Title: Landau Singularities and Symbology

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Abstract: <p>In previous work it has been observed that the singularity structure of multi-loop scattering amplitudes in planar N=4
 super-Yang-Mills theory is evidently dictated by cluster algebras. In my talk I will discuss the interplay between this mathematical
 observation and the physical principle that the singularities of Feynman integrals are encoded in the Landau equations.</p>

Landau Singularities & Symbology

S-matrix of $\mathcal{N}=4$

cluster algebras

MATH

What physics might dictate ?

One thing which has a chance is Landau singularities.

Landau Equations

$$I = \int \prod_{r=1}^{L} \frac{d^D \ell_r}{i\pi^{D/2}} \frac{1}{\prod_{j=1}^{N} (q_j^2 - m_j^2)} = \Gamma(N) \int \prod_{r=1}^{L} \frac{d^D \ell_r}{i\pi^{D/2}} \int_{\alpha_i \geq 0} d^N\alpha \; \delta\left(1-\sum \alpha\right) \frac{1}{\mathcal{D}^N}$$

L = loops

N = # of propagators

$$\mathcal{D} = \sum_i \alpha_i \left(-q_i^2 + m^2\right).$$

Landau Equations

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L = loops

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$$\mathcal{D} = \sum_i \alpha_i (-q_i^2 + m^2).$$

There are two distinct situations where I can develop a singularity

(1) Two zeroes of $\mathcal{D}$ collide & pinch a contour in all variables (ℓ_r^μ, α_i) simultaneously.

$\Rightarrow$ Leading Landau singularity LLS.

Condition $\Rightarrow$ $\dfrac{\partial D}{\partial \ell_r^\mu} = 0$ $\forall r, \mu$ $\Rightarrow$ Kirkhoff rule $\Rightarrow$ for each loop $\sum_{i \in \text{propagators in that loop}} \alpha_i q_i^\mu = 0$

$\dfrac{\partial D}{\partial \alpha_i} = 0$ $\forall i$ $\Rightarrow$ each propagator should be on shell. $-q_i^2 + m_i^2 = 0$

$\ell_r^\mu, \alpha_i)$ simultaneously.

$\Rightarrow$ for each loop $\sum_{i \in \text{propagators in that loop}} \alpha_i q_i^\mu = 0$

$-q_i^2 + m_i^2 = 0$

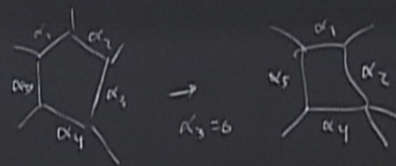
The set of possible solutions to these equations

$\Rightarrow$ The locus where I (in external kinematic space)

Can develop a branch cut.

(2) A zero of $\mathcal{D}$ hits the boundary of the contour, @ $\alpha_i = 0$ for one or more i's

$\Rightarrow$ subleading LS.

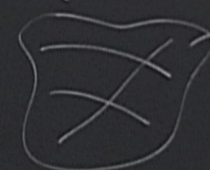


Leading singularities

4L-dimensional space of loop integration variables

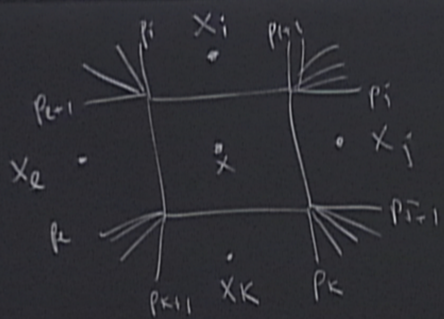
Points where integrand has poles.

