

Title: On the stable homotopy theory of stacks and elliptic cohomology

Date: Apr 21, 2016 09:30 AM

URL: <http://pirsa.org/16040082>

Abstract: In this talk, we'll discuss what it means to be a cohomology theory for topological stacks, using a notion of local symmetric monoidal inversion of objects in families. While the general setup is abstract, it specializes to many cases of interest, including Schwede's global spectra. We will then go on to discuss various examples with particular emphasis on elliptic cohomology. It turns out that  $TMF$  sees more objects as dualizable (or even invertible) than one might naively expect.

j/w/ Thomas Nikolaus

Basic idea: Many cohomology theories, top'l  $K$ -thy,  
would interesting answers on stacks.

Basic idea: Many cohomology theories, top  $k$  ( $k$ -fun),  
yield interesting answers on stacks.

Thm: (Atiyah- $\dots$ )

Basic idea: Many cohomology theories, top'l  $K$ -thy,  
yield interesting answers on stacks.

Thm: (Atiyah-Segal) Map  $BG \rightarrow BG = [ *^G ]$

gives interesting answers to stacks.

Thm: (Atiyah-Segal) Map

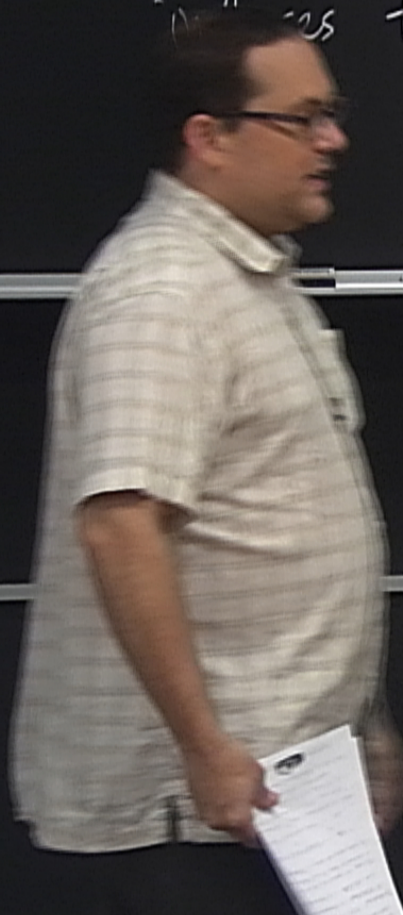
$$BG \longrightarrow TBG \cong [ *^{G^G} ]$$

induces the completion

$$KU^0(TBG) \longrightarrow KU^0(BG)$$

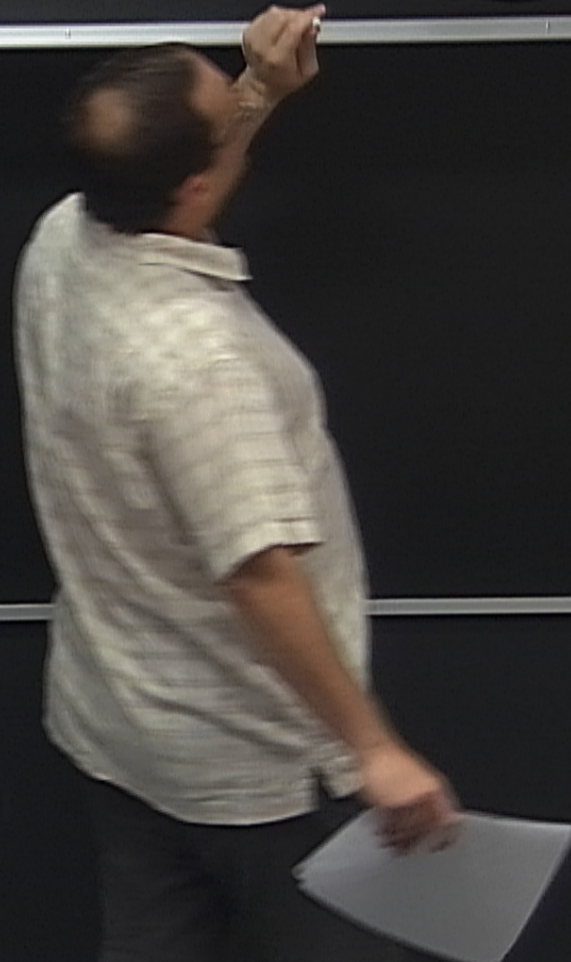
$$\text{Rep}(G)$$

$$\text{Rep}(G)_I^{\wedge}$$



CAUTION  
Do not touch the chalkboard  
when it is hot or when it is  
being cleaned.

Why we need coh. theories to  $\text{Rep}(G)$  take interesting  $\text{Rep}(G)^\wedge_I$  values on sites



... also interesting values on stks.

$$\text{Top} \longrightarrow \underline{N(\text{Top})[W^{-1}]} = \mathcal{S} = \text{Gpd}_{\infty} \longrightarrow \text{Stab}(\mathcal{S}) = \mathcal{S}_p$$

... interesting values on stks.

$$\text{Top} \longrightarrow \underline{N(\text{Top})[W^{-1}]} = \mathcal{S} = \text{Gpd}_{\infty} \longrightarrow \text{Stab}(\mathcal{S}) = \mathcal{S}p$$

↓

$$\text{Shv}_{\text{Gpd}}(\text{Top}) = \text{Stk}$$



... also interesting alues on stacks.

$$T_{0D} \longrightarrow \underline{N(Top)[W^{-1}]} = \mathcal{S} = Gpd_{\infty} \longrightarrow \text{Stab}(\mathcal{S}) = Sp$$



