Title: On level crossing in conformal field theories

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Abstract: <p>We study the properties of operators in a unitary conformal field theory whose scaling dimensions approach each other for some values of the parameters and satisfy von Neumann-Wigner non-crossing rule. We argue that the scaling dimensions of such operators and their OPE coefficients have a universal scaling behavior in the vicinity of the crossing point. We demonstrate that the obtained relations are in a good agreement with the known examples of the level-crossing phenomenon in maximally supersymmetric N=4 Yang-Mills theory, three-dimensional conformal field theories and QCD.</p>

Level crossing in conformal field theories

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Outline

- ✓ Level crossing phenomenon in CFT
- ✓ von Neumann–Wigner non-crossing rule
- ✓ Level crossing in $\mathcal{N} = 4$ SYM
- ✓ Level crossing in three-dimensional CFT
- ✓ Conclusions and speculations

Conformal data of CFT

Scalar conformal primary operators ($\alpha_{ij} = \Delta_i + \Delta_j - \Delta_k$)

$$\langle O_i(x) O_j(0) \rangle = \frac{\delta_{ij}}{(x^2)^{\Delta_i}}$$

$$\langle O_i(x_1) O_j(x_2) O_k(x_3) \rangle = \frac{C_{ijk}}{|x_{12}|^{\alpha_{ij}} |x_{23}|^{\alpha_{jk}} |x_{13}|^{\alpha_{ik}}}$$

Structure constants of the OPE

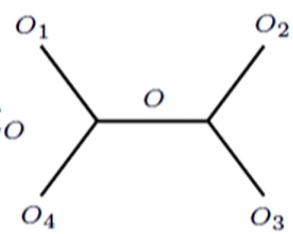
$$O_1(x) O_2(0) \sim \frac{1}{|x|^{\alpha_{12}}} C_{123} O_3(0) + \dots$$

Conformal data

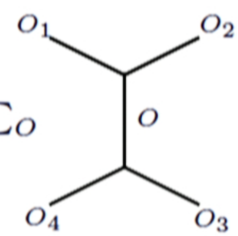
$$\{\Delta_i,\, C_{ijk}\} \quad = \quad \begin{pmatrix} \text{functions of coupling constants,} \\ \text{Casimirs of internal symmetry group etc.} \end{pmatrix}$$

Satisfy crossing symmetry conditions (recent progress in the conformal bootstrap program)

$\langle O_1 O_2 O_3 O_4 \rangle = \sum_O$



$= \sum_O$



Main question: *What happens when the scaling dimensions of two operators collide* $\Delta_1 \sim \Delta_2$*?*

Example: $\mathcal{N}=4$ super Yang-Mills theory

- (Super)conformal $SU(N)$ gauge theory in $D=4$ dimensions

$$L = \mathrm{tr}\left\{ -\frac{1}{2}F_{\mu\nu}^2 - D_\mu\Phi^I D^\mu\Phi^I + \frac{1}{2}g_{\mathrm{YM}}^2[\Phi^I,\Phi^J][\Phi^I,\Phi^J] + \text{fermions}\right\}$$

Gauge fields + six real scalars Φ^I + fermions

- Topological expansion ('t Hooft coupling constant $\lambda = g_{\mathrm{YM}}^2 N$)

$$\Delta_i(g_{\mathrm{YM}}^2, N) = \underbrace{\Delta_i^{(0)}(\lambda)}_{\text{planar}} + \frac{1}{N^2}\Delta_i^{(1)}(\lambda) + \frac{1}{N^4}\Delta_i^{(2)}(\lambda) + \cdots .$$

- *Planar* $\mathcal{N}=4$ SYM is exactly solvable for arbitrary λ

- Simplest conformal operators

$$O^{IJ} = \frac{1}{N}\mathrm{tr}[\Phi^{(I}\Phi^{J)}] \qquad \text{half-BPS operator}$$

$$K = \frac{1}{N}\mathrm{tr}[\Phi^I\Phi^I] \qquad \text{Konishi operator}$$

$$O_D = \frac{1}{N^2}\mathrm{tr}[\Phi^{(I}\Phi^{J)}]\,\mathrm{tr}[\Phi^{(I}\Phi^{J)}] \qquad \text{double trace operator}$$