Landau singularities



n-particle scattering

loci in the $3(n-5)$-dim'l space where the integral has singularities.



$$= \int d^4x \, \frac{1}{(x-x_i)^2(x-x_j)^2(x-x_k)^2(x-x_l)^2}$$

trivial $\sum p_a = 0$ by using

dual variables. $p_a = x_a - x_{a-1}$

solve $(x-x_i)^2=0$

$(x-x_j)^2=0$

$(x-x_k)^2=0$

$(x-x_l)^2=0$

$\alpha_i(x-x_i)+\alpha_j(x-x_j)+\alpha_k(x-x_k)+\alpha_l(x-x_l)=0.$

$\det\big((x-x_i) \quad (x-x_j$

$$\det\big((x-x_i) \quad (x-x_j) \quad (x-x_k) \quad (x-x_l) \big) = 0$$

$\ldots (x-x_l)^2$

$\ldots) + \alpha_k (x - x_k) + \alpha_l (x - x_l) = 0,$

$$\det\big((x-x_i) \quad (x-x_j) \quad (x-x_k) \quad (x-x_l) \big) = 0$$

$$\Rightarrow \left(x_{ij}^2\, x_{kl}^2 - x_{ik}^2\, x_{jl}^2 + x_{il}^2\, x_{jk}^2 \right)^2 - 4\, x_{ij}^2\, x_{jk}^2\, x_{kl}^2\, x_{il}^2 = 0$$

where $x_{ab} = x_a - x_b$

$$\ldots) + \alpha_k (x - x_k) + \alpha_l (x - x_l) = 0 .$$

Polylogs How do we see logarithmic singularities in amplitudes?

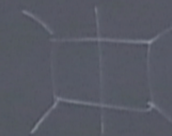
generalized polylogs $\Rightarrow$ generalize $Li_k(z) = \int_0^z \frac{dt}{t} Li_{k-1}(t)$ $Li_1(z) = -\log(1-z)$

A polylog function of weight k is one with the property that $df_k = \sum_i f_{k-1}^{(i)} \, d\log \phi_i$

$df_{k-1}^{(i)} = \sum_j f_{k-2}^{(ij)} \, d\log \phi_j$ etc.

The set of ϕ's which appear in a given amplitude is called its symbol alphabet trivial $\sum p_a = 0$ by using dual variables. $p_a = x_a - x_{a-1}$

$$(x-x_j)^2=0$$
$$(x-x_k)^2=0$$
$$(x-x_l)^2=0$$
$$\alpha_i(x-x_i)+\alpha_j(x-x_j)+\alpha_k(x-x_k)+\alpha_l(x-x_l)=0.$$



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The set of ϕ's which appear in a given amplitude is called its symbol alphabet

Amplitudes have branch points when some $\phi=0$ or $\phi=\infty$

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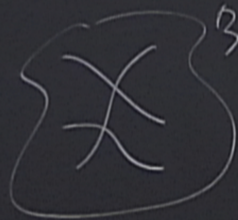
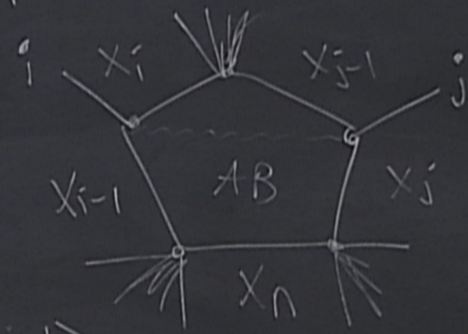
The symbol alphabet of any amplitude should have the property that the zeroes of the ϕ's should specify values of external kinematics where solutions of LE exist.



$3(n-5)$.

1-loop MHV amplitudes in $\mathcal{N}=4$

$$A^{1\text{-loop}}_{MHV} = \sum_{1<i<j<n}$$



$$= \frac{\langle AB\, \bar{i} \cap \bar{j} \rangle \langle i j n 1 \rangle}{\langle AB\, i-1\, i \rangle \langle AB\, i\, i+1 \rangle \langle AB\, j-1\, j \rangle \langle AB\, j\, j+1 \rangle \langle AB\, n\, 1 \rangle}$$

L = loops

N = # of propaga

There are two distinct

some linear comb. of Li_2 & $\log^2$ terms

The symbol entries are

- $\langle i\ i{+}1\ j\ j{+}1 \rangle \propto (x_i - x_j)^2$

first entry of the symbol
$\Rightarrow$ locations of branch points on the first sheet,

correspond to $(p_{i+1} + \cdots + p_j)^2 \longrightarrow 0$

- on the second sheet (ie second entry) you can get additional symbol entries
$\langle i\ \bar{j} \rangle$ $\quad \bar{j} = (j-1, j, j+1)$ & permutations.

