Title: Formal loop spaces

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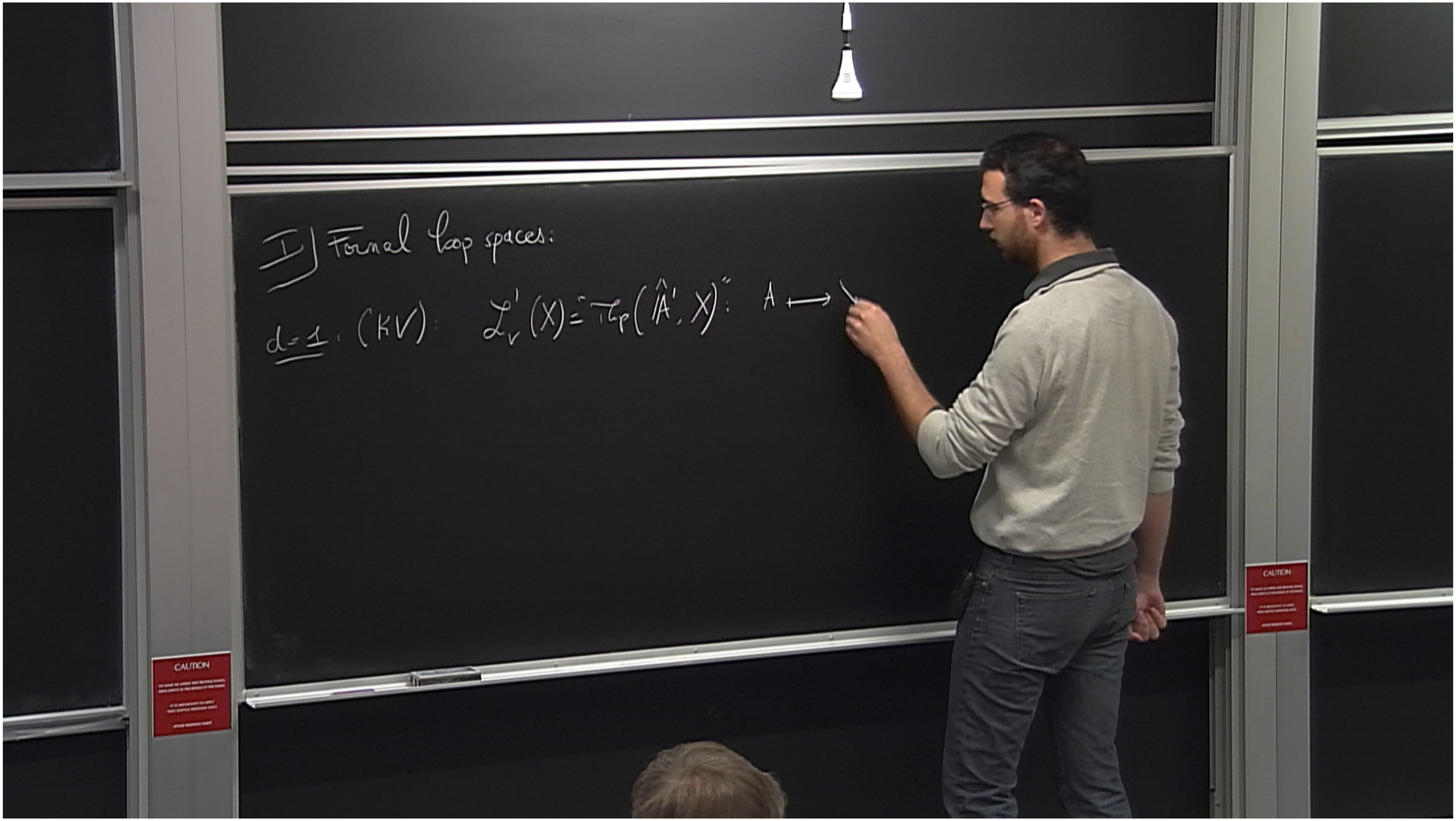
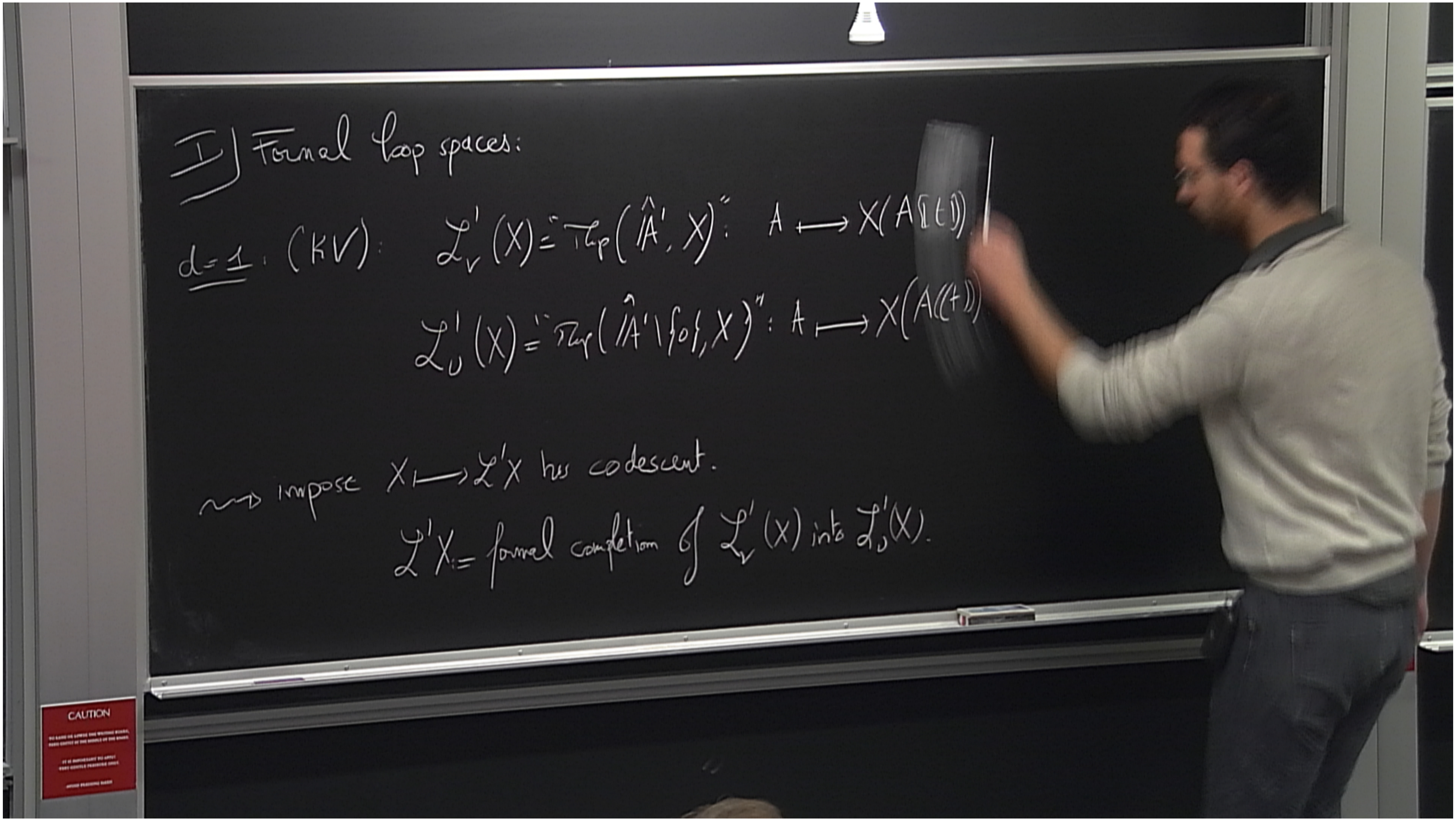
Abstract: Formal loop spaces are algebraic analogs to smooth loops. They were introduced and studied extensively in the 2000' by Kapranov and Vasserot for their link to chiral algebras.
In this talk, we will introduced higher dimensional analogs of K. and V. formal loop spaces. We will show how derived methods allow such a definition. We will then study their tangent complexes: even though formal loop spaces are "of infinite dimension", their tangent has enough structure so that we can speak of symplectic forms on them.


I) Formal loop spaces:
d=1 (KV): $\mathcal{L}^1_V(X) = $ "Map$(\hat{A}^1, X)$": A ⟼


I) Formal loop spaces:
d=1 (KV): $\mathcal{L}^1_v(X)$ = "Map($\hat{\mathbb{A}}^1$, X)": A ⟼ X(A[[t]])
$\mathcal{L}^1_u(X)$ = "Map($\hat{\mathbb{A}}^1 \setminus \{0\}$, X)": A ⟼ X(A((t)))
⇝ impose X ⟼ $\mathcal{L}^1$X has codescent.
$\mathcal{L}^1$X := formal completion of $\mathcal{L}^1_v$(X) into $\mathcal{L}^1_u$(X).
CAUTION

## I) Formal loop spaces:

$d=1$: (KV): $\mathcal{L}^1_V(X) = \text{"}\mathrm{Map}(\hat{\mathbb{A}}^1, X)\text{"}: A \longmapsto X(A[[t]])$

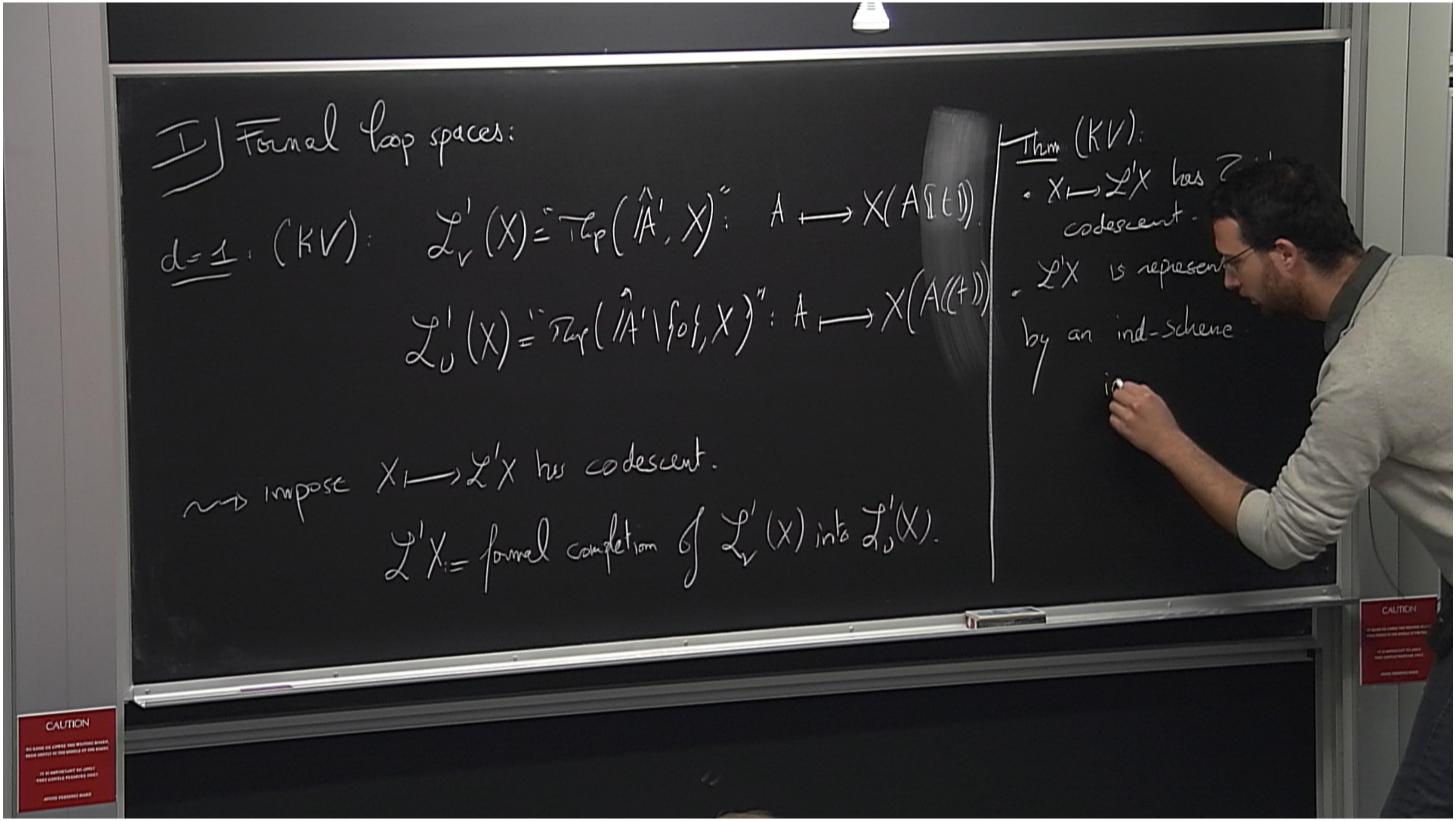
$\mathcal{L}^1_U(X) = \text{"}\mathrm{Map}(\hat{\mathbb{A}}^1 \setminus \{0\}, X)\text{"}: A \longmapsto X(A((t)))$

