Title: Free linear BV-quantization as an infinity-functor

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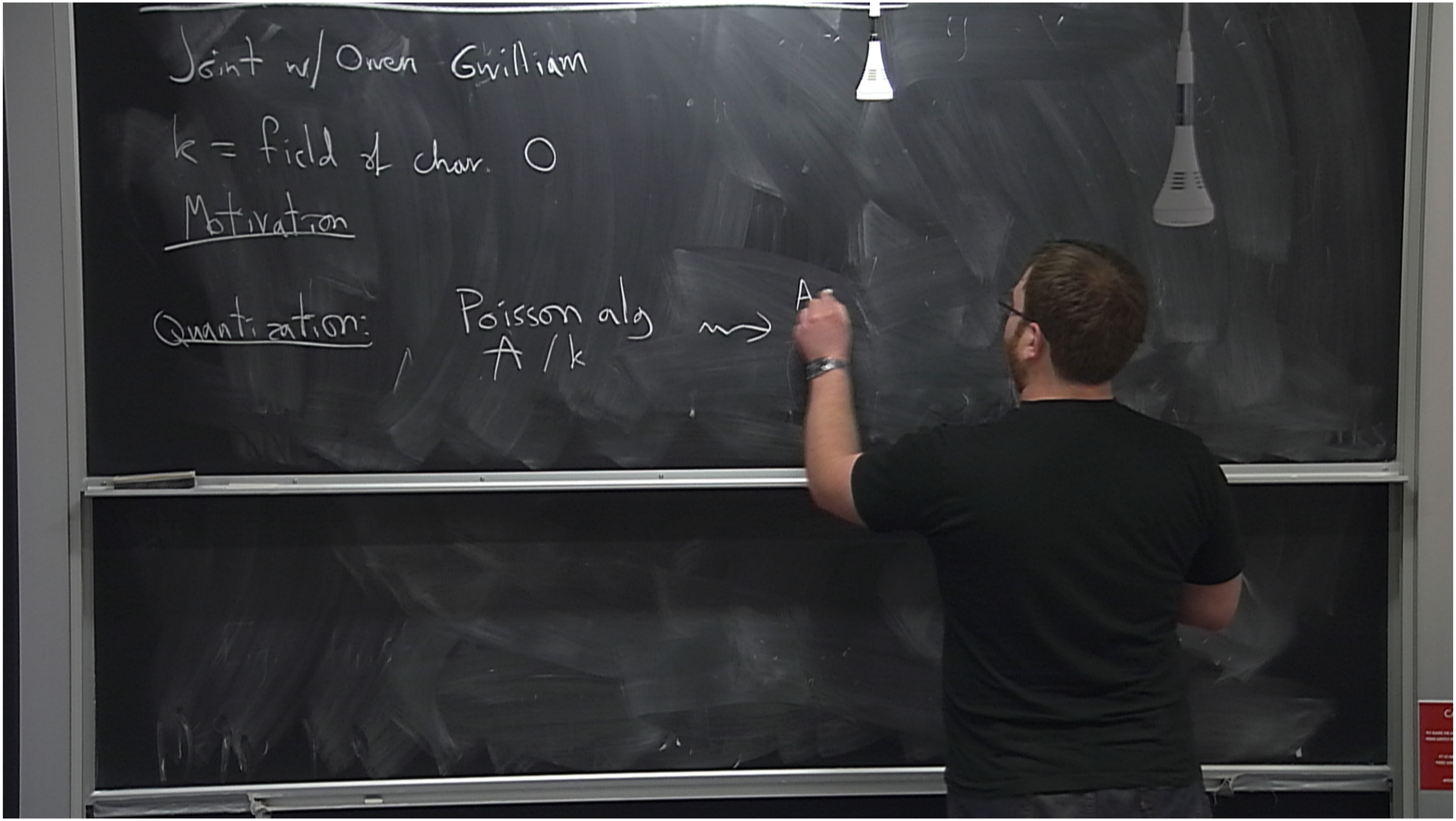
Abstract: I will describe a functorial construction of the free BV-quantization of chain complexes equipped with antisymmetric forms of degree 1 in the context of infinity-categories. This is joint work with Owen Gwilliam.


Joint w/ Owen Gwilliam
k = field of char. 0
Motivation
Quantization:
Poisson alg
A / k
A

Joint w/ Owen Gwilliam

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Motivation

Quantization: Poisson alg A / k $\rightsquigarrow$ Ass. alg $\hat{A} / k[\hbar]$

$\hbar = 0 \nearrow$ recover A

$\{a, b\} = $ " $\lim_{\hbar}$ $\frac{[a,b]}{\hbar}$ "

$V \longmapsto V \oplus V^{\vee}$ $= W \oplus kc$

$[v, v'] = \omega(v, v')c$

c central

$\Rightarrow$ Poisson alg

Free Linear BV-Quantization as an ∞-Functor

Joint w/ Owen Gwilliam

k = field of char. 0

Motivation

Quantization: Poisson alg A / k "Classical obs." $\rightsquigarrow$ Ass. alg $\hat{A} / k[\hbar]$ $\xrightarrow{\hbar = 1}$ Ass. alg $/k$ "Quantum obs."