= some linear comb. of Li_2 & $\log^2$ terms

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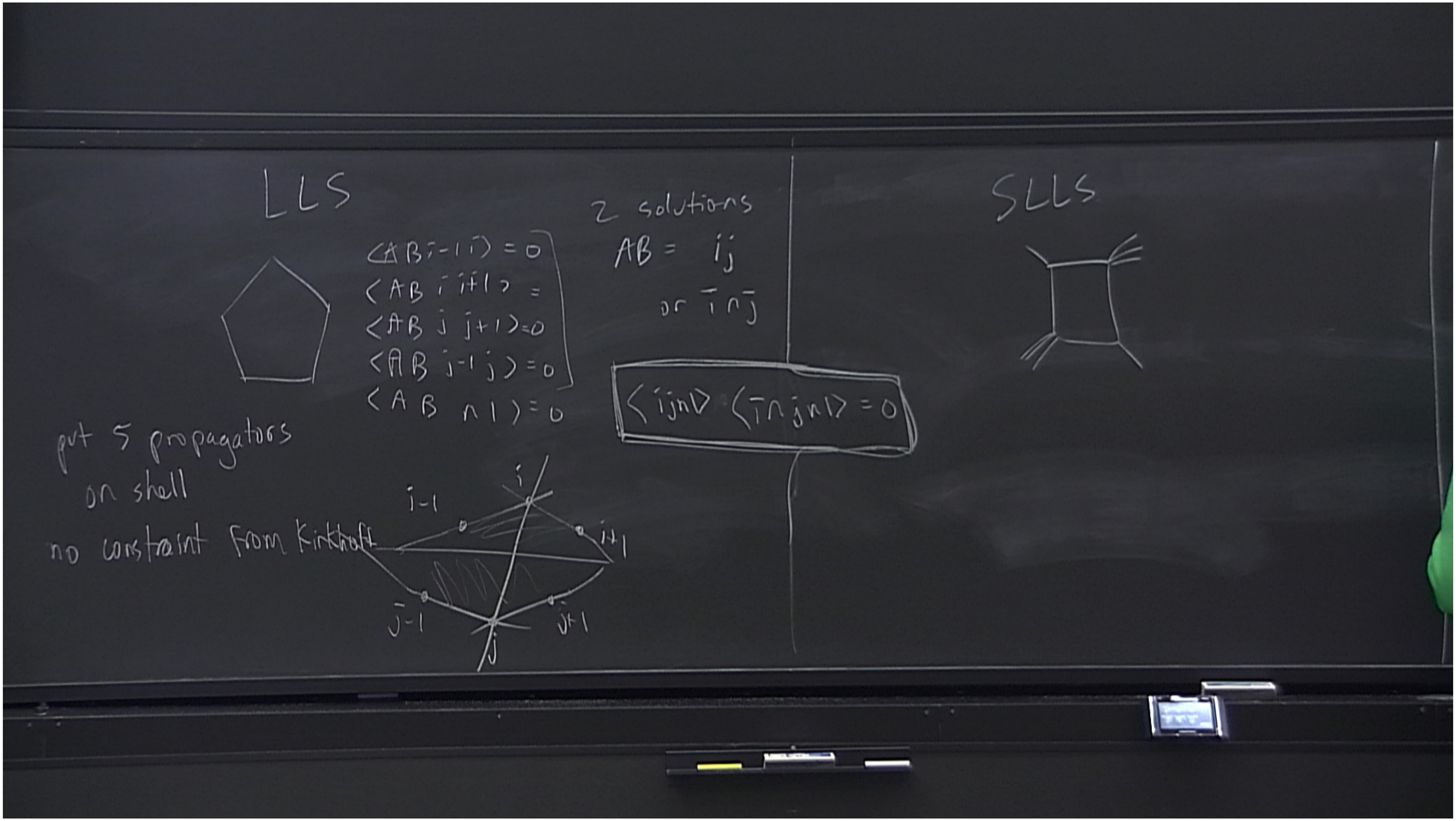
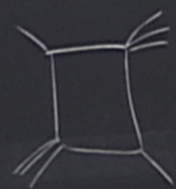
  $\langle i\; \bar{j}\rangle$ $\quad \bar{j} = (j-1, j, j+1)$ & permutations.