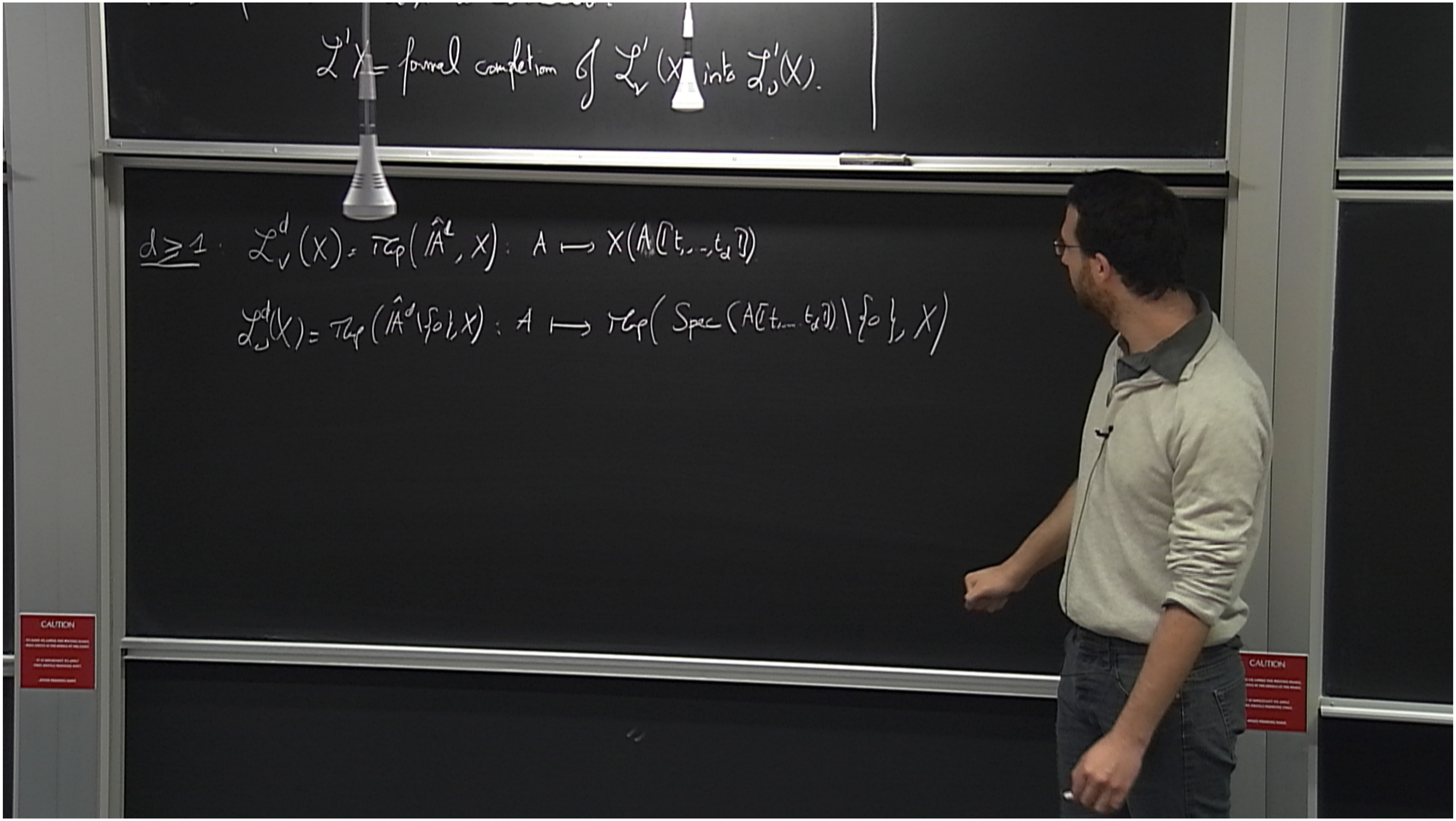
$\rightsquigarrow$ impose $X \longmapsto \mathcal{L}^1 X$ has codescent.

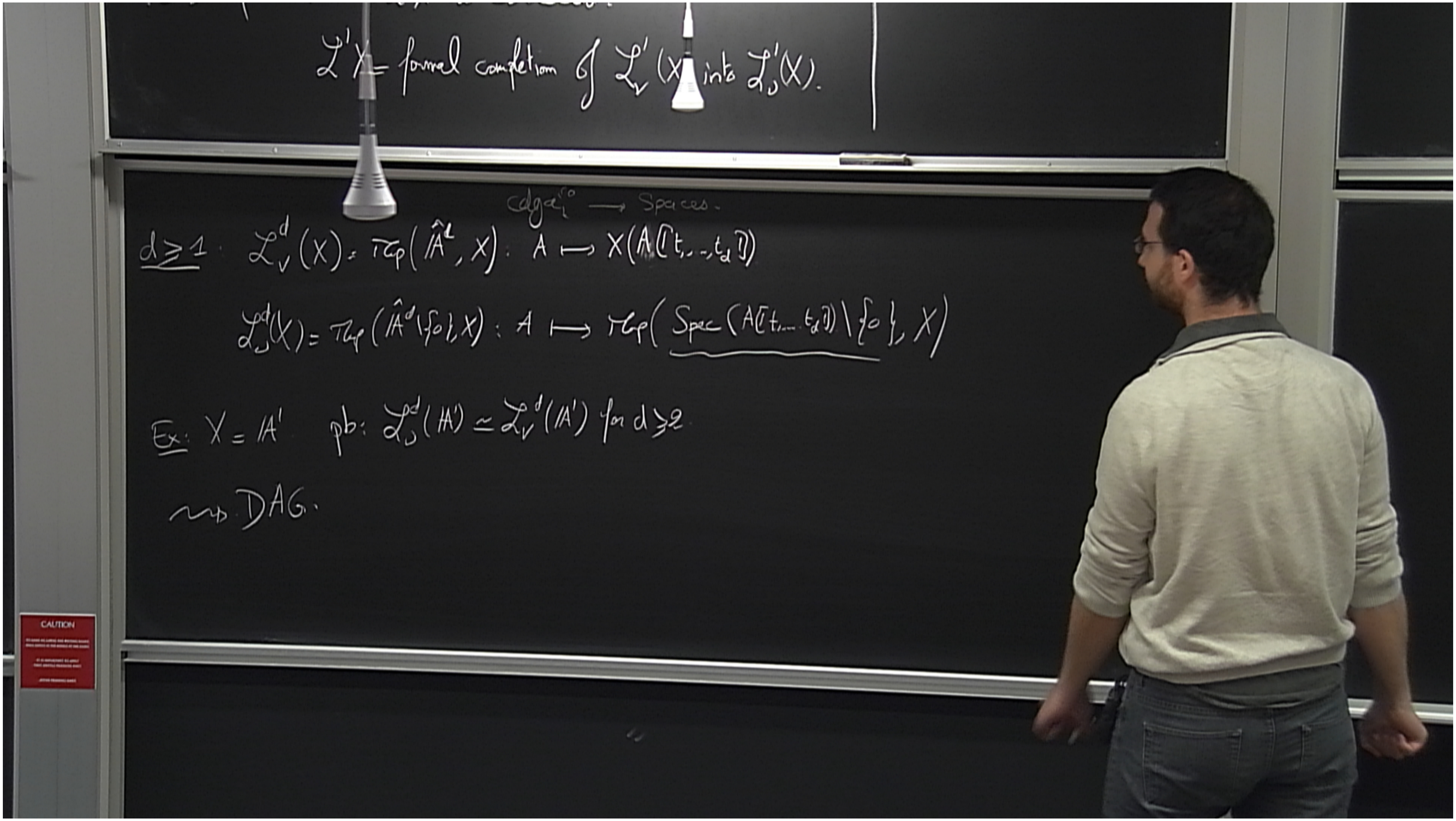
$\mathcal{L}^1 X :=$ formal completion of $\mathcal{L}^1_V(X)$ into $\mathcal{L}^1_U(X)$.

Thm (KV):

- $X \longmapsto \mathcal{L}^1 X$ has Zariski codescent.
- $\mathcal{L}^1 X$


I] Formal loop spaces:
d=1: (KV): $\mathcal{L}'_v(X) :=$ "Map$(\hat{A}^1, X)$": $A \longmapsto X(A[[t]])$
$\mathcal{L}'_u(X) :=$ "Map$(\hat{A}^1 \setminus \{0\}, X)$": $A \longmapsto X(A((t)))$
⇝ impose $X \longmapsto \mathcal{L}'X$ has codescent.
$\mathcal{L}'X :=$ formal completion of $\mathcal{L}'_v(X)$ into $\mathcal{L}'_u(X)$.
Thm (KV):
• $X \longmapsto \mathcal{L}'X$ has
codescent.
• $\mathcal{L}'X$ is represent
by an ind-scheme


$\mathcal{L}'X$ = formal completion of $\mathcal{L}'_v(X)$ into $\mathcal{L}'_\cup(X)$.
$d \geq 1$ : $\mathcal{L}^d_v(X) = \mathrm{Map}(\hat{\mathbb{A}}^d, X)$ : $A \longmapsto X(A[[t_1, \ldots, t_d]])$
$\mathcal{L}^d_\cup(X) = \mathrm{Map}(\hat{\mathbb{A}}^d \setminus \{0\}, X)$ : $A \longmapsto \mathrm{Map}(\mathrm{Spec}(A[[t_1 \ldots t_d]]) \setminus \{0\}, X)$
CAUTION
CAUTION


$\mathcal{L}'X$ = formal completion of $\mathcal{L}'_v(X)$ into $\mathcal{L}'_u(X)$.
$\mathrm{cdga}_k^{\leq 0} \longrightarrow$ Spaces.
$d \geq 1$: $\mathcal{L}^d_v(X) = \mathrm{Map}(\hat{\mathbb{A}}^d, X)$: $A \longmapsto X(A[[t_1,\dots,t_d]])$
$\mathcal{L}^d_u(X) = \mathrm{Map}(\hat{\mathbb{A}}^d \setminus \{0\}, X)$: $A \longmapsto \mathrm{Map}(\mathrm{Spec}(A[[t_1,\dots,t_d]]) \setminus \{0\}, X)$
Ex: $X = \mathbb{A}^1$ pb: $\mathcal{L}^d_u(\mathbb{A}^1) \simeq \mathcal{L}^d_v(\mathbb{A}^1)$ for $d \geq 2$
$\rightsquigarrow$ DAG.

$\mathcal{L}'X$ = formal completion of $\mathcal{L}'_v(X)$ into $\mathcal{L}'_u(X)$.

$cdga_k^{\leq 0} \longrightarrow$ Spaces.

