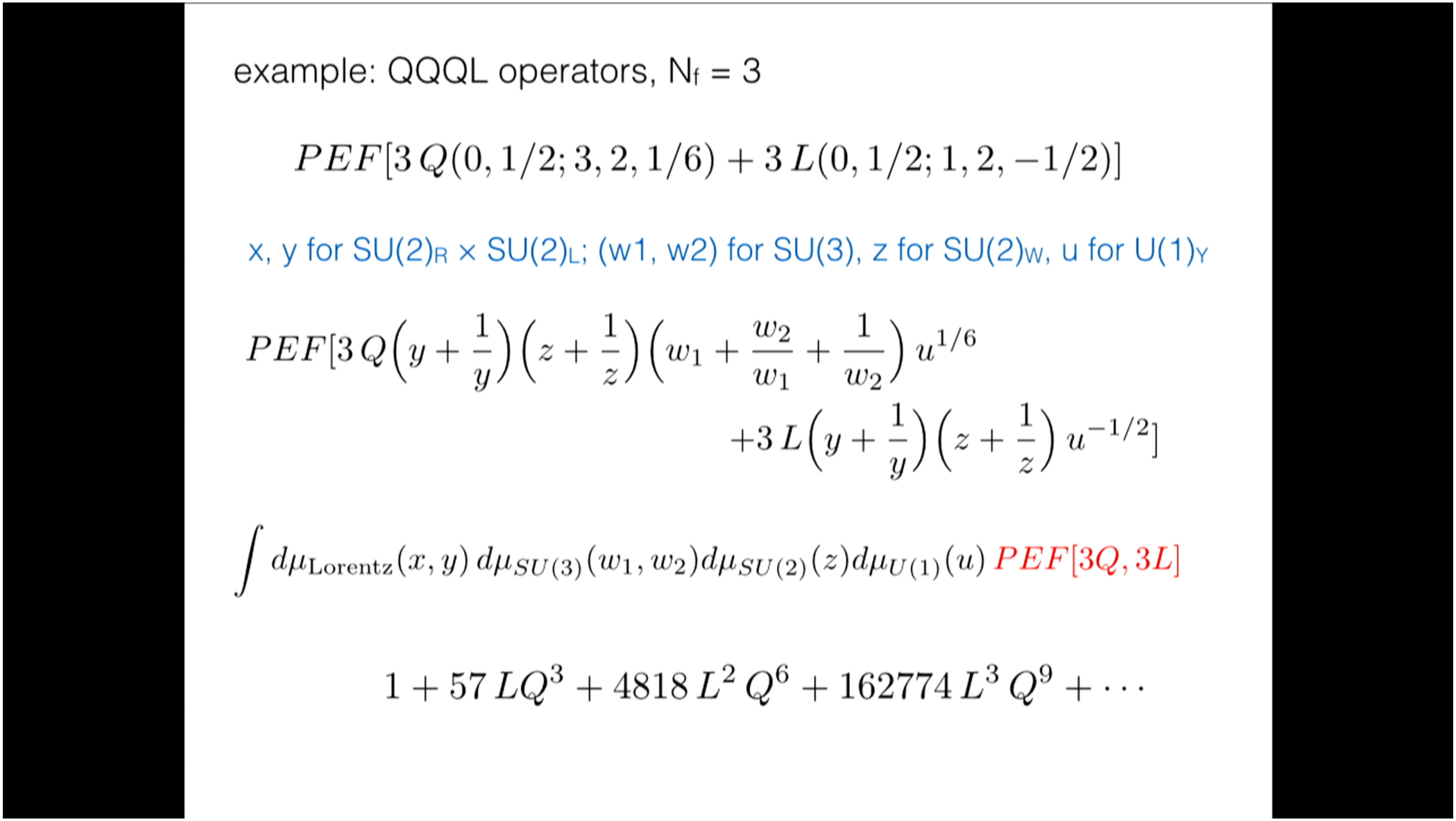
$$PEF[3\,Q(0,1/2;3,2,1/6) + 3\,L(0,1/2;1,2,-1/2)]$$

x, y for $SU(2)_R \times SU(2)_L$; (w1, w2) for SU(3), z for $SU(2)_W$, u for $U(1)_Y$