$\hat{A} / k[\hbar] \xrightarrow{\hbar = 0}$ recover A

$\{a, b\} =$ "$\lim_{\hbar \to 0} \frac{[a,b]}{\hbar}$"

$\|$ $\qquad\qquad\qquad\qquad\qquad$ $\|$

P_0-alg. $\qquad\qquad\qquad$ pointed chain cx

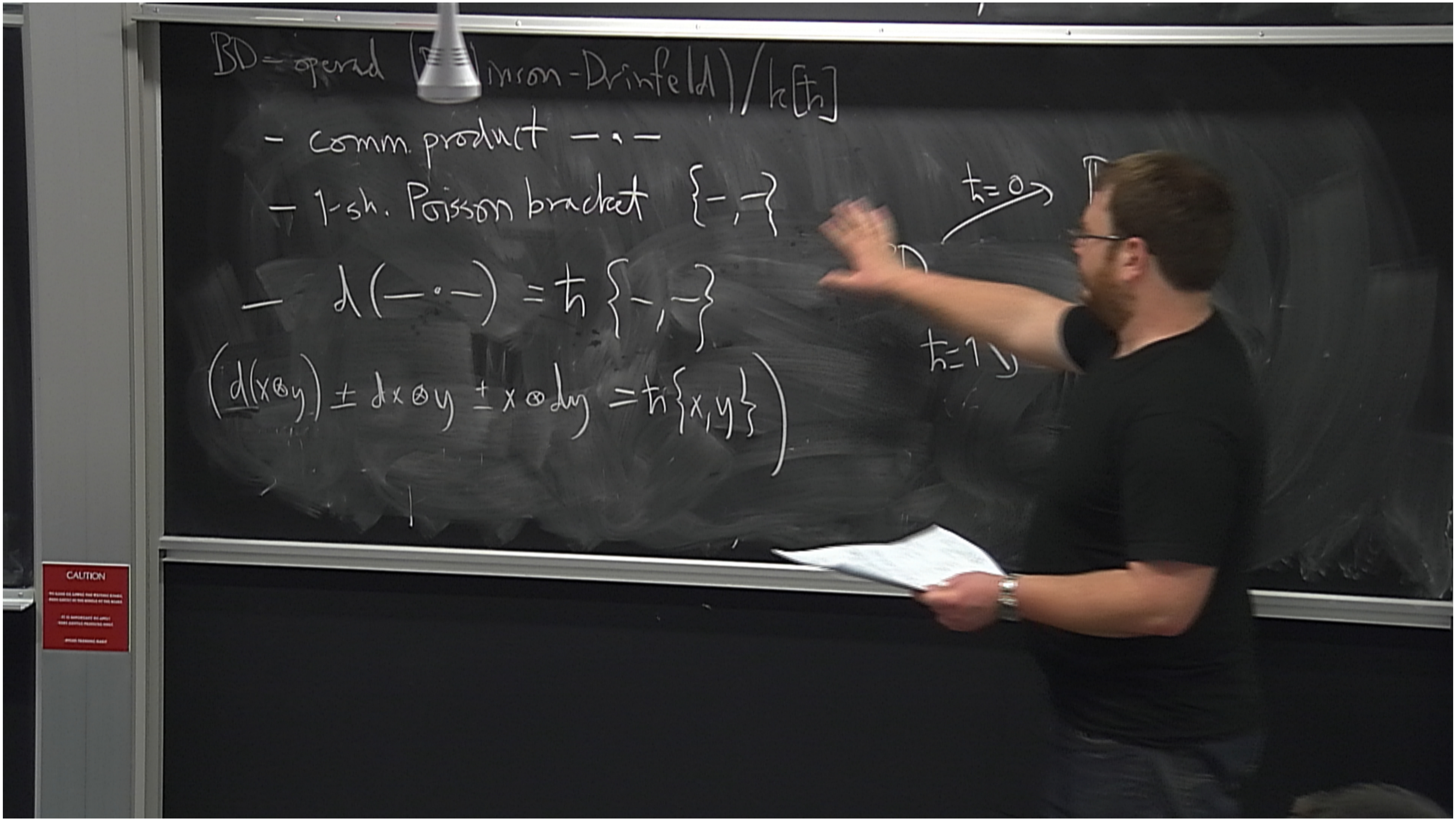
$$\mathrm{CE}\text{-chains} \xrightarrow[(c=1)]{} \mathbb{E}_0\text{-alg.}$$

Analogue of (*) should be:

$$\mathrm{Ch}_k \to \mathrm{Ch}_k \text{ w/ deg 1 antisymm. pairing} \to \mathrm{Lie}_1\text{-alg}$$

$$\| \quad \mathrm{Quad}_1(k)$$

$$V \mapsto V \oplus V^\vee[1] \qquad (W,\omega) \mapsto \mathrm{Heis}_1(W,\omega) = W \oplus kc$$


BD-operad (inson-Drinfeld)/$k[\hbar]$
- comm. product $- \cdot -$
- 1-sh. Poisson bracket $\{-,-\}$
- $d(- \cdot -) = \hbar\{-,-\}$
$(d(x \otimes y) \pm dx \otimes y \pm x \otimes dy = \hbar\{x,y\})$
$\hbar=0$
$\hbar=1$
CAUTION

"BV-quantization": 1-shifted Poisson alg $\rightsquigarrow$ $\mathbb{E}_0$-alg.

