Title: Stereoscopic CFT Tools for Holography

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Abstract: <p>I will discuss a natural basis of CFT operators for probing dual gravitational physics in a diffeomorphism-invariant manner. On the CFT side, these operators are already well-known: they are 'OPE Blocks' that contribute to the Operator Product Expansion with fixed Casimir. On the gravity side, I will show that these OPE blocks are dual to diff-invariant geodesic or surface operators. Our new entry to the holographic dictionary can be understood as an operator generalization of the Ryu-Takayanagi proposal and I will show it gives a unified description for a host of important results in holography. </p>

A Stereoscopic CFT Lens for Holography

JAMES SULLY

B. Czech, L. Lamprou, S.McCandlish, B. Mosk, JS arXiv: 1604.03110

B. Czech, L. Lamprou, S.McCandlish, B. Mosk, JS arXiv: 1605.XXXX

(Related Work: de Boer, Haehl, Heller, Myers)

Perimeter Institute
April 2016

Motivation

There is an uncomfortable asymmetry in our use of the holographic dictionary:

Boundary: *Gauge-invariance is sacrosanct.*

- We always use gauge-invariant observables (eg. $O = \mathrm{Tr}(X)$) to probe the bulk and vice-versa.

Bulk: *Diff-invariance is required...* *in the same way that flossing is required.*

- When we visit an expert twice a year, we acknowledge the importance of diff-invariance (naïve local observables require gravitational dressing).
- We really only worry about it once a month or so.
- We are usually happy to work with local quantities (eg. some field $\phi(x)$).

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Worrying about **diff-invariance** is largely seen as adding **unpleasant (and unnecessary) complications**.

Is this a fair assessment?

Aim of this talk: To convince you the answer is **NO**.

NO to what?

No, talking about bulk gravitational physics in naturally diff-invariant, non-local variables is **NOT** more complicated nor unpleasant.

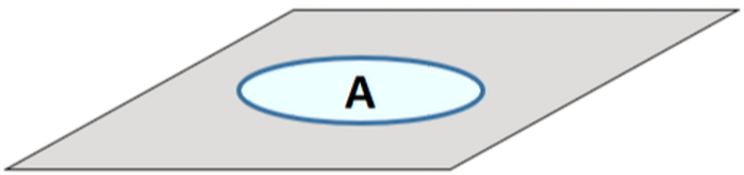
There is ample historical precedent that this might be true, or is at least conceptually pleasing.

- (Strings, branes, S-matrices, …)

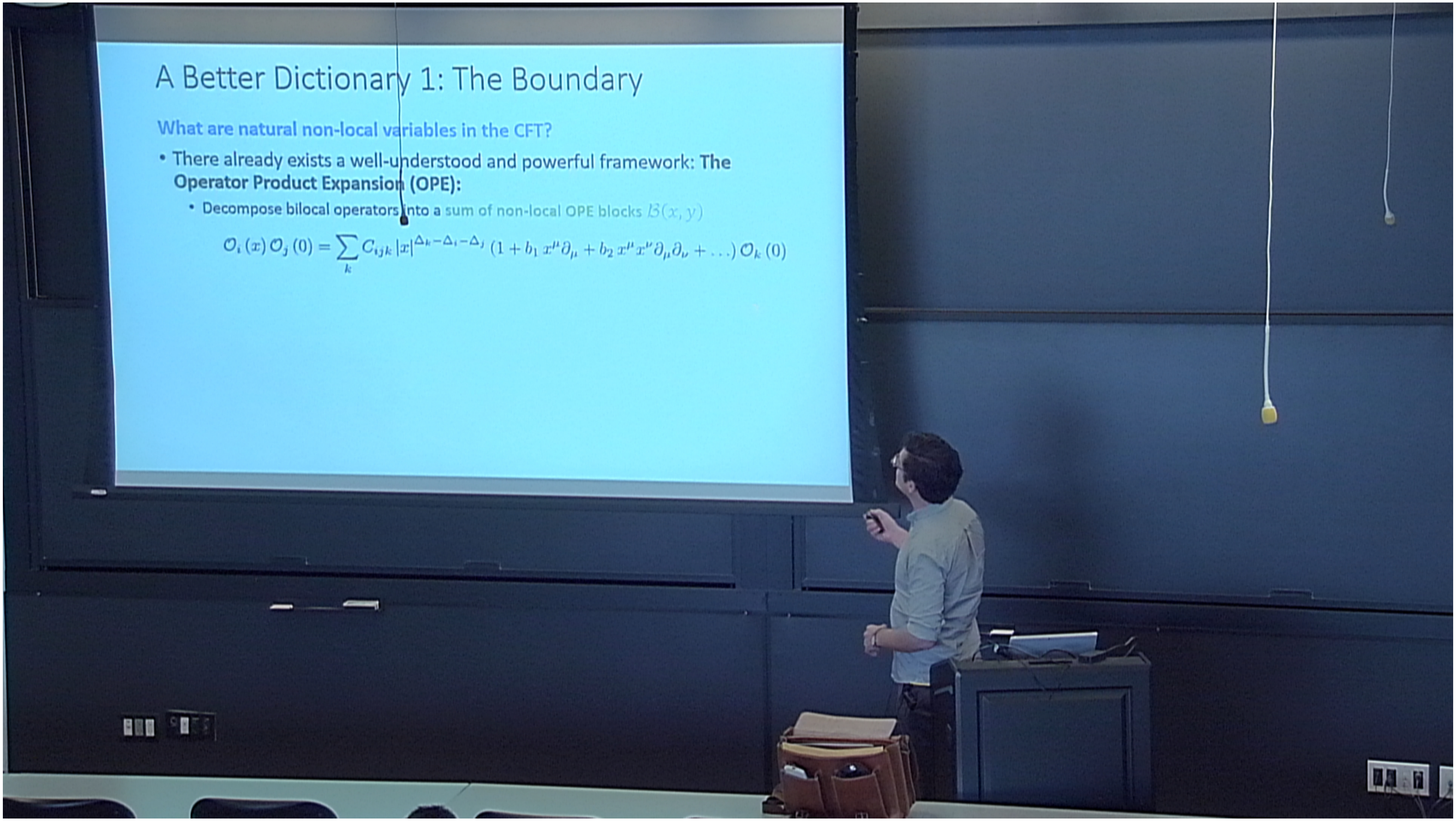
From amplitudes: have learned it can be important to be mindful of the symmetries we make manifest, and useful to get a little healthy distance from locality.

Prototypical Example

- We already have one *well-known*, *well-defined* example of what we're looking for:
- The **Ryu-Takayanagi (RT/HRT) proposal** connects:
  1) In the CFT:



The **entanglement entropy** of a region A on the boundary as well as its **modular Hamiltonian** H_{mod}
(Natural, nonlocal, Gauge-invariant)


A Better Dictionary 1: The Boundary
What are natural non-local variables in the CFT?
• There already exists a well-understood and powerful framework: The Operator Product Expansion (OPE):
• Decompose bilocal operators into a sum of non-local OPE blocks $\mathcal{B}(x,y)$
$\mathcal{O}_i(x)\,\mathcal{O}_j(0) = \sum_k C_{ijk}\,|x|^{\Delta_k-\Delta_i-\Delta_j}\,(1 + b_1\,x^\mu\partial_\mu + b_2\,x^\mu x^\nu\partial_\mu\partial_\nu + \ldots)\,\mathcal{O}_k(0)$

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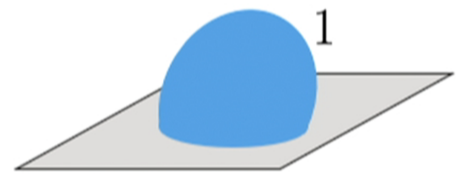
- The OPE block is a natural choice of **fundamental** variable for CFT
  - **Non-local, gauge-invariant**, transforms in **simple representation** of Conformal Group
  - *It already appears in our benchmark example:* we will see that the modular Hamiltonian has a simple expression in the language of OPE blocks.

A Better Dictionary 2: The Bulk

What are natural non-local variables in the CFT?

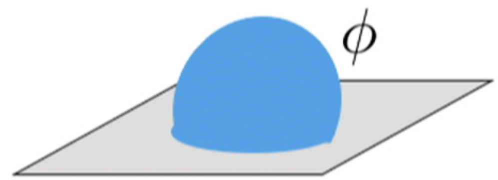
- Precisely as in RT, a natural set of non-local, diff-invariant objects are **minimal (extremal) surfaces and geodesics**.
- We can think of their areas as integrating the unit operator over the minimal surface:

$$A = \int d^n x \sqrt{h}(1)$$



- A natural generalization then is:

$$\tilde{\phi} = \int d^n x \sqrt{h}(\phi)$$



The Kinematic Dictionary

- In this talk I will **establish a correspondence**

$$\mathcal{B}(x,y) \leftrightarrow \tilde{\phi}\,(x,y)$$

between **OPE Blocks** and **geodesic operators** (and extensions to surface operators)

- This will be a **powerful framework**. It brings together many familiar ideas in holography, including:

1. The **Entanglement First Law** and **Einstein's Equations** [Faulkner, Guica, Hartman, Lashkari, McDermont, Myers, Swingle, Van Raamsdonk]
2. **Geodesic Witten diagrams** and **conformal blocks** [Hijano, Kraus, Perlmutter, Snively]
3. The **HKLL** construction of interacting **'local' bulk fields** [Hamilton, Kabat, Lifschytz, Lowe]+…
4. **de Sitter dynamics** for the variations of **EE** [de Boer, Heller, Myers, Neiman] [Nozaki, Numasawa, Prudenziati, Takayanagi], [Bhattacharya, Takayanagi]