- Scaling dimensions are eigenvalues of the dilatation operator

Level crossing

Scaling dimensions in planar $\mathcal{N} = 4$ SYM

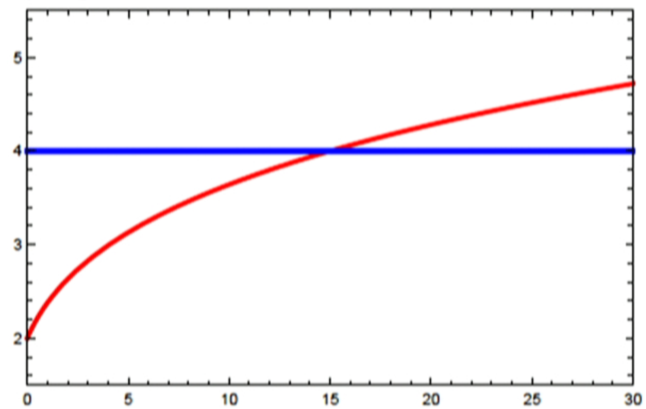
- ✓ Konishi operator:

$$\Delta_K = 2 + \gamma_K(\lambda)\,, \qquad \gamma_K(\lambda) = \begin{cases} \frac{3}{4\pi^2}\lambda + \ldots, & \lambda < 1 \\ 2\lambda^{1/4} + \ldots, & \lambda \gg 1 \end{cases}$$

- ✓ Double-trace operator:

$$\Delta_D = 4$$

Dependence on 't Hooft coupling constant



The scaling dimensions collide for $\lambda \sim 15 \implies$ *violation of the non-crossing rule!*

von Neumann–Wigner non-crossing rule

Über das Verhalten von Eigenwerten bei adiabatischen Prozessen

J. von Neumann und E. P. Wigner

Physikalische Zeitschrift *30*, 467–470 (1929)

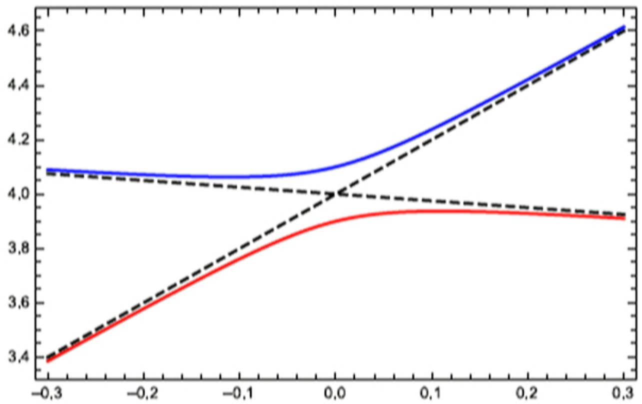
Two-level interacting system

$$H = H_0 + gV = \begin{bmatrix} E_1 & gV \\ gV & E_2 \end{bmatrix}$$

Energy levels

$$E_{\pm} = \frac{1}{2}(E_1 + E_2) \pm \sqrt{\frac{\Delta^2}{4} + g^2 V^2}$$

with $\Delta = E_1 - E_2 = O(g)$



The energy levels with the same symmetry cannot cross!