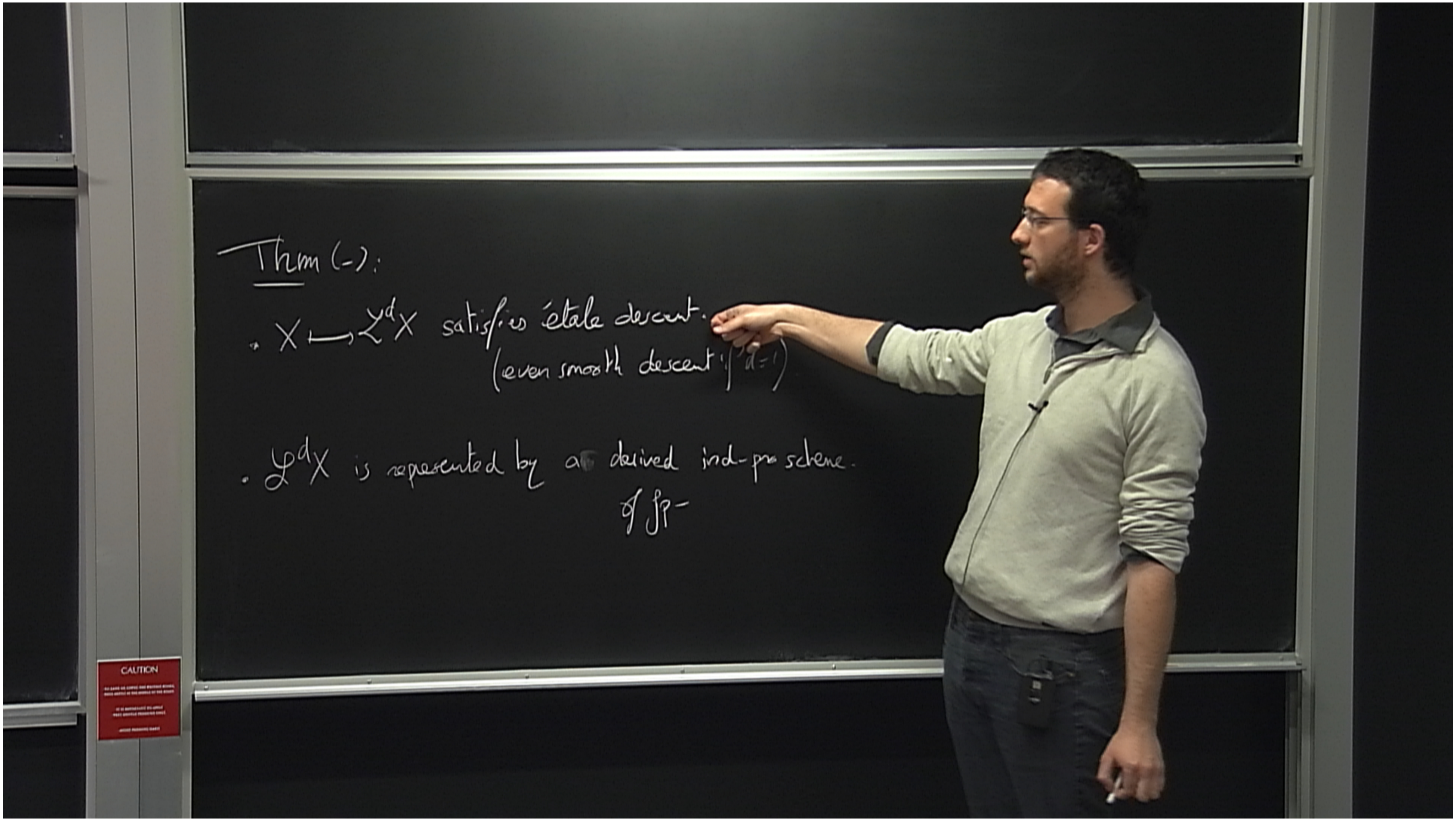
$\underline{d \geq 1}$. $\mathcal{L}^d_v(X) = \mathrm{Map}(\hat{\mathbb{A}}^d, X) : A \longmapsto X(A[[t_1, \dots, t_d]])$

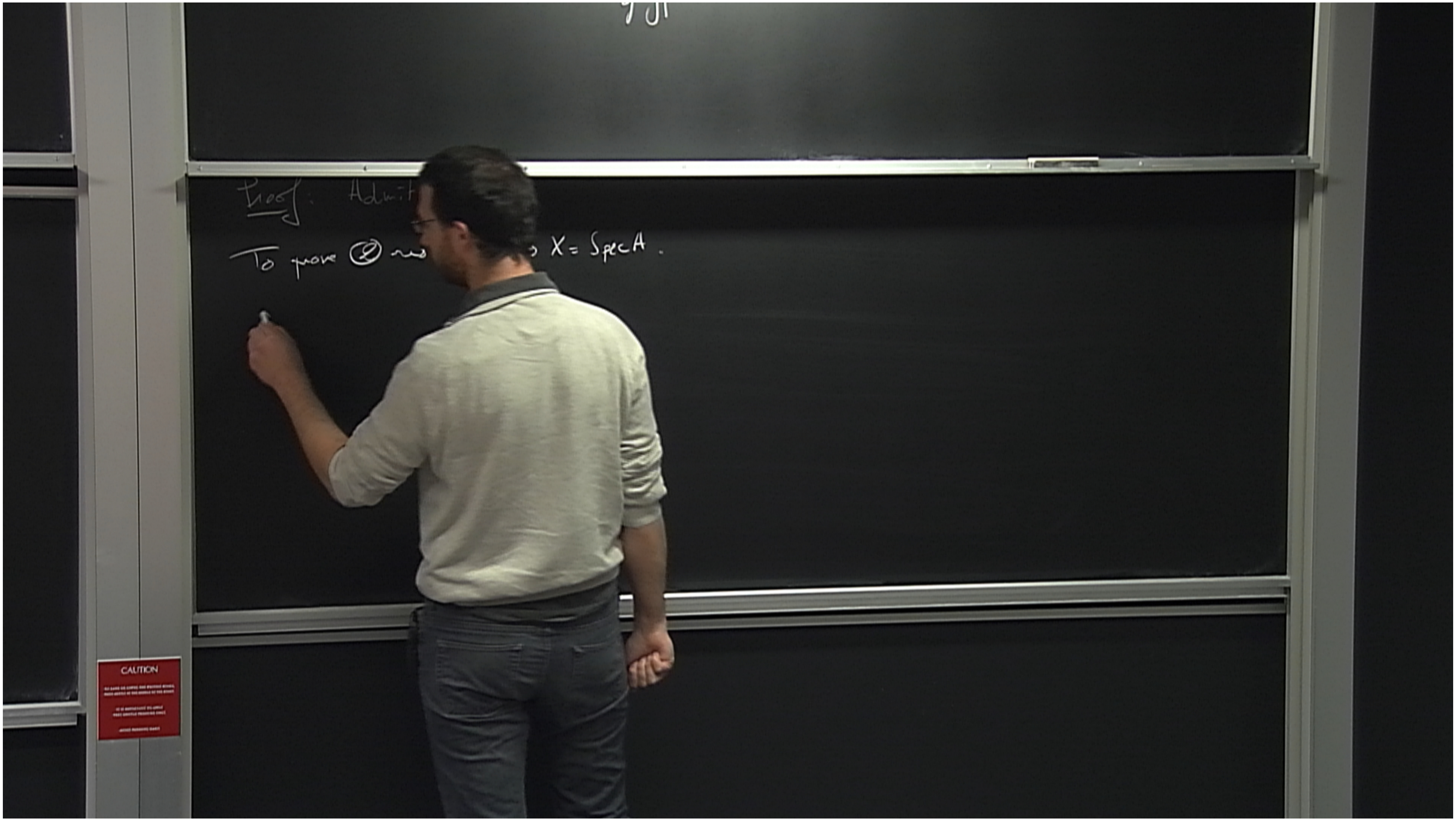
$\mathcal{L}^d_u(X) = \mathrm{Map}(\hat{\mathbb{A}}^d \setminus \{0\}, X) : A \longmapsto \mathrm{Map}(\underline{\mathrm{Spec}(A[[t_1 \dots t_d]]) \setminus \{0\}}, X)$

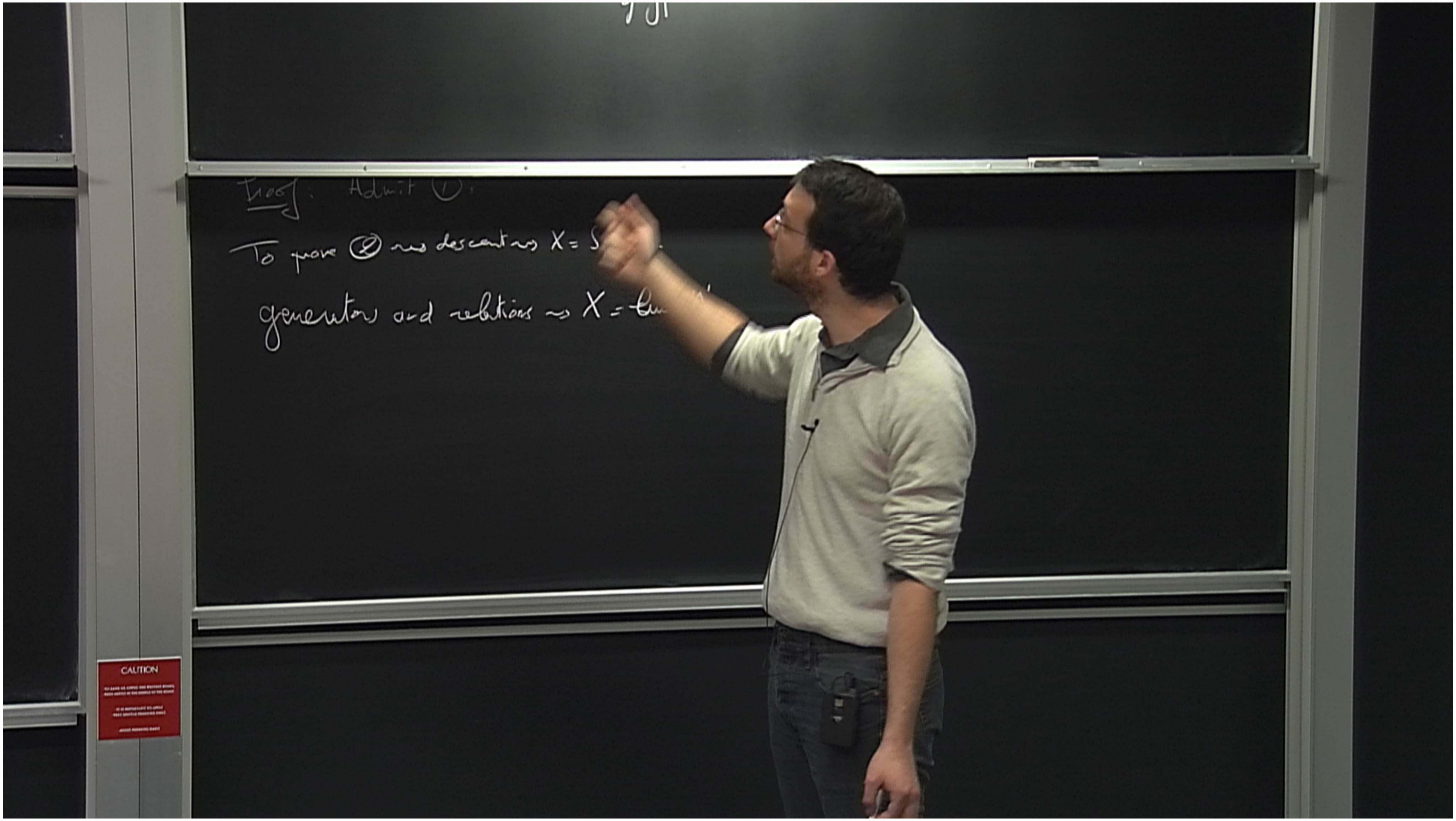
$\underline{\text{Ex}}$: $X = \mathbb{A}^1$ pb: $\mathcal{L}^d_u(\mathbb{A}^1) \simeq \mathcal{L}^d_v(\mathbb{A}^1)$ for $d \geq 2$ if not $\underline{\text{derived}}$

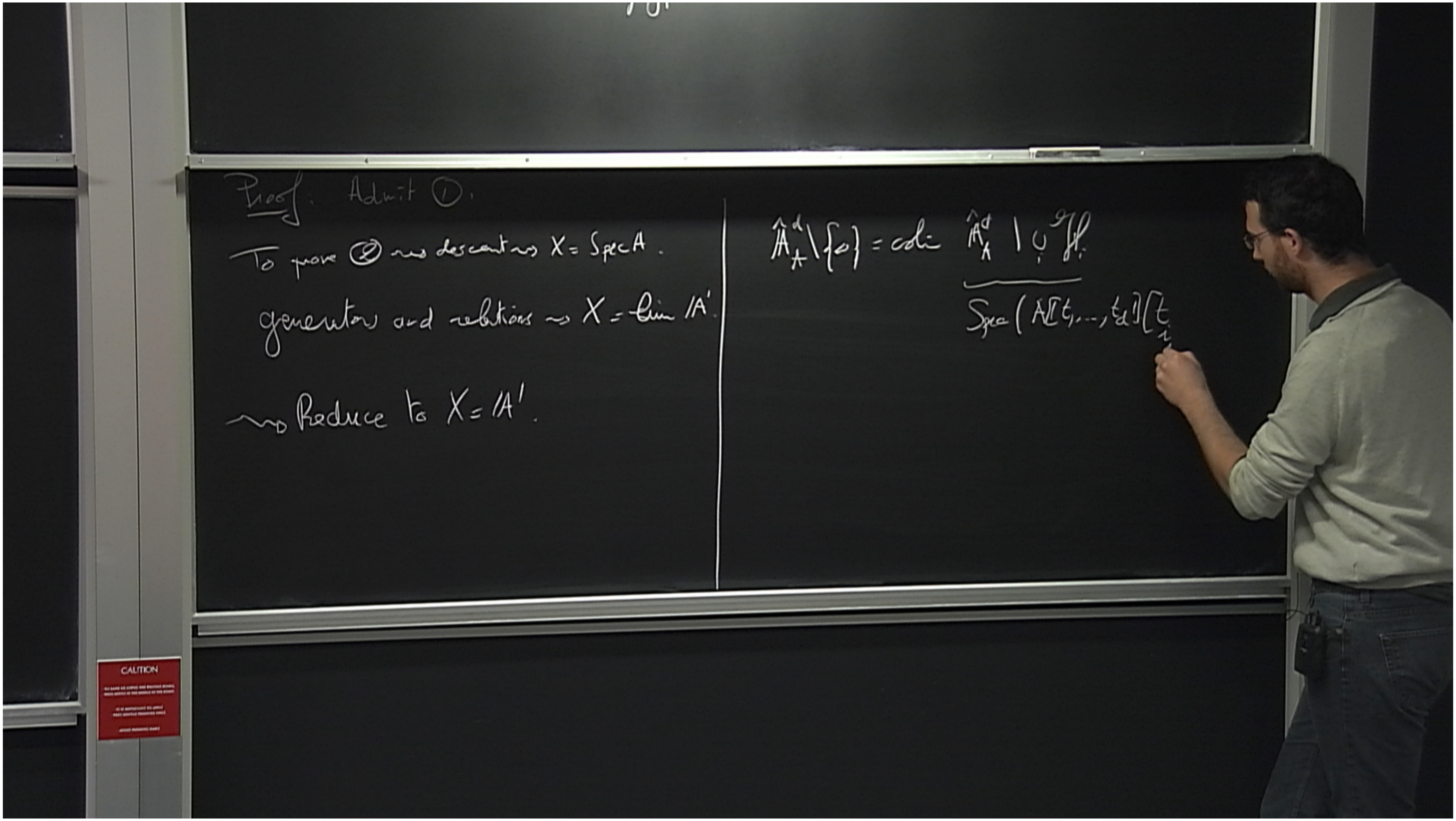
$\rightsquigarrow$ DAG.

