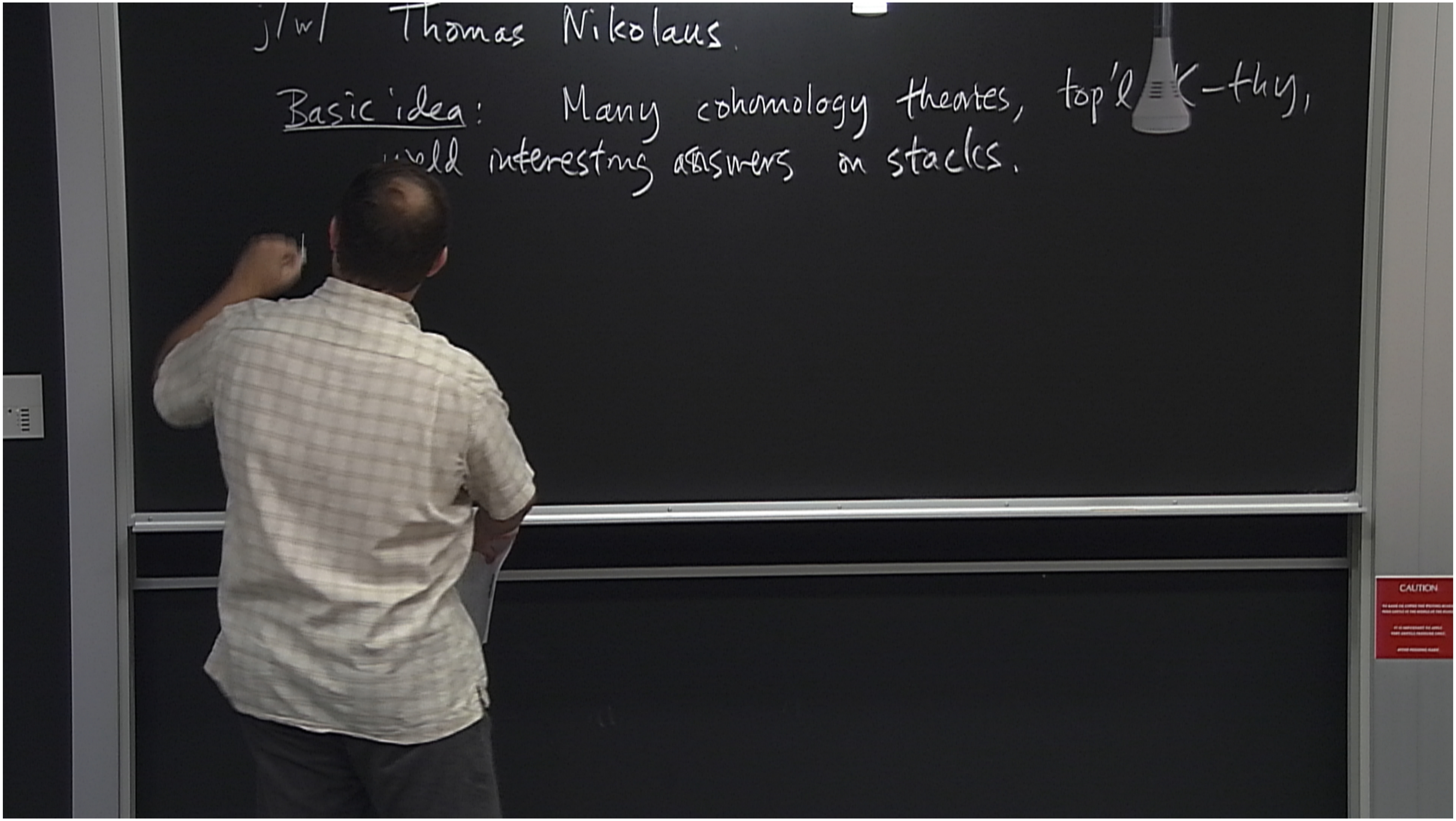
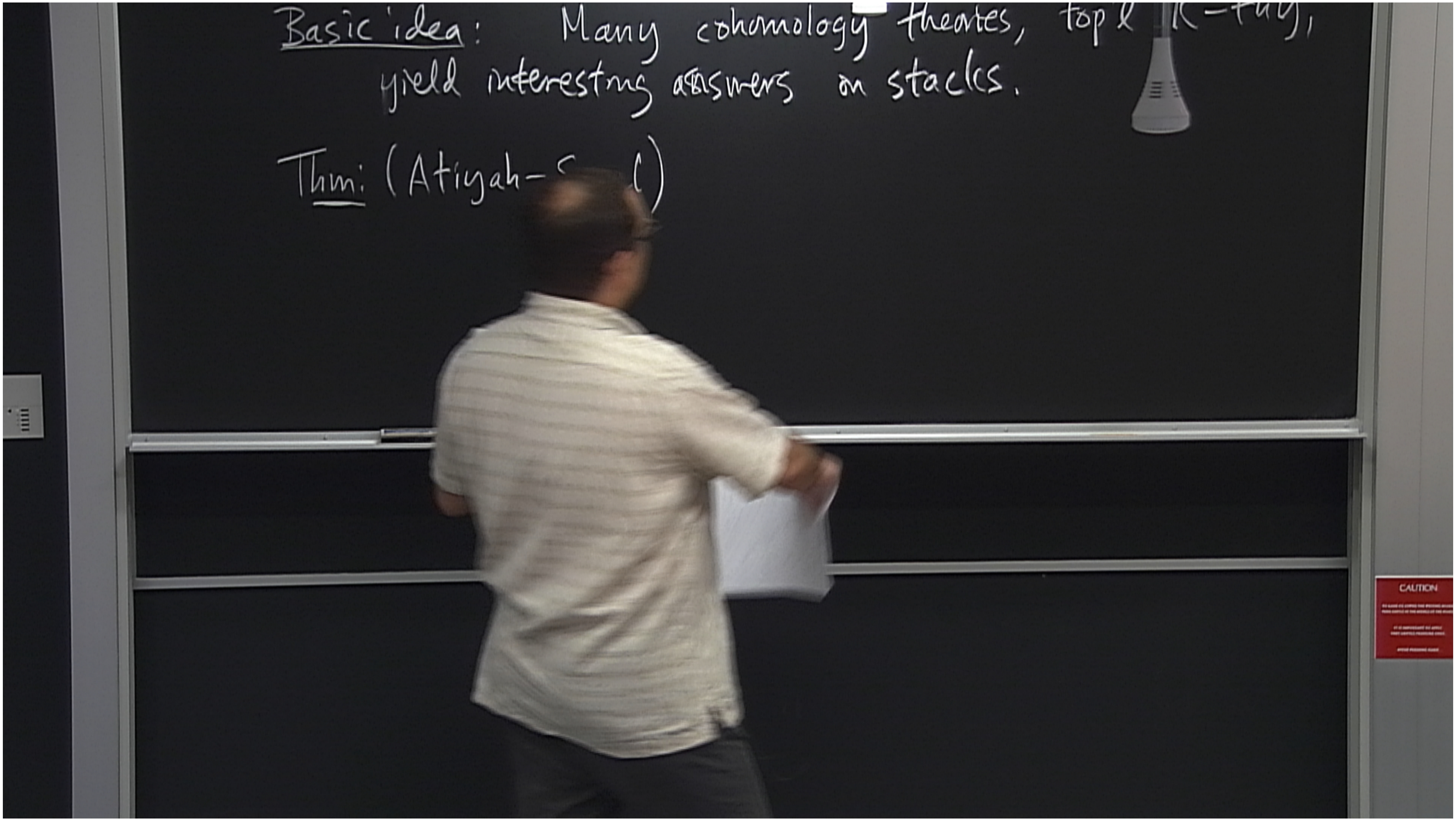
Title: On the stable homotopy theory of stacks and elliptic cohomology

Date: Apr 21, 2016 09:30 AM

URL: http://pirsa.org/16040082

Abstract: In this talk, we'll discuss what it means to be a cohomology theory for topological stacks, using a notion of local symmetric monoidal inversion of objects in families. While the general setup is abstract, it specializes to many cases of interest, including Schwede's global spectra. We will then go on to discuss various examples with particular emphasis on elliptic cohomology. It turns out that TMF sees more objects as dualizable (or even invertible) than one might naively expect.