1-shifted Poisson alg $= \mathbb{P}_0$-alg.

$\mathbb{E}_0$-alg. $=$ pointed chain cx

Analogue of $(*)$ should be:

$\mathrm{Ch}_k \to \mathrm{Ch}_k$ w/ deg 1 antisymm. pairing $\to$ Lie_1-alg

Ch_k w/ deg 1 antisymm. pairing $= \mathrm{Quad}_1(k)$

$V \longmapsto V \oplus V^\vee[1]$

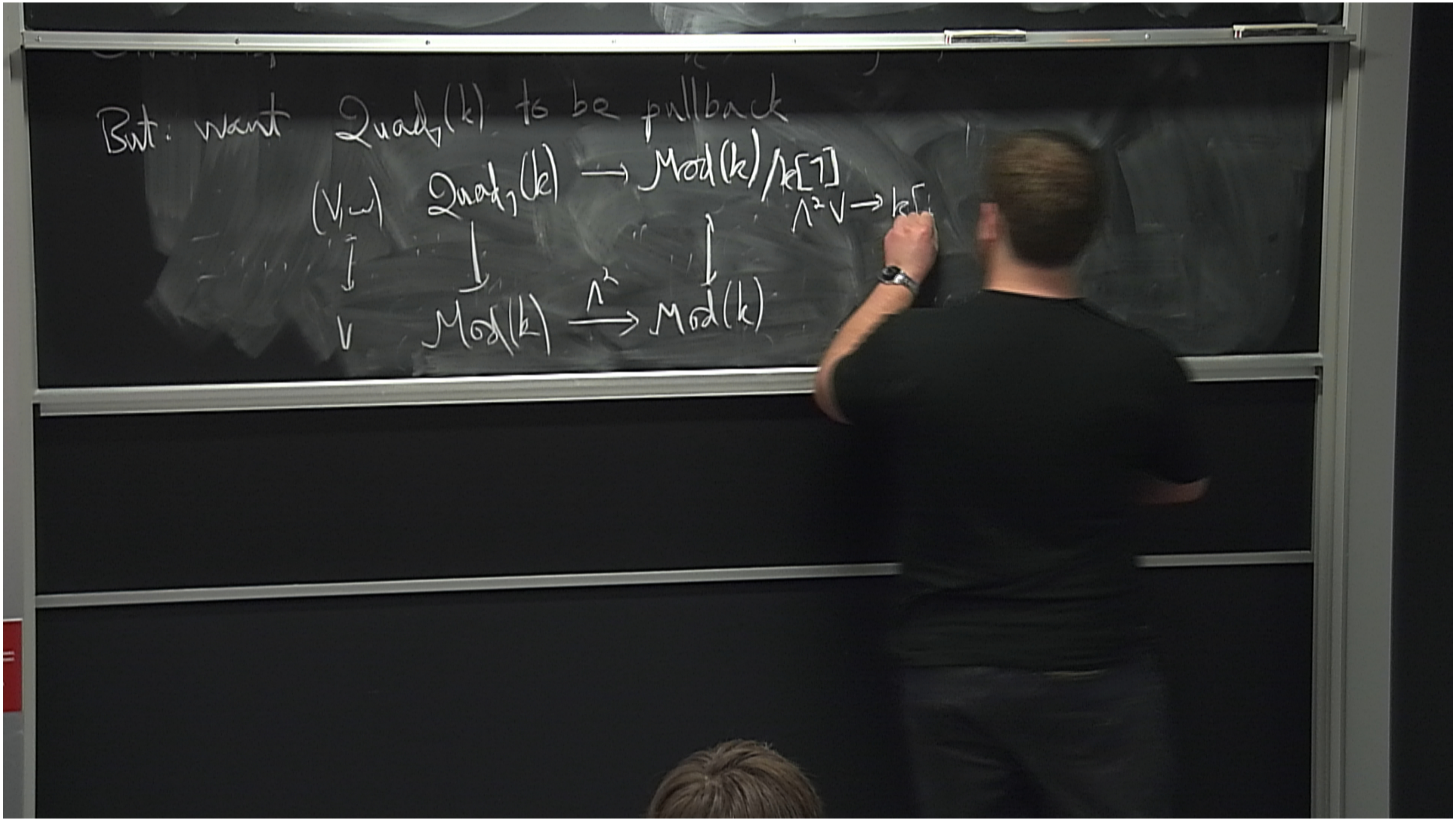
$(W,\omega) \mapsto \mathrm{Heis}_1(W,\omega) = W \oplus kc$

