

Title: Formal loop spaces

Date: Apr 21, 2016 02:00 PM

URL: <http://pirsa.org/16040084>

Abstract: Formal loop spaces are algebraic analogs to smooth loops. They were introduced and studied extensively in the 2000' by Kapranov and Vasserot for their link to chiral algebras.

In this talk, we will introduced higher dimensional analogs of K. and V. formal loop spaces. We will show how derived methods allow such a definition. We will then study their tangent complexes: even though formal loop spaces are "of infinite dimension", their tangent has enough structure so that we can speak of symplectic forms on them.

I) Formal top spaces:

$d=1, (KV): \mathcal{L}'_v(X) = \text{Top}(\hat{A}, X): A \leftrightarrow$

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I) Formal loop spaces:

$d=1$, (KV): $\mathcal{L}'_v(X) = \text{Top}(\hat{A}, X) : A \mapsto X(A[[t]])$

$\mathcal{L}'_0(X) = \text{Top}(\hat{A} \setminus \{0\}, X) : A \mapsto X(A[[t^+]])$

\leadsto impose $X \mapsto \mathcal{L}'X$ has codescent.

$\mathcal{L}'X :=$ formal completion of $\mathcal{L}'_v(X)$ into $\mathcal{L}'_0(X)$.

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Thm (KV):

• $X \mapsto \mathcal{L}'X$ has Zariski codescent.

• $\mathcal{L}'X$

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Thm (KV):

- $X \mapsto \mathcal{L}'X$ has codescent.
- $\mathcal{L}'X$ is represented by an ind-scheme

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$\mathcal{L}'_v X = \text{formal completion of } \mathcal{L}'_v(X) \text{ into } \mathcal{L}'_v(X).$

$d \geq 1$. $\mathcal{L}'_v(X) = \text{TC}_p(\hat{A}^L, X) : A \mapsto X(A[\![t_1, \dots, t_d]\!]])$

$\mathcal{L}'_v(X) = \text{TC}_p(\hat{A}^d \setminus \{0\}, X) : A \mapsto \text{TC}_p(\text{Spec}(A[\![t_1, \dots, t_d]\!]]) \setminus \{0\}, X)$

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$\mathcal{L}'X$ = formal completion of $\mathcal{L}'_v(X)$ into $\mathcal{L}'_0(X)$.

$\text{alg}_{\mathbb{Z}}^{\text{e}} \rightarrow \text{Spaces}$

$d \geq 1$. $\mathcal{L}'_v(X) = \text{Top}(\hat{A}^L, X)$. $A \mapsto X(A[t_1, \dots, t_d])$

$\mathcal{L}'_0(X) = \text{Top}(\hat{A}^L \setminus \{0\}, X)$. $A \mapsto \text{Top}(\text{Spec}(A[t_1, \dots, t_d]) \setminus \{0\}, X)$

Ex: $X = A'$ pb: $\mathcal{L}'_0(A) \simeq \mathcal{L}'_v(A)$ for $d \geq 2$

\leadsto DAG.

$\mathcal{L}^1 X =$ formal completion of $\mathcal{L}_V^1(X)$ into $\mathcal{L}_0^1(X)$.

$d \geq 1$. $\mathcal{L}_V^d(X) = \text{Top}(\hat{A}^d, X) : A \mapsto X(A[t_1, \dots, t_d])$

$\mathcal{L}_0^d(X) = \text{Top}(\hat{A}^d \setminus \{0\}, X) : A \mapsto \text{Top}(\text{Spec}(A[t_1, \dots, t_d]) \setminus \{0\}, X)$