Non-crossing rule for scaling dimensions

- ✓ Mixing between operators at weak coupling

$$\langle O_i(x) O_j(0)\rangle = \frac{1}{(x^2)^\Delta}\left[\delta_{ij} - g^2_{\rm YM} H^{(1)}_{ij}\ln(x^2\mu^2) + \dots\right]$$

Dilatation operator

$$-\mu\frac{d}{d\mu}O_i(0) = i[\mathbb{D}, O_i(0)] = (g^2 H^{(1)}_{ij} + \dots)\, O_j(0) \equiv H_{ij} O_j(0)$$

Conformal operators are eigenstates of $\mathbb{D}$

$$i[\mathbb{D}, O_i^{(\rm conf)}(0)] = \Delta_i O_i^{(\rm conf)}(0)\,, \qquad \Delta_i = \text{'energy' levels of } H$$

$$\langle O_i^{(\rm conf)}(x) O_j^{(\rm conf)}(0)\rangle = \frac{\delta_{ij}}{(x^2)^{\Delta_i}}$$

- ✓ H is a hermitian matrix in a unitary CFT:
  - × Δ_i take real values only
  - × Δ_i have to satisfy the non-crossing rule
- ✓ *Scaling dimensions of operators with the same quantum numbers cannot cross each other!*

  How to reconcile this with the fact that Δ_D and Δ_K collide for $\lambda \sim 1$ in the planar limit?
- ✓ $1/N$ *expansion breaks down in the vicinity of the crossing point!*

Resummed scaling dimensions

Let O_1 and O_2 be the eigenstates of the dilatation operator in the planar limit

- ✓ For $\Delta_1 \neq \Delta_2$ conformal symmetry implies

$$\langle O_1(x) O_2(0) \rangle = 0 \; + \underbrace{O(1/N^2)}_{\text{nonplanar correction}}$$

- ✓ For $\Delta_{12} \equiv \Delta_1 - \Delta_2 = O(1/N)$ the leading nonplanar correction is enhanced by $1/\Delta_{12}$

$$\langle O_1(x) O_2(0) \rangle = \frac{1}{N} \varphi(x^2) + \dots$$

- ✓ Conformal operators for $\varphi(x^2) \neq 0$

$$O_+ = O_1 + c_2 O_2 \,, \qquad O_- = O_2 + c_1 O_1 \,.$$

Require $\langle O_+(x) O_-(0) \rangle = 0$

$$\frac{1}{N} \varphi(x^2)(1 + c_1 c_2) = -c_1 (x^2)^{-\Delta_1} - c_2 (x^2)^{-\Delta_2} = -(x^2)^{-\Delta_1} \left[c_1 + c_2 + c_2 \, \Delta_{12} \ln x^2\right]$$

Matching $1/N$ on the both sides

$$\varphi(x^2) = \frac{\gamma \ln x^2}{(x^2)^{\Delta_1}} \,, \qquad c_1^2 + c_1 \frac{\Delta_{12} N}{\gamma} - 1 = 0 \,, \qquad c_2 = -c_1 + O(1/N)$$

γ is a new (nonperturbative) parameter

Resummed scaling dimensions II

- ✓ Scaling dimensions for $\Delta_{12} = \Delta_1 - \Delta_2 = O(1/N)$

$$\Delta_\pm = \frac{\Delta_1 + \Delta_2}{2} \pm \sqrt{\frac{\Delta_{12}^2}{4} + \frac{\gamma^2}{N^2}}$$

- ✓ Coincide with energies of a two-level system

$$H = \begin{bmatrix} \Delta_1 & \gamma/N \\ \gamma/N & \Delta_2 \end{bmatrix}$$

  Analogous to H at weak coupling with $1/N$ playing the role of 't Hooft coupling constant

- ✓ $\Delta_\pm$ do not intersect, the minimal distance between the two curves is $|\Delta_+ - \Delta_-| \geq 2\gamma/N$
- ✓ Large N expansion

$$\Delta_+ = \Delta_1 + \frac{\gamma^2/\Delta_{12}}{N^2} - \frac{\gamma^4/\Delta_{12}^3}{N^4} + \cdots$$

  - × Takes into account an infinite class of corrections $1/(\Delta_{12}^{2p-1} N^{2p})$ to all orders in $1/N$
  - × For $\Delta_{12} = O(1/N)$, the leading correction to $\Delta_\pm$ scales as $O(1/N)$ (and not $O(1/N^2)$!).
  - × Each term is singular for $\Delta_{12} \to 0$ but the sum is finite, $\Delta_+ = \Delta_1 + \gamma/N$.