j/w/ Thomas Nikolaus.
Basic idea: Many cohomology theories, top'l K-thy,
yield interesting answers on stacks.
CAUTION


Basic idea: Many cohomology theories, top'l K-thy,
yield interesting answers on stacks.
Thm: (Atiyah-
CAUTION

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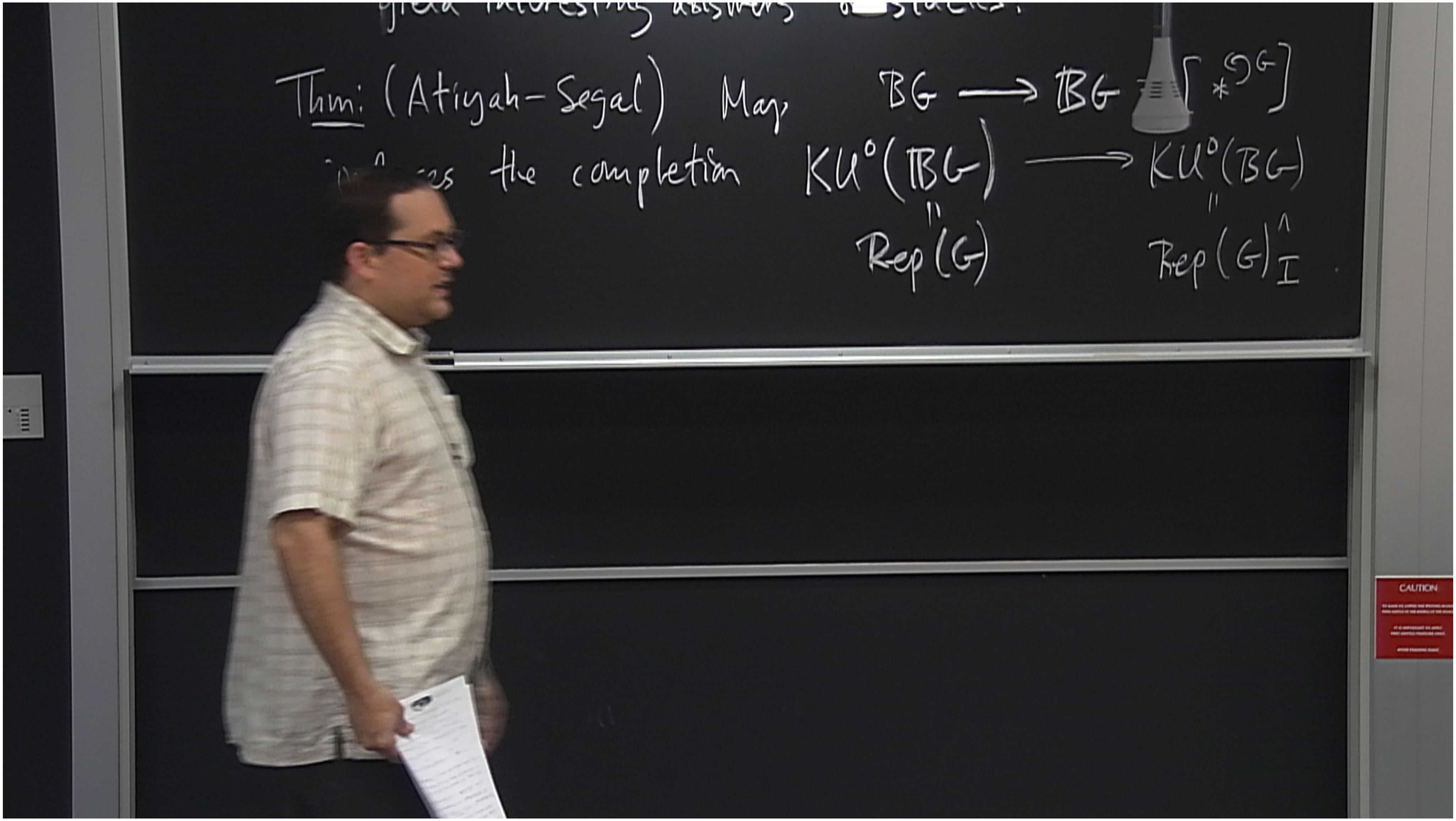
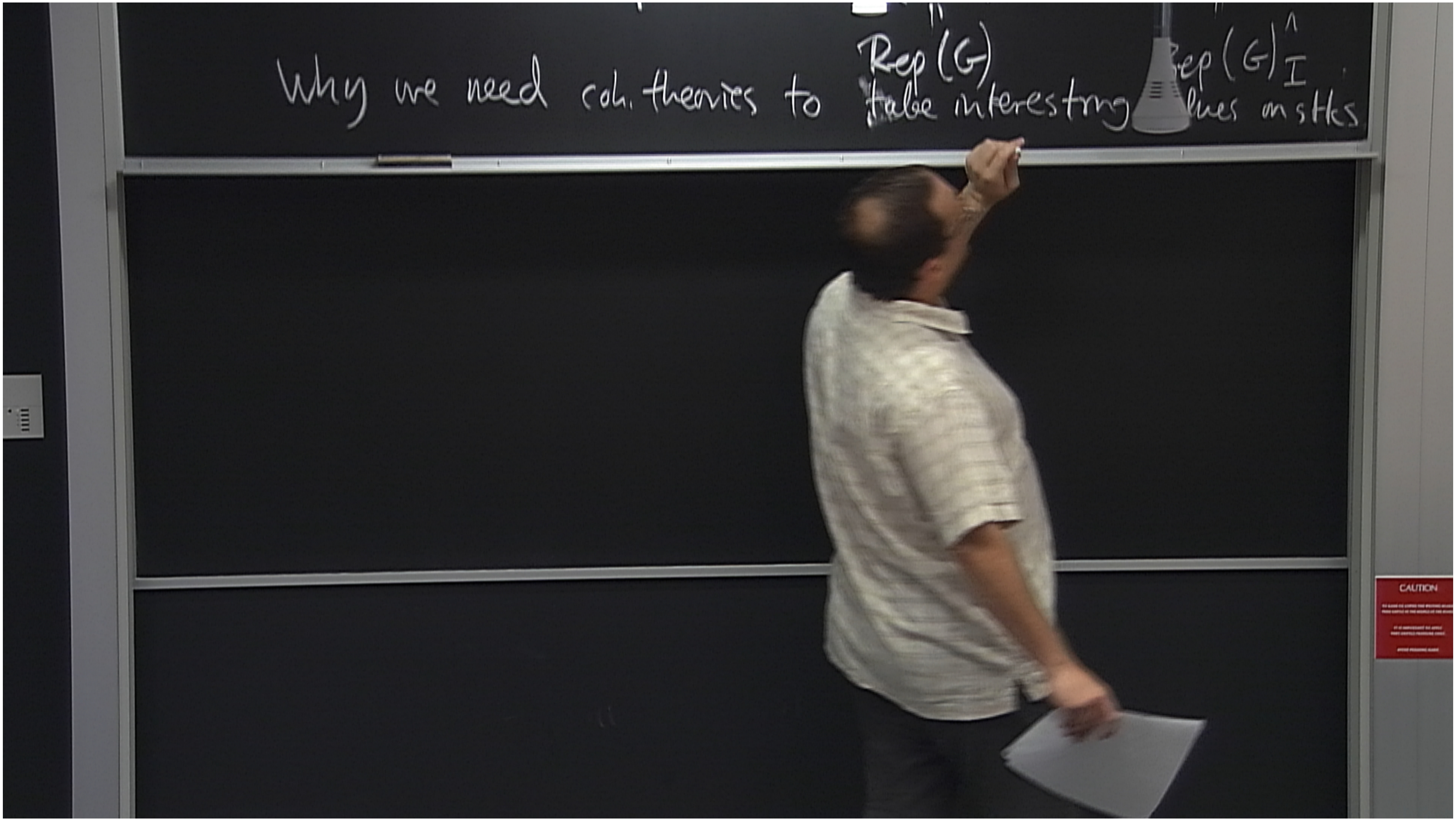
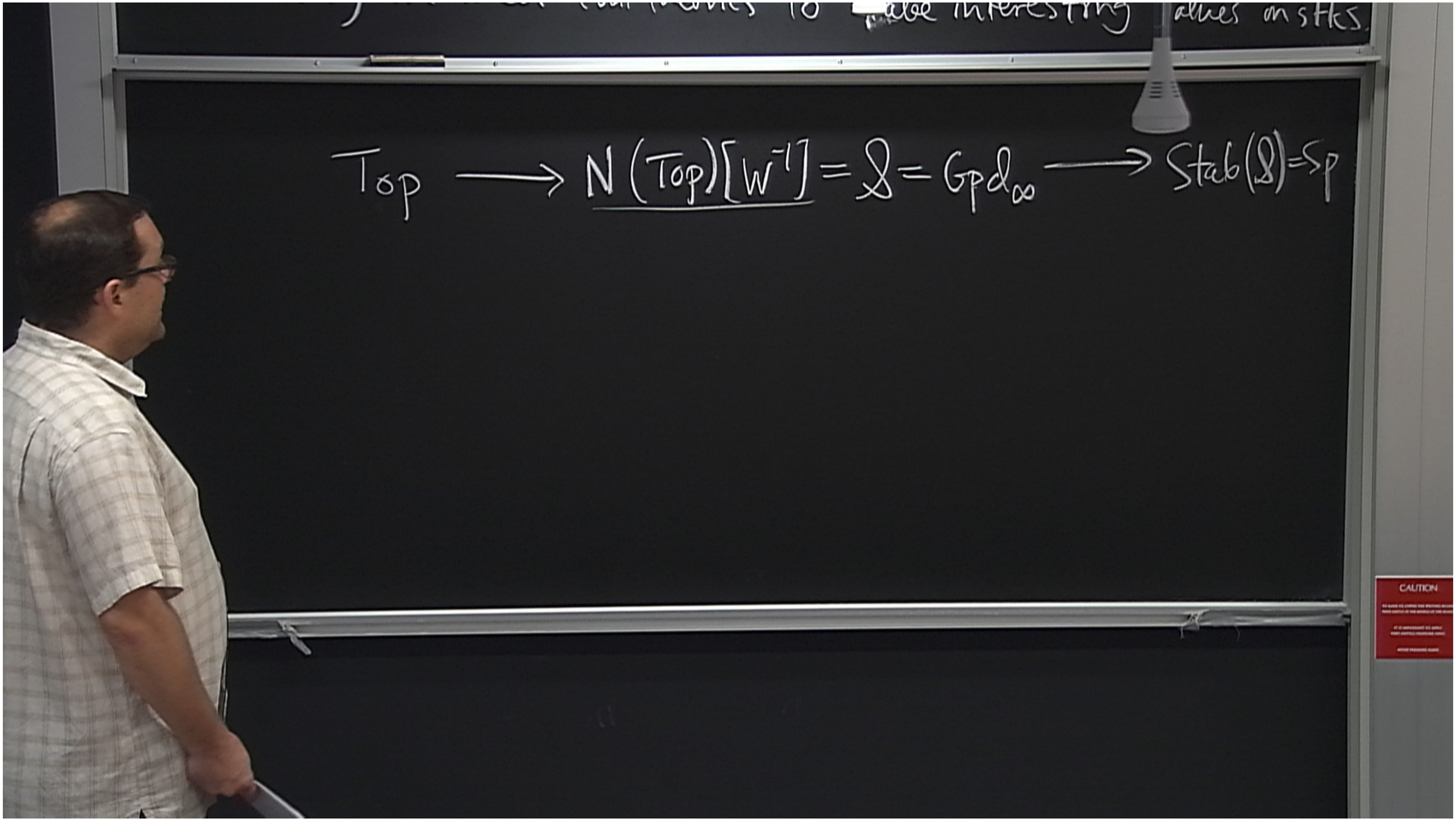