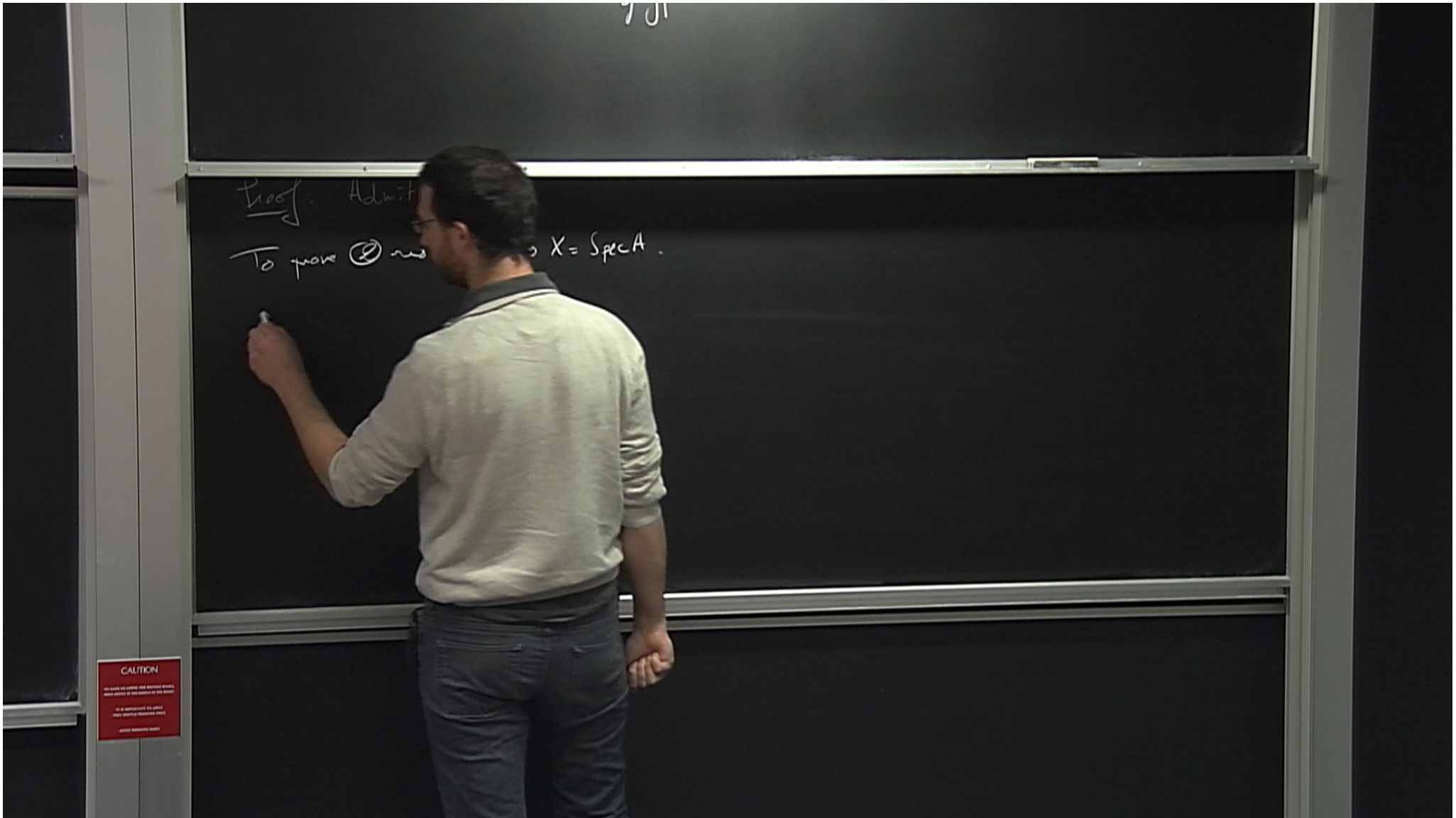
Ex: $X = A^1$ pb: $\mathcal{L}_0^d(A^1) \simeq \mathcal{L}_V^d(A^1)$ for $d \geq 2$ if not divided.