$$PEF[3\,Q\Big(y+\frac{1}{y}\Big)\Big(z+\frac{1}{z}\Big)\Big(w_1+\frac{w_2}{w_1}+\frac{1}{w_2}\Big)\,u^{1/6} \\ +3\,L\Big(y+\frac{1}{y}\Big)\Big(z+\frac{1}{z}\Big)\,u^{-1/2}]$$

$$\int d\mu_{\mathrm{Lorentz}}(x,y)\, d\mu_{SU(3)}(w_1,w_2) d\mu_{SU(2)}(z) d\mu_{U(1)}(u)\, PEF[3Q,3L]$$

$$1 + 57\,LQ^3 + 4818\,L^2\,Q^6 + 162774\,L^3\,Q^9 + \cdots$$



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derivatives:

general EFT expansion can have derivatives on fields as well as fields

$$\mathcal{L} \supset \phi^n,\ (\partial_\mu \phi)^n \phi^m, \text{etc}$$

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∂_μ ~ (1/2, 1/2) of Lorentz, so doesn’t look too terrible

but even at ∂^2 there are two possibilities:

$$\partial_{\{\mu,\nu\}}\phi, \quad \Box\phi$$

$$(1,1), \quad (0,0)$$

but any polynomial containing any $\Box\phi$ formed by the PE

i.e. $\phi^m \Box\phi$

always reduces via the EOM

$$\Box\phi = m^2\phi^2 + \lambda\phi^3$$ (for ϕ^4 theory)

form of RHS of EOM is not important. We only care that $\Box\phi$ can always be replaced by terms with fewer derivatives

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(all polynomials in ϕ and $\partial^2\phi$ = all polynomials in ϕ)

by same logic, at higher derivative order, only keep the fully symmetric term

$$PE[\phi] \to PE[\phi(0,0) + D\,\phi(1/2,1/2) + D^2\,\phi(1,1) + \cdots]$$

similar story for fermions:

derivative on LH field:

$$\overset{\partial_\mu}{(\frac{1}{2},\frac{1}{2})} \otimes \overset{Q}{(0,\frac{1}{2})} = (\frac{1}{2},0) \oplus (\frac{1}{2},1)$$

EOM: $\not{D} Q = y_u\, H\, u^{c\dagger} + y_d\, H^*\, d^{c\dagger}$

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& analogous story for field strengths

we can now ‘derivative-extend’ the SM PE.

$$H \to H\Big(0,0;\frac{1}{2},-\frac{1}{2}\Big) + DH\Big(\frac{1}{2},\frac{1}{2};\frac{1}{2},-\frac{1}{2}\Big) + D^2H\Big(1,1;\frac{1}{2},\frac{1}{2}\Big)\cdots$$

$SU(2)_R, SU(2)_L$ $\quad$ $SU(2)_W, U(1)_Y$

+ same for all other fields

Generates **all** invariant polynomials of fields & their derivatives, no EOM redundancy

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& analogous story for field strengths

comments:

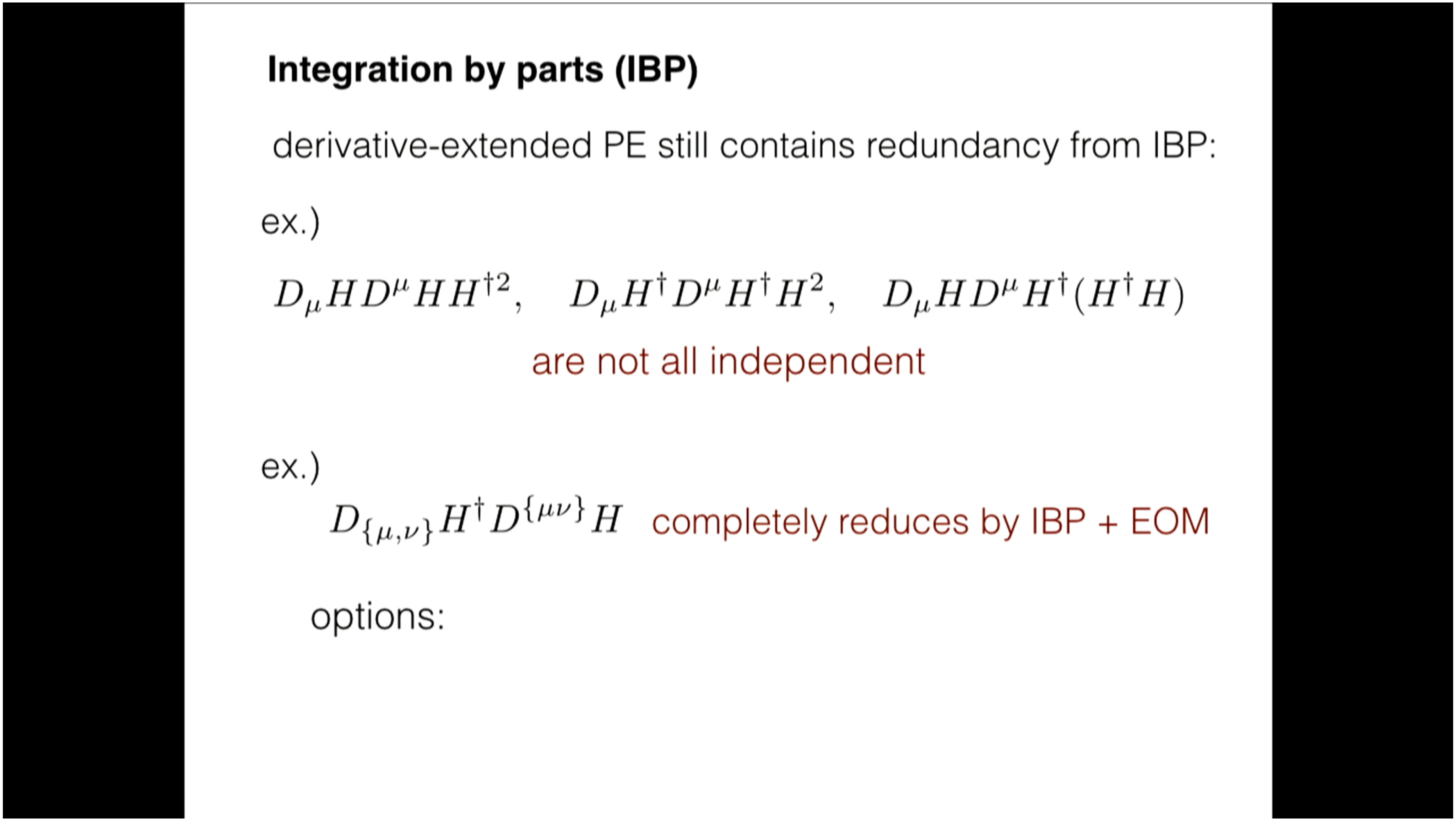
1.) for each derivative, we need a new entry in the PE/PEF. Can sum derivatives to all orders:

$$\frac{(1-D^2)}{(1-D\,x\,y)(1-D/(xy))(1-D\,x/y)(1-D\,y/x)} \qquad \text{(scalars)}$$

$\{x, y\}$ parametrize $SU(2)_L \times SU(2)_R$

2.) **D** is a separate spurion; i.e. **D**ϕ is two separate objects, not one. This allows us to keep track of the total # derivatives in an invariant

3.) At a given mass dimension, don't need all orders (at dim=8, know <3 derivs)



Integration by parts (IBP)

derivative-extended PE still contains redundancy from IBP:

ex.)

$$D_\mu H D^\mu H H^{\dagger 2}, \quad D_\mu H^\dagger D^\mu H^\dagger H^2, \quad D_\mu H D^\mu H^\dagger (H^\dagger H)$$