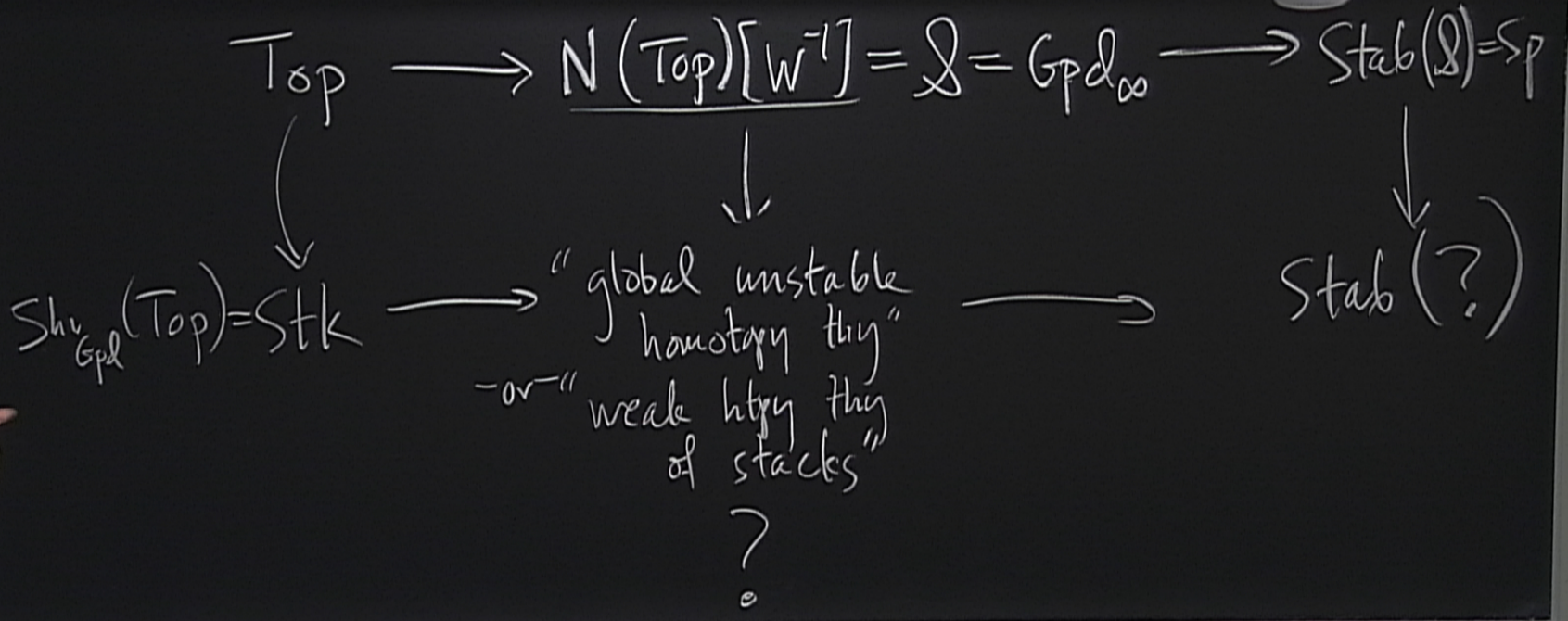
$Shv_{Gpd}(Top)$

→ "global unstable homotopy thry"  
-or- "weake hty thry of stacks"

→  $\text{Stab}(?)$

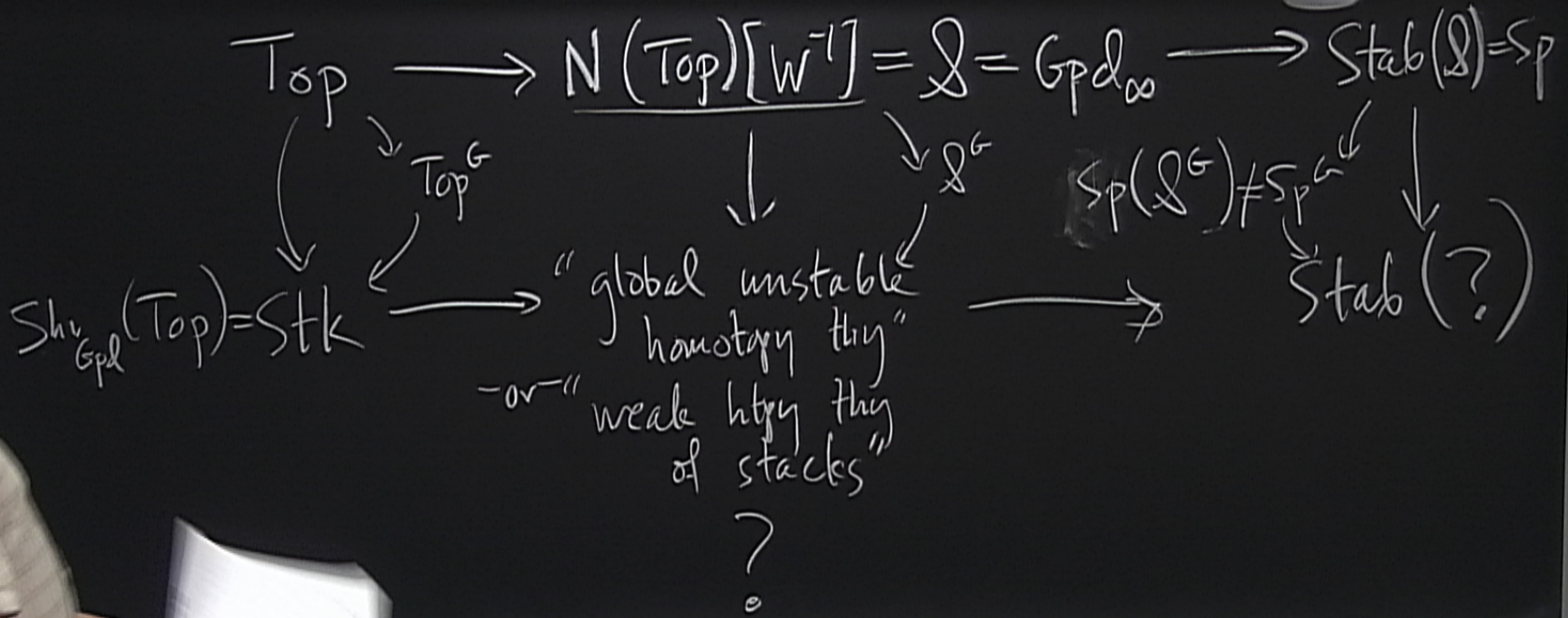
?

... also interesting alues on stks.