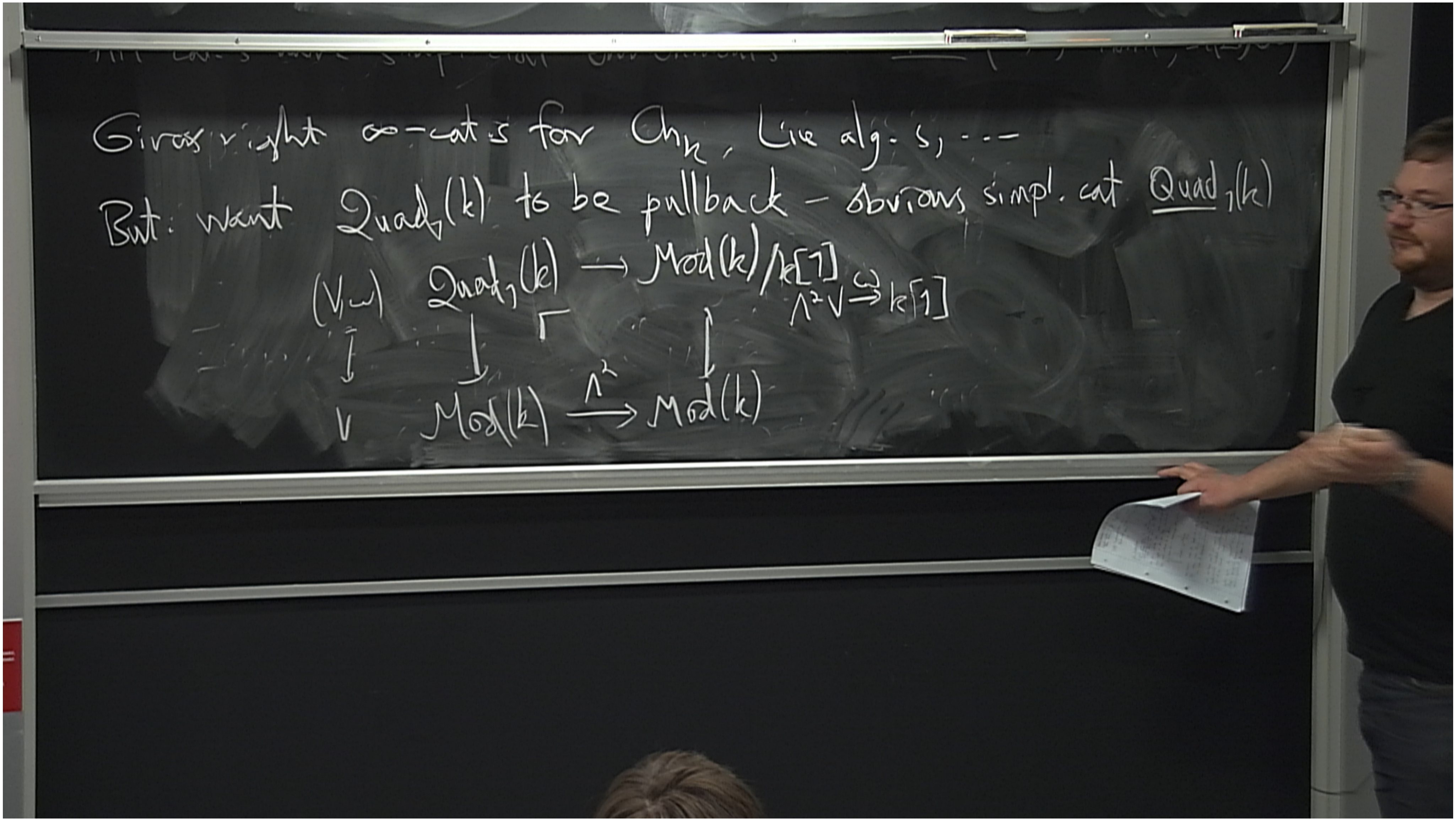
$$\mathrm{Lie}_1\text{-alg} \xrightarrow[(c=1)]{\mathrm{CE\text{-}chains}} \mathbb{E}_0\text{-alg}.$$

$$\mathrm{Lie}_1\text{-alg} \xrightarrow[(c=1)]{\mathrm{Sym}} \mathbb{P}_0\text{-alg}$$

$$\mathrm{BD\text{-}alg}/k[\hbar] \xrightarrow{\hbar=1} \mathbb{E}_0\text{-alg}., \qquad \mathrm{BD\text{-}alg}/k[\hbar] \xrightarrow{\hbar=0} \mathbb{P}_0\text{-alg}$$

$\mathrm{Sym}(\mathfrak{g})$, $d(x\otimes y) = dx \otimes y + x \otimes dy + [x,y]$


But: want Quad$_1$(k) to be pullback
(V, ω) Quad$_1$(k) ⟶ Mod(k)/k[1]
Λ²V ⟶ k[
V Mod(k) $\xrightarrow{\Lambda^2}$ Mod(k)