A Survey of Kinematic Space

Kinematic Space

- Our non-local, gauge-invariant entries to the holographic dictionary are best organized by a `**kinematic space**':
  - Consider **ordered pairs of spacelike separated points** on the CFT_2 cylinder

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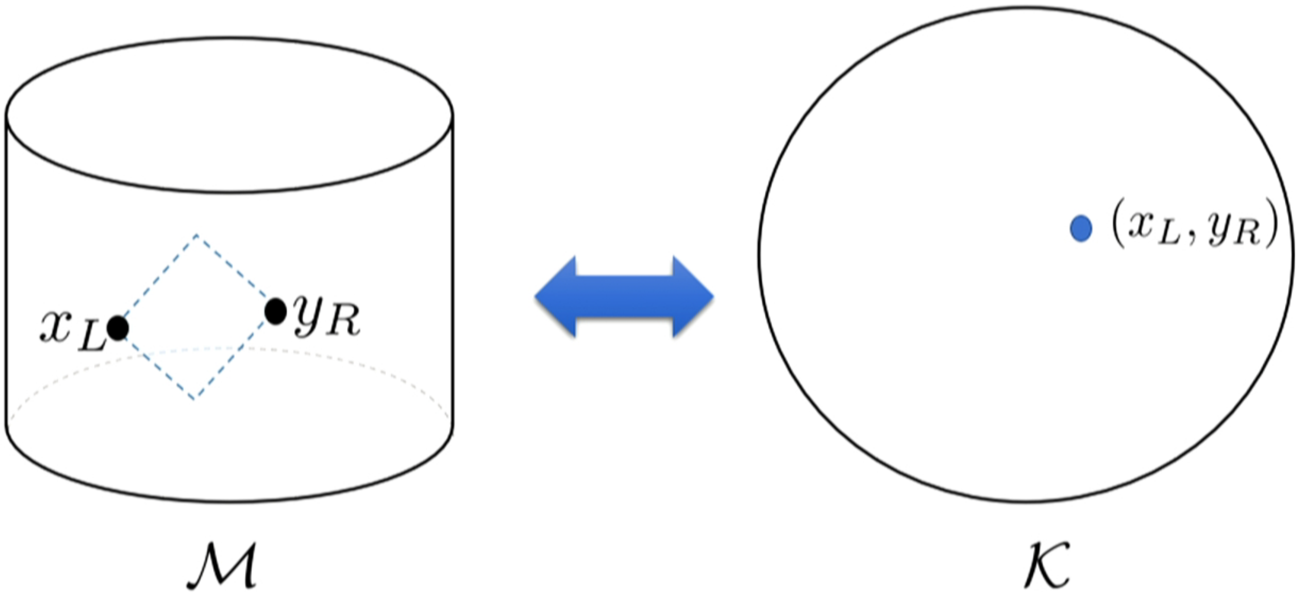
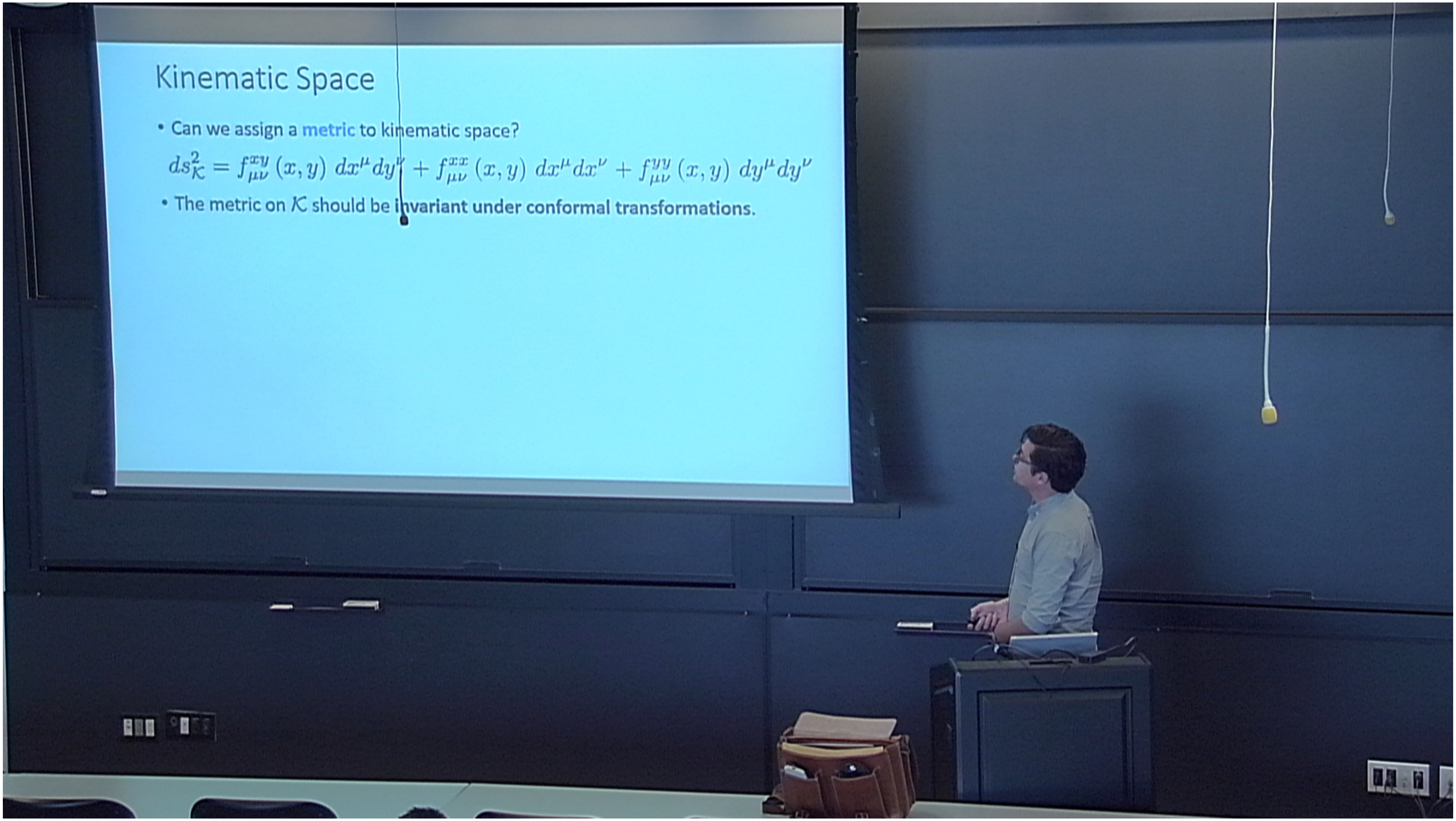


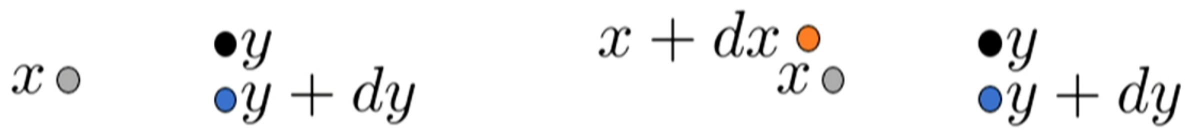

Kinematic Space
• Can we assign a metric to kinematic space?
$ds^2_{\mathcal{K}} = f^{xy}_{\mu\nu}(x,y)\, dx^\mu dy^\nu + f^{xx}_{\mu\nu}(x,y)\, dx^\mu dx^\nu + f^{yy}_{\mu\nu}(x,y)\, dy^\mu dy^\nu$
• The metric on $\mathcal{K}$ should be invariant under conformal transformations.

Kinematic Space

- Can we assign a **metric** to kinematic space?

$$ds^2_{\mathcal{K}} = f^{xy}_{\mu\nu}(x,y)\, dx^\mu dy^\nu + f^{xx}_{\mu\nu}(x,y)\, dx^\mu dx^\nu + f^{yy}_{\mu\nu}(x,y)\, dy^\mu dy^\nu$$

- The metric on $\mathcal{K}$ should be **invariant under conformal transformations.**
  - No terms like $dx^\mu dx^\nu$ or $dy^\mu dy^\nu$:



  - Metric transforms like a pair of operators of conformal dimension (1,1)

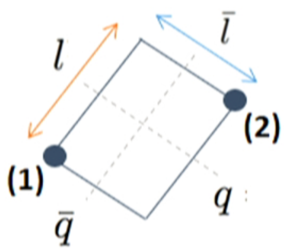
$$f_{\mu\nu}(x,y) \to \frac{dx'^\alpha}{dx^\mu}\frac{dy'^\beta}{dx^\nu} f_{\alpha\beta}(x',y')$$

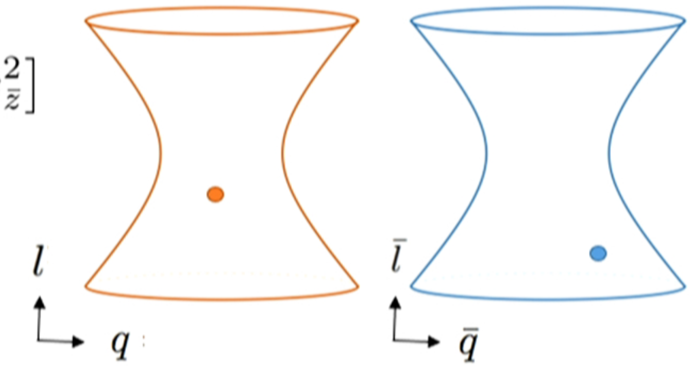
Kinematic Space for $\mathrm{AdS}_3/\mathrm{CFT}_2$