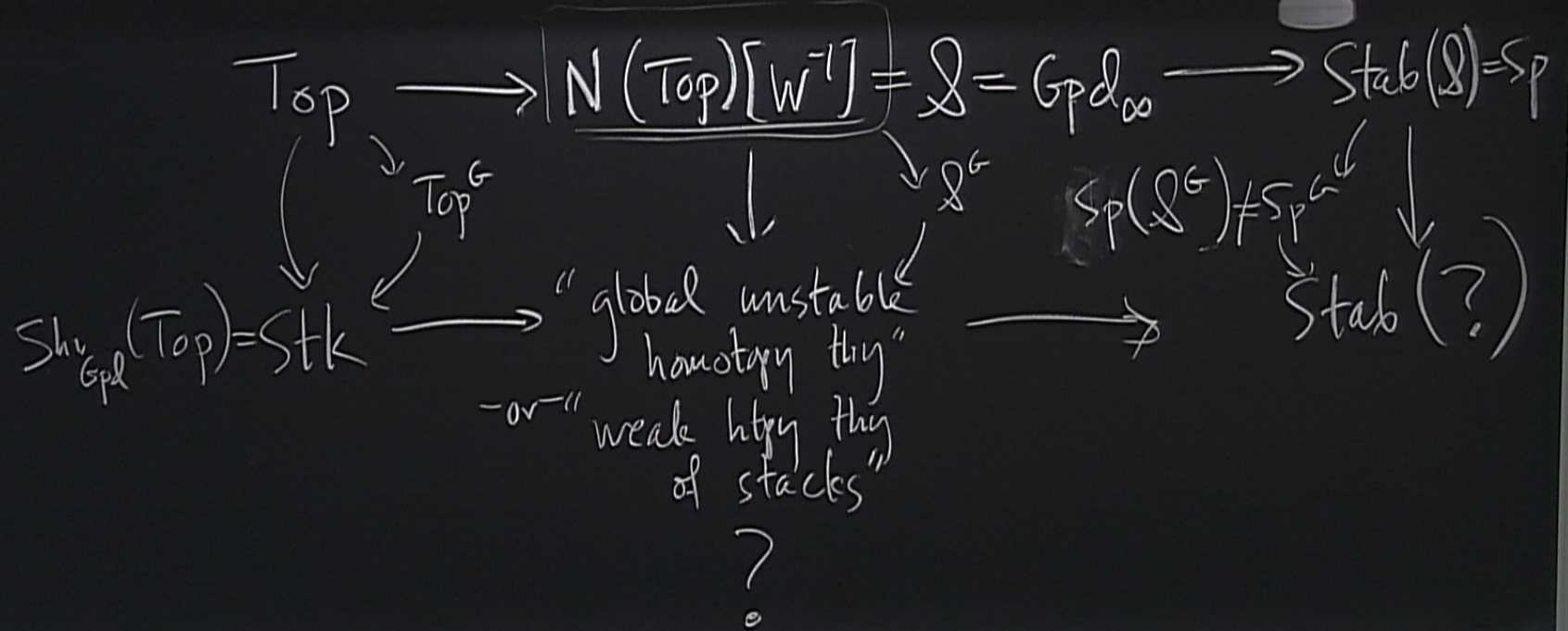
CAUTION

... also interesting algebras on stacks.



CAUTION  
 NO SMOKING OR OPEN FLAMES PERMITTED IN THE VICINITY OF THE BOARD  
 IT IS UNLAWFUL TO SMOKE OR USE OPEN FLAMES IN THIS AREA  
 THANK YOU FOR YOUR COOPERATION

... also interesting alues on stks.



What's so good about Gpdos?

"algebraic"

1) A map  $f: X \rightarrow Y$  is an equiv if  $\pi_n(X, x) \xrightarrow{f_*} \pi_n(Y, f(x)) \cong \pi_n(Y, y)$   
 $\forall n, x \in X.$

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What's so good about  $Gpd_{\infty}$ ?

"algebraic"

1) A map  $f: X \rightarrow Y$  is an equiv if  $\pi_n(X, x) \rightarrow \pi_n(Y, f(x)) \cong \mathbb{Z} \oplus$

2)  $X \rightarrow \lim_{\leftarrow} \tau_{\geq 0} X$ , slices of which are  $K(A, n) \forall n, x \in X$ .

A discrete group

What's so good about  $\text{Gpd}_\infty$ ?

"algebraic"

- 1) A map  $f: X \rightarrow Y$  is an equiv if  $\pi_n(X, X) \rightarrow \pi_n(Y, Y)$
  - 2)  $X \rightarrow \lim_{\leftarrow} \tau_{\geq 0} X$ , slices of which are  $K(A, n) \neq \emptyset$
- 
- $\text{Top} \rightarrow \text{Top}_\infty = \begin{cases} \text{objects same} \\ \text{Map}(X, Y) \end{cases}$  A discrete group



What's so good about  $\text{Gpd}_\infty$ ?

"algebraic"

- 1) A map  $f: X \rightarrow Y$  is an equiv if  $\pi_n(X, x) \rightarrow \pi_n(Y, f(x)) \cong \mathbb{Z}^k$
- 2)  $X \rightarrow \lim_{\leftarrow} \tau_{\geq 0} X$ , slices of which are  $K(A, n) \forall n, x \in X$ .

$$\text{Top} \rightarrow \text{Top}_\infty = \left\{ \begin{array}{l} \text{objects same} \\ \text{Map}(X, Y) = |n| \rightarrow \text{Hom}(\Delta^n \times X, Y) \end{array} \right. \quad \text{A discrete group}$$

$\text{Stk}_{X,Y}$  is also naturally an  $\infty$ -cat  $\text{Stk}_{\infty}$ .  
 $\text{Map}(X,Y) = |n| \rightarrow \underbrace{\text{Hom}(\Delta^n \times X, Y)}_{\text{gpd}}$

Cellularize it an subcategory  $\text{Orb} \hookrightarrow \text{Stk}_{\infty}$   
 full on TBG,

$\text{Stk}_{X,Y}$  is also naturally an  $\infty$ -cat  $\text{Stk}_{\infty}$ .

$$\text{Map}(X, Y) = \left| n \mapsto \text{Hom}_{\text{Spd}}(n \times X, Y) \right|$$

Cellularize it as subcategory  $\text{Orb} \hookrightarrow \text{Stk}_{\infty}$   
 Check:  $\text{Map}(\text{TBH}, \text{TBG}) \simeq \text{Map}_{\text{Orb}}(\text{TBG}, \text{G}_{\text{cpt}})$

$\text{Stk}_{X,Y}$  is also naturally an  $\infty$ -cat  $\text{Stk}_{\infty}$ .  
 $\text{Map}(X,Y) = |n| \rightarrow \text{Hom}(\Delta^n \times X, Y)$

Cellularize it an subcategory  $\text{Orb} \hookrightarrow \text{Stk}_{\infty}$

Check:  $\text{Map}(\text{TB}H, \text{TB}G) \simeq \text{Hom}_{\text{Sim}}(H,G) // G$ . full on  $\text{TB}G$ ,  $G$  cpt Lie.  
 $\pi_0 \text{Map}(\text{TB}H, \text{TB}G) = [H,G]$ .

$\text{Stk}_{X,Y}$  is also naturally an  $\infty$ -cat  $\text{Stk}_{\infty}$ .  
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Cellularize it an subcategory  $\text{Orb} \hookrightarrow \text{Stk}_{\infty}$   
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 $\pi_0 \text{Map}(\text{TB}H, \text{TB}G) = [H,G]$

$\text{Cell}(*) = \mathcal{S} = \left\{ \begin{array}{l} \text{full subcat } g\text{-n under} \\ \text{colims } b, * \end{array} \right.$

$\text{Stk}_k$  is also naturally an  $\omega$ -cat  $\text{Stk}_\infty$   
 $X, Y \quad \text{Map}(X, Y) = \left\{ n \mapsto \underbrace{\text{Hom}(\Delta^n \times X, Y)}_{\text{gpl}} \right\}$

Cellularize it an subcategory  $\text{Orb} \hookrightarrow \text{Stk}_\infty$

Check:  $\text{Map}(\text{TB}H, \text{TB}G) \simeq \text{Hom}_{\text{Sim}}(H, G) // G$ . full on  $\text{TB}G, G$  crt lre.