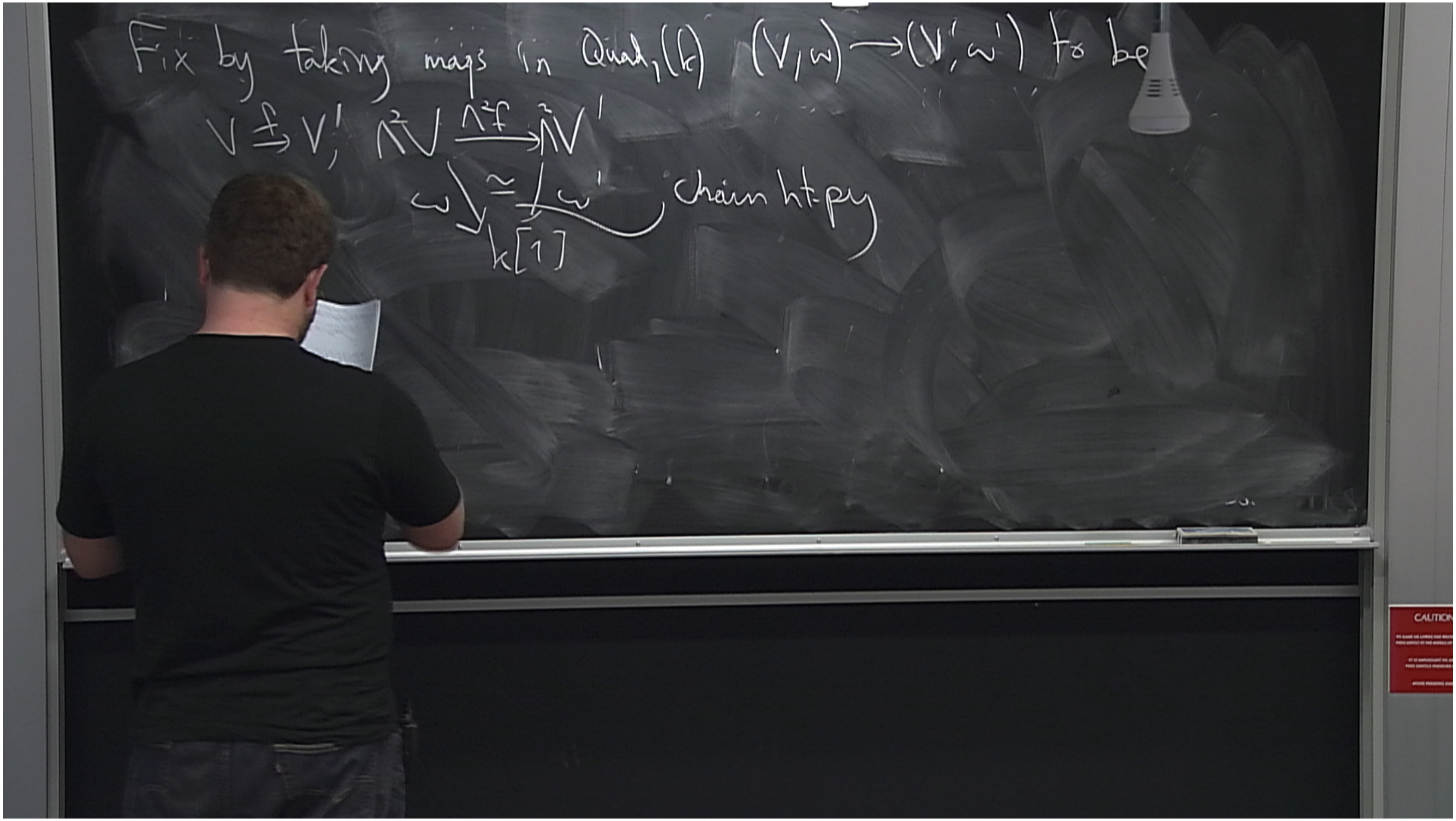


Gives right ∞-cat.s for Ch_k, Lie alg.s, ...
But: want $Quad_1(k)$ to be pullback - obvious simpl. cat $\underline{Quad}_1(k)$
(V,ω)
$Quad_1(k) \longrightarrow Mod(k)/k[1]$
$\Lambda^2 V \xrightarrow{\omega} k[1]$
V
$Mod(k) \xrightarrow{\Lambda^2} Mod(k)$

Naive version: $(W, \omega) \longmapsto \mathrm{Env}_{BD}(\mathrm{Heis}_1(W, \omega))$.