Resummed OPE coefficients

$$C^2_{\phi\phi O_\pm} \sim \frac{\langle \phi(1)\phi(2)O_\pm(3)\rangle^2}{\langle O_\pm O_\pm\rangle}$$

ϕ scalar operator with the dimension Δ_ϕ separated from $\Delta_\pm$ by a finite gap

$$C^2_{\phi\phi O_+} = \frac{\left(C_{\phi\phi O_1} - c_1 C_{\phi\phi O_2}\right)^2}{1+c_1^2}\,, \qquad C^2_{\phi\phi O_-} = \frac{\left(C_{\phi\phi O_2} + c_1 C_{\phi\phi O_1}\right)^2}{1+c_1^2}$$

$C_{\phi\phi O_i}$ are the OPE coefficients for the operators O_i in the planar limit

$$c_1 = -\frac{N\Delta_{12}}{2\gamma} + \sqrt{\frac{N^2\Delta_{12}^2}{4\gamma^2}+1} = \frac{\gamma}{N\Delta_{12}} - \left(\frac{\gamma}{N\Delta_{12}}\right)^3 + \dots\,,$$

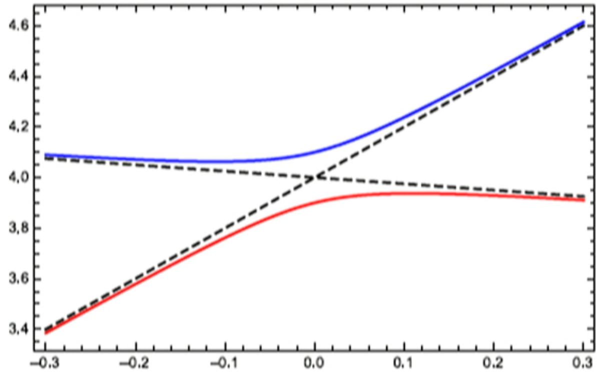
Large N expansion of $C^2_{\phi\phi O_\pm}$ runs in $1/(N\Delta_{12})$
... but large N expansion of $\langle\phi\phi\phi\phi\rangle$ should run in $1/N^2$

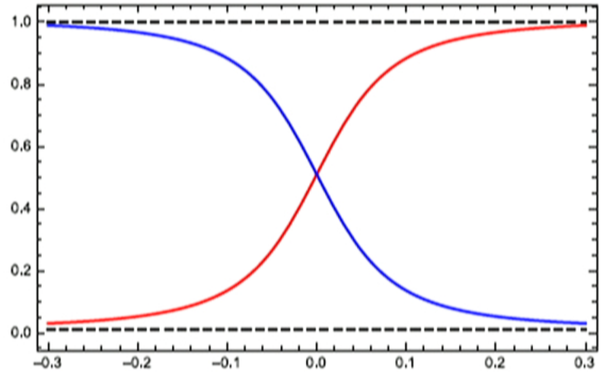
To eliminate all terms with odd powers of $1/N$, the OPE coefficients have to satisfy

$$\frac{C_{\phi\phi O_2}}{C_{\phi\phi O_1}} = O(1/N)$$

The operators O_1 and O_2 can collide only if their OPE coefficients differ by a factor of N!