$\text{Stk}_\infty \xrightarrow{L} \text{Pre}_g(\text{Orb})$   
 $\text{Stk}_\infty \xrightarrow{R} \text{Pre}_g(\text{Orb})$   
 $\pi_0 \text{Map}(\text{TB}H, \text{TB}G) = [H, G]$   
 $X \mapsto \text{Map}(\text{TB}G, X)$

$\text{Cell}(*) = \mathcal{S} = \left\{ \begin{array}{l} \text{Full subcat } \mathcal{G}\text{-n under} \\ \text{colims } b \triangleleft * \end{array} \right.$

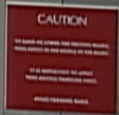
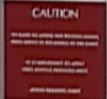
a weak equiv if  
 $\pi_n^G(X, x) \rightarrow \pi_n^G(Y, f(x))$  is  
 iso for all  $n, x: BG \rightarrow X$ .

$\text{Stk}_k$  is also naturally an  $\infty$ -cat  $\text{Stk}_\infty$ .  
 $X, Y \quad \text{Map}(X, Y) = \left[ n \mapsto \underbrace{\text{Hom}(\Delta^n \times X, Y)}_{\text{gpd}} \right]$

Cellularize it an subcategory  $\text{Orb} \hookrightarrow \text{Stk}_\infty$

Check:  $\text{Map}(BG, BG) \simeq \text{Hom}_{S^1}(\mathbb{H}, G) // G$ . full on  $BG, G$  crt Lie.

$\text{Stk}_\infty \xrightarrow{R} \text{Preq}(\text{Orb})$   $\xrightarrow{L}$   
 $X \mapsto \text{Map}(BG, X)$   $\pi_0 \text{Map}(BG, BG) = [\mathbb{H}, G]$



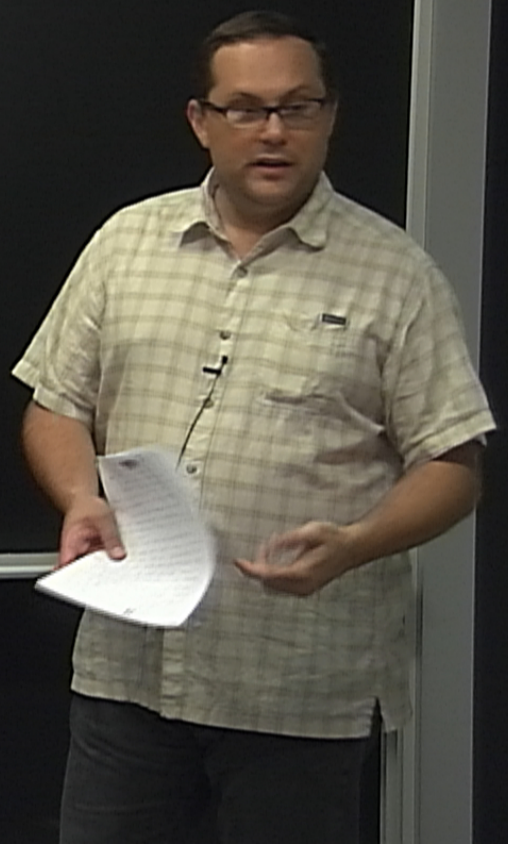
$f(x)$  is  
 $BG \rightarrow X$

$$\text{Orb}_{BG}^{\text{rep}} \cong \text{Orb}_G$$

(  $BH \rightarrow BG$  is rep  
if  $\rho \in [H, G]$  is injective )

$\int$  cpt  
lre.

CAUTION  
DO NOT TOUCH THE BOARD  
OR THE MARKERS ON THE BOARD





$f(x)$  is  
 $BG \rightarrow X$

$$Ov_{BG}^{rep} \simeq Ov_{BG}$$

$$\mathcal{Q}^G \simeq Pre(Ov_{BG}) \simeq Pre(Ov_{BG}^{rep})$$

(  $TBH \xrightarrow{f} BG$  is rep  
if  $f \in [H, G]$  is injective )

$$\begin{array}{c} \uparrow \downarrow \\ Pre(Ov_{BG}^{rep}) \simeq Pre(Ov_{BG}) \end{array}$$

$f$  cpt  
Lie.

CAUTION  
DO NOT TOUCH THE BOARD OR THE MARKERS  
OR THE ERASER

$f(x)$  is  
 $BG \rightarrow X$

$$\text{Orb}_{/BG}^{\text{rep}} \simeq \text{Orb}_G$$

(  $TBH \xrightarrow{f} BG$  is rep  
if  $f \in [H, G]$  is injective )

$$\mathcal{L}^G \simeq \text{Pre}(\text{Orb}_G) \simeq \text{Pre}(\text{Orb}_{/BG}^{\text{rep}}) \simeq \mathcal{L}^G$$

$$\begin{array}{c} \uparrow \downarrow \\ \text{Pre}(\text{Orb}_{/BG}) \simeq \text{Pre}(\text{Orb}_{/BG}) \end{array}$$