  $\langle i\,(i-1, i+1)(j, j+1)(n1)\rangle$


LLS
⟨AB i-1 i⟩ = 0
⟨AB i i+1⟩ =
⟨AB j j+1⟩=0
⟨AB j-1 j⟩ = 0
⟨AB n 1⟩ = 0
put 5 propagators
on shell
no constraint from Kirchhoff
i-1
i
i+1
j̄-1
j
j̄+1
2 solutions
AB = ij
or ī∩j̄
⟨ijn1⟩ ⟨ī∩j̄n1⟩ = 0
SLLS

$\boxed{\ldots j\, n\, 1\rangle = 0}$

SLLS



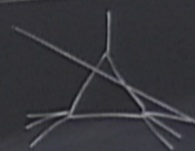
$\Rightarrow$ all possible second entries

$\langle i\, j\rangle = 0$

$\langle i\,(i-1, i+1)(j, j+1)(n\,1)\rangle = 0.$

SSLLS

or

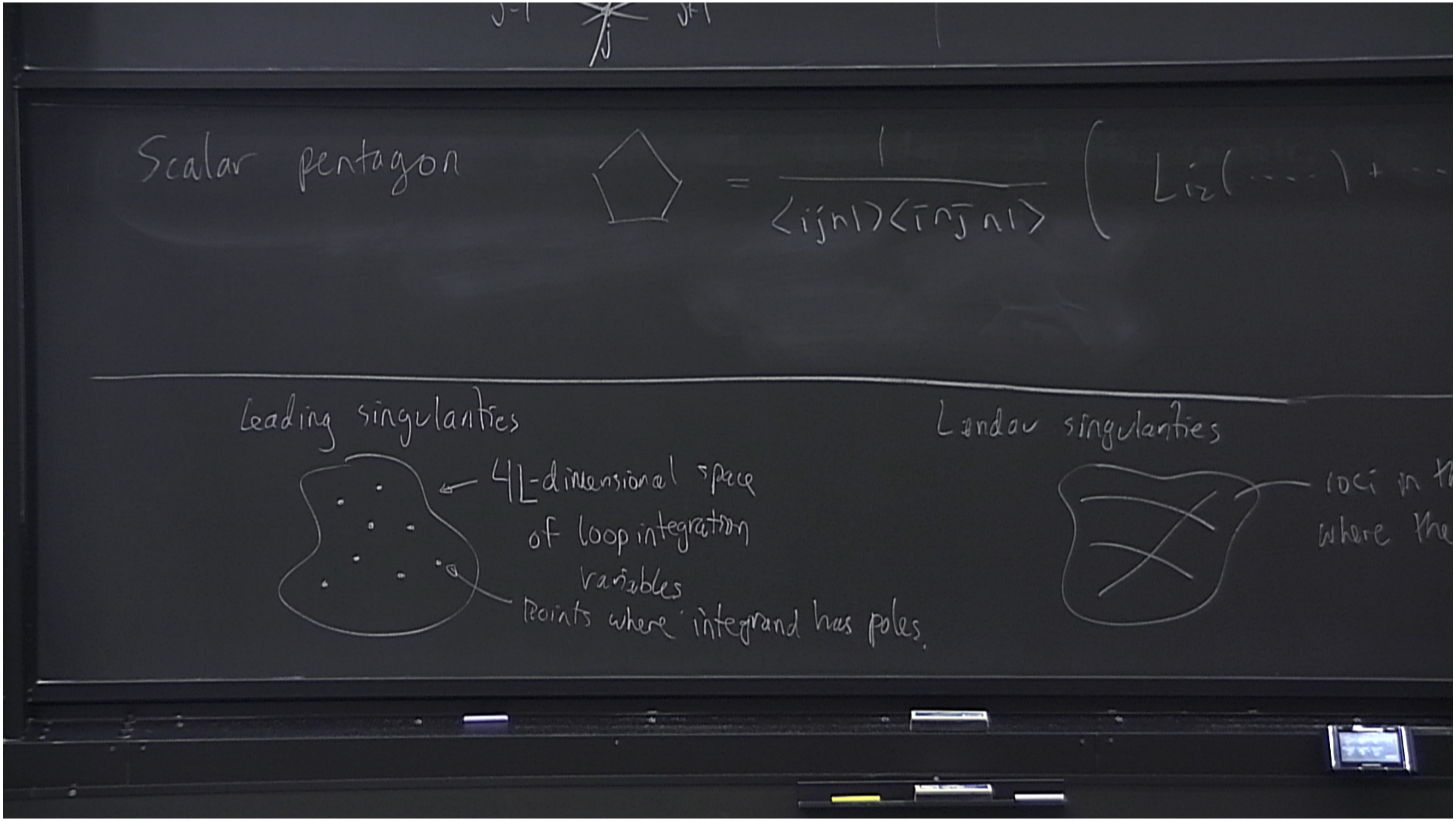


~~allows landau singularities for any kinematics~~

$\Rightarrow$ $\langle a, a+1, b, b+1\rangle = 0$

for all choices of $a, b \in \{i, j, k, l\}$

$\Rightarrow$ first entries of the symbol


Scalar pentagon
= 1 / ⟨ijn1⟩⟨īnjn1⟩ (Li2(. . .) + . . .
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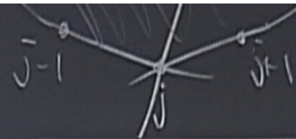
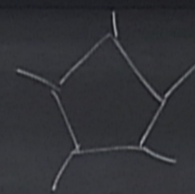