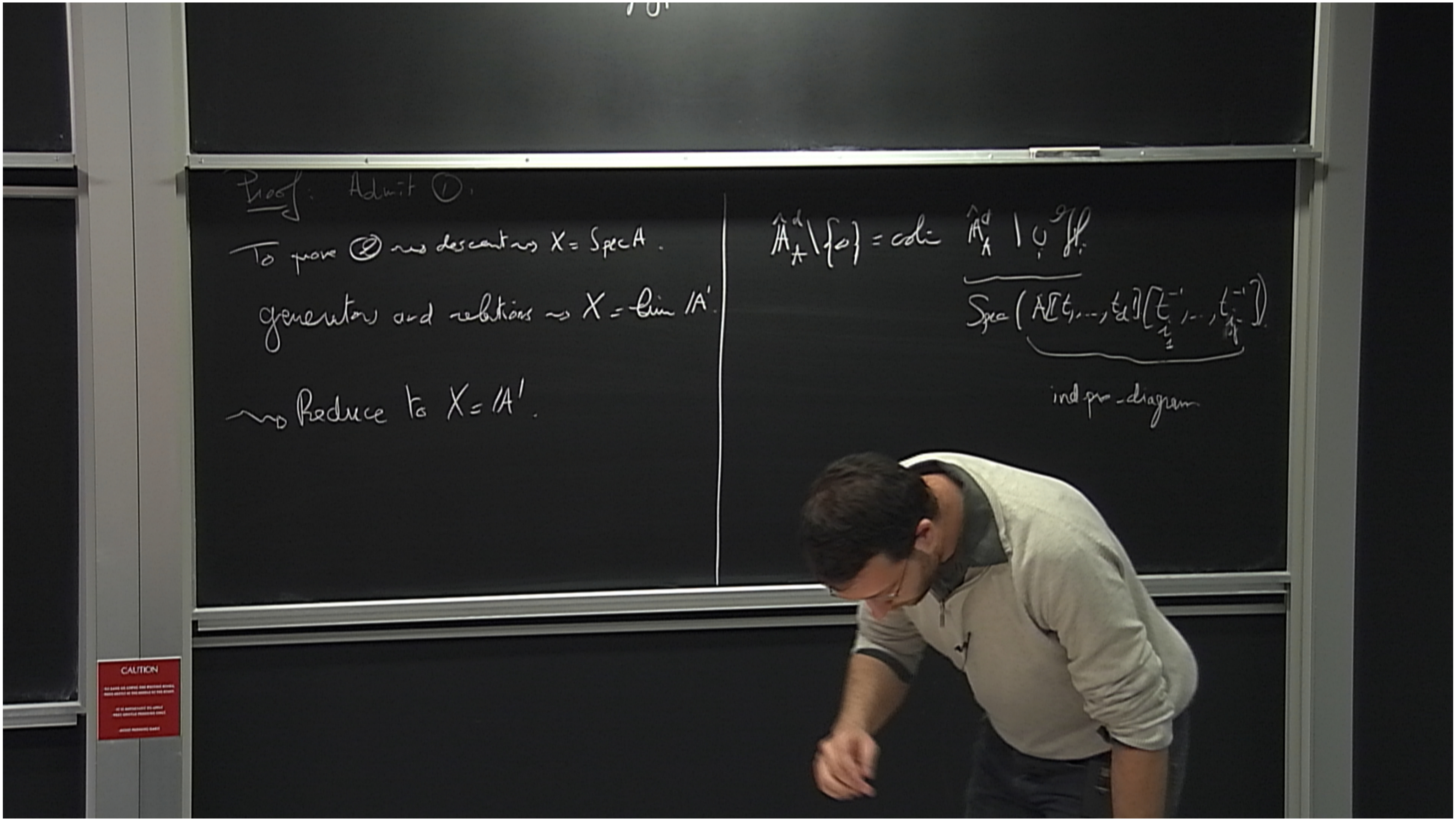
$\underline{\text{Def}}$: $\mathcal{L}^d X$ = formal neighborhood of $\mathcal{L}^d_v X$ into $\mathcal{L}^d_u X$.


Thm (-):
• X ⟼ ℒ^d X satisfies étale descent.
(even smooth descent if d=1)
• ℒ^d X is represented by a derived ind-pro scheme.
CAUTION


Proof:
To prove
X = Spec A .


Proof: Admit ①.
To prove ② ⇝ descent ⇝ X =
generators and relations ⇝ X =


Proof. Admit ①.
To prove ② ⇝ descent ⇝ X = Spec A.
generators and relations ⇝ X = lim A¹.
⇝ Reduce to X = A¹.
Â^d_A \ {0} = coli Â^d_A \ ...
Spec ( A[[t_1, ..., t_d]][t_i


Proof: Admit 1.
To prove 2 ⇝ descent ⇝ X = Spec A.
generators and relations ⇝ X = lim A'
⇝ Reduce to X = A'.
ind pro-diagram

Proof: Admit (1).

To prove (2) $\rightsquigarrow$ descent $\rightsquigarrow$ $X = \operatorname{Spec} A$.

generators and relations $\rightsquigarrow$ $X = \lim \mathbb{A}^1$.

$\rightsquigarrow$ Reduce to $X = \mathbb{A}^1$.

$\hat{\mathbb{A}}^d_A \setminus \{0\} = \operatorname{colim} \hat{\mathbb{A}}^d_A \setminus \cup \ldots$

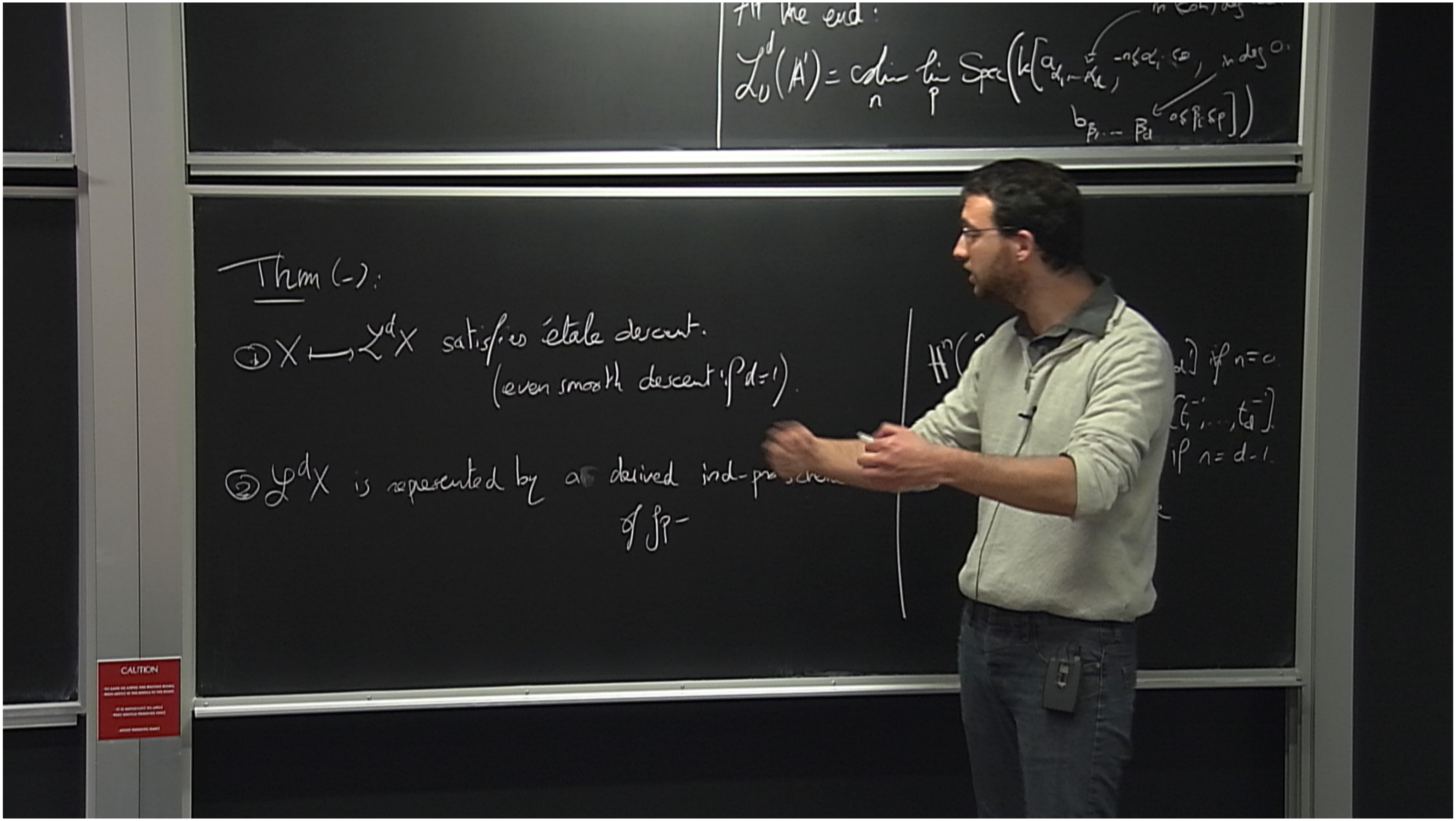
$\operatorname{Spec}\left( A[[t_1, \ldots, t_d]][t_1^{-1}, \ldots, t_d^{-1}] \ldots \right)$

ind pro-diagram

At the end:

$\mathcal{L}^d_U(\mathbb{A}^1) = \operatorname{colim}_n \lim_p \operatorname{Spec}\left(k[a_{\alpha_1, \ldots, \alpha_d}, -n \leq \alpha_i \leq \infty, \ldots b_{\beta_1 \ldots \beta_d} \ldots]\right)$

in (coh) deg


At the end:
$\mathcal{L}^d_U(A') = \operatorname{colim}_n \lim_p \operatorname{Spec}(k[a_{\alpha_1 \ldots \alpha_d}, -n \le \alpha_i \le 0, \text{ in deg } 0,$
$b_{\beta_1 \ldots \beta_d}, 0 \le \beta_i \le p])$
Thm (-):
① $X \longmapsto \mathcal{L}^d X$ satisfies étale descent.
(even smooth descent if $d=1$).
② $\mathcal{L}^d X$ is represented by a derived ind-pro-scheme
of fp-
if $n=0$
if $n=d-1$

At the end:

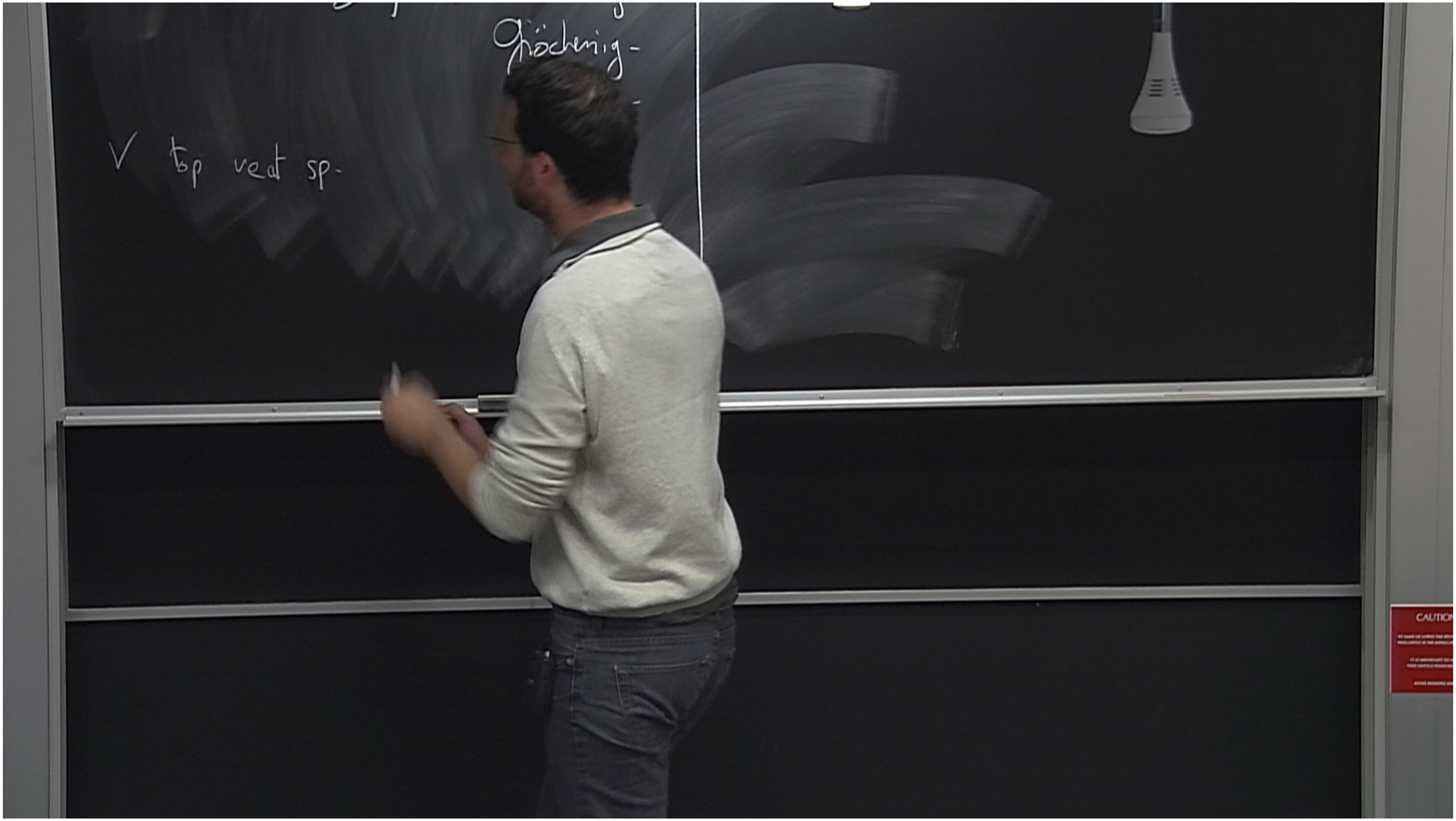
$$\mathcal{L}^d_U(\hat{\mathbb{A}}^d) = \operatorname{colim}_n \lim_p \operatorname{Spec}\left(k\left[a_{\alpha_1 \ldots \alpha_d},\ -n \leq \alpha_i \leq 0,\ \text{in deg } 0.\ b_{\beta_1 \ldots \beta_d}\ 0 \leq \beta_i \leq p\right]\right)$$

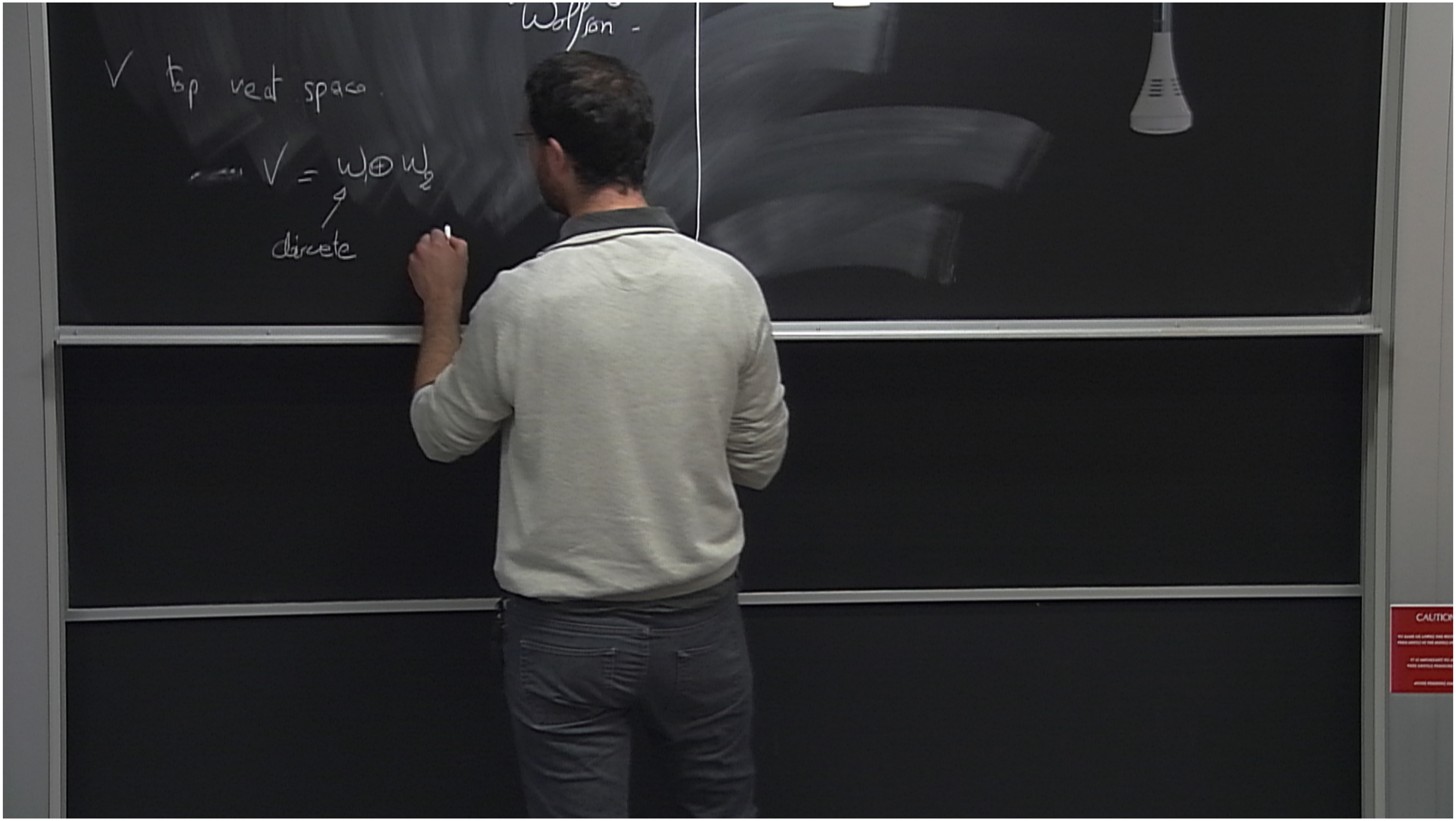
Thm (-):

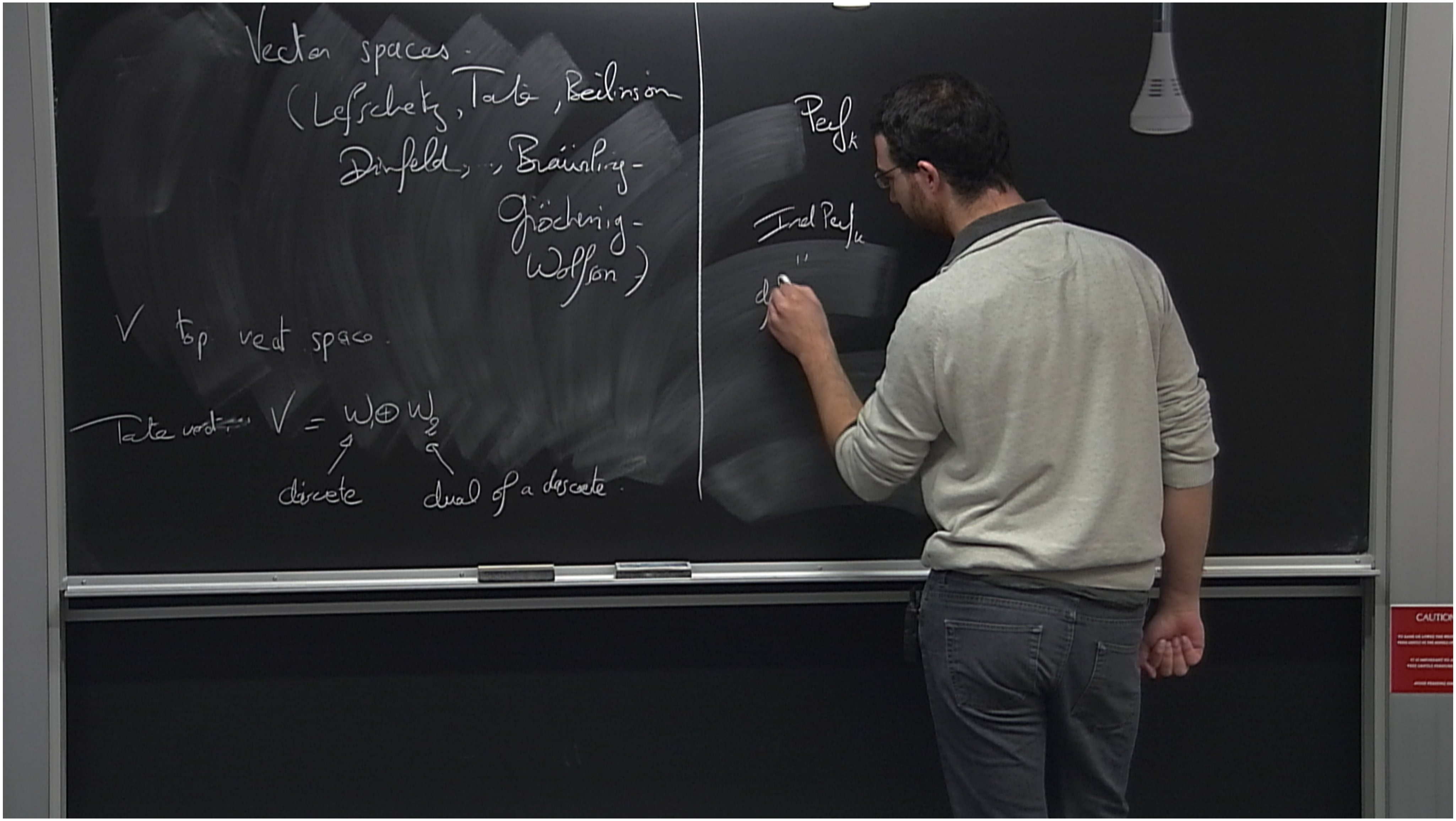
① $X \longmapsto \mathcal{L}^d X$ satisfies étale descent.
(even smooth descent if $d=1$).