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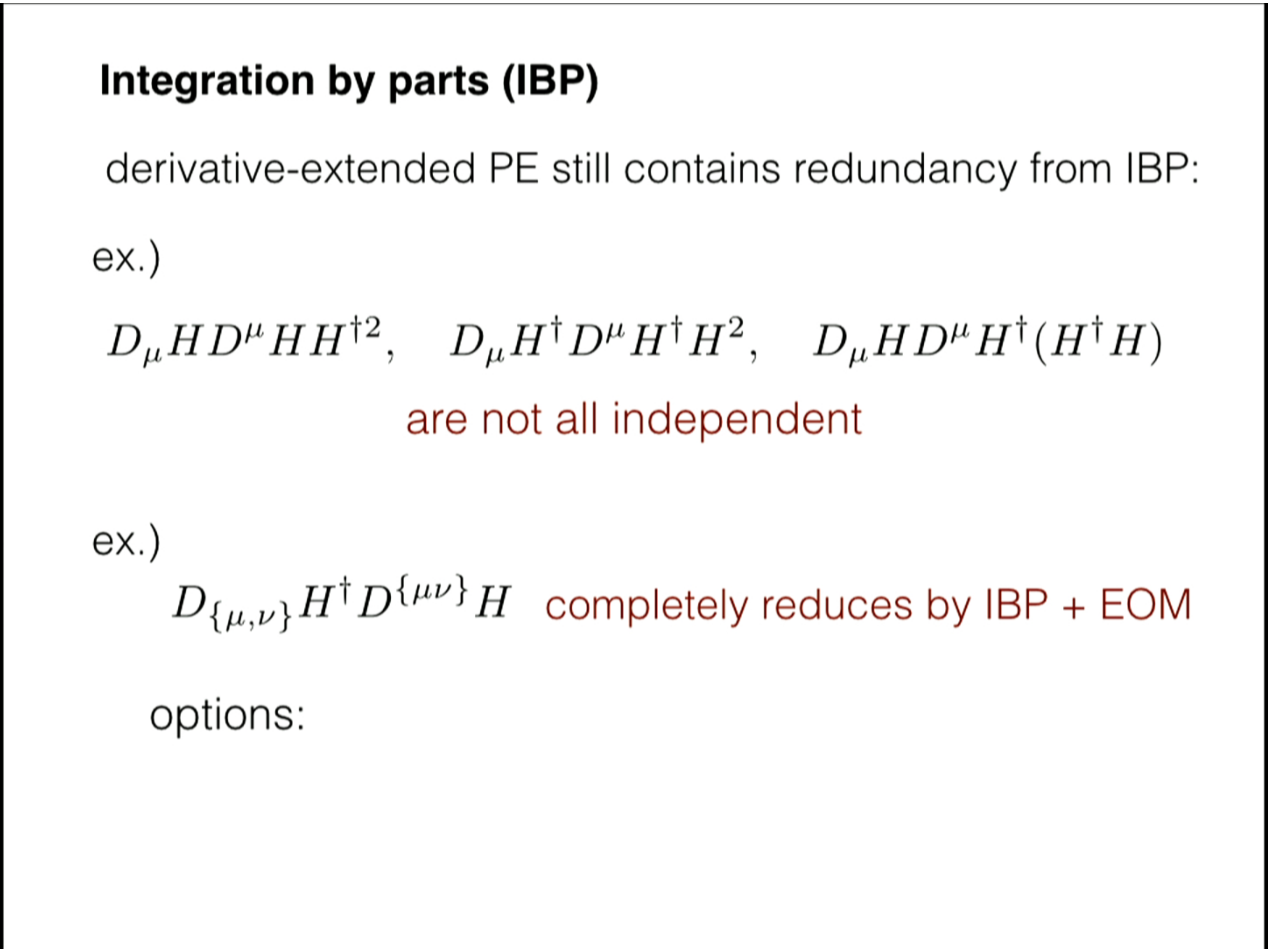

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options:

explore using a toy theory of a single real scalar

$$h_\phi = \int d\mu_{SU(2)_L} d\mu_{SU(2)_R} PE[\phi(1 + D(\frac{1}{2}, \frac{1}{2}) + D^2(1,1) + \cdots)]$$

we can group terms in h_φ by the number of D and ϕ

$\mathcal{O}(D^m \phi^n)$ = m derivs, n scalars

idea:

all $\mathcal{O}(D^m \phi^n)$ must come from $D \times \mathcal{O}(D^{m-1} \phi^n)$

if we can count the number of $\mathcal{O}(D^{m-1} \phi^n)$, thats a set of constraints on the $\mathcal{O}(D^m \phi^n)$