Resummed OPE coefficients II





The scaling dimensions $\Delta_\pm$ (left) and the OPE coefficients $(C_{\phi\phi O_\pm}/C_{\phi\phi O_1})^2$ (right)

$$\frac{C^2_{\phi\phi O_\pm}}{C^2_{\phi\phi O_1}} = \frac{1}{2}\left[1 \pm \frac{\Delta_{12}}{\sqrt{\Delta_{12}^2 + 4\gamma^2/N^2}}\right]$$

Large N expansion

$$\frac{C_{\phi\phi O_+}}{C_{\phi\phi O_1}} = 1 - \frac{\gamma^2}{2N^2\Delta_{12}^2} + \dots\,, \qquad \frac{C_{\phi\phi O_-}}{C_{\phi\phi O_1}} = \frac{\gamma}{N\Delta_{12}}\left(1 - \frac{3\gamma^2}{2N^2\Delta_{12}^2} + \dots\right)$$

The leading correction to $C_{\phi\phi O_-}$ contains a pole at $\Delta_{12} = 0$

Resummed expressions for $C_{\phi\phi O_\pm}$ are finite for $\Delta_{12} = 0$

Level crossing in $\mathcal{N} = 4$ SYM

The Konishi and double-trace operators

$$\mathcal{O}_K = \frac{1}{N}\,\mathrm{tr}[\Phi^I\Phi^I]\,,$$
$$\mathcal{O}_D = \frac{1}{N^J}\,\mathrm{tr}[\Phi^{(I_1}\ldots\Phi^{I_J)}]\,\mathrm{tr}[\Phi^{(I_1}\ldots\Phi^{I_J)}]$$

Φ^I (with $I = 1,\ldots,6$) real scalar fields, $\mathrm{tr}[\Phi^{(I_1}\ldots\Phi^{I_J)}]$ symmetric traceless $SO(6)$ tensor

Normalization of the two-point correlation functions $\langle\mathcal{O}(1)\mathcal{O}(2)\rangle = O(N^0)$

Scaling dimensions at weak coupling

$$\Delta_K = 2 + O(\lambda)$$
$$\Delta_D = 2J + O(\lambda/N^2)$$

Scaling dimensions at strong coupling from the AdS/CFT correspondence

$$\Delta_K = 2\lambda^{1/4} - 2 + \frac{2}{\lambda^{1/4}} + \ldots\,,$$
$$\Delta_D = 2J - \frac{2(J-1)J(J+2)}{N^2} + \ldots$$

The two levels cross each other at strong coupling $\lambda \approx J^4$

Level crossing in $\mathcal{N}=4$ SYM II

Three-point functions with protected half-BPS operators

$$O_J = \frac{1}{N^{J/2}}\,\mathrm{tr}[\Phi^{(I_1}\dots\Phi^{I_J)}]$$

Carries the $R-$charge J and scaling dimension $\Delta_{O_J} = J$

$$\langle O_J(1)O_J(2)O_K(0)\rangle = \frac{C_{JJK}}{x_1^{\Delta_K}x_2^{\Delta_K}x_{12}^{2J-\Delta_K}}$$

$$\langle O_J(1)O_J(2)O_D(0)\rangle = \frac{C_{JJD}}{x_1^{\Delta_D}x_2^{\Delta_D}x_{12}^{2J-\Delta_D}}$$

Large N limit

$$\langle O_J(1)O_J(2)O_D(0)\rangle = \langle O_J(1)O_J(0)\rangle\langle O_J(2)O_J(0)\rangle + O(1/N^2)$$

The OPE coefficient

$$C_{JJD} = 1 + O(1/N^2)$$

The OPE coefficient C_{JJK} at strong coupling from AdS/CFT [Minahan,Pereira 2014]

$$C_{JJK} = \frac{\sqrt{J^7/12}}{2N(J-\lambda^{1/4})} + \dots\,, \qquad \text{for } J\sim\lambda^{1/4}$$

develops a pole at $\Delta_K = \Delta_D$

Level crossing in $\mathcal{N}=4$ SYM III

What we expect from the level crossing formulae

$$C_{JJD} = 1 + O(1/N^2) = 1 - \frac{\gamma^2}{2N^2\Delta_{12}^2} + \dots ,$$

$$C_{JJK} = \frac{\sqrt{J^7/12}}{2N(J-\lambda^{1/4})} + \dots = \frac{\gamma}{N\Delta_{12}}\left(1 - \frac{3\gamma^2}{2N^2\Delta_{12}^2} + \dots\right)$$