- This metric simplifies for **d=2** because SO(2,2) = SO(2,1) X SO(2,1):
  - Change to left-moving and right-moving coordinates $\{\, \Delta z = l\,,\; z_c = q\,,\;\; z \leftrightarrow \bar{z}\,\}$ for the CFT

$$ds^2 = \frac{1}{2}\left[\frac{dq^2 - dl^2}{l^2} + \frac{d\bar{q}^2 - d\bar{l}^2}{\bar{l}^2}\right] = \frac{1}{2}\left[ds_z^2 + ds_{\bar{z}}^2\right]$$

dS_z $\quad$ $\mathrm{dS}_{\bar{z}}$

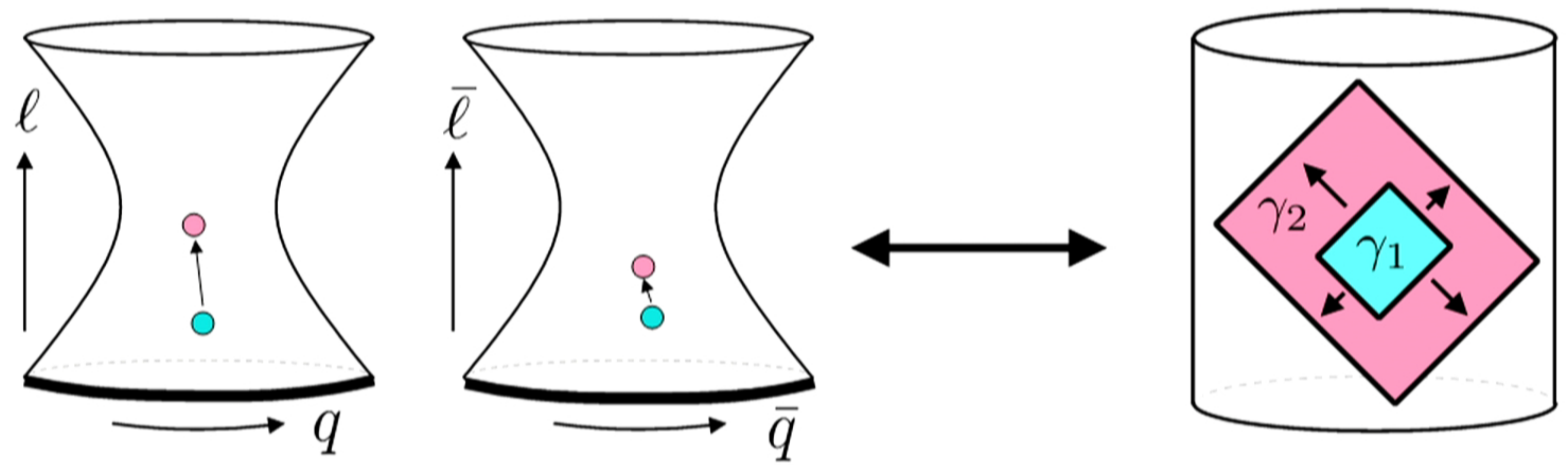




- We can also restrict to **pairs of points that lie on a time slice**, say t=0, (or geodesics that lie in a 2-dimensional hyperbolic plane). Then fix $z = \bar{z}$, and take the **diagonal space** in the two coordinates

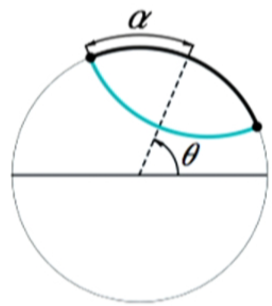
Kinematic Metric

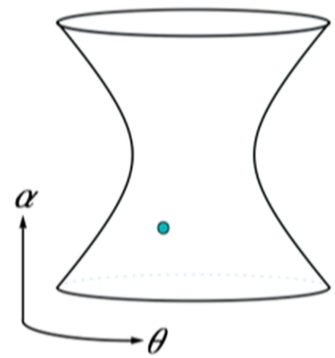
- Has a notion of **causality**: **Partial order** of **causal diamonds**.

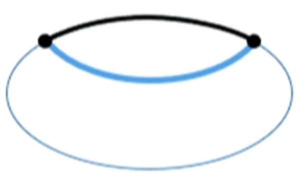


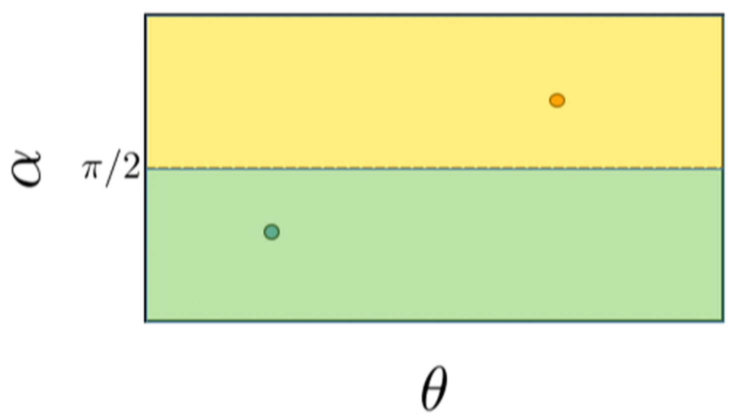
Kinematic Space for $\mathrm{AdS}_3/\mathrm{CFT}_2$

- Our coordinates on the plane cover a patch of de Sitter. In global coordinates:









$$\theta \to \theta + \pi$$
$$\alpha \to \pi - \alpha$$

Summary

- Kinematic Space is the space of:

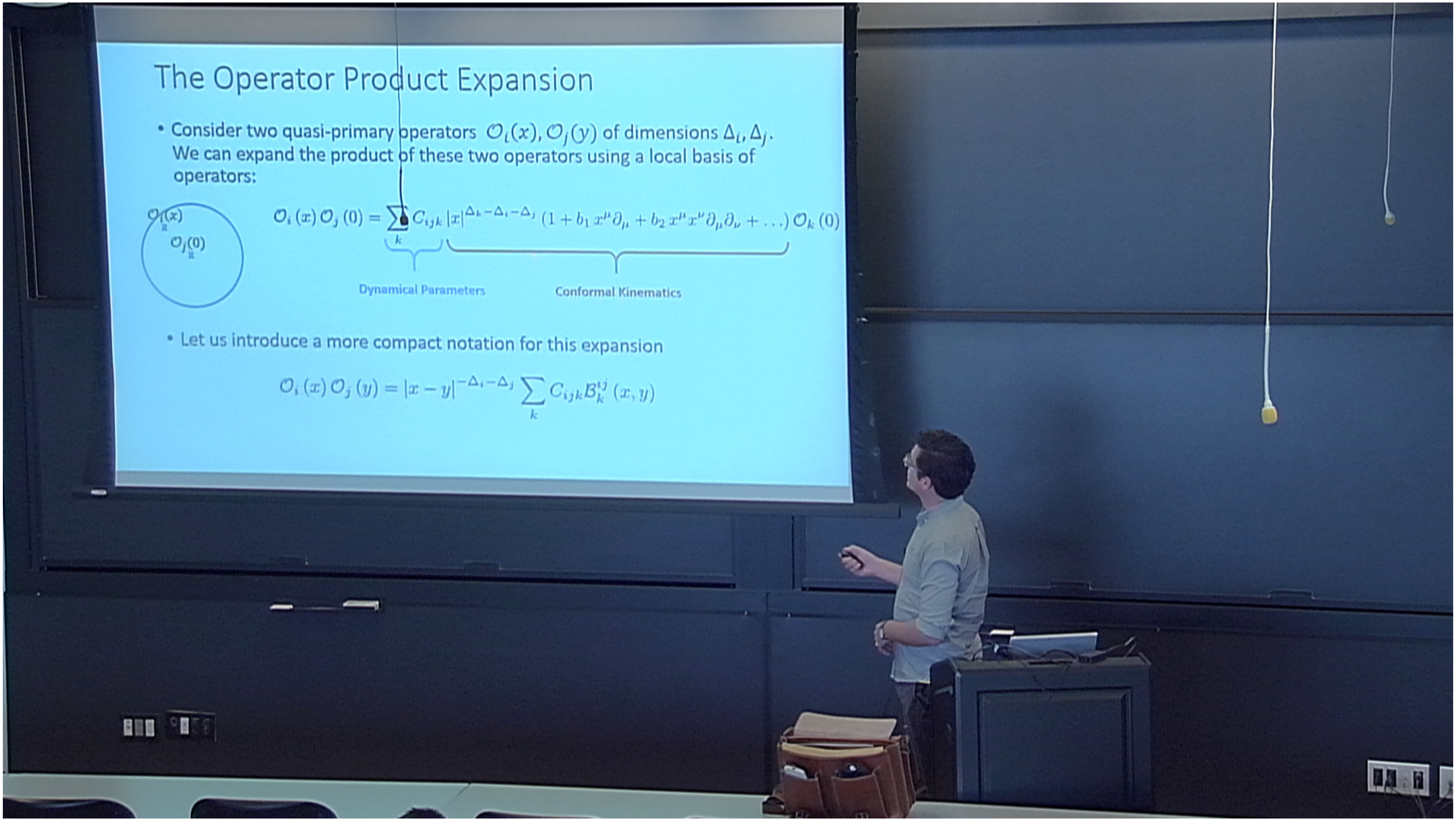
  1. **Ordered pairs of points** (x_L, y_R) in the CFT

  2. **Oriented geodesics** $\gamma_{x_L y_R}$ in the AdS_3

- Kinematic space is Lorentzian metric space of signature (2,2), with a **causal ordering**

- For $\boldsymbol{AdS_3}$ Kinematic space is $\boldsymbol{dS_2} \times \overline{\boldsymbol{dS_2}}$
  - For the spatial slice, $\boldsymbol{H_2}$, it is $\boldsymbol{dS_2}$

OPE Blocks


The Operator Product Expansion
• Consider two quasi-primary operators $\mathcal{O}_i(x), \mathcal{O}_j(y)$ of dimensions Δ_i, Δ_j. We can expand the product of these two operators using a local basis of operators:
$\mathcal{O}_i(x)$
$\mathcal{O}_j(0)$
$\mathcal{O}_i(x)\,\mathcal{O}_j(0) = \sum_k C_{ijk}\,|x|^{\Delta_k-\Delta_i-\Delta_j}\left(1 + b_1\,x^\mu\partial_\mu + b_2\,x^\mu x^\nu\partial_\mu\partial_\nu + \ldots\right)\mathcal{O}_k(0)$
Dynamical Parameters
Conformal Kinematics
• Let us introduce a more compact notation for this expansion
$\mathcal{O}_i(x)\,\mathcal{O}_j(y) = |x-y|^{-\Delta_i-\Delta_j}\sum_k C_{ijk}\mathcal{B}_k^{ij}(x,y)$

OPE Blocks as Kinematic Fields

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Consider a scalar block $(\Delta_k, l = 0)$. Let's characterize this field:

1) What type of field is an OPE block on KS?