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to get # $\mathcal{O}(D^{m-1}\phi^n)$:

recall PE generates **all** combinations, not just invariants. we just need to pick out the part of the PE in the (1/2, 1/2) representation

$$PE[\phi] = \sum_R A_{\phi,R}\,\chi_R(z_i)$$

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$$h_{D\phi} = \int d\mu_{SU(2)_L}d\mu_{SU(2)_R}\left(\frac{1}{2},\frac{1}{2}\right)PE[\phi]\ = A_{(1/2,1/2)}$$

picks out just the term we want

conjecture:

number of invariants from a set of fields in linear representations of symmetry groups, including EOM and IBP redundancy

$$h_{\phi_i,\psi_i} - D\, h_{D,\phi_i,\psi_i}$$

Hilbert series,
projecting out
invariants

gets
dimension
correct

Hilbert series,
projecting out
(1/2, 1/2)

in gory detail: parametrize $SU(2)_L \times SU(2)_R$ with $\{x, y\}$

$$\overbrace{\frac{1}{(2\pi\, i)^2}\int dx\,\frac{(1-x^2)}{x}\int dy\,\frac{(1-y^2)}{y}}^{d\mu_G} \times \Big(1 - D(\frac{1}{x}+x)(\frac{1}{y}+y)\Big)$$

$$\times\, PE\Big[\frac{(1-D^2)\,\phi}{(1-D\,x\,y)(1-D/(x\,y))(1-D\,x/y)(1-D\,y/x)}\Big]$$

some dim-8, according to this algorithm:

$$\begin{array}{cccc}
(d_c^\dagger d_c)(e_c^\dagger e_c)F^L & (u_c^\dagger u_c)(e_c^\dagger e_c)F^L & 2\,(d_c^\dagger d_c)(u_c^\dagger u_c)F^L & (d_c^\dagger d_c)(L^\dagger L)F^L \\
(u_c^\dagger u_c)(L^\dagger L)F^L & (e_c^\dagger e_c)(L^\dagger L)F^L & (e_c^\dagger e_c)(Q^\dagger Q)F^L & (d_c Q)(e_c^\dagger L^\dagger)F^L \\
(d_c Q)(e_c^\dagger L^\dagger)F^R & 2\,(L^\dagger L)(Q^\dagger Q)F^L & 2\,(d_c^\dagger d_c)(Q^\dagger Q)F^L & 2\,(u_c^\dagger u_c)(Q^\dagger Q)F^L \\
3\,(e_c\,L)(u_c\,Q)F^L & 3\,(u_c\,d_c)\,Q^2\,F^L & (d_c^\dagger d_c)(L^\dagger L)W^L & (e_c^\dagger e_c)(L^\dagger L)W^L \\
(e_c^\dagger e_c)(Q^\dagger Q)W^L & (u_c^\dagger u_c)(L^\dagger L)W^L & (L^\dagger L)^2 W^L & (e_c^\dagger L^\dagger)(d_c\,Q)W^L \\
(e_c\,L)(d_c^\dagger Q^\dagger)W^L & 2\,(d_c^\dagger d_c)(Q^\dagger Q)W^L & 2\,(u_c^\dagger u_c)(Q^\dagger Q)W^L & 3\,(L^\dagger L)(Q^\dagger Q)W^L \\
2\,(Q^\dagger Q)^2 W^L & 3\,(e_c\,L)(u_c\,Q)W^L & 3\,(u_c\,d_c)\,Q^2\,W^L & (d_c^\dagger)^2\,d_c^2\,G^L \\
(u_c^\dagger)^2\,u_c^2\,G^L & (d_c^\dagger d_c)(e_c^\dagger e_c)\,G^L & (u_c^\dagger u_c)(e_c^\dagger e_c)\,G^L & 4\,(d_c^\dagger d_c)(u_c^\dagger u_c)\,G^L \\
(Q^\dagger Q)(e_c^\dagger e_c)\,G^L & (d_c^\dagger d_c)(L^\dagger L)\,G^L & (u_c^\dagger u_c)(L^\dagger L)\,G^L & 2\,(Q^\dagger Q)(L^\dagger L)\,G^L \\
4\,(d_c^\dagger d_c)(Q^\dagger Q)\,G^L & 4\,(u_c^\dagger u_c)(Q^\dagger Q)\,G^L & 2\,(Q^\dagger)^2\,Q^2 G^L & (d_c\,Q)(e_c^\dagger L^\dagger)\,G^L \\
(d_c\,Q)(e_c^\dagger L^\dagger)\,G^R & 3\,(e_c\,L)(u_c\,Q)\,G^L & 6\,(d_c\,u_c)\,Q^2\,G^L &
\end{array}$$