A perfect agreement for

$$\Delta_{12} = \Delta_D - \Delta_K \approx 2(J-\lambda^{1/4}), \qquad \gamma = \frac{J^{7/2}}{\sqrt{12}}$$

Resummed expressions in the transition region $|J-\lambda^{1/4}| = O(1/N)$

$$\Delta_\pm = J + \lambda^{1/4} \pm \sqrt{(J-\lambda^{1/4})^2 + \frac{J^7}{12N^2}}$$

$$C^2_{JJO_\pm} = \frac{1}{2}\left[1 \pm \frac{J-\lambda^{1/4}}{\sqrt{(J-\lambda^{1/4})^2 + J^7/(12N^2)}}\right]$$

The levels do not cross, the OPE coefficients and scaling dimensions remain finite at $J \approx \lambda^{1/4}$

Take into account an infinite class of nonplanar corrections in $1/[N(J-\lambda^{1/4})]^2$

Level crossing for large spin operators

Twist-2 operators $O_2(S) = \mathrm{tr}[\Phi^I D^S \Phi^I]$ with large Lorentz spin $S \gg 1$

$$\Delta_2 = 2 + S + \gamma_S\,, \qquad \gamma_S = 2\Gamma_{\rm cusp}(\lambda)\ln S + O(S^0)$$

The leading twist-four operators $O_4(S) = O\,\partial^S O$ with large spin

$$\Delta_4 = 4 + S - c(N)/S^2$$

The two levels cross each other at weak coupling

$$\gamma_S = 2 + O(1/S^2) \quad\Longrightarrow\quad \lambda \sim 1/\ln S < 1$$

The three-point correlation functions $C_{\rm tw} \sim \langle O_{\frac12{\rm BPS}} O_{\frac12{\rm BPS}} O_{\rm tw}(S)\rangle$ [Alday,Bissi'2014]

$$C_2(S) \sim \frac{1}{N}\Gamma\left(1 - \tfrac12\gamma_S(\lambda)\right)\,, \qquad C_4(S) = 1 + O(1/N^2)$$

The twist-two OPE coefficient develops a pole at $\epsilon = \Delta_4 - \Delta_2 \to 0$

$$\frac{C_2(S)}{C_4(S)} \sim \frac{\gamma}{\epsilon\, N}\,, \qquad \gamma = 2\,\mathrm{e}^{-\gamma_{\rm E}}/S = 2\,\mathrm{e}^{-4\pi^2/\lambda + O(\lambda)}$$

The interaction energy between levels is exponentially small at weak coupling

Resummed expressions for $\Delta_{2,4}$ and $C_{2,4}(S)$ are smooth functions in the transition region $|\Delta_4 - \Delta_2| = O(1/N)$.

Level crossing in 3d CFT's

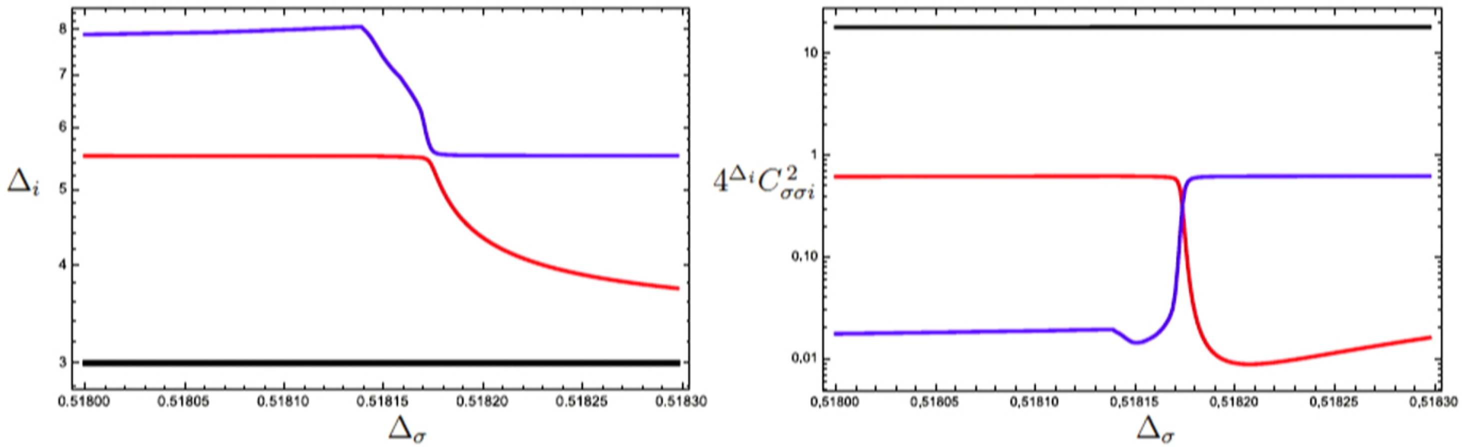
The crossing symmetry + unitarity restrict the possible values of the scaling dimensions in 3d CFT