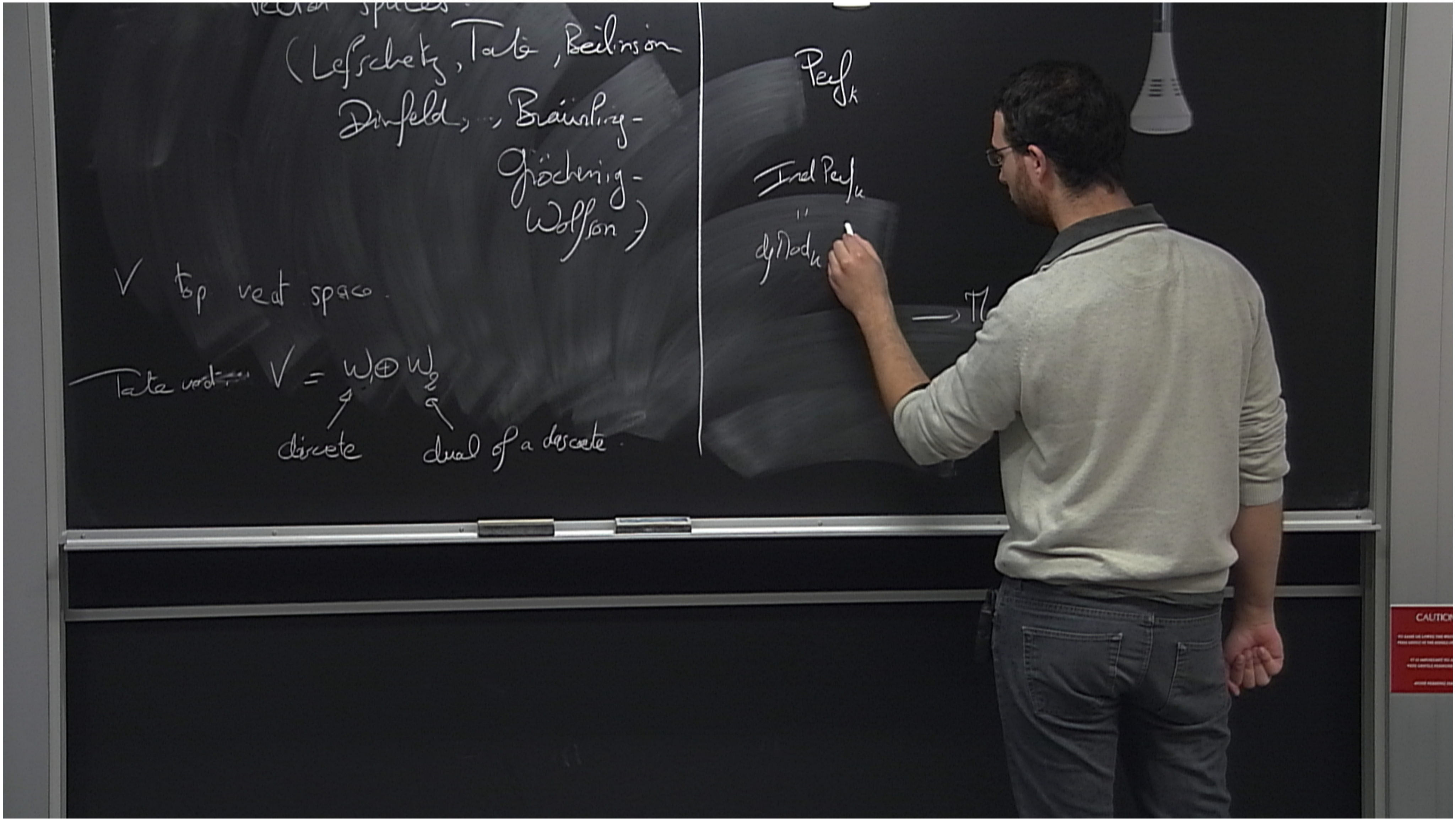
② $\mathcal{L}^d X$ is represented by a derived ind-pro scheme.
of fp -

$$H^n(\hat{\mathbb{A}}^d \setminus \{0\}) = \begin{cases} k[t_1, \ldots, t_d] & \text{if } n=0 \\ (t_1 \ldots t_d)^{-1} k[t_1^{-1}, \ldots, t_d^{-1}] & \text{if } n = d-1 \\ 0 & \text{else} \end{cases}$$


Gröchenig-
CAUTION


Wolfson -
V top vect space.
V = W1 ⊕ W2
discrete


Vector spaces.
(Lefschetz, Tate, Beilinson
Drinfeld, .., Bräunling-
Gröchenig-
Wolfson)
V top vect space.
Tate vect: V = W1 ⊕ W2
discrete
dual of a discrete.
Perf_k
Ind Perf_k


(Lefschetz, Tate, Beilinson
Drinfeld, .., Bräunling-
Gröchenig-
Wolfson)
V top vect space.
Tate vect: V = W1 ⊕ W2
discrete
dual of a discrete
Perfk
Ind Perfk
dgModk

Vector spaces.

(Lefschetz, Tate, Beilinson, Drinfeld, .., Bräunling-Gröchenig-Wolfson)

$V$ top vect space.

Tate vect $V = W_1 \oplus W_2$

discrete

dual of a discrete.

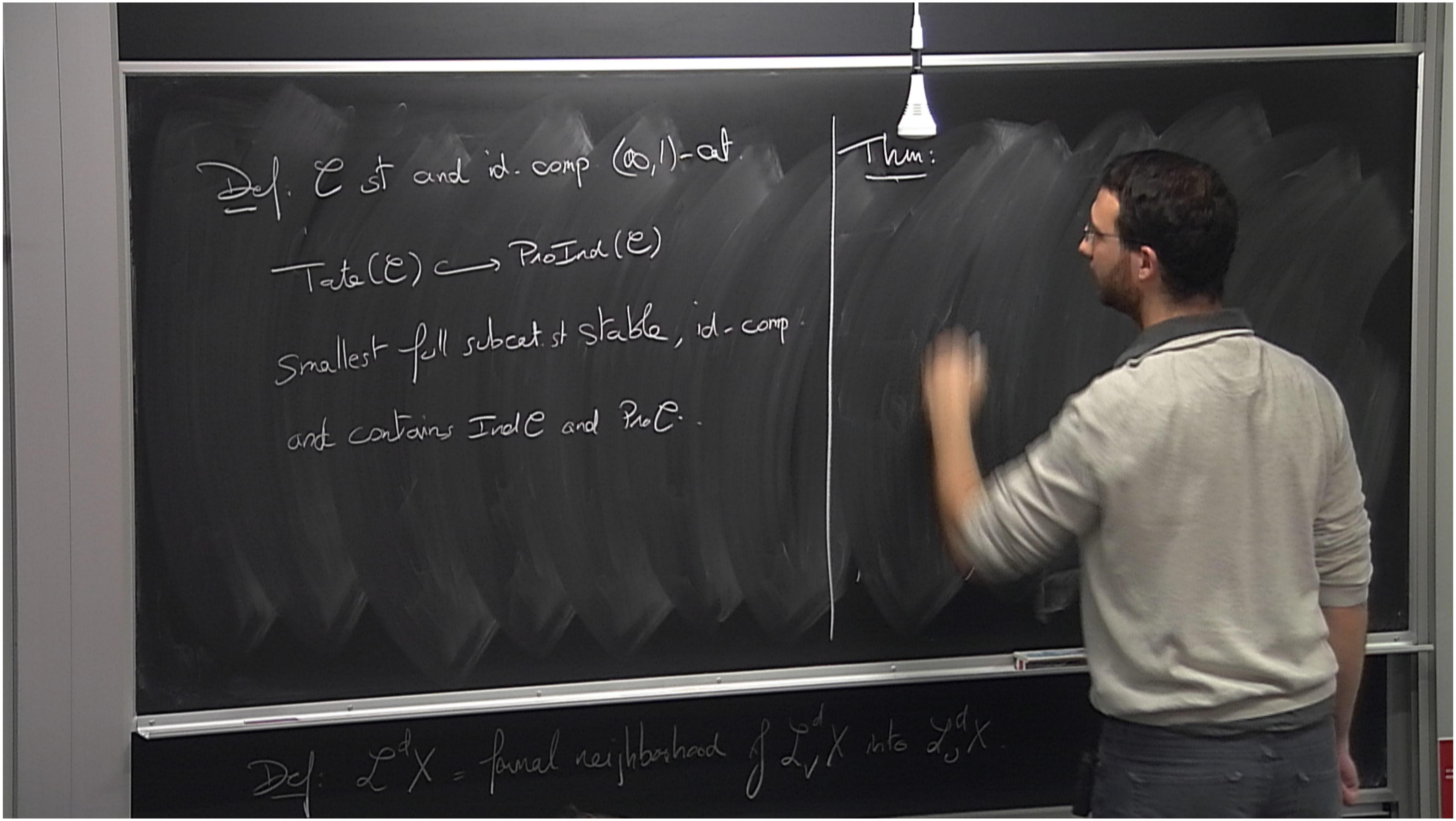
$\mathrm{Perf}_k$

$\mathrm{Ind}\,\mathrm{Perf}_k$ $\qquad\qquad$ $\mathrm{Pro}\,\mathrm{Perf}_k$

$\|$

$\mathrm{dgMod}_k$ $\rotatebox{90}{$\in$}$ $\qquad\qquad$ $\rotatebox{90}{$\in$}$

$$T_1 \longrightarrow T \longrightarrow T_2$$


Def. $\mathcal{C}$ st and id. comp $(\infty,1)$-cat.
$\mathrm{Tate}(\mathcal{C}) \hookrightarrow \mathrm{ProInd}(\mathcal{C})$
Smallest full subcat. st stable, id-comp.
and contains Ind $\mathcal{C}$ and Pro $\mathcal{C}$.
Thm:
Def: $\mathcal{L}^d X$ = formal neighborhood of $\mathcal{L}^d_V X$ into $\mathcal{L}^d_U X$.

Def: $\mathcal{C}$ st and id-comp $(\infty,1)$-cat.

$\mathrm{Tate}(\mathcal{C}) \hookrightarrow \mathrm{ProInd}(\mathcal{C})$

smallest full subcat st stable, id-comp

and contains $\mathrm{Ind}\,\mathcal{C}$ and $\mathrm{Pro}\,\mathcal{C}$.

Thm:

- $\mathrm{Tate}\,\mathcal{C}$ has a univ. prop $\leadsto$

$\mathrm{Tate}(\mathcal{C}) \hookrightarrow \mathrm{Ind}\,\mathrm{Pro}(\mathcal{C})$

- $K^{nc}(\mathrm{Tate}\,\mathcal{C}) \simeq \Sigma K^{nc}(\mathcal{C})$

Def: $\mathcal{L}^d X$ = formal neighborhood of $\mathcal{L}^d_V X$ into $\mathcal{L}^d_U X$.

Def: $\mathcal{C}$ st and id.-comp. $(\infty,1)$-cat.

$$\mathrm{Tate}(\mathcal{C}) \hookrightarrow \mathrm{ProInd}(\mathcal{C})$$

smallest full subcat. st stable, id.-comp.

and contains $\mathrm{Ind}\,\mathcal{C}$ and $\mathrm{Pro}\,\mathcal{C}$.

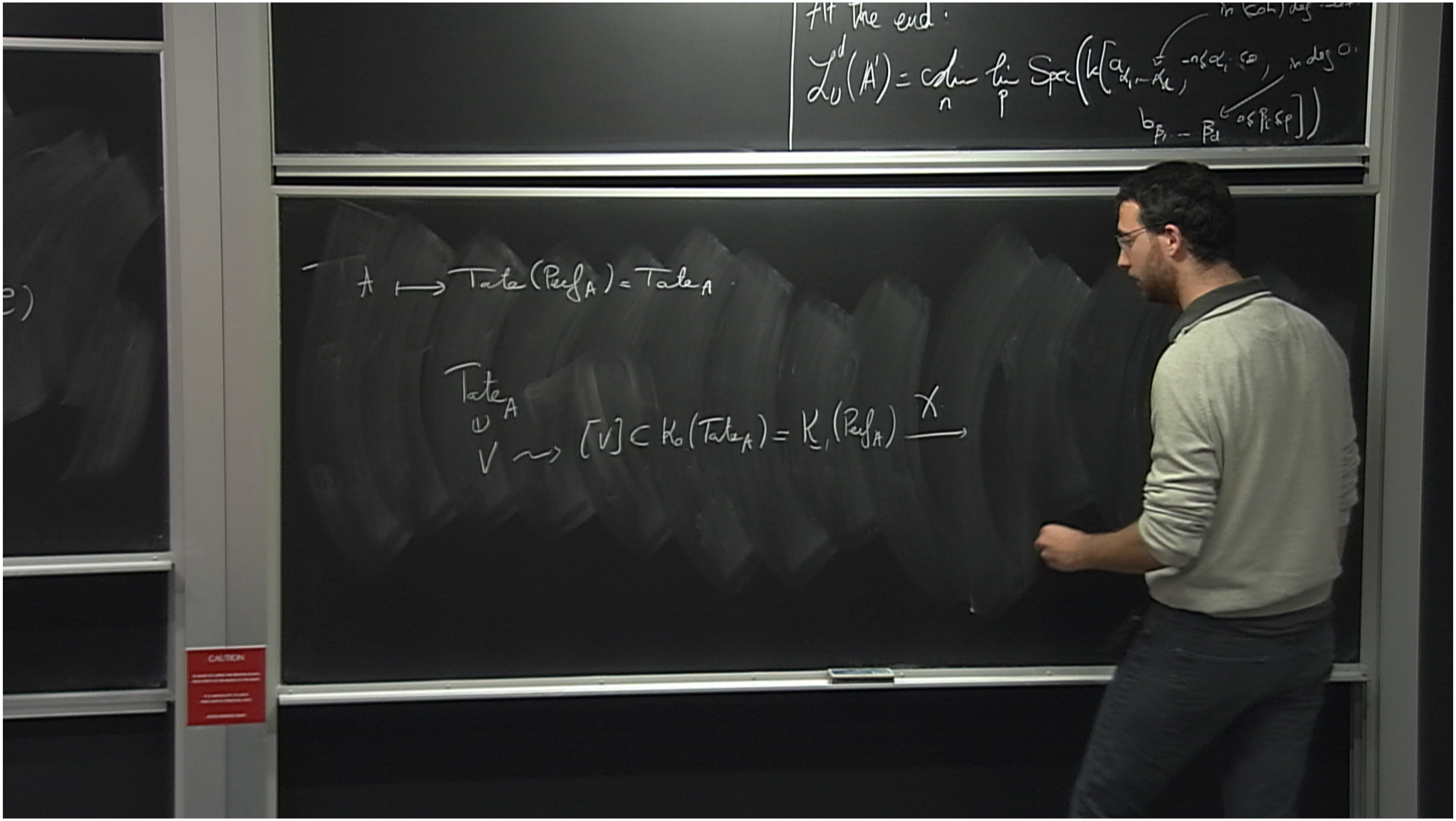
Thm:

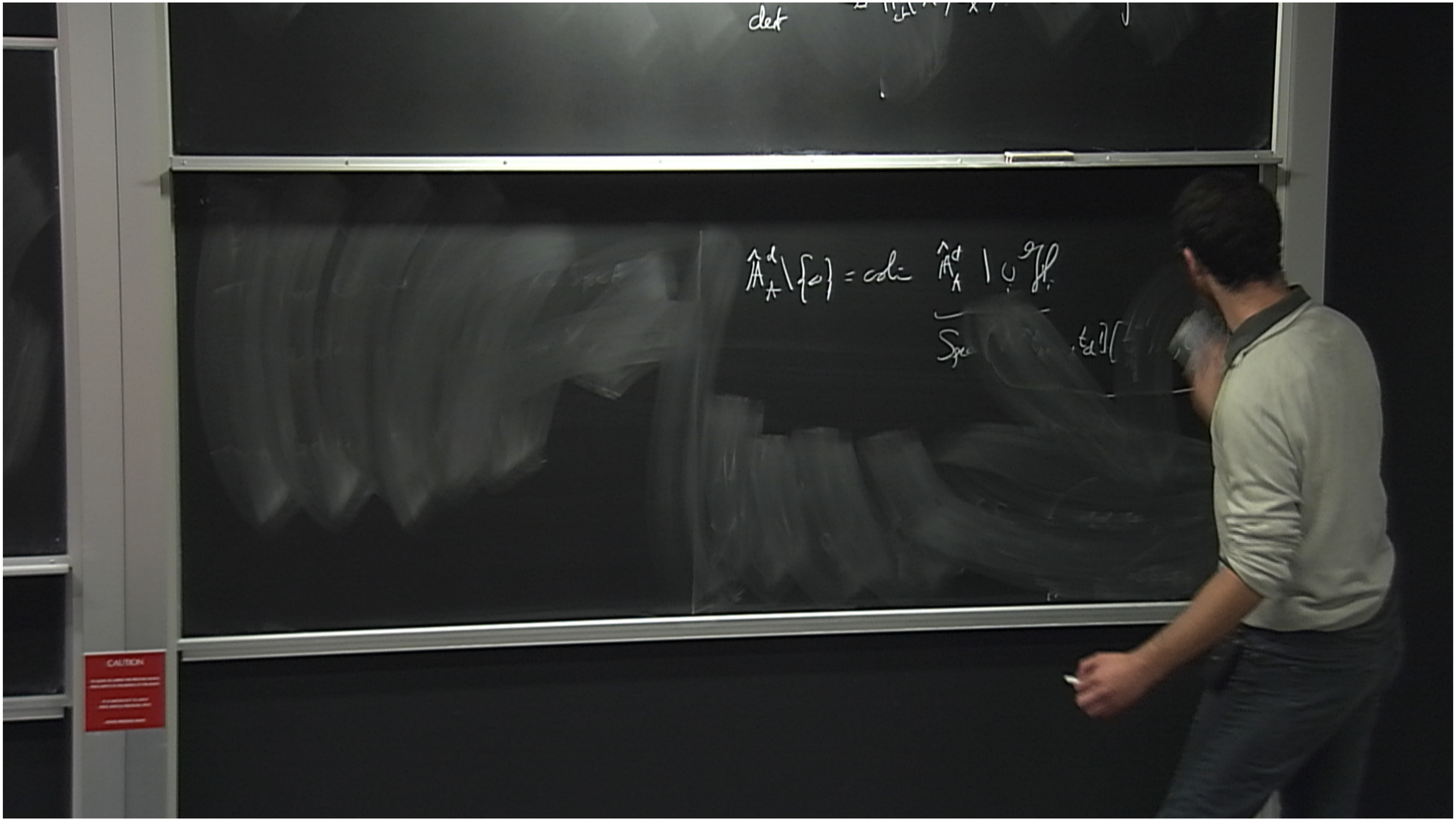
- $\mathrm{Tate}\,\mathcal{C}$ has a univ. prop $\rightsquigarrow$

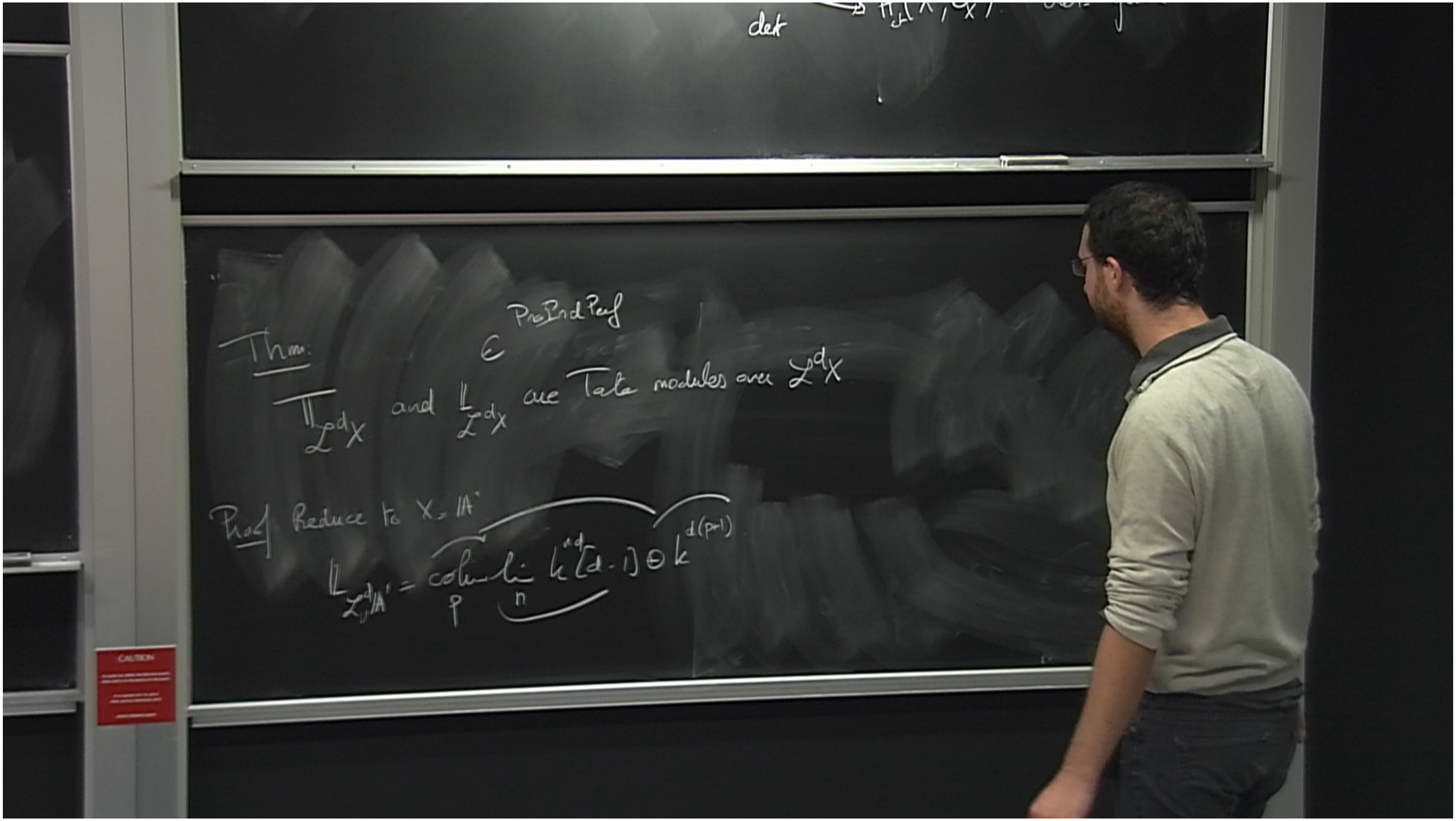
$$\mathrm{Tate}(\mathcal{C}) \hookrightarrow \mathrm{IndPro}(\mathcal{C})$$

- $K^{nc}(\mathrm{Tate}\,\mathcal{C}) \simeq \Sigma K^{nc}(\mathcal{C})$

Def: $\mathcal{L}^d X$ = formal neighborhood of $\mathcal{L}^d_V X$ into $\mathcal{L}^d_U X$.


At the end:
A ⟼ Tate(Perf_A) = Tate_A
Tate_A
V ⇝ [V] ⊂ K_0(Tate_A) = K_{-1}(Perf_A)
CAUTION


det


det
Thm:
ProIndPerf
∈
$\mathbb{T}_{\mathcal{L}^d X}$ and $\mathbb{L}_{\mathcal{L}^d X}$ are Tate modules over $\mathcal{L}^d X$
Proof Reduce to $X = \mathbb{A}^1$
$\mathbb{L}_{\mathcal{L}^d_u \mathbb{A}^1} = \operatorname{colim}_p \lim_n k^{nd}[d-1] \oplus k^{d(p+1)}$

$\det$ $\quad H^1_{et}(X, \mathcal{O}_X^*)$

**Thm:** $\qquad \in \mathrm{ProIndPerf}$

$\mathbb{T}_{\mathcal{L}^d X}$ and $\mathbb{L}_{\mathcal{L}^d X}$ are Tate modules over $\mathcal{L}^d X$

**Cor:** $[\mathbb{T}_{\mathcal{L}^d X}] \in H^2(\mathcal{L}^d X, \mathcal{O}^*_{\mathcal{L}^d X})$

**Proof:** Reduce to $X = \mathbb{A}^1$

$$\mathbb{L}_{\mathcal{L}^d_\nu \mathbb{A}^1} = \operatorname{colim}_p \lim_n k^{nd}[d-1] \oplus k^{d(p+1)}$$

III) Kac-Moody: $G$ red alg group. | $\mathfrak{g}$ tangent Lie algebra.

d=1

$$\mathcal{L}'(BG) \longrightarrow K(\mathbb{G}_m, 2)$$

determinantal class $[\Pi_{\mathcal{L}'BG}]$

KV

$\mathfrak{g}((t))$ central ext:

$$\sum x_n t^n \otimes \sum y_n t^n \longrightarrow \sum_n n\, K(x_n,$$

$$\mathfrak{g}((t)) \oplus k$$

$$\mathfrak{g}((t)) \longrightarrow k[1] \rightsquigarrow \text{ext. cl}$$

joint w/ Faonte, Kapranov.

idea:

$d \geq 2$:

$$\mathcal{L}^d(BG) \longrightarrow K(\mathbb{G}_m, 2)$$

Def: $\mathcal{L}^d X$ = formal neighborhood of $\mathcal{L}^d_V X$ into $\mathcal{L}^d_0 X$.

$BG) \longrightarrow K(\mathbb{G}_m, 2)$

determinantal class $[T_{\Sigma, BG}]$

KV

$\mathfrak{g}((t))$ central ext.:

$$\sum x_n t^n \otimes \sum y_n t^n \longrightarrow \sum_n n\, K(x_n, y_{-n}).$$

$\mathfrak{g}((t)) \oplus k$.

$\mathfrak{g}((t)) \longrightarrow k[1]. \rightsquigarrow$ ext. class in $C^2(\mathfrak{g}((t)))$.

$\mathrm{End}(\mathfrak{g}((t))) \longrightarrow k[1]$

$\sum x_n t^n \otimes \sum y_n t^n \longrightarrow \sum_n n\, h(x_n, y_{-n})$.

$g((t)) \oplus k$.

KV

$g((t)) \longrightarrow k[1]$. $\rightsquigarrow$ ext class in $C^2(g((t)))$.

defines. $HC^1(\mathrm{End}(g((t))))$

$\mathrm{End}(g((t))) \longrightarrow k[1]$

Take trace.