112 at $O(D^0)$

$$\begin{array}{cccc}
3\,D\,(d_c^\dagger d_c)(L\,H\,e_c) & D\,(e_c^\dagger e_c)(L\,H\,e_c) & 3\,D\,(L^\dagger L)(L\,H\,e_c) & 3\,D\,(d_c^\dagger d_c)(Q\,H\,d_c) \\
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6\,D\,(Q^\dagger Q)(Q\,H^\dagger u_c) & 3\,D\,(u_c^\dagger u_c)(L\,H\,e_c) & 6\,D\,(u_c^\dagger u_c)(Q\,H\,d_c) & 3\,D\,(u_c^\dagger u_c)(Q\,H^\dagger u_c)
\end{array}$$

181 at $O(D)$

535 total (931 counting +h.c. separately)

issues:

method breaks down if # $\mathcal{O}(D^{m-1}\phi^n)$ terms are not independent..

problems arise when there are multiple ways to partition derivatives on fields

m=6, n=4:

$$(\partial^2\phi)^2(\partial\phi)^2$$

$$(\partial_{\{\mu,\nu\}}\partial^{\{\mu,\nu\}}\phi)\,(\partial^\rho\phi\,\partial_\rho\phi)$$

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Henning, Lu, Melia, Murayama 1512.03433

instead of $\int \prod d\mu_G \left(1 - D(1/2,1/2)\right) PE[\phi\, P(1 - D^2)]$:

$$P = \left(1 - D(1/2,1/2) + D^2((1,0) + (0,1)) - D^3(1/2,1/2) + D^4\right)^{-1}$$

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works, free of issues

- extend d=8 SMEFT set to 992 (+62 from Lehman, AM)
- count d=9,10,11,12 SMEFT operators (560, 15456, 11962..)
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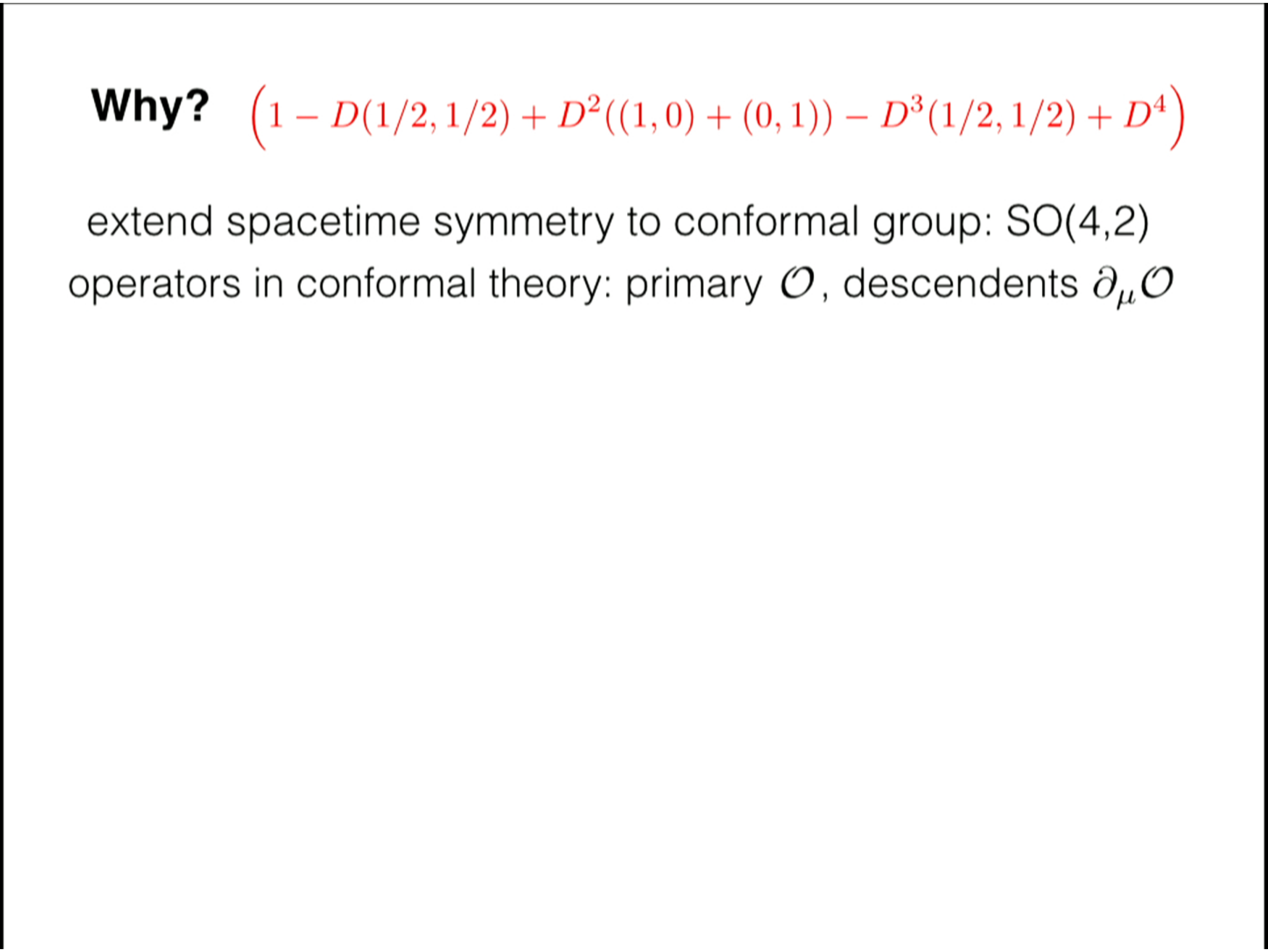