[Rattazi et al'2008] [El-Showk et el'2012]

Examine extremal solutions to these constraints living on the boundary of the allowed region of Δ_i

The spectrum of the extremal 3d CFT's is parameterized by the dimension Δ_σ of the leading $\mathbb{Z}_2-$odd scalar operator

The scaling dimensions of the lowest spin-2 operators and their OPE coefficients: [El-Showk et el 2012]



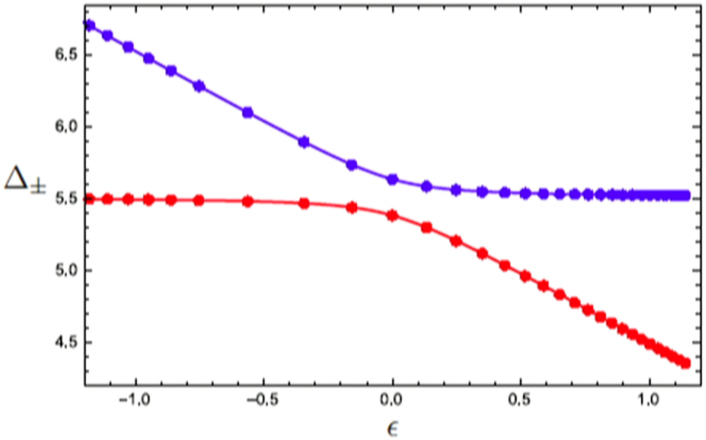
Sharp transition at $\Delta_\sigma^\star = 0.518154(15)$, expected critical point of three-dimensional Ising model.

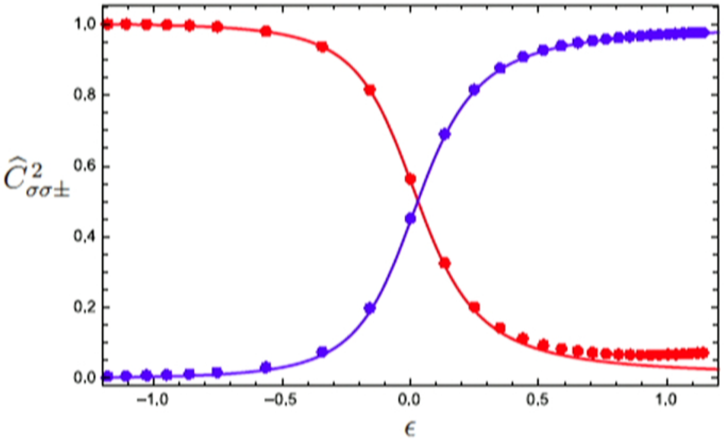
Some operators (red) disappear from the spectrum, $C_{\sigma\sigma O} \to 0$