- Consider a conformal transformation $x \to x'$ and $\Omega(x') = \det\left(\frac{\partial x'^{\mu}}{\partial x^{\nu}}\right)$. Then

$$\mathcal{B}_k^{ij}(x, y) \to \left(\frac{\Omega(x')}{\Omega(y')}\right)^{(\Delta_i - \Delta_j)/2} \mathcal{B}_k^{ij}(x', y')$$

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- In the simplest **case where $\Delta_i = \Delta_j$**, this transforms as a **scalar field** under isometries of KS.

Let's keep this simplifying assumption for the purposes of this talk: $\mathcal{B}_k(x, y)$

OPE Blocks as Kinematic Fields

2) What is its equation of motion?

- A quasi-primary operator and its descendants have the same conformal Casimir so the OPE block satisfies the equation

$$\left[L^2, \mathcal{B}_k\left(x,y\right)\right] = C_{\mathcal{O}_k}\mathcal{B}_k\left(x,y\right)$$

$$C_{\mathcal{O}_k} = -\Delta\left(\Delta - d\right) - \ell\left(\ell + d - 2\right)$$

- The OPE block transforms in a bi-local representation of the conformal group, so the Casimir is represented

$$\mathcal{L}^2_{(B)} = \left(\mathcal{L}_x + \mathcal{L}_y\right)^2 \quad = \quad 2\left[\ell^2\left(-\partial_\ell^2 + \partial_q^2\right) + \bar{\ell}^2\left(-\partial_{\bar{\ell}}^2 + \partial_{\bar{q}}^2\right)\right]$$

$$\ell = \frac{z_y - z_x}{2},\; q = \frac{z_y + z_x}{2}$$

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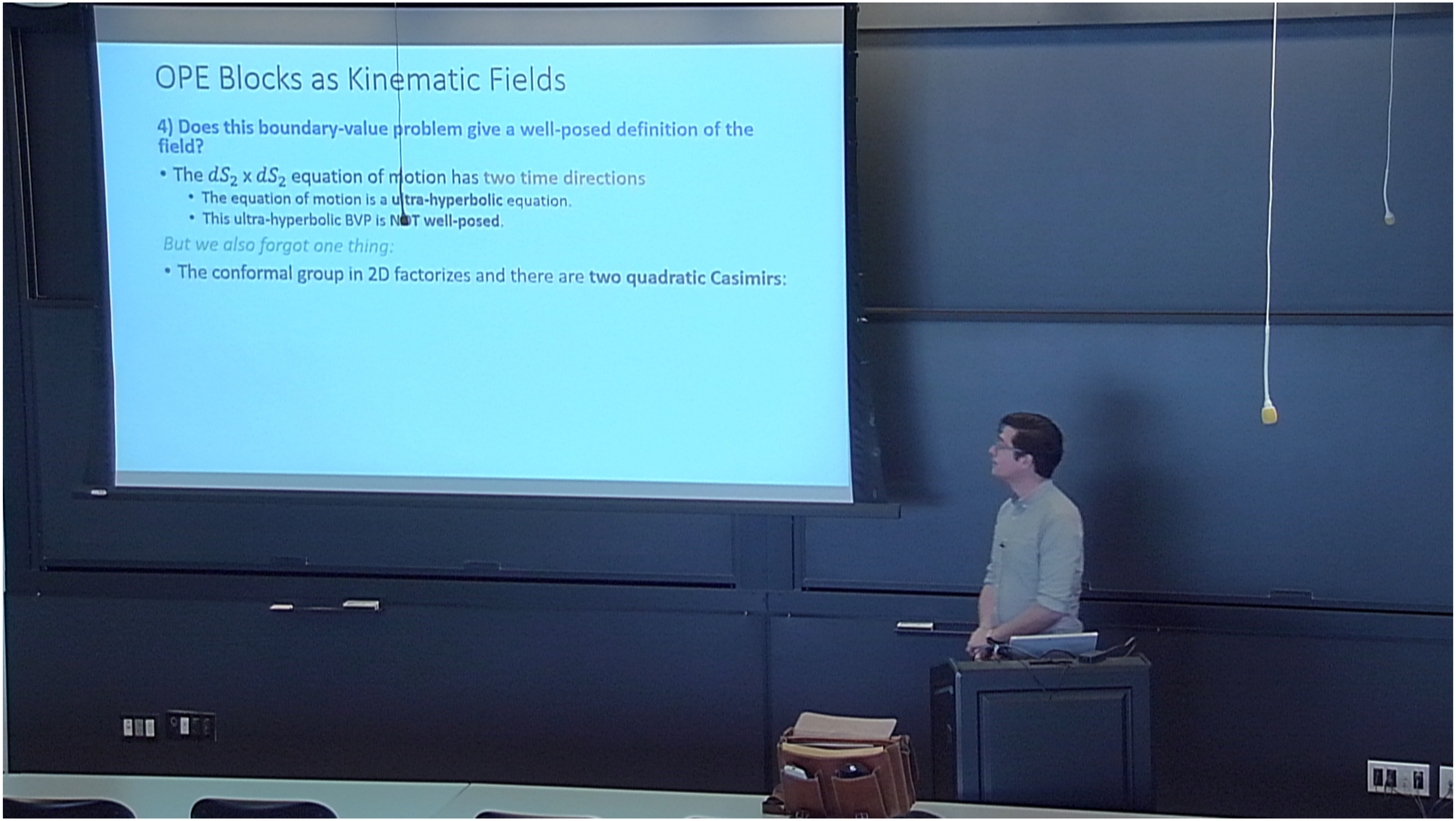
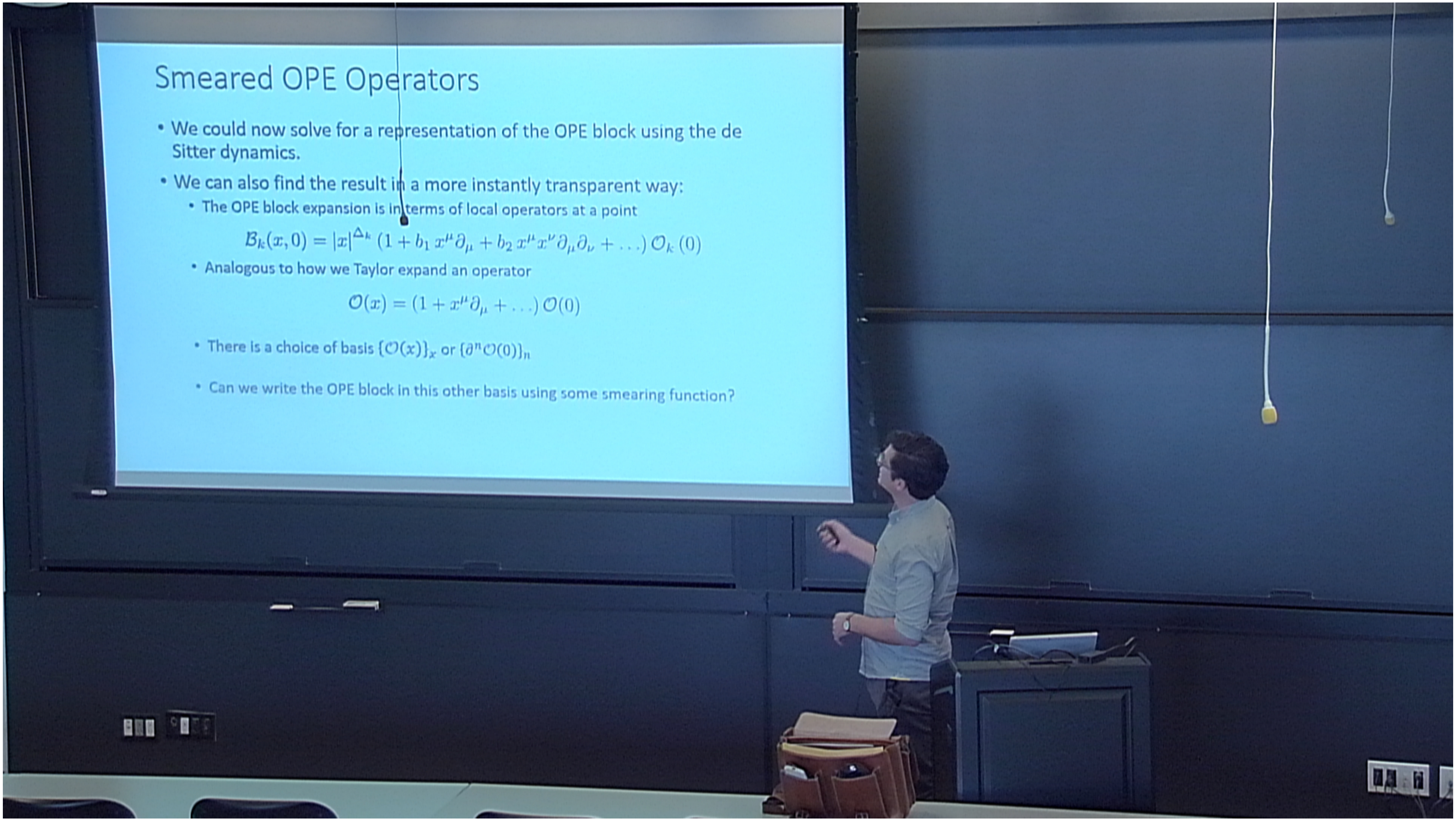
$$\boxed{\mathcal{L}^2_{(B)} = 2\left[\Box_{\mathrm{dS}_2} + \bar{\Box}_{\mathrm{dS}_2}\right]}$$