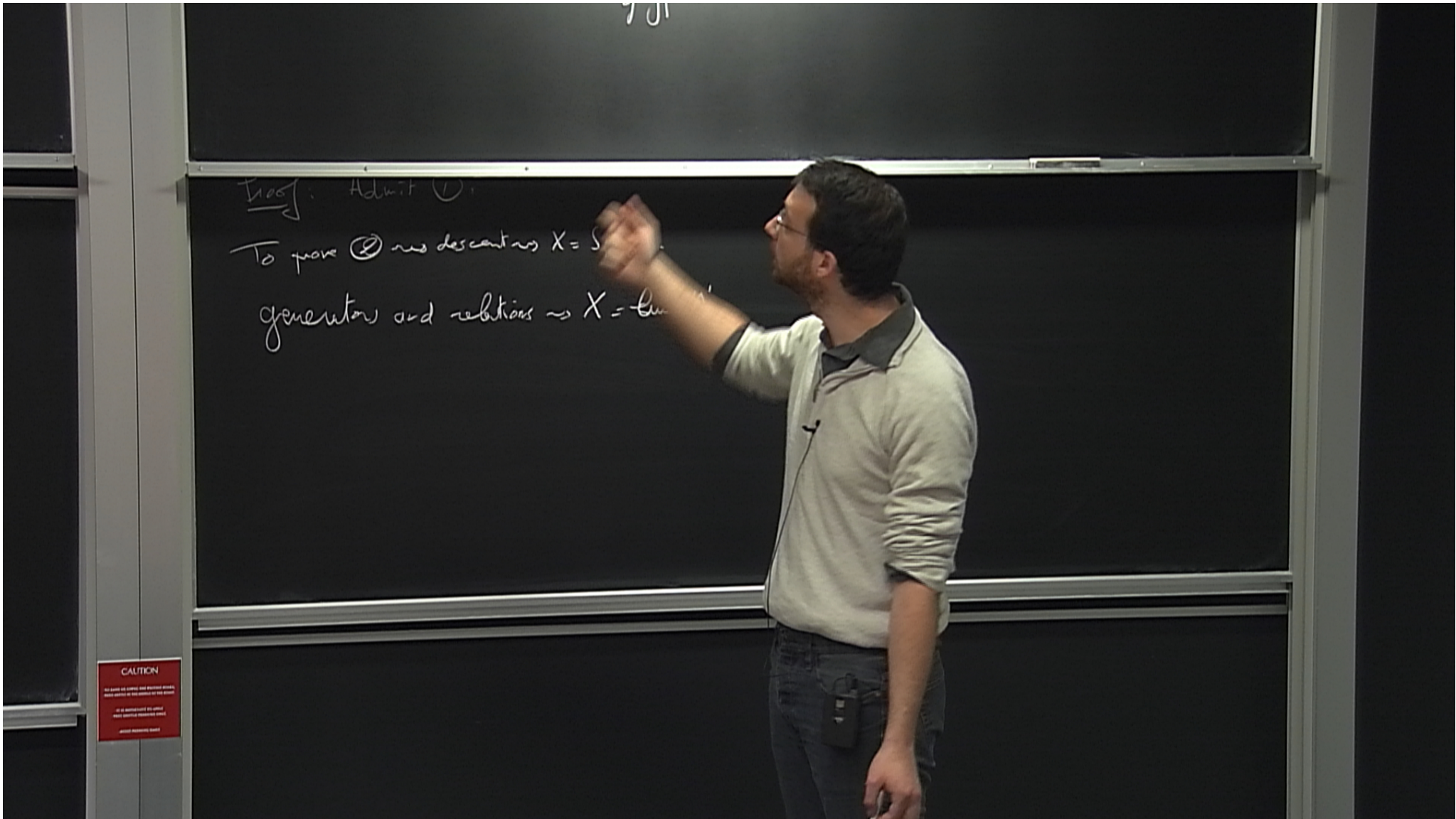
\rightsquigarrow DAG.

Def: $\mathcal{L}^d X =$ formal neighborhood of $\mathcal{L}_V^d X$ into $\mathcal{L}_0^d X$.

Thm (-):

- $X \mapsto \mathcal{Y}^d X$ satisfies étale descent.
(even smooth descent if $d=1$)
- $\mathcal{Y}^d X$ is represented by a derived ind-pro scheme of fp-





Proof. Admit ①.

To prove ② no descantors $X = \text{Spec } A$.

generators and relations $\leadsto X = \text{lim } A'$.

\leadsto Reduce to $X = A'$.

$$\hat{A}_+^d \setminus \{0\} = \text{cd} \hat{A}_X^d \setminus \{0\}^{\text{gl}}$$

$$\text{Spec}(A[t_1, \dots, t_n][t_1^{-1}])$$

Proof: Admit ①.

To prove ② no descent $X = \text{Spec } A$.

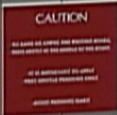
generators and relations $\leadsto X = \text{lim } A_i$.

\leadsto Reduce to $X = A'$.

$$\hat{A}_+^d \setminus \{0\} = \text{cd} = \hat{A}_+^d \setminus \{0\}^{\text{gp}}$$

$$\text{Spec} \left(A[t_1, \dots, t_n] \left[\frac{t_1^{-1}}{t_2}, \dots, \frac{t_n^{-1}}{t_1} \right] \right)$$

ind pro-diagram



Proof. Admit ①.

To prove ② no descantors $X = \text{Spec } A$.

generators and relations $\leadsto X = \text{lim } A'$

\leadsto Reduce to $X = A'$.

$$\hat{A}_+^d \setminus \{0\} = \text{cd} \hat{A}_X^d \cup \text{pt}$$

$$\text{Spec}(A[t_1, \dots, t_n][\frac{1}{t_1}, \dots, \frac{1}{t_n}])$$

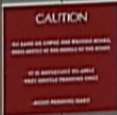
ind pr - diagram

in (coh) deg - d

At the end:

$$Z_U^d(A') = \text{cd} \lim_n \text{Spec}(k[a_{d_1, \dots, d_n}])$$

$\begin{matrix} \swarrow -n \alpha_i \text{ co, in } \\ \searrow \text{asso } J) \\ b_{P_i} = \bar{P}_i \end{matrix}$



At the end:

$$\mathcal{L}_U^d(A) = \text{colim}_n \text{lin}_P \text{Spec}(k[a_{d-1}, \dots, a_0])$$