CAUTION  
DO NOT TOUCH THE BOARD  
OR THE MARKERS ON THE BOARD OR THE FRAME

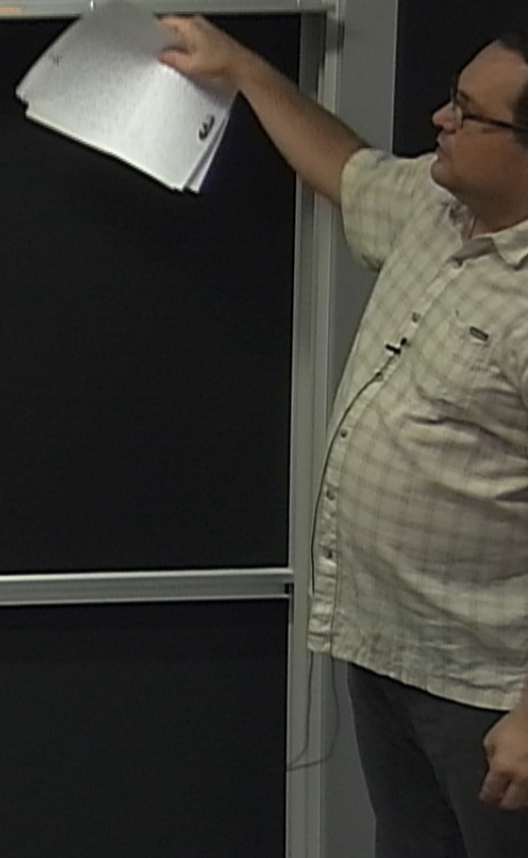
$f(x)$  is  
 $BG \rightarrow X$

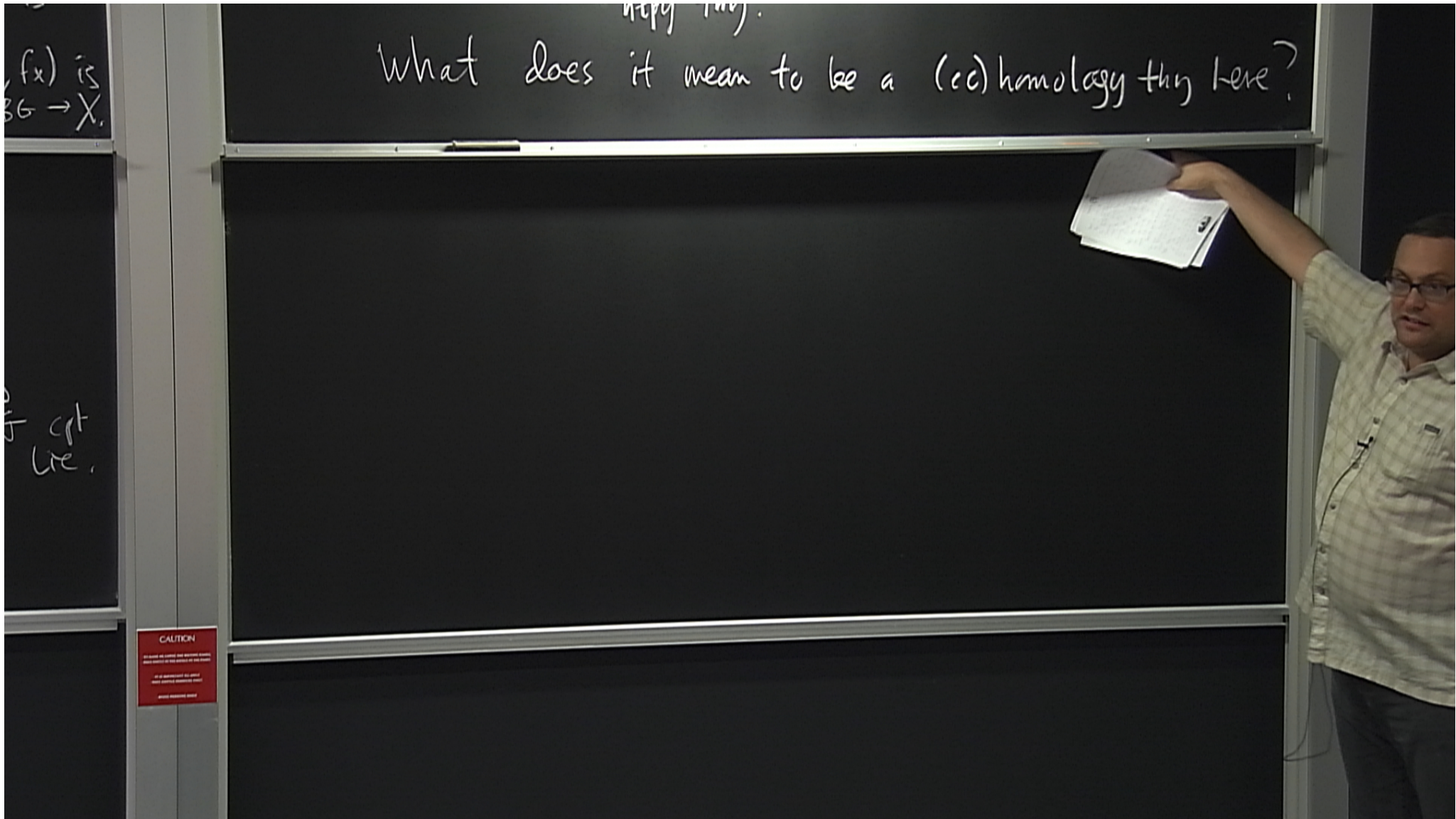
$\text{Pre}(Orb) \simeq$  "unstable global  
hopy thg".

What does it mean to be a (co)homology thg here?

$\int$  cpt  
lre.

CAUTION  
DO NOT RE-ENTER THE ROOM UNTIL THE ALL CLEAR IS GIVEN.  
IF YOU HEAR THE ALL CLEAR, LEAVE THE ROOM IMMEDIATELY.  
THANK YOU FOR YOUR COOPERATION.





$f_x$  is  
 $BG \rightarrow X$ .

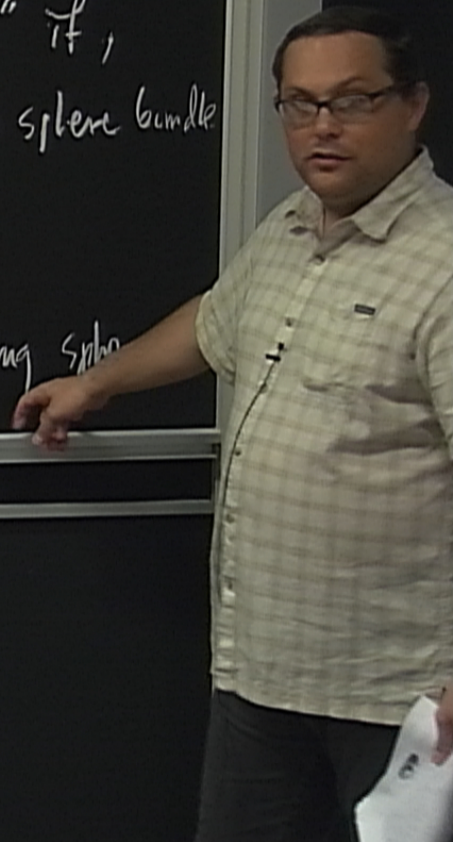
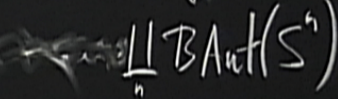
$\mathcal{F}$  cpt  
lre.

map  $\mathcal{F}$ .