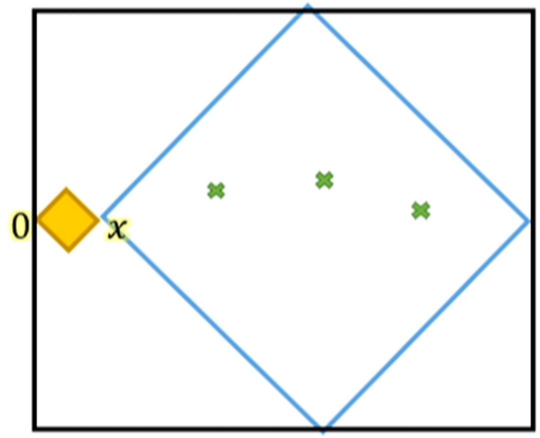

OPE Blocks as Kinematic Fields
4) Does this boundary-value problem give a well-posed definition of the field?
• The dS_2 x dS_2 equation of motion has two time directions
• The equation of motion is a ultra-hyperbolic equation.
• This ultra-hyperbolic BVP is NOT well-posed.
But we also forgot one thing:
• The conformal group in 2D factorizes and there are two quadratic Casimirs:


Smeared OPE Operators
We could now solve for a representation of the OPE block using the de Sitter dynamics.
We can also find the result in a more instantly transparent way:
The OPE block expansion is in terms of local operators at a point
$\mathcal{B}_k(x,0) = |x|^{\Delta_k} \left(1 + b_1\, x^\mu \partial_\mu + b_2\, x^\mu x^\nu \partial_\mu \partial_\nu + \ldots\right) \mathcal{O}_k(0)$
Analogous to how we Taylor expand an operator
$\mathcal{O}(x) = (1 + x^\mu \partial_\mu + \ldots)\, \mathcal{O}(0)$
There is a choice of basis $\{\mathcal{O}(x)\}_x$ or $\{\partial^n \mathcal{O}(0)\}_n$
Can we write the OPE block in this other basis using some smearing function?

Smeared OPE Operators

- To choose the integration region, we match the boundary conditions:

$$\lim_{x \to 0} \mathcal{B}_k(x, 0) = x^{\Delta_k} \mathcal{O}_k(0)$$



Summary

- We have established that the **OPE block** $\mathcal{B}_k\,(x, y)$ behaves like a scalar field on the kinematic space of pairs of spacelike points.
  1. **EOM**
  2. **Constraint**
  3. **Boundary conditions**

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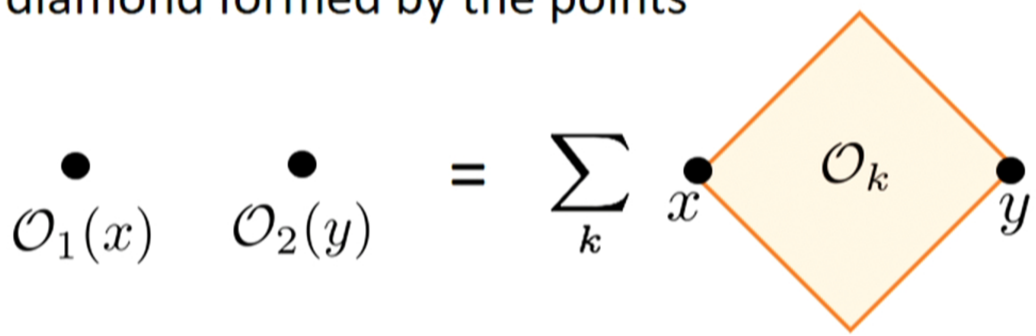
We also wrote down a representation of this operator as a smeared operator over the causal diamond formed by the points



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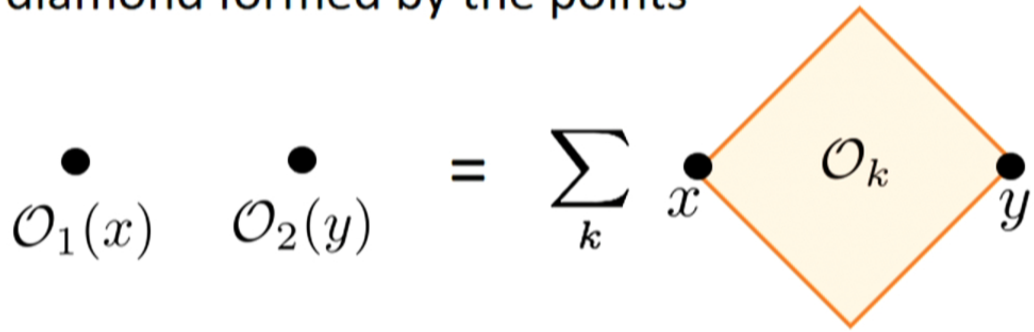
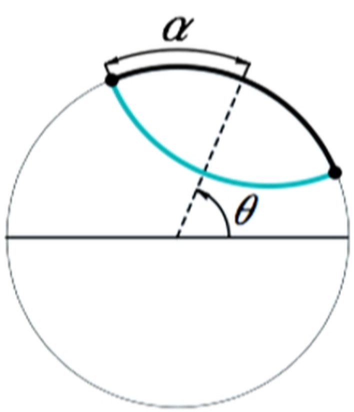


The OPE block is just what we have been looking for: a suitably **invariant, nonlocal** building block in the CFT.

X-Ray Transform

- The analogue of this problem for functions in Hyperbolic space is a well-studied field of integral geometry: it's the **X-ray transform**

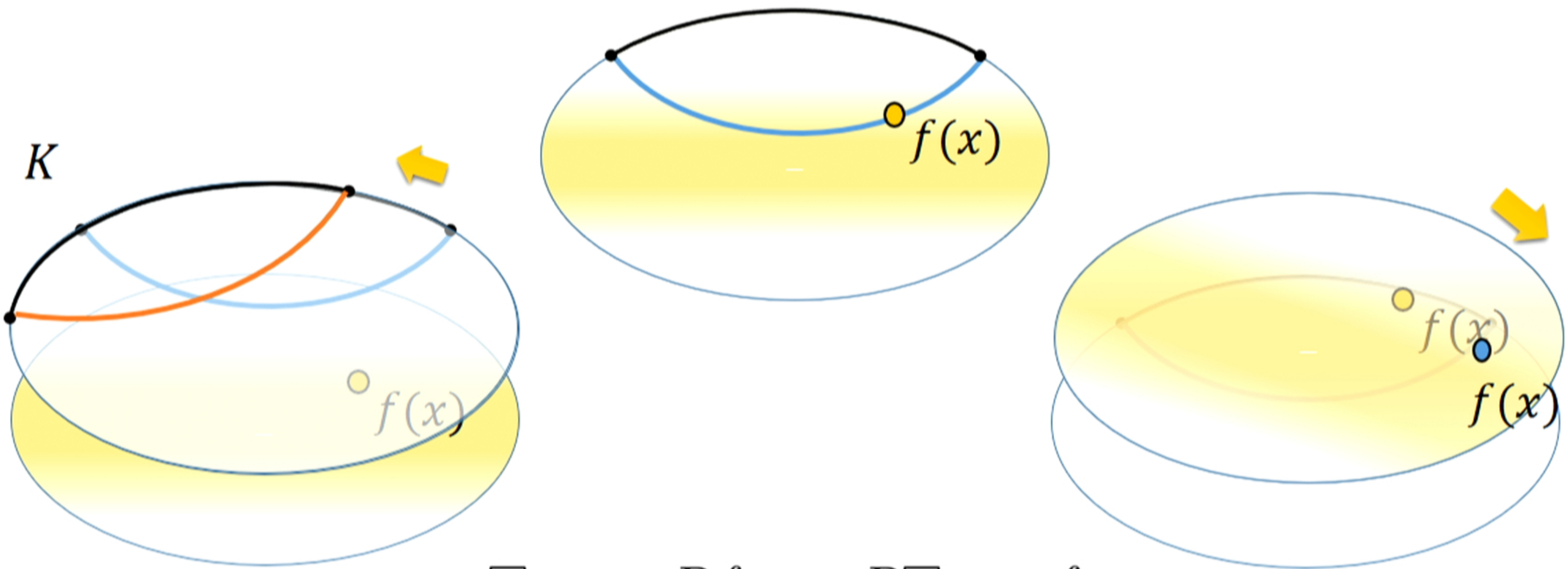
$$f(x)$$



X-Ray Transform

X-Ray Transform

- The X-ray transform has nice properties under isometries of the geometry:



$$\Box_{dS\times dS} Rf = -R\Box_{AdS_3} f$$

“Intertwining Operators”

X-Ray Transform

- $dS_2 \times dS_2$ is a **4-dimensional** space AdS_3 is **3-dimensional**.
  - KS is a redundant description of the real geometry and functions that live on it

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  - Many more functions $\tilde{f}$ we can specify on KS than are are consistent images Rf of the transform

What constraints do the ray transforms obey?