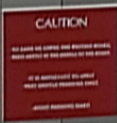
$\begin{matrix} \swarrow & \searrow \\ -n \leq d_i \leq 0 & n \leq \deg 0 \\ \downarrow & \downarrow \\ b_{\beta_i} = \beta_i & \leftarrow \text{of } \beta_i \in \mathbb{P} \end{matrix}$

Thm (-):

① $X \hookrightarrow \mathcal{L}^d X$ satisfies étale descent.
 (even smooth descent if $P^d=1$)

② $\mathcal{L}^d X$ is represented by a derived ind-proscheme
 of \mathbb{P}^d

$H^i(\dots)$ if $n=0$
 $[t_0^{-1}, \dots, t_d^{-1}]$
 if $n=d-1$



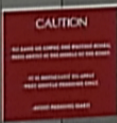
At the end:
 $\mathcal{L}_U^d(A) = \text{colim}_n \text{lin}_P \text{Spec}(k[a_{d_1, \dots, d_n}])$
in $\text{Coh}(U)$
 $-n \leq d_i \leq 0, n \leq \deg 0$
 $b_{\beta_1, \dots, \beta_d} \in \mathbb{P}^d$

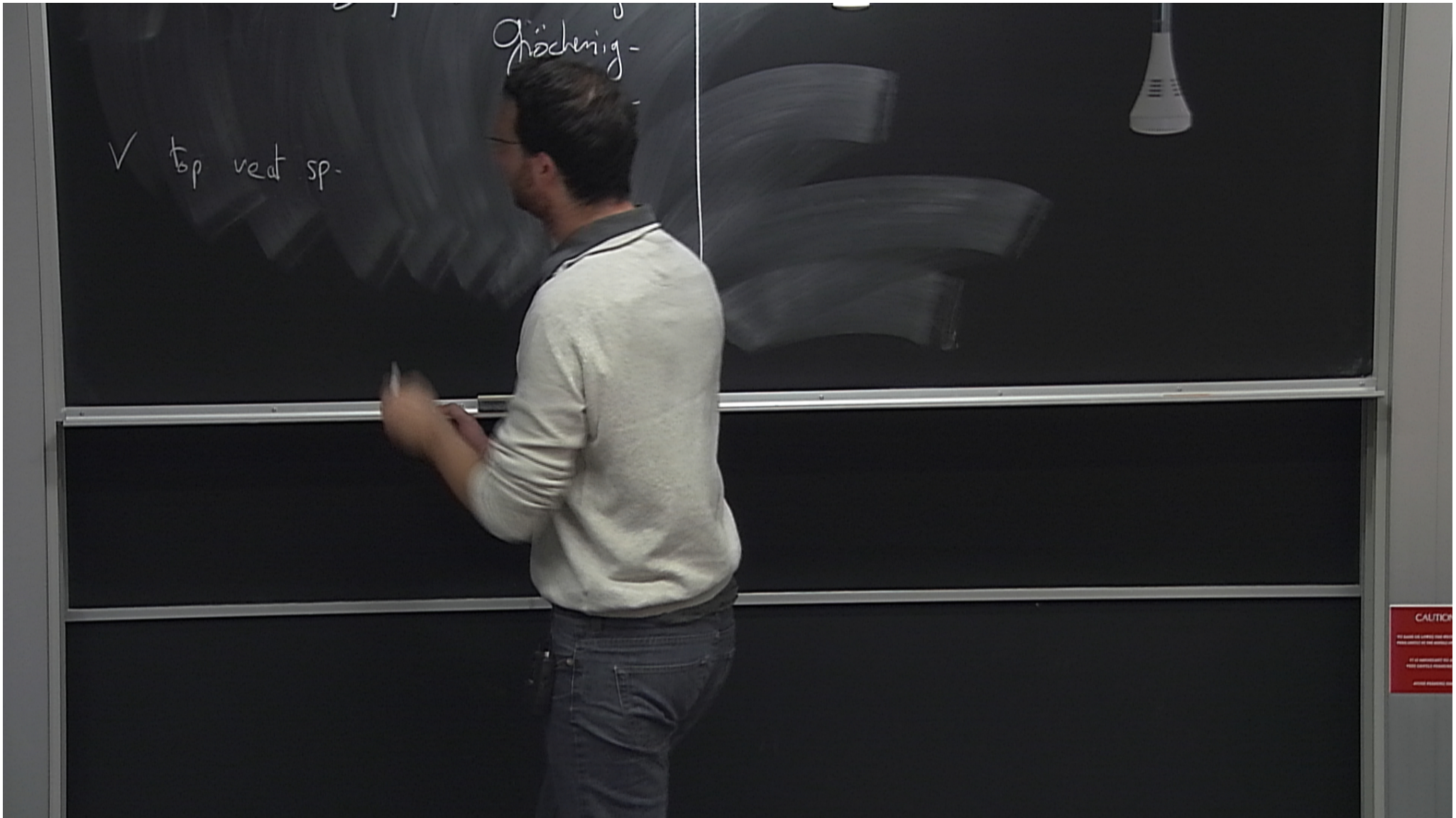
Thm (-):