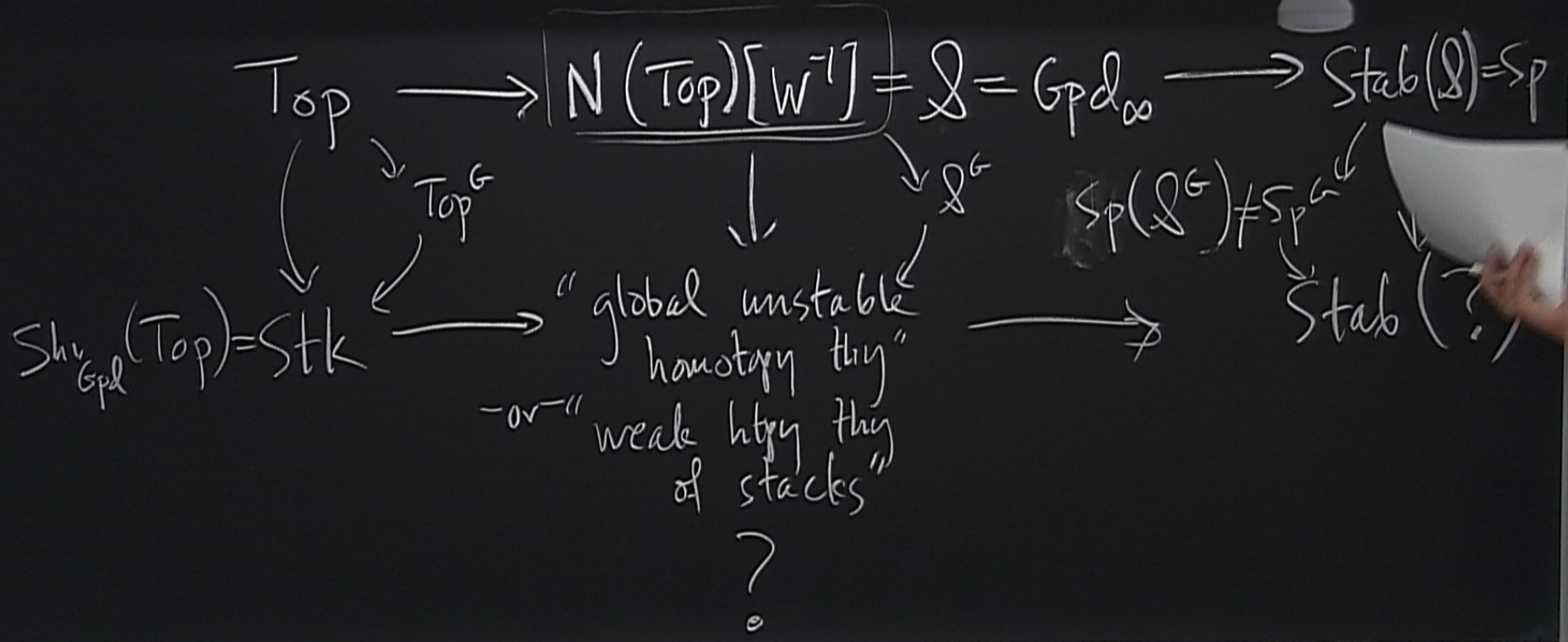
What does it mean to be a (co)homology theory here?

let  $\mathcal{E}$  be a topos  
(e.g. an  $\omega$ -topos)  
Def'n: A map  $f: Y \rightarrow X$  in  $\mathcal{E}$  is a "sphere bundle" if,  
for all  $T \in \mathcal{E}$ ,  $Y(T) \rightarrow X(T)$  is a sphere bundle.

Equip  $\mathcal{E}$  with a class  $\mathcal{F} \hookrightarrow \mathcal{E}^{\Delta^1}$  consisting of "spherical fibrations" including sphere bundles.



$$A \left( \begin{array}{c} \downarrow \\ \mathbb{Z} \end{array} \right) = A(\mathcal{Y}) = K(S(\mathcal{Y})), \quad A \left( \begin{array}{c} \downarrow \\ X \end{array} \right) = \mathcal{V}(X).$$



$$A \left( \begin{array}{c} \downarrow \\ \uparrow \end{array} \right) = A(y) = K(S(y)), \quad A \left( \begin{array}{c} \downarrow \\ \uparrow \end{array} \right) = \nabla(X)$$

$$\bullet \text{ } \text{Biv}^{\text{exc}}(\mathcal{C}) \hookrightarrow \text{Biv}(\mathcal{C})$$

Full on the

$$\bullet \text{ } F \left( \begin{array}{c} y = \text{colim } y_i \\ \downarrow \\ X = \text{colim } X_i \end{array} \right) = \lim F \left( \begin{array}{c} y_i \\ \downarrow \\ X_i \end{array} \right)$$

$$\bullet \text{ } F \left( \begin{array}{c} y = \text{colim } y_i \\ \downarrow \\ X \end{array} \right) \simeq \text{colim } F \left( \begin{array}{c} X_i \\ \downarrow \\ y_i \end{array} \right)$$

Eq.

$$A\left(\begin{array}{c} \downarrow \\ \mathbb{Z} \end{array}\right) = A(\mathbb{Z}) = K(S(\mathbb{Z})), \quad A\left(\begin{array}{c} \downarrow \\ X \end{array}\right) = \mathbb{Z}(X).$$

$$\circ \text{BN}^{\text{ex}}(\mathcal{C}) \hookrightarrow \text{BN}(\mathcal{C})$$

Full on the

$F$  st

$$\circ F\left(\begin{array}{c} y = \text{colim } y_i \\ \downarrow \\ X = \text{colim } X_i \end{array}\right) = \lim F\left(\begin{array}{c} y_i \\ \downarrow \\ X_i \end{array}\right)$$

$$\circ F\left(\begin{array}{c} y = \text{colim } y_i \\ \downarrow \\ X \end{array}\right) \simeq \text{colim } F\left(\begin{array}{c} y_i \\ \downarrow \\ X_i \end{array}\right)$$

Eg. twisted (co)homology:  $\text{BN}^{\text{ex}}(\mathcal{A}) \simeq \text{Sp}$ .