- **John's equation:**

$$\left(\Box_{dS} - \Box_{\bar{dS}}\right)\tilde{g} = 0 \iff \tilde{g} = Rf$$

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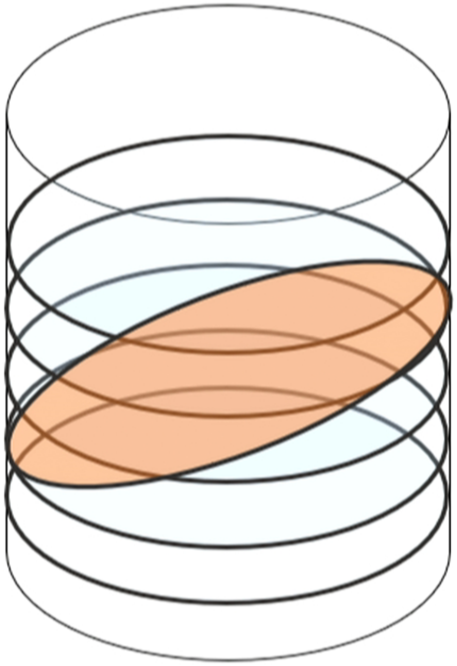
  - *Intuitive meaning:* we can determine function from flat slicing geodesics. Boosted geodesics are a redundant description.



Geodesic Operators

- We can apply the same transform to a 'local' bulk operator ϕ:

$$\tilde{\phi}(\alpha, \theta_c) = \int_{\gamma(\alpha,\theta_c)} ds\phi(x)$$

- Let's assume ϕ is a bulk operator such that $\left(\Box_{AdS} - m^2\right)\phi(x) = 0$

Intertwinementof the EOM: $(\Box_{dS} + \Box_{\bar{dS}} + m^2)\tilde{\phi}(\gamma) = 0$

John's equation: $(\Box_{dS} - \Box_{\bar{dS}})\tilde{\phi}(\gamma) = 0$

- Let's also assume that ϕ is dual to the bulk field $\mathcal{O}$ ($\phi(x,z) \sim z^{\Delta}\mathcal{O}_k(x)$). Then

$$\lim_{\alpha\to 0} \tilde{\phi}(\alpha, \theta_c) \sim \alpha^{\Delta}\mathcal{O}_k(\theta_c)$$

Gauge-Invariant Kinematic Dictionary

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$$\left[\Box_{\mathrm{dS}_2} + \bar{\Box}_{\mathrm{dS}_2} + m^2\right] \mathcal{B}_k\,(x,y) = 0$$
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  3. The same boundary conditions

$$(\Box_{dS} + \Box_{\bar{dS}} + m^2)\tilde{\phi}(\gamma) = 0 \qquad \left[\Box_{\mathrm{dS}_2} + \bar{\Box}_{\mathrm{dS}_2} + m^2\right]\mathcal{B}_k(x,y) = 0$$

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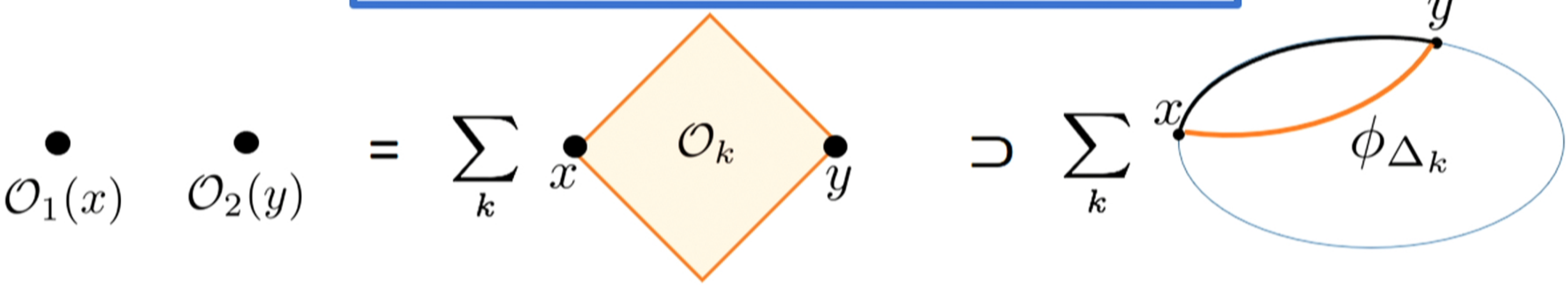
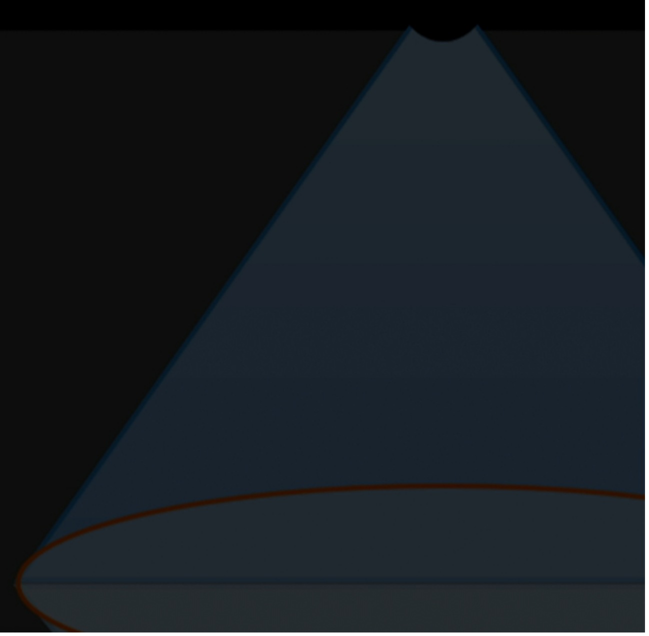
$$\lim_{\alpha\to 0}\tilde{\phi}(\alpha,\theta_c) \sim \alpha^\Delta \mathcal{O}_k(\theta_c) \qquad \lim_{x\to 0}\mathcal{B}_k(x,0) \sim x^{\Delta_k}\mathcal{O}_k(0)$$

$$\mathcal{B}_k(x,y) = \tilde{\phi}(\gamma) = \int ds\ \phi(x,z)\Big|_\gamma$$



Extensions



Higher Dimensions

Most of what I've said extends to higher dimensions.

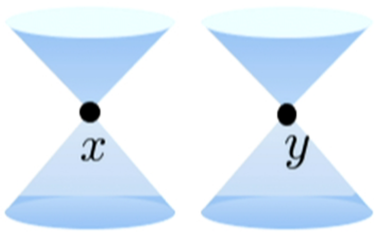
1. There is a **good kinematic metric** on pairs of points:

$$ds^2_{\mathcal{K}_g} = \frac{I_{\mu\nu}(x-y)}{|x-y|^2} dx^\mu dy^\nu$$

2. Geodesics and OPE blocks are still both described as **scalar fields in K**.
3. **But**, we **no longer have a nice smearing function** because there is no compact conformally invariant integration region.

Higher Dimensions

- In higher dimensions, there is a difference between specifying two **spacelike points** and two **timelike points:**



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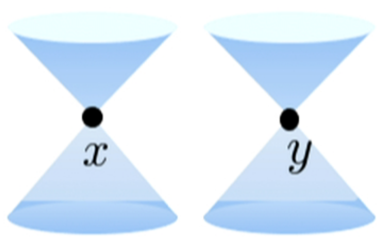


- The kinematic space of pairs of **timelike separated points** is now the **kinematic space of bulk minimal surfaces** for the boundary region.
  - For d-dim Hyperbolic space, this kinematic space is d-dim de Sitter [de Boer, Heller, Myers, Neiman]
- Instead of **X-ray transform**, have **Radon transform** over minimal surface of bulk operator: **bulk surface operators**
- We can study to contribution of operators of fixed Casimir to the expansion of loop/surface operators: **surface operator OPE.**

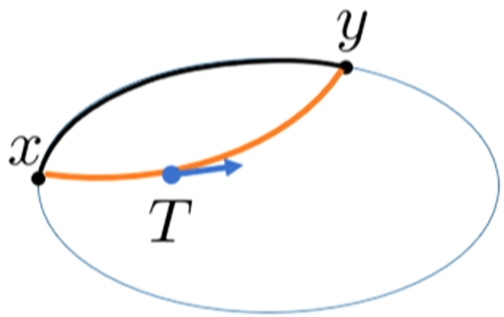
Operators with Spin

- Our **smearing functions** can equally be written for **operators with spin**:

$$(x-y)^{-2\Delta_0}\mathcal{B}_k(x,y) = \int d^d z \, \langle \mathcal{O}_{\Delta_0}(x)\mathcal{O}_{\Delta_0}(y)\tilde{\mathcal{O}}^{\mu\ldots\nu}_{\Delta_k}(z)\rangle \; \mathcal{O}_{\mu\ldots\nu,\,\Delta_k}(z)$$

- We can correspondingly write **X-Ray transforms of bulk tensor fields**

$$\tilde{T}(\alpha,\theta_c) = \int_{\gamma(\alpha,\theta_c)} ds T^{\mu_1\ldots\nu_k}\dot{x}_{\mu_1}\ldots\dot{x}_{\mu_k}$$



Stress Tensor OPE

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- Let's consider the **smearing function** for the **stress tensor** $T(z)$, $\overline{T}(\bar{z})$:

$$\mathcal{B}_T(x_1, x_2) = 6\int_{z_1}^{z_2} \frac{(z_2 - z)(z - z_1)}{z_2 - z_1} T(z)$$

$$\mathcal{B}_{\bar{T}}(x_1, x_2) = 6\int_{\bar{z}_1}^{\bar{z}_2} \frac{(\bar{z}_2 - \bar{z})(\bar{z} - \bar{z}_1)}{\bar{z}_2 - \bar{z}_1} \bar{T}(\bar{z})$$

- Adding the two we find:

$$\mathcal{B}_{T_{00}} = -12\pi \int_{x_1}^{x_2} dx \frac{(x_2 - x)(x - x_1)}{x_2 - x_1} T_{00}(x)$$

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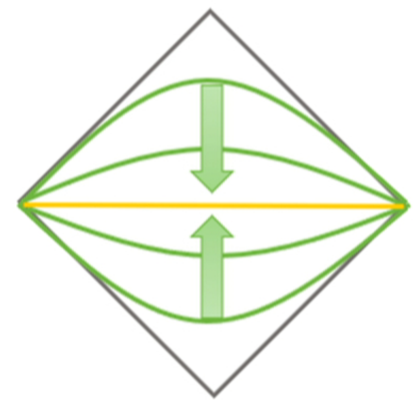


- This may look familiar, because:

$$H_{\text{mod}} = -\frac{1}{6}\mathcal{B}_{T_{00}}$$

- So the **modular Hamiltonian** is just an **OPE block** (and is a field on KS).