Why? $\left(1 - D(1/2,1/2) + D^2((1,0)+(0,1)) - D^3(1/2,1/2) + D^4\right)$

extend spacetime symmetry to conformal group: SO(4,2)

operators in conformal theory: primary $\mathcal{O}$, descendents $\partial_\mu \mathcal{O}$

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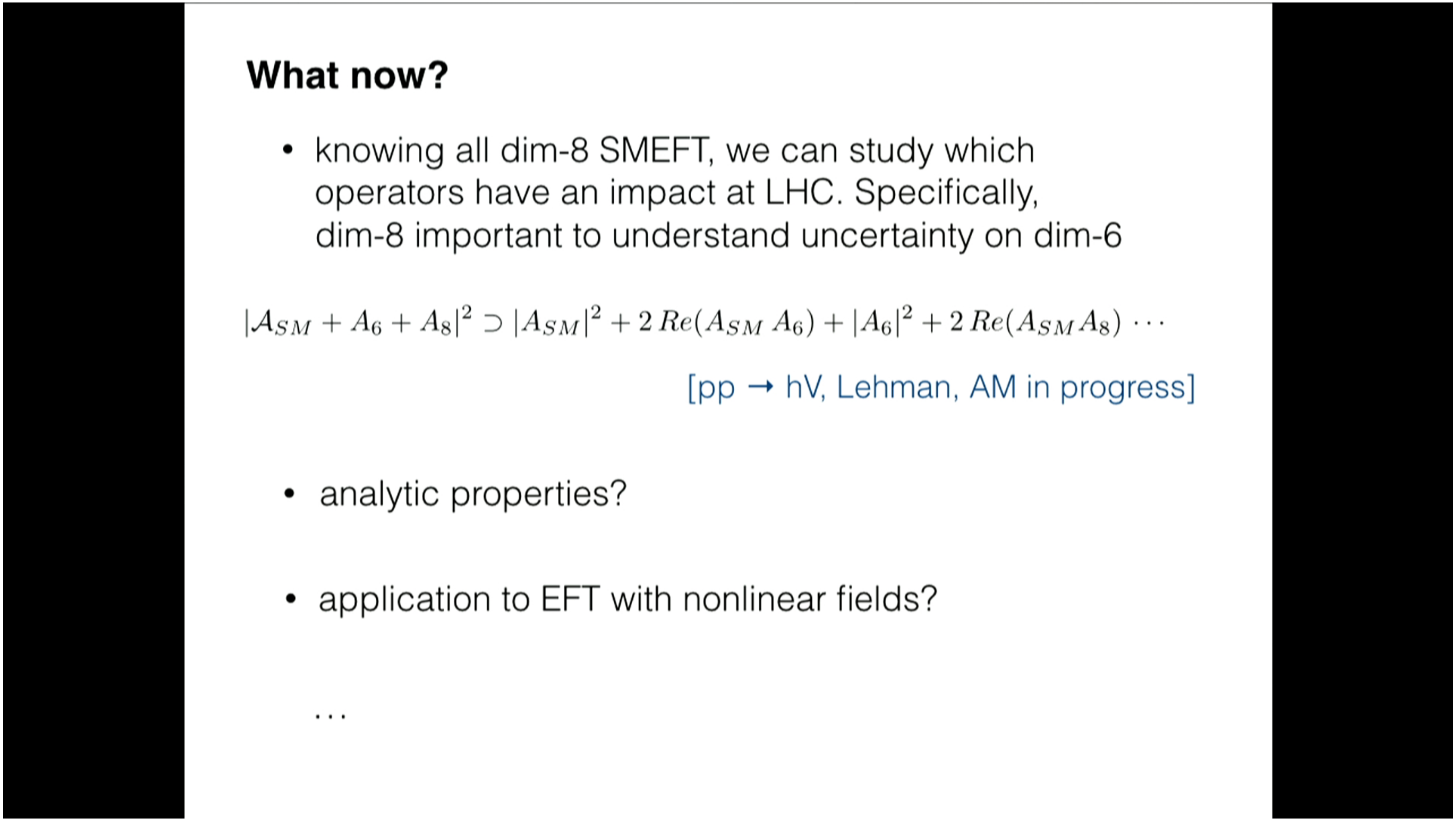
removing IBP redundancy = eliminating all operators that are descandants of other ops.