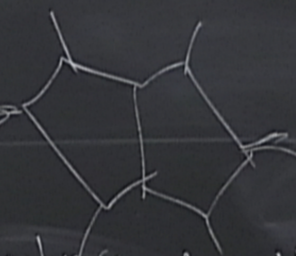


Scalar pentagon  $= \dfrac{1}{\langle i j n 1\rangle \langle \bar{i} \cap \bar{j}\, n 1\rangle} \Big(\mathrm{Li}_2(\cdots) + \cdots \log^2$

We did a 2-loop analysis $\Rightarrow$  and found $\{$list of sy

Caveats $\Rightarrow$

(1) Landau analysis doesn't know about numerators

(2) 2^{nd} class Landau singularities $\Rightarrow$ poles @ ∞ in loop momenta.

(3). ...

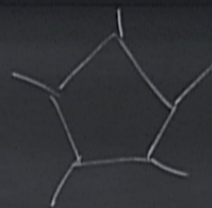
$\text{i}_2(\cdots) + \cdots \log^2(\;) \Big)$.

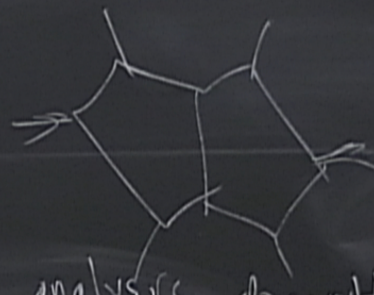
$\{\text{list of symbol entries}\} \subsetneq \left\{ \begin{array}{l} \text{set of brackets which vanish} \\ \text{on solutions of Landau equations} \end{array} \right\}$.

ators

in loop momenta.

$j-1 \quad j \quad j+1$

Scalar pentagon  $= \dfrac{1}{\langle i j n 1\rangle\langle \bar{i} \bar{n} \bar{j}\, n 1\rangle}\left(Li_2(\cdots) + \cdots \right.$