[de Boer, Myers, Heller, Neiman]

Einstein's Equations

- Since $\boldsymbol{H}_{\mathbf{mod}}$ is a kinematic operator, we know it **obeys a de Sitter wave equation**:

$$(\Box_{\mathcal{K}} + 2d)H_{\rm mod} = 0$$

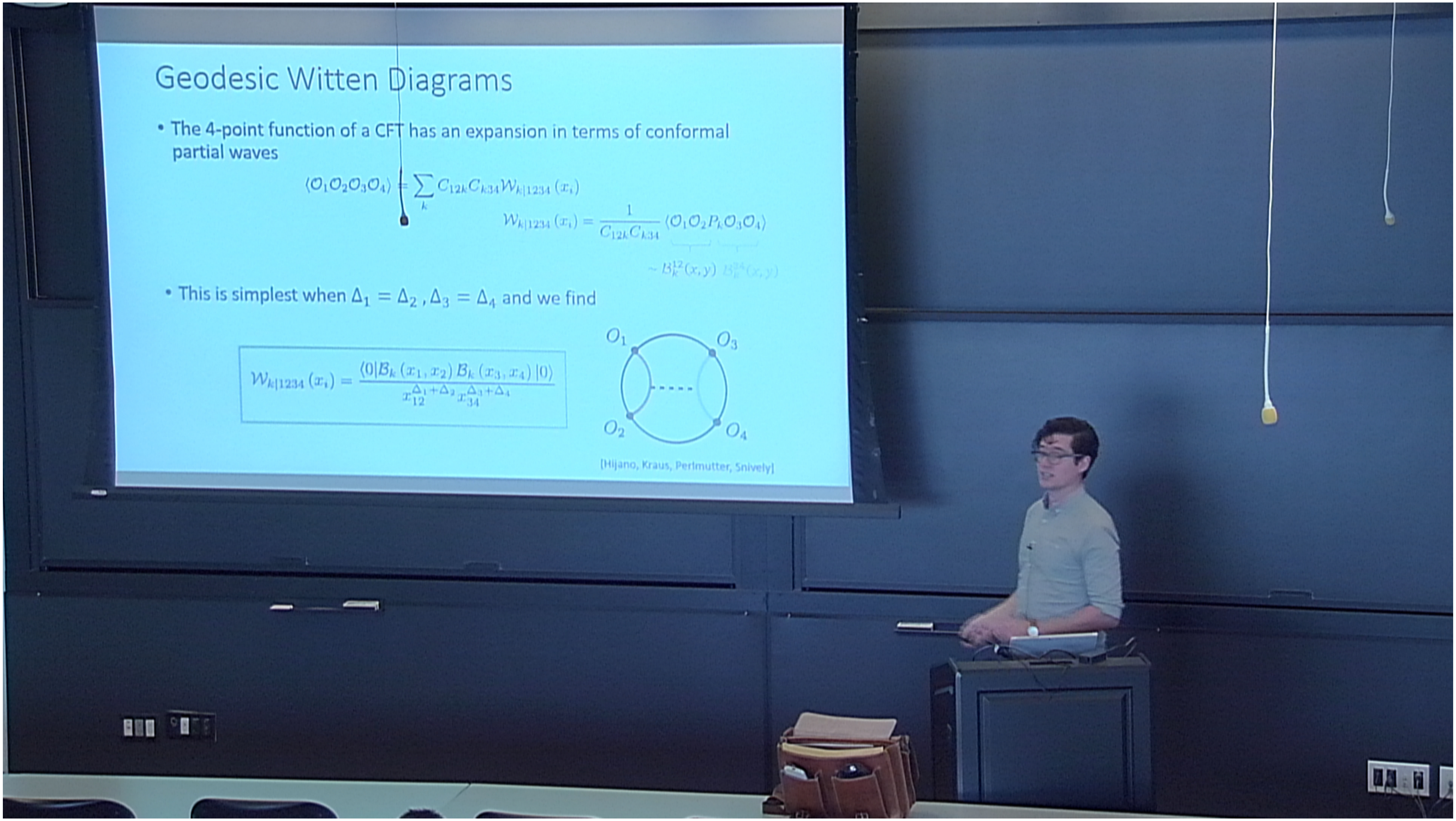
- The geodesic operator that is dual to the perturbation in the modular Hamiltonian is just the perturbation in the entanglement entropy

$$\delta S = \delta\langle H_{\rm mod}\rangle \qquad \delta S = \int_{\gamma} \delta g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}$$

- Using intertwinement of the Laplacian, we can find

$$(\Box_{\mathcal{K}} + 2d)\delta H = -\int_{\gamma}(\Box_{AdS} - 2d)\delta g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu} = 0$$


Geodesic Witten Diagrams
• The 4-point function of a CFT has an expansion in terms of conformal partial waves


Geodesic Witten Diagrams
• The 4-point function of a CFT has an expansion in terms of conformal partial waves
$\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \mathcal{O}_4 \rangle = \sum_k C_{12k} C_{k34} \mathcal{W}_{k|1234}(x_i)$
$\mathcal{W}_{k|1234}(x_i) = \frac{1}{C_{12k} C_{k34}} \langle \mathcal{O}_1 \mathcal{O}_2 P_k \mathcal{O}_3 \mathcal{O}_4 \rangle$
$\sim \mathcal{B}_k^{12}(x,y)\ \mathcal{B}_k^{34}(x,y)$
• This is simplest when $\Delta_1 = \Delta_2$, $\Delta_3 = \Delta_4$ and we find
$\mathcal{W}_{k|1234}(x_i) = \frac{\langle 0 | \mathcal{B}_k(x_1, x_2)\, \mathcal{B}_k(x_3, x_4) | 0 \rangle}{x_{12}^{\Delta_1+\Delta_2} x_{34}^{\Delta_3+\Delta_4}}$
O_1
O_3
O_2
O_4
[Hijano, Kraus, Perlmutter, Snively]

Local Bulk Operators

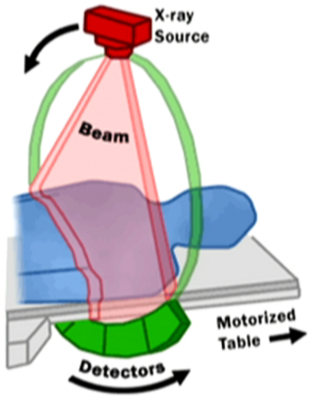
So far:

Geodesic operators give non-local, diff-invariant probes of bulk physics.

Nevertheless:

Would still like to understand the **emergence** of (approximate**) local effective field theory** in a gravitational background.

- Of course, we don't have just one projection of the bulk data—**we have the projections in all 'angles'**
- Reconstructing a complete 3D image from all of these projections (geodesic integrals) is a well-understood problem. It's what allows this:



Inversion Formulae for the Radon Transform

What data do we need to reconstruct a function in the bulk geometry?

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[Helgason]

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- For the **geodesic transform in** H^2, this formula is singular and **inversion requires an integral over all geodesics**:

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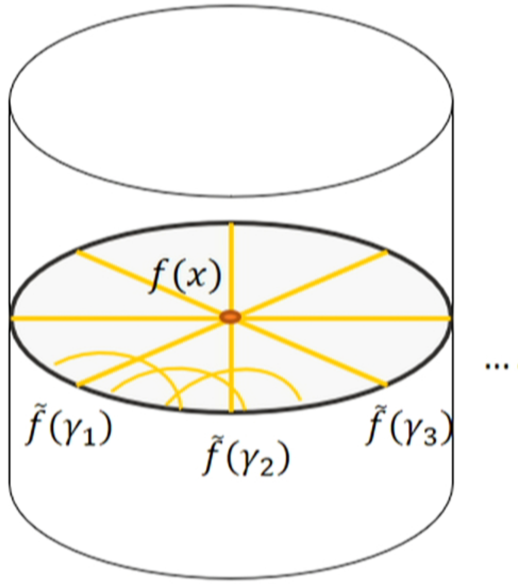