$M \in \text{Sp}, M(y \rightarrow X)$

$$F \longmapsto F(\text{pt} \rightarrow \text{pt})$$



$$A\left(\begin{array}{c} \downarrow \\ \mathbb{Z} \end{array}\right) = A(\mathbb{Z}) = K(S(\mathbb{Z})), \quad A\left(\begin{array}{c} \downarrow \\ X \end{array}\right) = \mathbb{Z}(X).$$

$$\bullet \text{ } BN^{ex}(\mathcal{C}) \hookrightarrow BN(\mathcal{C})$$

Full on the  $F$  st  $\bullet$   $F\left(\begin{array}{c} y = \text{colim } y_i \\ \downarrow \\ X = \text{colim } X_i \end{array}\right) = \lim F\left(\begin{array}{c} y_i \\ \downarrow \\ X_i \end{array}\right)$

$$\bullet F\left(\begin{array}{c} y = \text{colim } y_i \\ \downarrow \\ X \end{array}\right) \simeq \text{colim } F\left(\begin{array}{c} y_i \\ \downarrow \\ X_i \end{array}\right)$$

E.g. twisted (co)homology:  $BN^{ex}(\mathcal{A}) \simeq Sp.$

$$M \in Sp, \quad M(y \rightarrow X) = T\left(\sum_{i=1}^{\infty} y_i \wedge M\right), \quad F \longmapsto F(pt \rightarrow pt)$$

Defn:  $(\mathcal{C}, \mathcal{F})$ ,  $F \in \mathcal{B}_N^{\mathcal{F}\text{-exc}}(\mathcal{C}) \hookrightarrow \mathcal{B}_N^{\text{exc}}(\mathcal{C})$

the theories which satisfy "Thm iso". I.e.

$$\text{for all } Y \xrightarrow{P} X \in \mathcal{F},$$

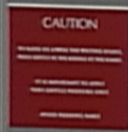
$$F(Y \xrightarrow{P} X) \oplus F(Y = Y) \xrightarrow[\sim]{P^* \oplus \delta_+} F\left(\begin{array}{c} Y \times Y \\ \xrightarrow{X} \end{array} Y\right).$$

Defn:  $(\mathcal{C}, \mathcal{Y})$ ,  $F \in \text{Biv}^{\text{exc}}(\mathcal{C}) \hookrightarrow \text{Biv}(\mathcal{C})$

the theories which satisfy "Thom iso". I.e.  
 for all  $Y \xrightarrow{P} X \in \mathcal{Y}$ ,

$$F(Y \xrightarrow{P} X) \oplus F(Y \rightarrow Y) \xrightarrow{\sim} F(Y \times_X Y \rightarrow Y)$$

In the case of  $\mathcal{C} = \text{Pre}(\text{orb})$ ,  $\mathcal{Y}$  sphere bundles,  
 $\text{Biv}^{\text{exc}}(\text{Pre}(\text{orb})) \hookrightarrow$  "global htpy, stable thy".



$\text{colim } b, *$

$\tau_n(X, x) \rightarrow \tau_n(Y, y)$  is  
 iso for all  $n, x: BG \rightarrow X$ .

for  $y \rightarrow X \in \mathcal{Y}$

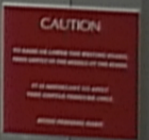
$$F(y \rightarrow X) \oplus F(y = y) \xrightarrow{P^+ \oplus \delta_*} F(y \times_X y \rightarrow y)$$

In the case of  $\mathcal{C} = \text{Pre}(\text{orb})$ ,  $\mathcal{Y}$  sphere bundles,

$$\text{Biv}^{\text{exc}}(\text{Pre}(\text{orb})) \leftarrow$$

"global hty, stabl thm"

$$F(BG \rightarrow pt) \rightarrow F(BG \rightarrow pt)$$



$\text{colim } b, *$

$\tau_n(X, x) \rightarrow \tau_n(Y, y)$  is iso for all  $n, x: BG \rightarrow X$ .

Defn:  $(\mathcal{C}, \mathcal{Y}), F \in \text{Biv}^{\mathcal{F}\text{-exc}}(\mathcal{C}) \hookrightarrow \text{Biv}^{\text{exc}}(\mathcal{C})$

the theories which satisfy "Thom iso". I.e.

for all  $Y \xrightarrow{P} X \in \mathcal{Y}$ ,

$$F(Y \xrightarrow{P} X) \oplus F(Y = Y) \xrightarrow[\sim]{P^* \oplus \delta_*} F(Y \times_X Y \rightarrow Y)$$

In the case of  $\mathcal{C} = \text{Pre}(\text{orb}), \mathcal{Y}$  sphere bundles,

$\text{Biv}^{\mathcal{F}\text{-exc}}(\text{Pre}(\text{orb})) \hookrightarrow$

"global htyg, stable thy"

$$F((Y \rightarrow X)^{\text{reg}}) \rightarrow F(Y \rightarrow X)$$

Equip  $\mathcal{C}$  with a class  $\mathcal{F}$  consisting of "spherical fibrations" including spheres.

Geometrically, there's lots of br. theories.  $(X, \mathcal{O})$ ,

Say that a map  $f: Y \rightarrow X$  is spherical wrt  $\mathcal{O}$  if  $f_* \mathcal{O}_Y \rightarrow \mathcal{O}_X$  has invertible fiber as  $\mathcal{O}_X$ -mod.

What does it mean

Equip  $\mathcal{C}$  with a class  $\mathcal{F} \subset \mathcal{C}$   
consisting of "spherical fibrations" including spheres.

Geometrically, there's lots of br. theories.  $(X, \mathcal{O})$ ,

Say that a map  $f: Y \rightarrow X$  is spherical wrt  $\mathcal{C}$

if  $\mathcal{O}_X \rightarrow f_* \mathcal{O}_Y$  has invertible fiber as  $\mathcal{O}_Y$ .

check:  $T(Y \rightarrow X) = T(\mathbb{P}^1, \mathcal{O}_Y)$ .

CAUTION  
Do not touch the board  
or the projector or the  
screen or the board  
or the projector or the  
screen or the board

Equip  $\mathcal{C}$  with a class  $\mathcal{F}$  of "..."  
Geometrically, there's lots of br. theories.  $(X, \mathcal{O})$ ,

Say that a map  $f: Y \rightarrow X$  is spherical wrt  $\mathcal{O}$  if  $\mathcal{O}_X \rightarrow f_* \mathcal{O}_Y$  has invertible fiber as  $\mathcal{O}_X$ -mod.

Check:  $T(Y \rightarrow X) = T(\pi_1 \mathcal{O}_Y)$ .

This is  $\mathcal{F}$ -exc. for the  $\mathcal{O}$ -spherical maps.

What does it mean

CAUTION  
Do not touch the screen  
Do not touch the screen



Equip  $\mathcal{C}$  with a class  $\mathcal{F}$  of "..."  
Geometrically, there's lots of br. theories.  $(X, \mathcal{O})$ ,

Say that a map  $f: Y \rightarrow X$  is spherical wrt  $\mathcal{O}$  if  $\mathcal{O}_X \rightarrow f_* \mathcal{O}_Y$  has invertible fiber as  $\mathcal{O}_X$ -mod.

Check:  $T(Y \rightarrow X) = T(\pi_1 \mathcal{O}_Y)$ .

This is  $\mathcal{F}$ -exc. for the  $\mathcal{O}$ -spherical maps.

with a class  $\mathcal{F}$  of "..."