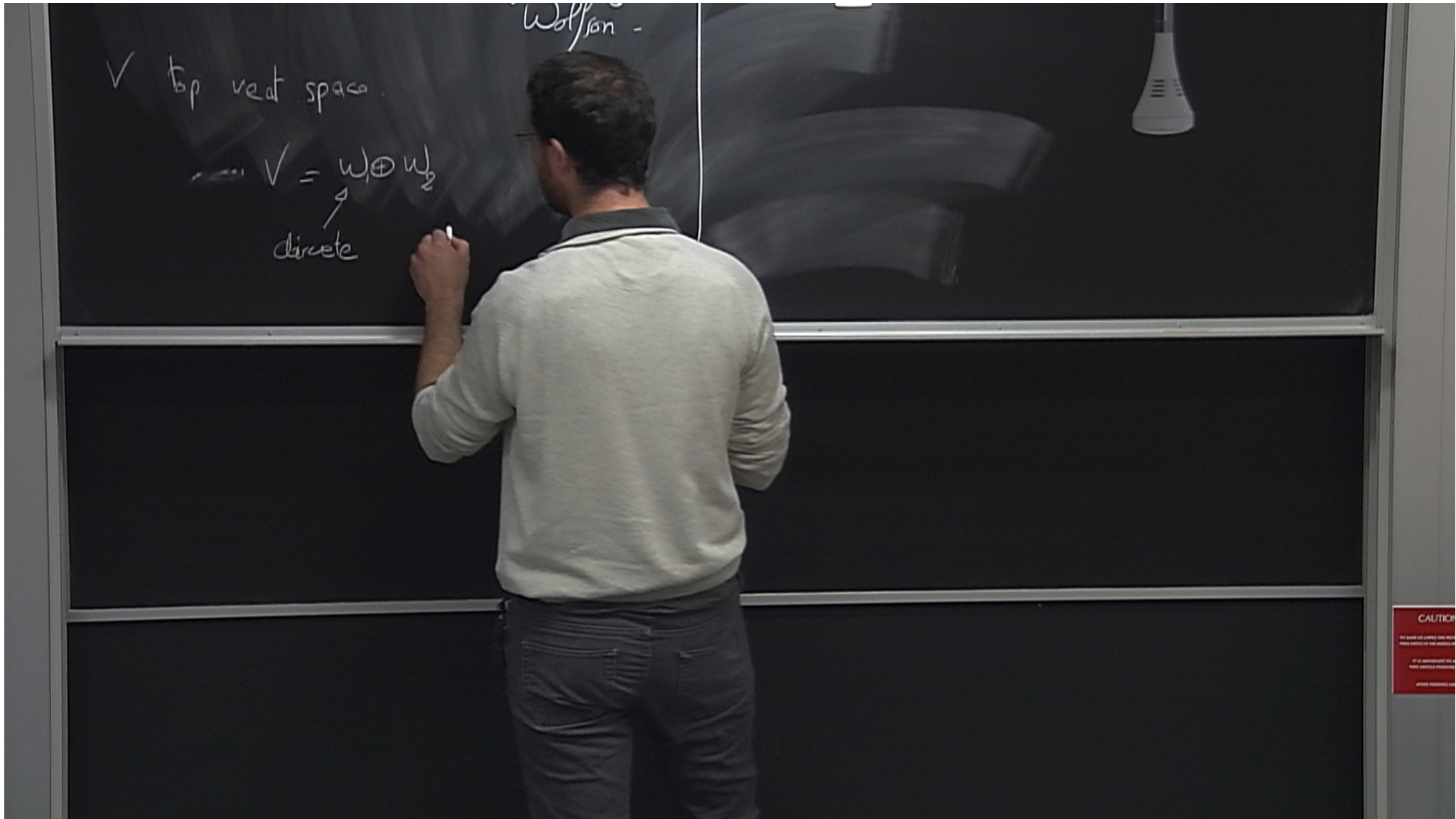
① $X \hookrightarrow \mathcal{L}^d X$ satisfies étale descent.
 (even smooth descent if $P^d = 1$).

② $\mathcal{L}^d X$ is represented by a derived ind-pro scheme of \mathbb{P}^d .

$$H^n(\hat{A}^d \setminus \{0\}) = \begin{cases} k[t_1, \dots, t_d] & \text{if } n=0 \\ (t_1, \dots, t_d)^{-1} k[t_1, \dots, t_d] & \text{if } n=d-1 \\ 0 & \text{else} \end{cases}$$







Vector spaces -
 (Lefschetz, Tate, Beilinson
 Deligne, ..., Brauerling-
 Groeching-
 Wolfson)

V top vect space.