We did a 2-loop analysis $\Rightarrow$  and found $\{$ list

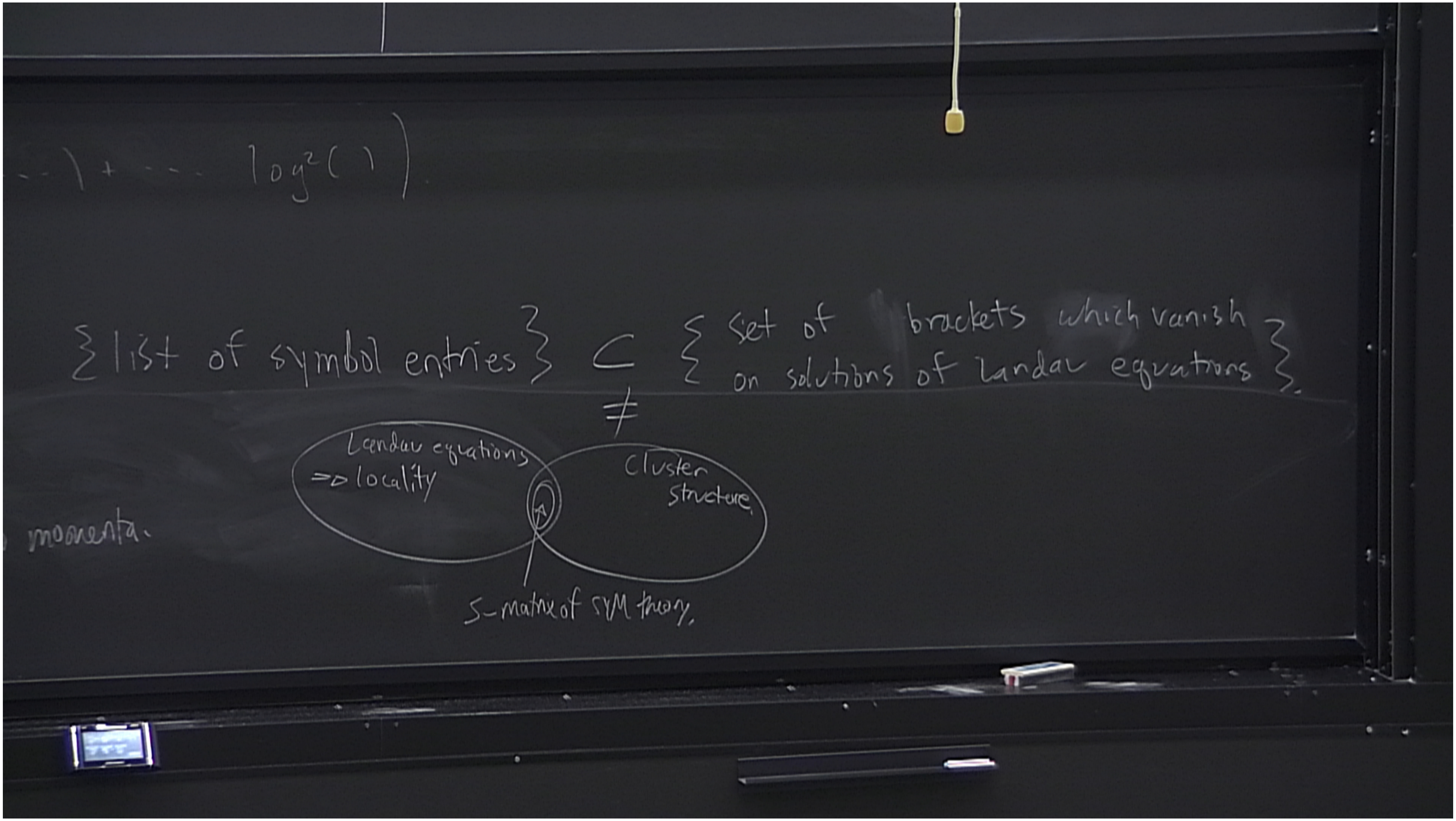
Caveats $\Rightarrow$

(1) Landau analysis doesn't know about numerators

(2) 2^{nd} class Landau singularities $\Rightarrow$ poles @ ∞ in loop momenta.

(3). lucky $\Rightarrow$ brackets emerge naturally.

(4) possible cancellation between different diagrams.


...) + ... log²( ))
{ list of symbol entries } ⊂ { set of brackets which vanish on solutions of Landau equations }.
≠
Landau equations
⇒ locality
Cluster structure.
momenta.
S-matrix of SYM theory.