CAUTION

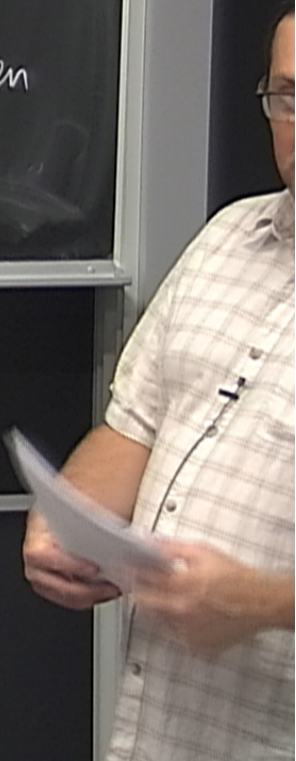
Do not touch the screen or the board.  
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Do not touch the screen or the board.

Say that a map  $f: Y \rightarrow X$  is spherical wrt  $\mathcal{O}$  if  $\mathcal{O}_X \rightarrow f_* \mathcal{O}_Y$  has invertible fiber as  $\mathcal{O}_X$ -mod.

Check:  $T(Y \rightarrow X) = T(\pi_! \mathcal{O}_Y)$ .

This is  $\mathcal{O}$ -exc. for the  $\mathcal{O}$ -spherical maps.

Fact: If  $\varphi^*: X' \rightarrow (X, \mathcal{O})$  preserves spherical fibrs, then



CAUTION  
 Do not touch the bottom board  
 or the board at the end  
 of the board. Do not  
 touch the board.

Say that a map  $f: Y \rightarrow X$  is spherical wrt  $\mathcal{O}$  if  $\mathcal{O}_X \rightarrow f_* \mathcal{O}_Y$  has invertible fiber as  $\mathcal{O}_X$ -mod.

check:  $T(Y \rightarrow X) = T(\pi_1 \mathcal{O}_Y)$ .

This is  $\mathcal{O}$ -exc. for the  $\mathcal{O}$ -spherical maps.

Fact: If  $\varphi^*: X' \rightarrow (X, \mathcal{O})$  preserves spherical fibrs, then pull back  $T$  to a  $\mathcal{O}$ -exc. thy on  $X'$  by  $T(Y \rightarrow X) = T(\varphi^* Y \rightarrow X')$

CAUTION  
do not touch the bottom board  
do not touch the top board  
do not touch the middle board

pull back  $\mathbb{P}^1$  to a  $\mathbb{P}^1$  by  $T(y \rightarrow X) = T(y^* \rightarrow X)$

$\mathcal{X}' = \text{Pre}(\text{Orb}) \dashrightarrow (\mathcal{X}, \mathcal{O}) = \text{Shv}_{\text{ét}}/\mathcal{M} \leftarrow \text{Lurie's moduli stack of oriented der. elliptic curves}$

CAUTION  
 Do not touch the screen  
 Do not touch the screen  
 Do not touch the screen

pull back  $\mathcal{P}$  to a  $\mathcal{P}'$  on  $\mathcal{X}'$  by  $T(y \rightarrow X) = T(y' \rightarrow X')$

$$\mathcal{X}' = \text{Pre}(\text{Orb}) \dashrightarrow (\mathcal{X}, \mathcal{O}) = \text{Shv}_{\mathcal{M}}^{\text{ét}} \leftarrow \begin{array}{l} \text{Lurie's} \\ \text{moduli stack of} \\ \text{oriented der. elliptic} \\ \text{curves.} \end{array}$$

$$\downarrow \tilde{z}^*$$

$$\text{Pre}(\text{Orb}^{ab})$$

$$\tilde{z}: \text{Orb}^{ab} \hookrightarrow \text{Orb}$$

CAUTION  
 Do not touch the blackboard  
 or the boards of the board  
 Do not touch the board  
 or the boards of the board  
 Do not touch the board

$$\text{Ell}(\mathbb{B}\mathbb{T} \rightarrow \mathbb{B}\mathbb{T}) = T(\mathcal{O}_C), \quad \text{Ell}(pt \rightarrow pt) = T(\mathcal{O}_u)$$

$(G, N, \text{Meier})$        $\text{TMF}^{\mathbb{Z}_2} \oplus \Sigma, \text{TMF}$        $\text{TMF}$

Geometrically, there's lots of br. theories.  $(\mathcal{X}, \mathcal{O})$ ,

Say that a map  $f: Y \rightarrow X$  is spherical wrt  $\mathcal{O}$  if  $\mathcal{O}_X \rightarrow f_* \mathcal{O}_Y$  has invertible fiber as  $\mathcal{O}_X$ -mod.

Check:  $T(Y \rightarrow X) = T(\pi_! \mathcal{O}_Y)$ .

This is  $\mathcal{O}$ -exc. for the  $\mathcal{O}$ -spherical maps.

Fact: If  $\varphi^*: \mathcal{X}' \rightarrow (\mathcal{X}, \mathcal{O})$  preserves spherical fibrs, then pull back  $T$  to a  $\mathcal{O}$ -exc. thry on  $\mathcal{X}'$  by  $T(Y \rightarrow X) = T(\varphi^* Y \rightarrow \mathcal{X}')$ .

We want the homological completion (?) map

$$\text{Ell}(B\mathbb{T} \rightarrow \text{pt}) \longrightarrow \text{Ell}(B\mathbb{T} \xrightarrow{\mathbb{I}} \text{pt})$$

$$\begin{aligned} & \text{CP}^\infty \\ & \parallel \\ & T(\text{CP}^\infty \otimes \mathcal{O}_{\mathcal{M}}) \\ & \parallel \\ & \text{Tmf}_*(\text{CP}^\infty) \end{aligned}$$

$$T(\mathbb{D}_{P^*} \otimes \mathcal{O}_C)$$

$$\text{Ell}(B\mathbb{Z} \rightarrow pt) \longrightarrow \text{Ell}(B\mathbb{Z} \rightarrow pt)$$

$$\begin{aligned} & \xrightarrow{\cong} \text{CP}^\infty \\ & \cong \\ & T(\text{CP}^\infty \oplus \mathcal{O}_M) \\ & \cong \\ & \text{TMF}_*(\text{CP}^\infty) \end{aligned}$$

not an equiv.

$$\begin{aligned} & \cong \\ & T(\mathcal{D}_{\mathbb{Z}_2} P^* \mathcal{O}_C) \\ & \cong \\ & T(\mathcal{D}_{\mathbb{Z}_2} (\mathcal{O}_M \oplus \Sigma, \mathcal{O}_M)) \\ & \cong \\ & \text{TMF} \oplus \Sigma^{-1} \text{TMF} \end{aligned}$$