Tate vector $V = W_1 \oplus W_2$
 \uparrow discrete \uparrow dual of a discrete.

P_{eff}^k

$\text{Ind } P_{\text{eff}}^k$

d

vector spaces

(Lefschetz, Tate, Beilinson)

Deligne, ... Broussier-

Gröchenig-Wolfson)

V top vect space

Tate work: $V = W_1 \oplus W_2$
 $\swarrow \quad \nwarrow$
 discrete dual of a discrete

P_{eff}^k

$\text{Ind } P_{\text{eff}}^k$

"
 $d\text{Mod}_k$

$\rightarrow \mathbb{T}$

Vector spaces

(Lefschetz, Tate, Beilinson
 Deligne, Brylinski-
 Grothendieck-
 Wolfson)

V top vector space

Tate mod $V = W \oplus W^*$
 \uparrow discrete \uparrow dual of a discrete

Pic_k

$Ind Pic_k$

$dymod_k \cup$

$Pro Pic_k$

\cup

$$\pi_1 \rightarrow \pi_2 \rightarrow \pi_3$$

Def. \mathcal{C} st and id-comp $(\infty, 1)$ -cat.

$$\text{Tate}(\mathcal{C}) \hookrightarrow \text{ProInd}(\mathcal{C})$$

Smallest full subcat. st stable, id-comp
and contains $\text{Ind}\mathcal{C}$ and $\text{Pro}\mathcal{C}$.

Thm:

Def. $L^d X =$ formal neighborhood of $L_V^d X$ into $L_U^d X$.

Def. \mathcal{C} st and id. comp. $(\infty, 1)$ -cat.

$$\text{Tate}(\mathcal{C}) \hookrightarrow \text{ProInd}(\mathcal{C})$$

Smallest full subcat. st stable, id. comp.
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Thm:

• $\text{Tate} \mathcal{C}$ has a univ. prop \simeq
 $\text{Tate}(\mathcal{C}) \hookrightarrow \text{IndPro}(\mathcal{C})$

• $K^{nc}(\text{Tate} \mathcal{C}) \simeq \sum K^{nc}(\mathcal{C})$

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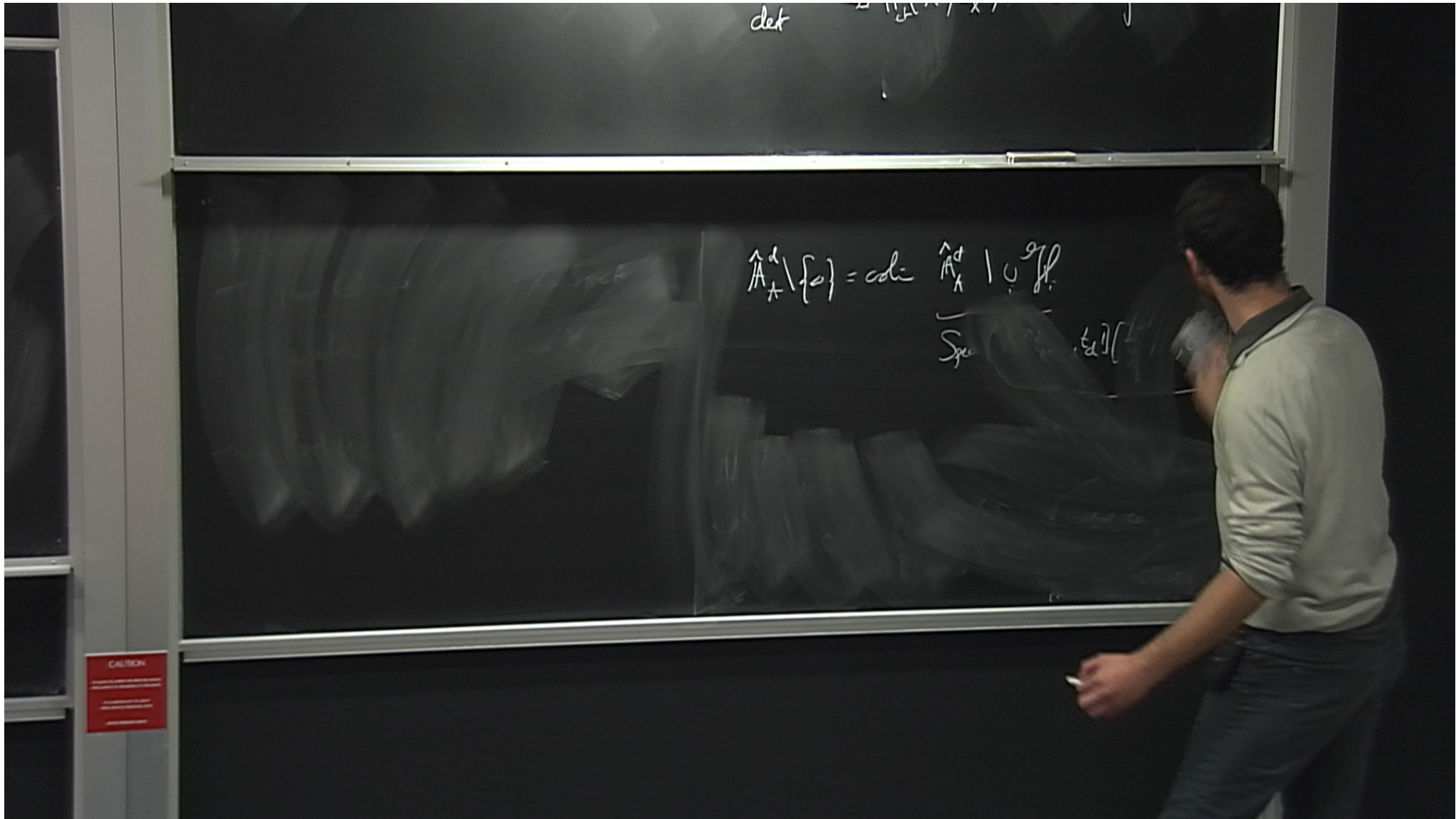
At the end:

$$L_U^d(A) = \text{col}_n \text{ lin}_p \text{ Spec}(k[a_{d_i - k_i}])$$
in $(\text{col}) \text{ deg} = \dots$
 \swarrow $-n \text{ deg}_i \text{ row}$, in $\text{deg } 0$
 \searrow $\text{of } \beta_i \text{ (sp)}$
 $b_{\beta_i} - \beta_i$

$$A \mapsto \text{Tate}(\text{Perf } A) = \text{Tate } A$$

$$\begin{matrix} \text{Tate} \\ \cup \\ A \\ \cup \\ V \end{matrix} \rightsquigarrow [V] \in K_0(\text{Tate } A) = K_{-1}(\text{Perf } A) \xrightarrow{X}$$

CAUTION



det $\rightarrow H^1(X, \mathcal{O}_X)$

Thm: $\mathbb{T}_{\mathcal{L}^d X}$ and $\mathbb{L}_{\mathcal{L}^d X}^{\text{ProIndProf}} \in$ are Tate modules over $\mathcal{L}^d X$

Proof: Reduce to $X = \mathbb{A}^1$
 $\mathbb{L}_{\mathcal{L}^d \mathbb{A}^1}^{\text{ProIndProf}} = \text{coker} \left(\begin{matrix} h^{\text{ad}} & \text{ad} \\ \downarrow & \downarrow \\ \mathbb{P} & \mathbb{N} \end{matrix} \right) \oplus k^{d(p-1)}$

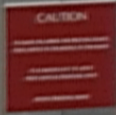
CAUTION

det $\rightarrow H^1(X, \mathcal{O}_X^*)$

Thm: \mathbb{P}^1 and \mathbb{A}^1 are Tate modules over \mathbb{A}^1

Proof: Reduce to $X = \mathbb{A}^1$
 $H^1(\mathbb{A}^1, \mathcal{O}_X^*) = \text{coker} \left(\begin{matrix} \text{ad} \\ \mathbb{Z} \xrightarrow{p} \mathbb{Z} \end{matrix} \right) \oplus \mathbb{Z}^{d(p-1)}$

Ca: $[H^1(\mathbb{A}^1, \mathcal{O}_X^*)] \subset H^1(\mathbb{A}^1, \mathcal{O}_X^*)$



III Kac-Moody:

G reductive group. \mathfrak{g} tangent Lie algebra.

$d=1$

$$\mathcal{L}'(BG) \longrightarrow K(G_m, 2)$$

determinantal

class $[\frac{T_{BG}}{Z_{BG}}]$

KV

$\mathfrak{g}((t))$ central ext:

$$\sum x_n t^n \otimes \sum y_n t^n \longrightarrow \sum_n k(x_n, y_n)$$

$$\mathfrak{g}((t)) \oplus k$$

$$\mathfrak{g}((t)) \longrightarrow k[[t]] \xrightarrow{\text{ext. de}}$$

joint w/ Faonte, Kapranov.

idea

$d \geq 2$:

$$\mathcal{L}^d(BG) \rightarrow K(K_m, 2)$$

Def: $\mathcal{L}^d X$ = formal neighborhood of $\mathcal{L}_1^d X$ into $\mathcal{L}_0^d X$.