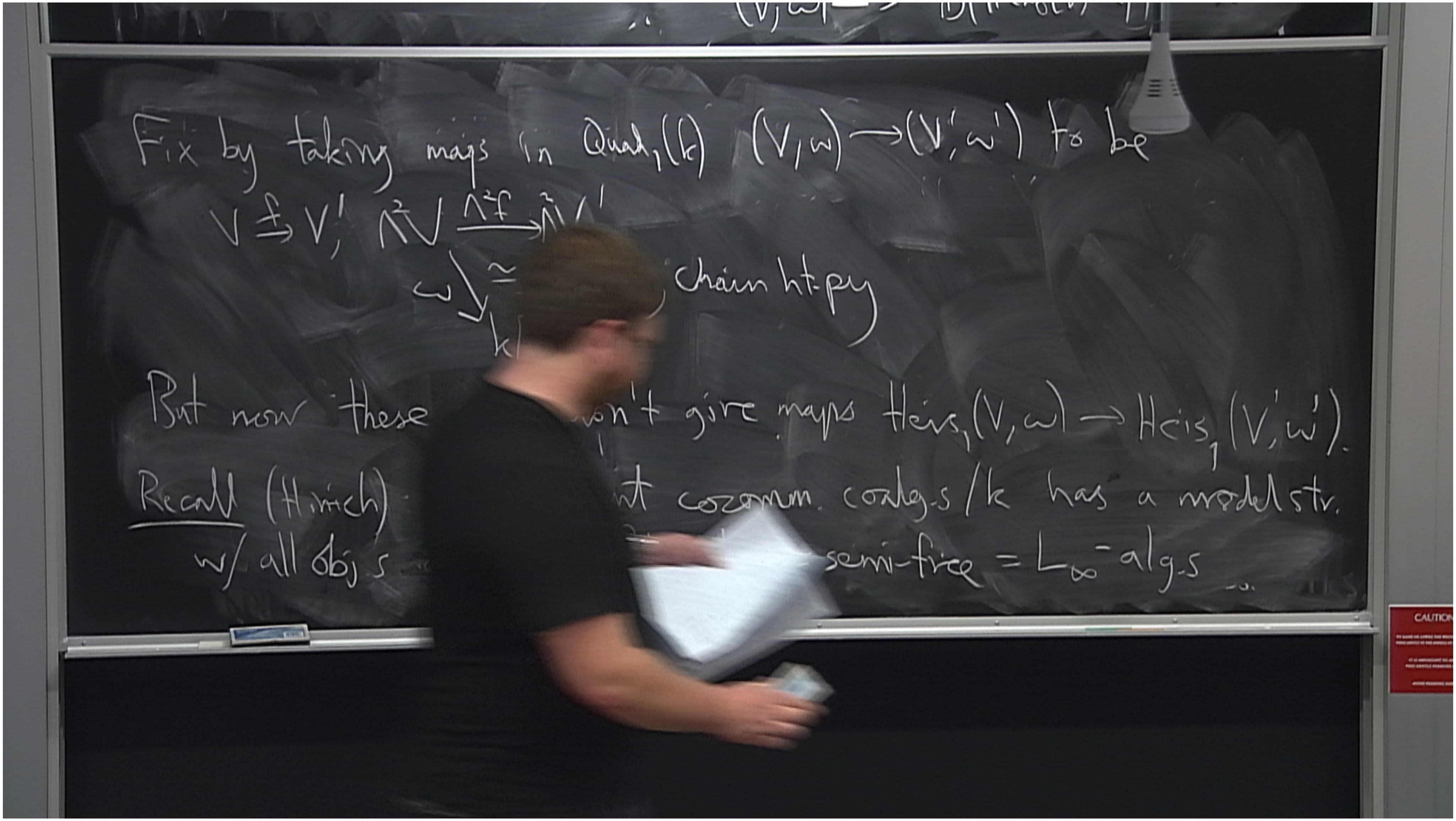
Goal: Implement this construction as symm mon ftrs of ∞-cats

$$\begin{tikzcd}
 & & & \mathcal{E}_0(k) \\
\mathrm{Quad}_1(k) \arrow[r] & \mathcal{L}ie_1(k) \arrow[r] & \mathbb{BD}(k) \arrow[ur] \arrow[dr] & \\
\oplus & \oplus & \otimes & \mathcal{P}_0(k)
\end{tikzcd}$$


Fix by taking maps in $\mathrm{Quad}_1(k)$ $(V,\omega)\to(V',\omega')$ to be
$V \xrightarrow{f} V'$; $\Lambda^2 V \xrightarrow{\Lambda^2 f} \Lambda^2 V'$
$\omega \searrow \simeq \swarrow \omega'$ chain htpy
$k[1]$
CAUTION

But now these maps don't give maps $\mathrm{Heis}_1(V,\omega) \to \mathrm{Heis}_1(V',\omega')$.