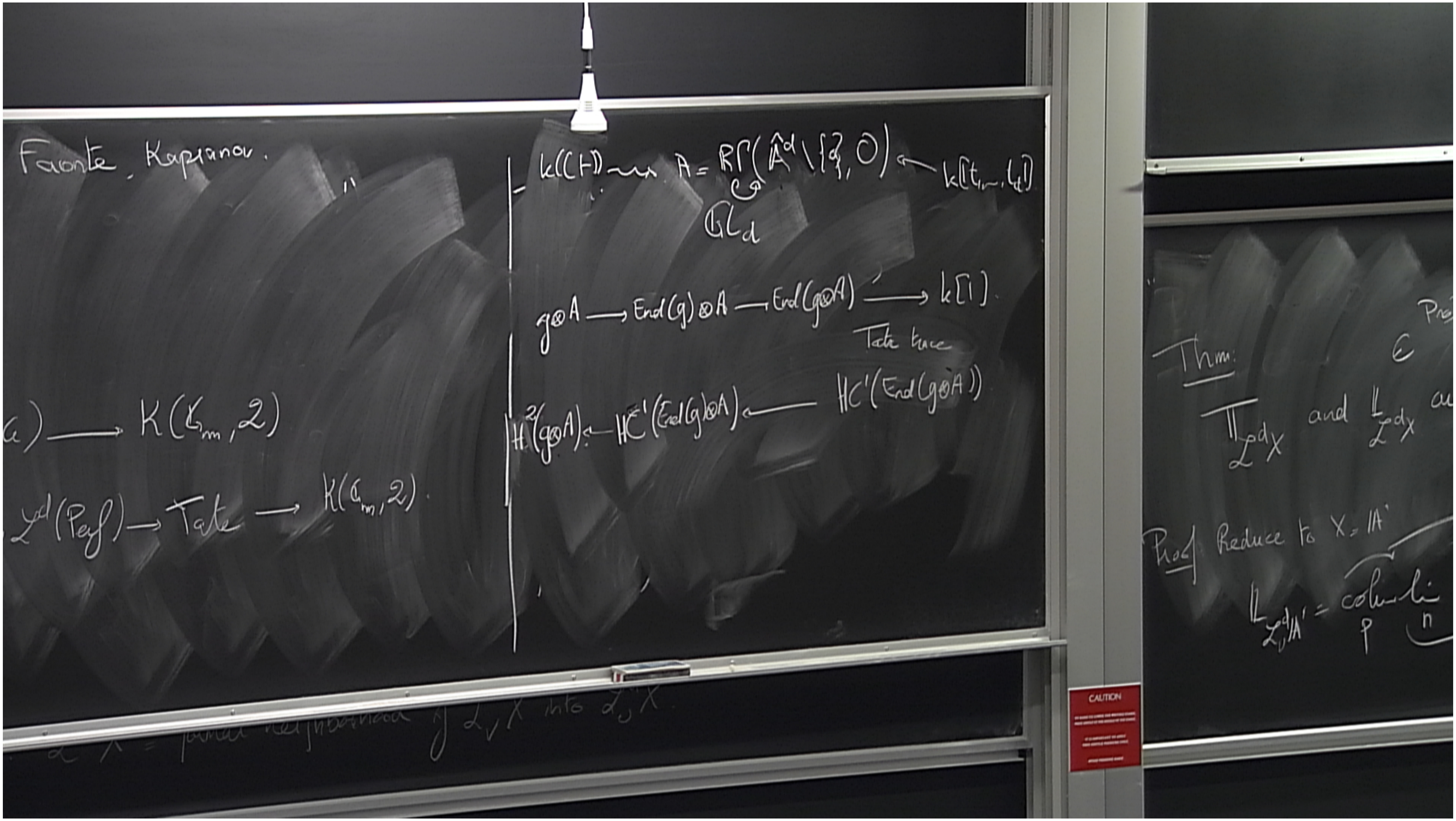
joint w/ Faonte.

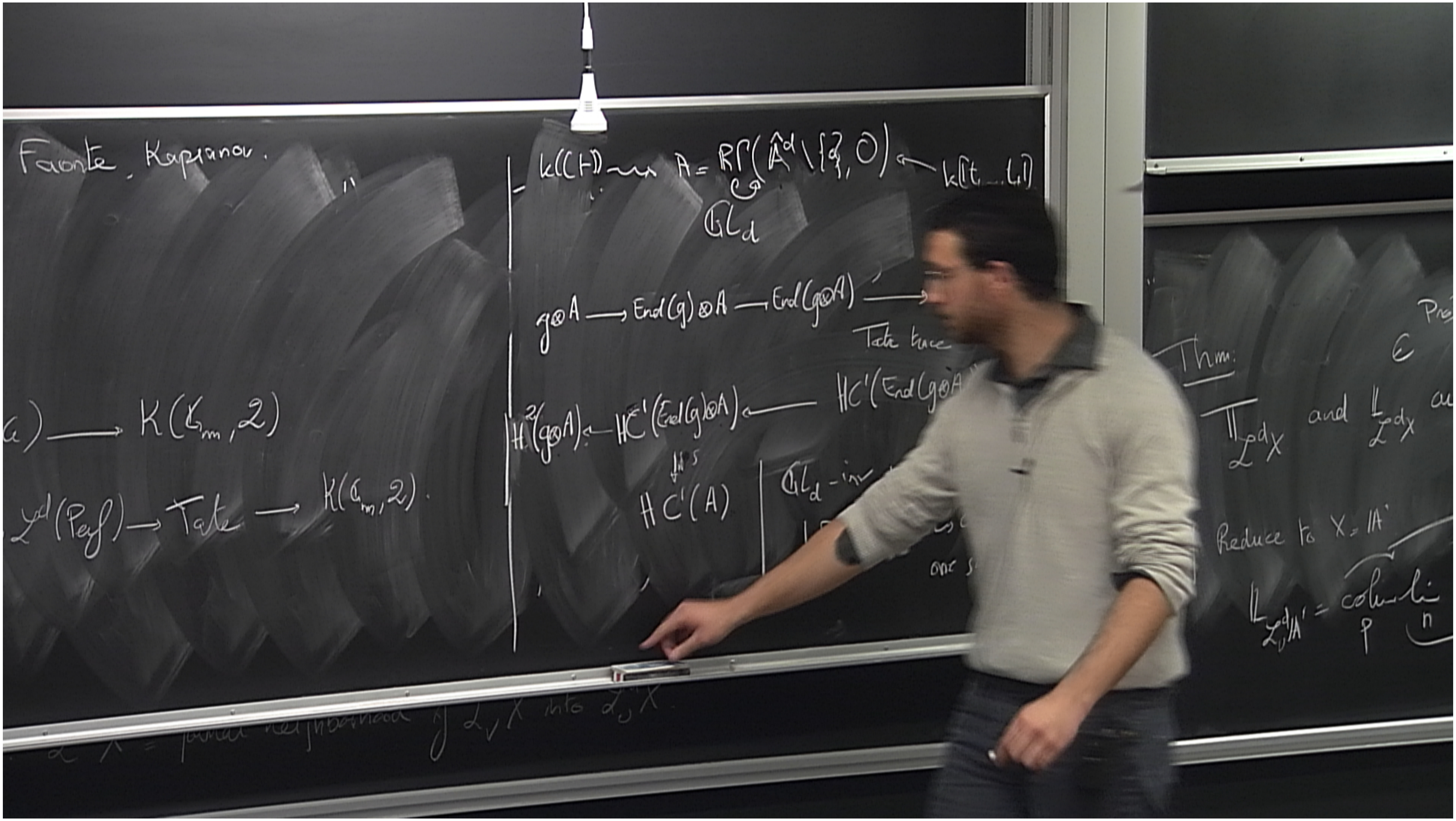
idea:

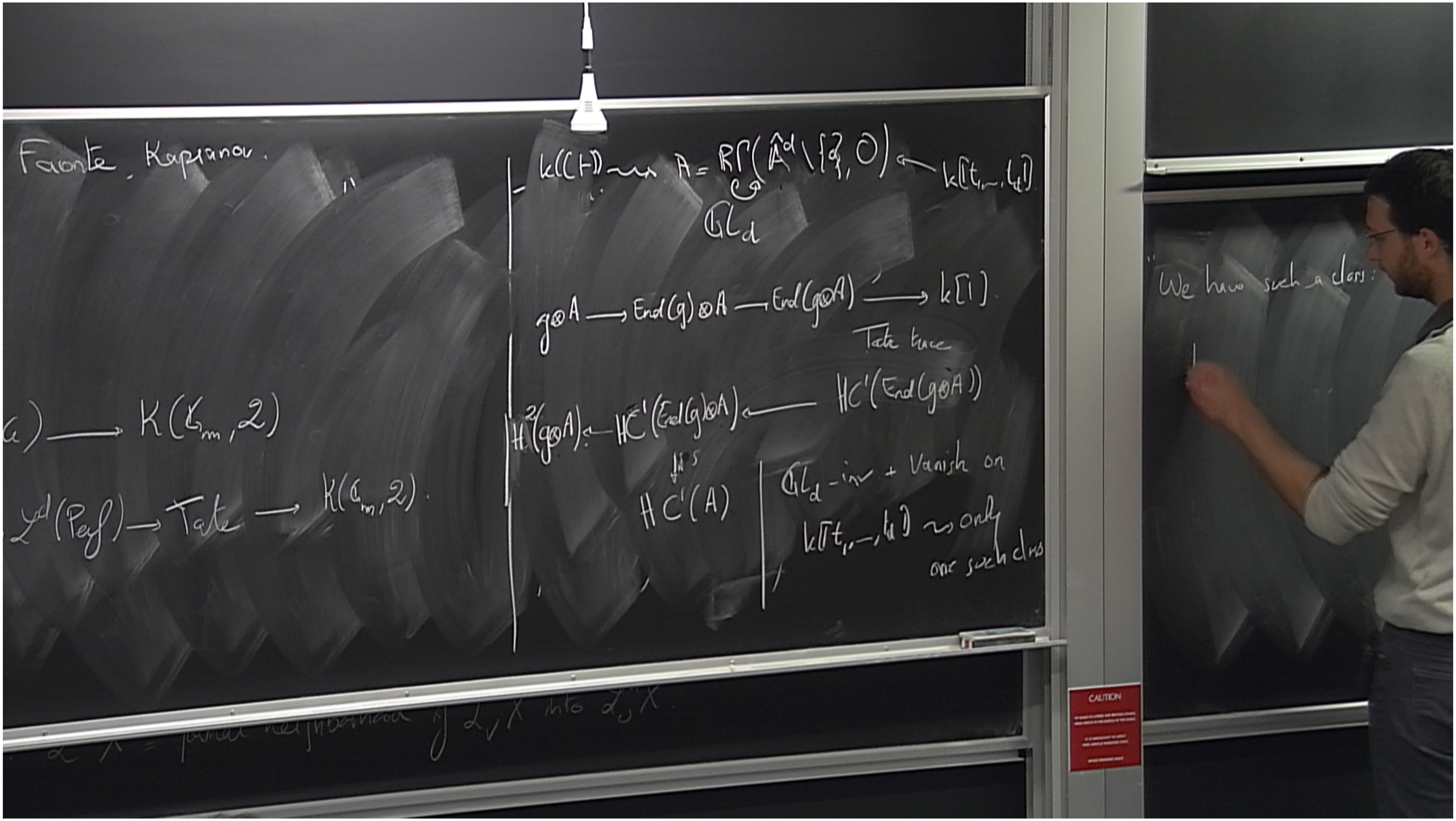
$d \geq 2$:

$\mathcal{L}^d(BG) \longrightarrow k$

Def: $\mathcal{L}^d X =$


Faonte, Kapranov.
$k((t)) \rightsquigarrow A = R\Gamma(\hat{\mathbb{A}}^d \setminus \{0\}, \mathcal{O}) \leftarrow k[[t_1, \dots, t_d]]$
$GL_d$
$\mathfrak{g} \otimes A \to \mathrm{End}(\mathfrak{g}) \otimes A \to \mathrm{End}(\mathfrak{g} \otimes A) \to k[1]$
Tate trace
$H^2(\mathfrak{g} \otimes A) \leftarrow HC^1(\mathrm{End}(\mathfrak{g}) \otimes A) \leftarrow HC^1(\mathrm{End}(\mathfrak{g} \otimes A))$
$) \to K(\mathbb{G}_m, 2)$
$\mathcal{L}^d(\mathrm{Perf}) \to \mathrm{Tate} \to K(\mathbb{G}_m, 2)$
Thm:
$\Pi_{\mathcal{L}^d} X$ and $\mathbb{L}_{\mathcal{L}^d} X$
Proof Reduce to $X = \mathbb{A}^1$
$\mathbb{L}_{\mathcal{L}^d_v \mathbb{A}^1} = \mathrm{colim}_p \lim_n$


Faonte, Kapranov.
g⊗A → End(g)⊗A → End(g⊗A)
Tate trace
HC'(A)
Thm:
and
Reduce to X = A'


Faonte, Kapranov.
k((t)) ⇝ A = RΓ(Â^d \ {0}, O) ← k[[t_1, …, t_d]]
GL_d
g⊗A → End(g)⊗A → End(g⊗A) → k[1]
Tate trace
H^2(g⊗A) ← HC^1(End(g)⊗A) ← HC^1(End(g⊗A))
HC^1(A)
GL_d-inv + Vanish on
k[[t_1, …, t_d]] ⇝ only one such class
K(G_m, 2)
Tate → K(G_m, 2)
We have such a class:
CAUTION