$$BG \longrightarrow K(G_m, \mathbb{Z})$$

determinantal
class $\left[\frac{T}{\Sigma BG} \right]$

$g((t))$ central ext:

$$\sum x_n t^n \otimes \sum y_n t^n \longrightarrow \sum_n K(x_n, y_n)$$

$$g((t)) \oplus k$$

KV

$$g((t)) \longrightarrow k[[t]] \rightsquigarrow \text{ex. class in } C^2(g((t)))$$

$$\text{End}(g((t))) \longrightarrow k[[t]]$$

$\sum x_n e \otimes \sum y_n t \rightarrow \sum x_n (x_n, y_n)$
 $g(t) \oplus k$
 $\underline{KV} \rightarrow g(t) \rightarrow k[t] \rightarrow \text{ext class in } C^2(g(t))$
 $\text{defines } HC^1(\text{End}(g(t)))$
 $\text{End}(g(t)) \rightarrow k[t]$
 Take trace.

joint w/ Faonte
 idea
 $d \geq 2$
 $L^d(BG) \rightarrow k$

Def: $L^d X =$

Favorte, Kapranov.

$$k((t)) \sim A = \mathbb{R}P(\hat{A}^d \setminus \{2, 0\}) \leftarrow k((t_m, t))$$

\mathbb{G}_m

$$g \otimes A \rightarrow \text{End}(g) \otimes A \rightarrow \text{End}(g \otimes A) \xrightarrow{\text{Tate trace}} k[[t]]$$

$$H^2(g \otimes A) \leftarrow H^1(\text{End}(g) \otimes A) \leftarrow H^1(\text{End}(g \otimes A))$$

$$a) \rightarrow K(\mathbb{G}_m, 2)$$

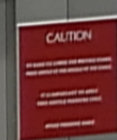
$$\mathcal{L}(P_{\text{eff}}) \rightarrow \text{Tate} \rightarrow K(\mathbb{G}_m, 2)$$

$\mathcal{L}_1 X = \text{punct neighborhood of } \mathcal{L}_1 X \text{ into } \mathcal{L}_0 X$

Thm. $\prod_{\mathcal{L}_1 X} \text{ and } \prod_{\mathcal{L}_0 X}$

Proof: Reduce to $X = \mathbb{A}^1$

$\prod_{\mathcal{L}_1 \mathbb{A}^1} = \text{column } \begin{pmatrix} i \\ p \\ n \end{pmatrix}$



Favorte, Kapranov.

$$k((t)) \xrightarrow{\sim} A = \mathbb{R}P^1(\hat{A}^d \setminus \{2, 0\}) \leftarrow k((t, \dots, t))$$

\mathbb{G}_m

$$g \otimes A \rightarrow \text{End}(g) \otimes A \rightarrow \text{End}(g \otimes A) \rightarrow$$

Take trace

$$H^2(g \otimes A) \leftarrow H^1(\text{End}(g) \otimes A) \leftarrow H^1(\text{End}(g \otimes A))$$

$\downarrow \text{tr}$

$$H^1(A) \quad \mathbb{G}_m\text{-inv}$$

$$a) \rightarrow K(\mathbb{G}_m, 2)$$

$$\mathcal{L}(P_{\text{eff}}) \rightarrow \text{Take} \rightarrow K(\mathbb{G}_m, 2)$$

Thm. $\prod_{\mathbb{Z}/p\mathbb{Z}} \text{and } \prod_{\mathbb{Z}/p\mathbb{Z}}$

Reduce to $X = \mathbb{A}^1$

$\mathbb{Z}/p\mathbb{Z} \setminus \{0\} = \text{columns } p \times n$

$\mathbb{Z}/p\mathbb{Z} \setminus \{0\} = \text{punctured neighborhood of } \mathbb{Z}/p\mathbb{Z} \text{ into } \mathbb{Z}/p\mathbb{Z}$

Favorte, Kapranov.

$$k((t)) \xrightarrow{\sim} A = \mathbb{R}P^1(\hat{A}^d \setminus \{2, 0\}) \xleftarrow{\sim} k((t, t^{-1}))$$

\mathbb{G}_m

$$g \otimes A \rightarrow \text{End}(g) \otimes A \rightarrow \text{End}(g \otimes A) \xrightarrow{\text{Tate trace}} k[[t]]$$

$$H^2(g \otimes A) \leftarrow H^1(\text{End}(g) \otimes A) \leftarrow H^1(\text{End}(g \otimes A))$$

$\downarrow \text{H}^1$

$$H^1(A)$$

\mathbb{G}_m -inv + vanish on $k((t, t^{-1})) \rightsquigarrow$ only one such class

$$a) \rightarrow K(\mathbb{G}_m, 2)$$

$$\text{yd}(\text{Paf}) \rightarrow \text{Tate} \rightarrow K(\mathbb{G}_m, 2)$$

$d_1 X =$ formal neighborhood of $d_1 X$ into $d_1 X$



CAUTION