Recall (Hinich): Conilpotent cocomm. coalg.s / k has a model str. w/ all obj's cofibrant, fibrant ob's = semi-free = L_∞-alg.s


Fix by taking maps in Quad1(k) (V,ω) → (V',ω') to be
V →f V', Λ²V →Λ²f Λ²V'
ω ≃
chain htpy
k
But now these
on't give maps Heis1(V,ω) → Heis1(V',ω').
Recall (Hinich)
nt cocomm. coalg.s /k has a model str.
w/ all obj.s
semi-free = L∞-alg.s

Map of operads $\mathrm{Lie}_1 \otimes k[\hbar] \longrightarrow BD$ induces Quillen adj.

$$\mathrm{Env}_{BD} = i_! : \mathrm{Lie}_1(k) \rightleftarrows BD(k) : i^*$$

<u>Propn</u> (Beilinson-Drinfeld/folklore) $\mathrm{Env}_{BD}(\mathfrak{g}) = \mathrm{Sym}(\mathfrak{g})[\hbar]$

w/ $d(x \otimes y) = dx \otimes y + x \otimes dy + \hbar[x,y]$

Thm (G–): On the ∞-cat. level,

(1) $\mathrm{Quad}_1(k)_{,\oplus} \longrightarrow \mathrm{Lie}_1(k), \oplus$ is lax mon.

$$\begin{tikzcd}
\mathrm{Quad}_1(k)_{,\oplus} \arrow[r] \arrow[dr] & \mathrm{Lie}_1(k), \oplus \\
 & \mathrm{Mod}_{kc}(\mathrm{Lie}_1(k))^{\oplus_{kc}} \arrow[u]
\end{tikzcd}$$

$\mathrm{Mod}_{kc}(\mathrm{Lie}_1(k))$ is strong mon.

(2) $\mathrm{Lie}_1(k), \oplus \to \mathcal{BD}(k), \otimes$ is strong mon.

Cor: $\exists$ strong symm. mon. fctrs

$$\mathrm{Quad}_{\mathbb{Z}}(k) \longrightarrow \mathrm{Mod}_{k_c}(\mathcal{L}ie_c(k)) \longrightarrow \mathrm{Mod}_{k[c,\hbar]}(\mathcal{BD}(k))$$

$$\xrightarrow[\hbar=0]{c=1} \mathcal{P}_0(k)$$

Works not just for k, but for dg-mod's over any cdga A/k

Cor ... strongly symm. mon. ftr.s

$$\mathrm{Quad}_\gamma(k) \longrightarrow \mathrm{Mod}_{k_c}(\mathcal{L}ie_c,(k)) \longrightarrow \mathrm{Mod}_{k[c,\hbar]}(\mathbb{BD}(k)) \xrightarrow[\hbar=0]{c=1} \mathcal{P}_0(k)$$

Works not just for k, but for dg-mod.s over any cdga A /
∞-cat.s all satisfy étale descent

Cor: X derived stack, symm. mon. ftr.s — — — —

q.coh. sheaves of Quad_γ's $\longrightarrow$ q.coh. of $\mathcal{L}ie_c$-alg.s,

q. coh. sheaves on Ind_{q} $\longrightarrow$ q. coh. of L_{∞}-alg s, ——

Thm (Gwilliam) A an Artinian cdga $/k$ and (V, ω) is 1-symplectic w/ V perfect, then $BVQ_{E_0}(V, \omega)$ is invertible n-mor. = bimod. in bimod's ... in bimod