Apply resummed formulae in the vicinity of the crossing point

- p. 16/19

Level crossing in 3d CFT's II





Scaling dimensions of spin-2 operators (left) and the OPE coefficients $\widehat{C}^2_{\sigma\sigma\pm} = C^2_{\sigma\sigma\pm}/C^2_{\sigma\sigma 1}$ (right)

$$\Delta_\pm(\epsilon) = \Delta_1 - \frac{\epsilon}{2} \pm \sqrt{\frac{\epsilon^2}{4} + v^2}\,, \qquad \widehat{C}^2_{\sigma\sigma\pm}(\epsilon) = \frac{[\widehat{C}_{\sigma\sigma 2} + c_\pm]^2}{1 + c_\pm^2}$$

with $\epsilon = \Delta_1 - \Delta_2$ and $c_\pm = \left(\epsilon \pm \sqrt{\epsilon^2 + 4v^2}\right)/(2v)$

Very good agreement with 'experimental data' for

$$v = 0.126\,, \qquad \widehat{C}_{\sigma\sigma 2} = 0.057$$

The OPE coefficient $C^2_{\sigma\sigma -}$ decreases by the order of magnitude as expected

Conclusions and speculations

- ✓ Scaling dimensions respect the level noncrossing rule
- ✓ Scaling dimensions and OPE coefficients have a universal scaling behavior in the vicinity of the crossing point
- ✓ The resummed formulae are in a good agreement with the known examples of the level-crossing phenomenon
- ✓ Implications for $S-$duality in $\mathcal{N}=4$ SYM
  - × For finite N, the spectrum of the scaling dimensions should be invariant under

$$h \to 1/h \quad \text{or} \quad \phi \to \pi/2 - \phi\,, \qquad \text{with } h = g^2/(4\pi) = \tan\phi$$

Two possible scenario for Konishi operator K:

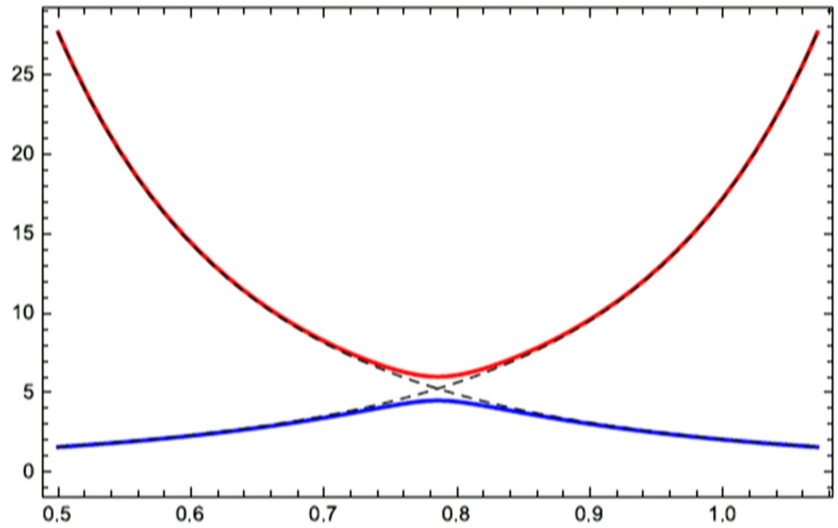
  - × K is invariant under the $S-$duality $\Delta_K(1/h) = \Delta_K(h)$ (very unlikely)
  - × K transforms into another (nonperturbative) operator K'

$$\Delta_{K'}(h) = \Delta_K(1/h) \quad \text{or} \quad \Delta_{K'}(\phi) = \Delta_K(\pi/2-\phi)$$

The functions $\Delta_{K'}(h)$ and $\Delta_K(h)$ ought to cross at $h=1$ or $\phi=\pi/4$

Speculations and open questions

The flow of the scaling dimensions $\Delta_{K'}(h)$ and $\Delta_K(h)$



The scaling dimensions as functions of $0 \leq \phi \leq \pi/2$

Both curves are invariant under the $S-$duality

What is the meaning of the operator K'?

Is it possible to test this picture using integrability/conformal bootstrap?