accomplished by keeping only the highest conformal weight of operator products

integrate over **SO(4,2)/SO(3,1)** (dilations, conformal trans)

results in **P** prefactor



What now?

- knowing all dim-8 SMEFT, we can study which operators have an impact at LHC. Specifically, dim-8 important to understand uncertainty on dim-6

$$|\mathcal{A}_{SM} + A_6 + A_8|^2 \supset |A_{SM}|^2 + 2\,Re(A_{SM}\,A_6) + |A_6|^2 + 2\,Re(A_{SM}\,A_8)\,\cdots$$

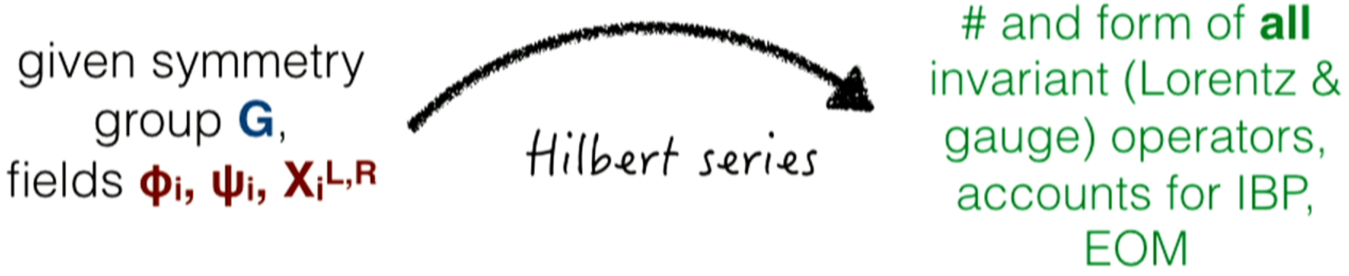
[pp → hV, Lehman, AM in progress]

- analytic properties?

- application to EFT with nonlinear fields?

…

conclusions:



given symmetry group **G**, fields $\boldsymbol{\phi_i}$, $\boldsymbol{\psi_i}$, $\boldsymbol{X_i^{L,R}}$

and form of **all** invariant (Lorentz & gauge) operators, accounts for IBP, EOM

- generates all possible combinations of operators, uses character orthonormality to pick out invariants
- derivatives tricky, but issues recently overcome

lots of interesting directions to explore!