<u>Cor</u>: $\exists$ strong symm. mon. ftr.s

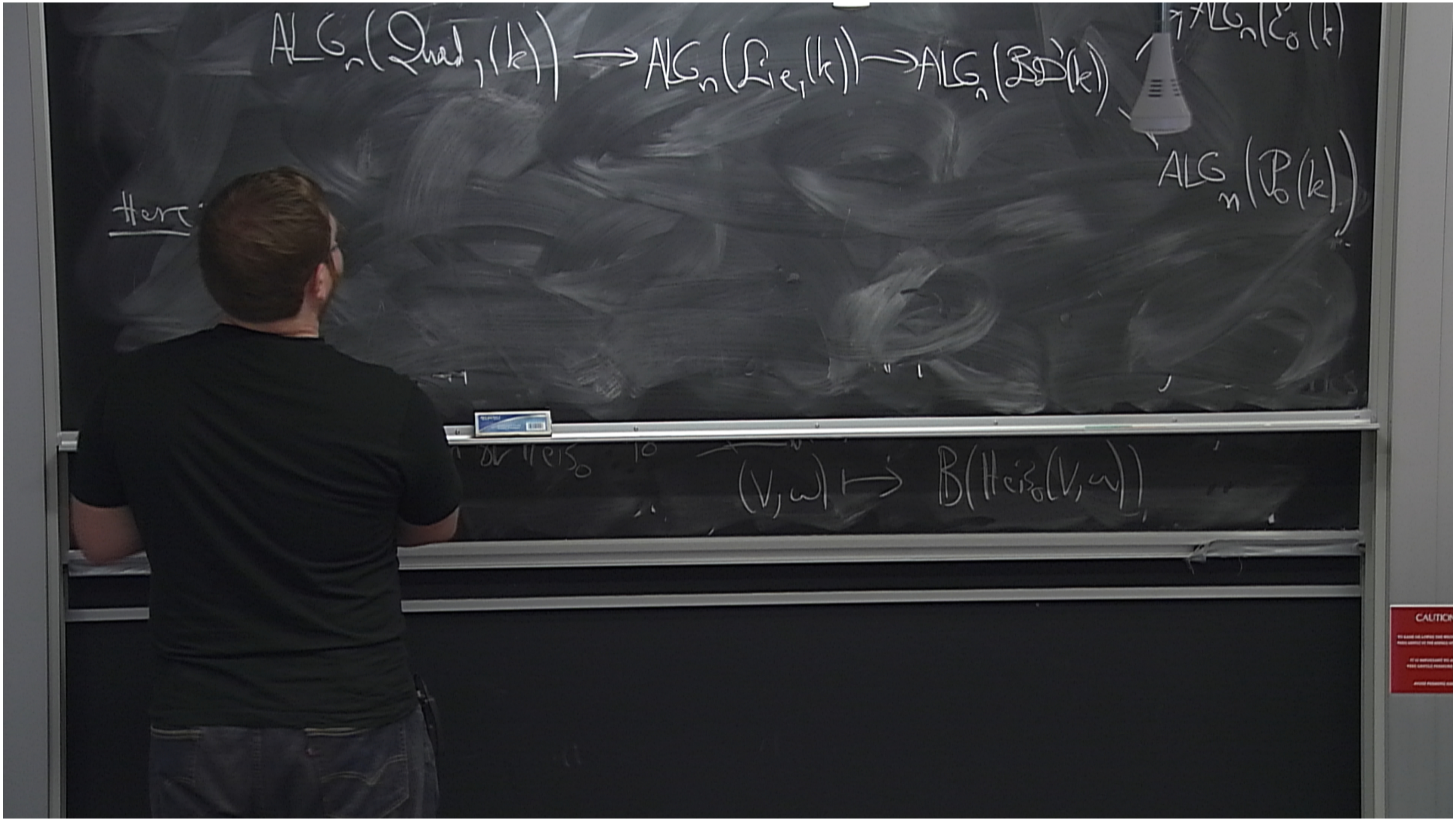
$$\begin{tikzcd}[column sep=large]
\mathrm{Quad}_\gamma(k) \arrow[r] \arrow[rrr, bend left, "BVQ_{E_0}"] & \mathrm{Mod}_{k_c}(\mathrm{Lie}_c(k)) \arrow[r] & \mathrm{Mod}_{k[c,\hbar]}(\mathbb{BD}(k)) \arrow[r, "c=\hbar=1"] \arrow[dr, "c=1,\ \hbar=0"'] & E_0(k) \\
 & & & P_0(k)
\end{tikzcd}$$

Works not just for k, but for dg-mod's over any cdga A/k

∞-cat.s all satisfy étale descent

<u>Cor</u>: X derived stack, symm. mon. ftr.s

q. coh. sheaves of Quad_γ's $\longrightarrow$ q. coh. of Lie_c-alg s, - - - -


ALG_n(Quad_1(k)) → ALG_n(Lie_1(k)) → ALG_n(BD_1(k))
ALG_n(E_0(k))
ALG_n(P_0(k))
Heis_0
(V, ω) ↦ B(Heis_0(V, ω))

- E_n-alg's in $\mathrm{Quad}_{\gamma}(k) \simeq \mathrm{Quad}_{\gamma-n}(k)$
- bimodules in $\mathrm{Quad}_{\gamma}(k) \simeq$ coisotropic cospan

E_n-alg's in $\mathrm{Quad}_1(k) \simeq \mathrm{Quad}_{1-n}(k)$

bimodules in $\mathrm{Quad}_1(k) \simeq$ coisotropic cospan

Induced ftr. $\mathrm{Lie}_{1-n}(k) \longrightarrow \mathcal{E}_n(k)$ shd be enveloping E_n-alg
(Knudsen)

$\mathrm{Quad}_{1-n}(k) \longrightarrow \mathcal{E}_n(k)$ $\longmapsto$ Weyl n-alg
(Markarian)

Joint w/ Calaque, Scheimbauer:

$\exists$ symm mon. (∞,n)-cat $\mathrm{Lag}^{k,\mathrm{lin}}_{(\infty,n)}$ w/ ob.s k-sympl. ch. cx.s

1-mor.s Lag. corr.s

$\vdots$

n-fold Lag. corr.s

<u>Hope</u>: taking duals should give

$$\mathrm{Lag}^{n-1,\mathrm{lin}}_{(\infty,n)} \longrightarrow \mathrm{ALG}_n(\mathrm{Quad}_1(k))$$

We get for V (n-1)-sympl. a "classical" TQFT

$$\underbrace{\underbrace{\mathrm{Bord}^{or}_{(\infty,n)} \longrightarrow \mathrm{Lag}^{n-1,\,lin}_{(\infty,n)}}_{\text{"classical"}} \dashrightarrow \mathrm{ALG}_n(\mathrm{Quad}_n(k)) \longrightarrow \mathrm{ALG}_n(\mathcal{E}_0(k))}_{\text{"quantized" TQFT}}$$