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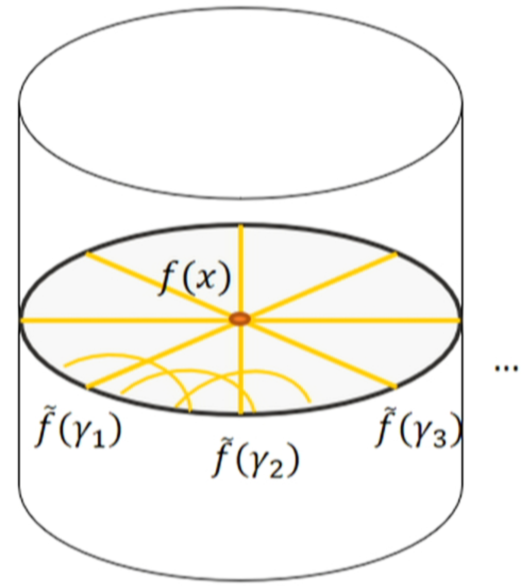
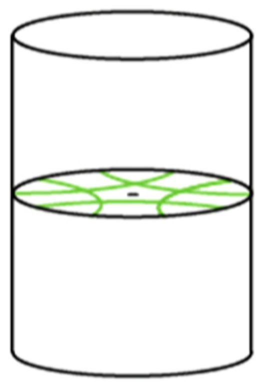
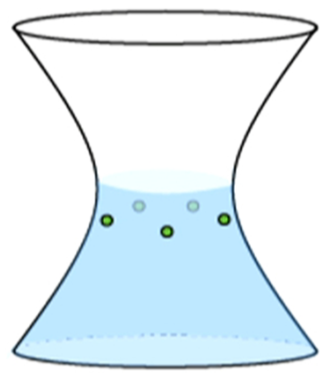


- We **integrate over the parameters** $(\boldsymbol{\alpha}, \boldsymbol{\theta_c})$, while leaving the boundary integrals in $\mathcal{B}(\alpha, \theta_c)$ unintegrated.



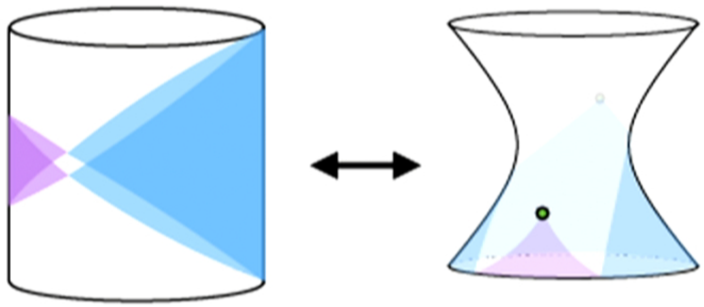


- We **integrate over the parameters** $(\boldsymbol{\alpha}, \boldsymbol{\theta_c})$, while leaving the boundary integrals in $\mathcal{B}(\alpha, \theta_c)$ unintegrated.
  - This gives the **bulk operator** as an **integral over the spacelike separated boundary region**

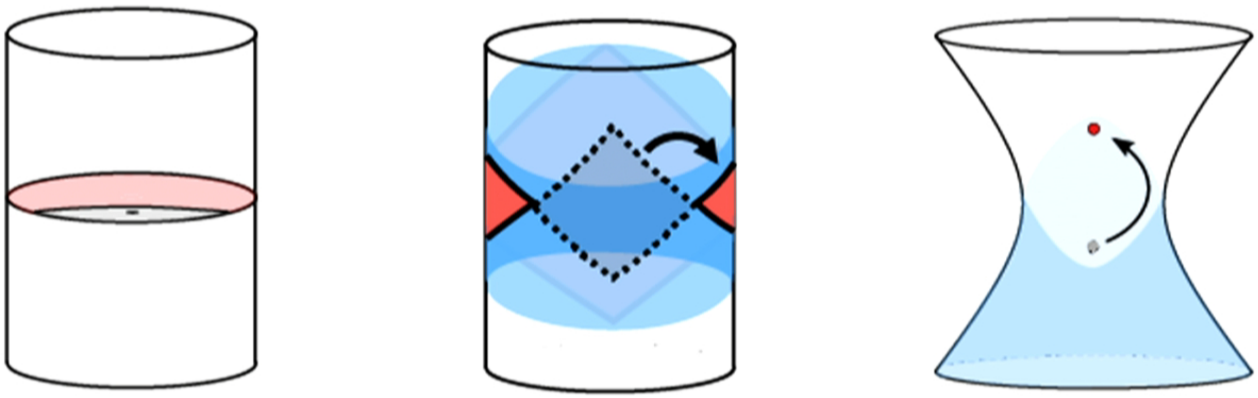
$$\phi(\rho = 0) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dt \int_0^{2\pi} d\theta \, K_\Delta(t) \; \mathcal{O}_\Delta(t, \theta) \qquad K_\Delta(t) = -\frac{k}{\pi} (\cos t)^{\Delta - 2} \log \cos t$$

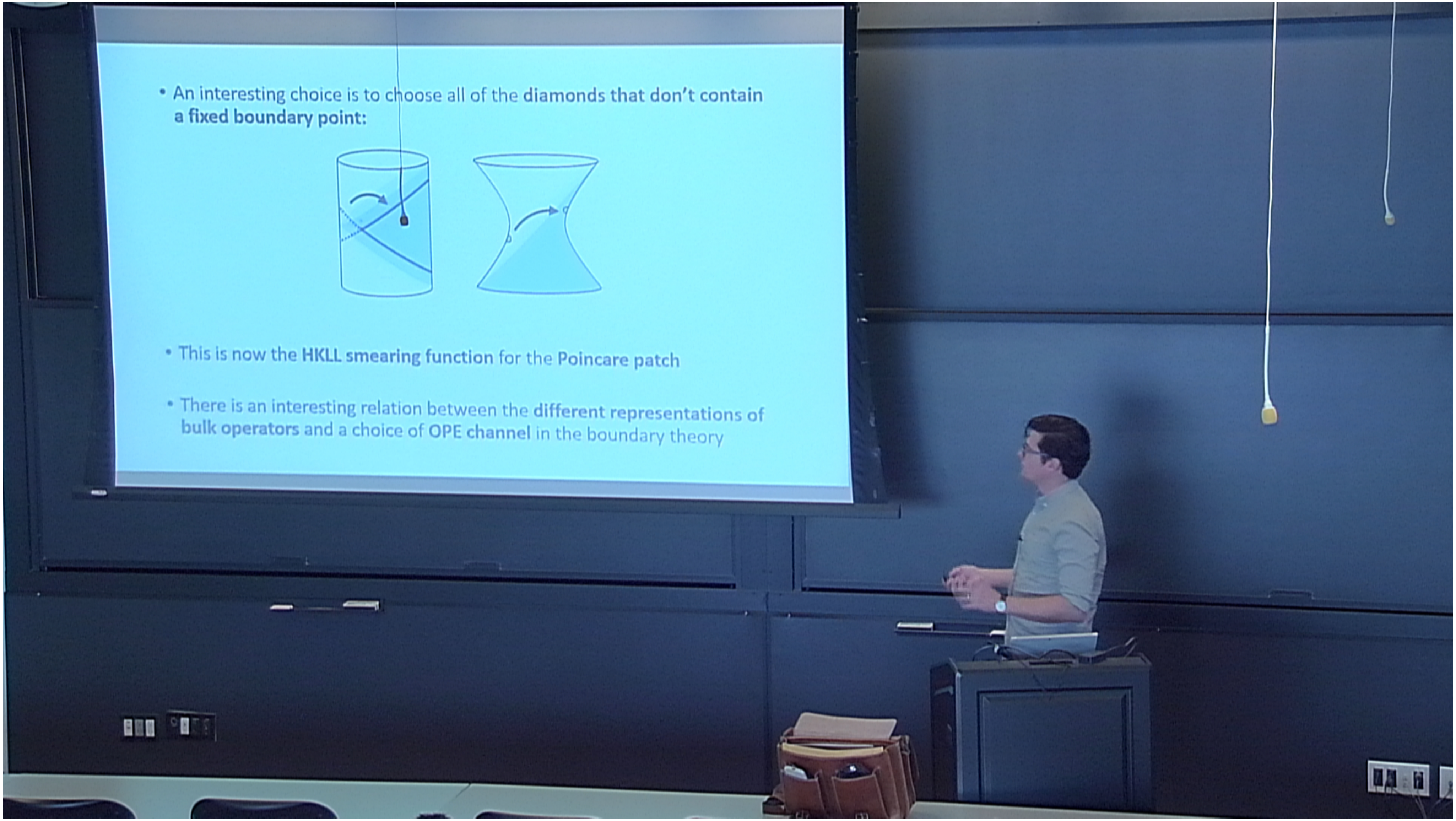
- We have recovered the well-known **HKLL global smearing function**

- Our domain of integration is over **½ of the oriented KS**.



- We made a **choice of which orientation/causal diamond** to choose for each geodesic.
- We have the apparent **freedom to swap one causal diamond for another**…




• An interesting choice is to choose all of the diamonds that don't contain a fixed boundary point:
• This is now the HKLL smearing function for the Poincare patch
• There is an interesting relation between the different representations of bulk operators and a choice of OPE channel in the boundary theory

Interactions

- The construction of local bulk operators can be **extended to include interactions:**

$$\phi(x,z) = \int d^d x' K(x,z|x')\mathcal{O}(x') + \frac{1}{N}\sum_n a_n^{CFT} \int d^d x' K_n(x,z|x')\mathcal{O}_n(x')$$

[Kabat,Lifschytz,Lowe; Heemskerk,Marolf,Polshinski,JS]

- The explicit corrections were computed up to **O(1/N)** for *specific conformal dimensions* (and conjectured more generally).
- We are able to **compute these to higher order** (much more easily) and for **arbitrary conformal dimension,** confirming the conjectured form at O(1/N).
- The simplification comes by **not solving for the local operator, but solving for the geodesic operator:**

$$\tilde{\phi}_\Delta(\gamma) = \mathcal{B}_\Delta(\gamma) + \frac{1}{N}\sum_n a_n^{CFT} \mathcal{B}_n(\gamma)$$