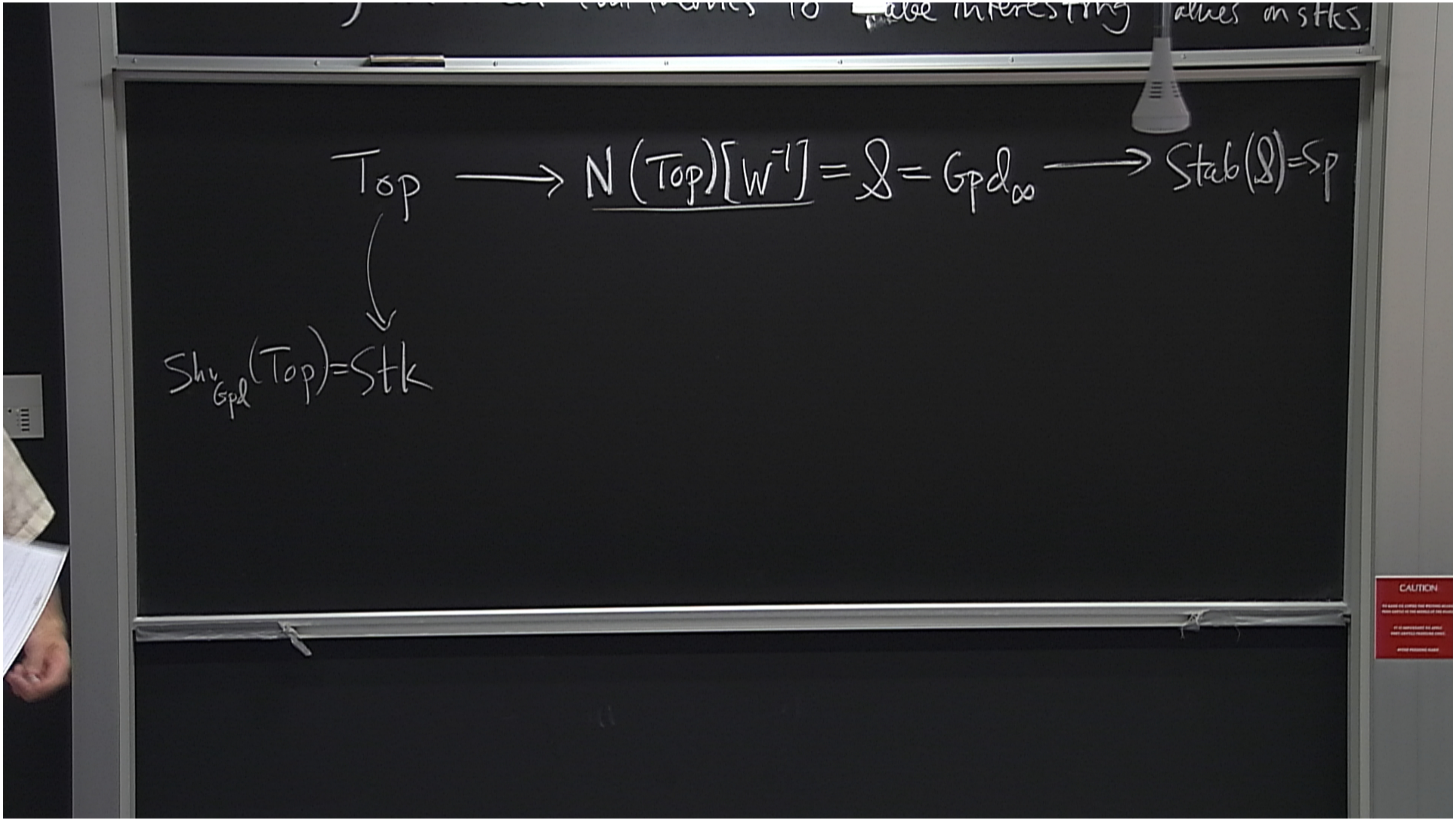
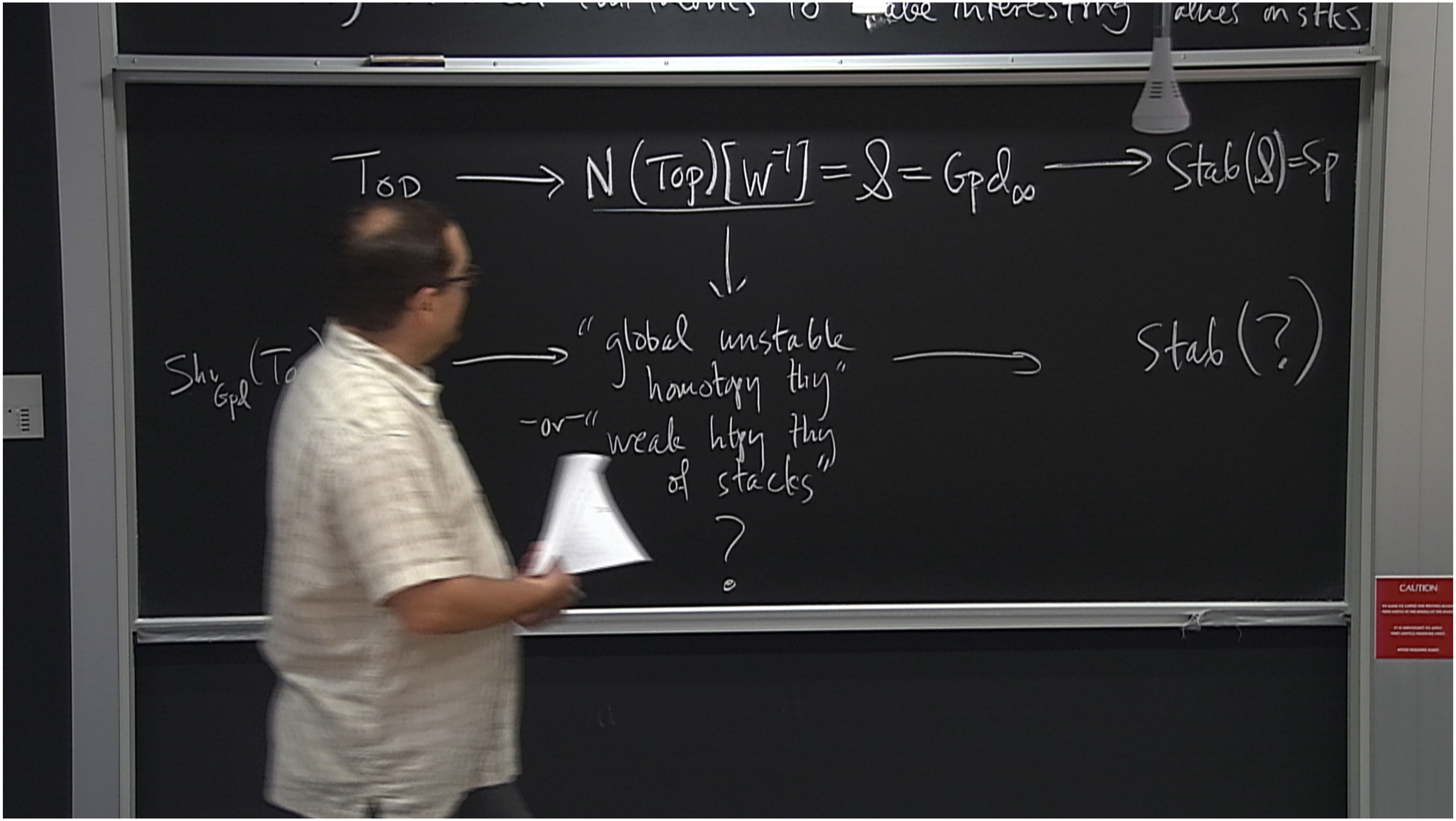
Thm: (Atiyah-Segal) May, $BG \longrightarrow \mathbb{B}G = [*^{\circlearrowleft G}]$

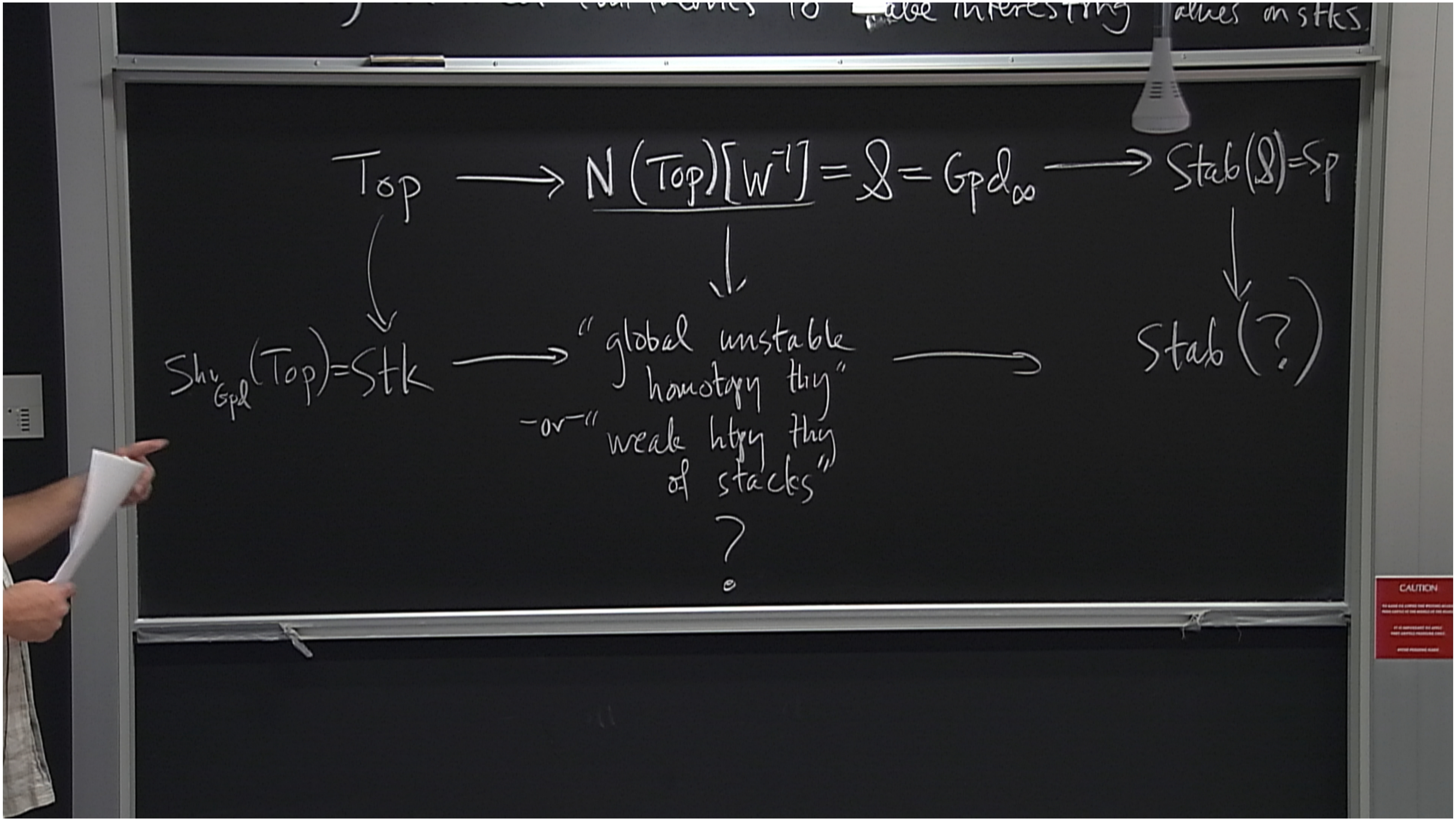

Thm: (Atiyah-Segal) Map BG → 𝔹G = [*/G]
induces the completion KU°(𝔹G) → KU°(BG)
Rep(G)
Rep(G)^I


Rep(G)
Rep(G)
I
Why we need coh. theories to take interesting lues on stks
CAUTION


$\mathrm{Top} \longrightarrow \underline{N(\mathrm{Top})[W^{-1}]} = \mathcal{S} = \mathrm{Gpd}_\infty \longrightarrow \mathrm{Stab}(\mathcal{S}) = \mathrm{Sp}$
CAUTION


Top ⟶ N(Top)[W^-1] = S = Gpd_∞ ⟶ Stab(S) = Sp
Shv_Gpd(Top) = Stk
CAUTION


Top → N(Top)[W^-1] = S = Gpd∞ → Stab(S) = Sp
"global unstable homotopy thy"
-or- "weak htpy thy of stacks"
Stab(?)
?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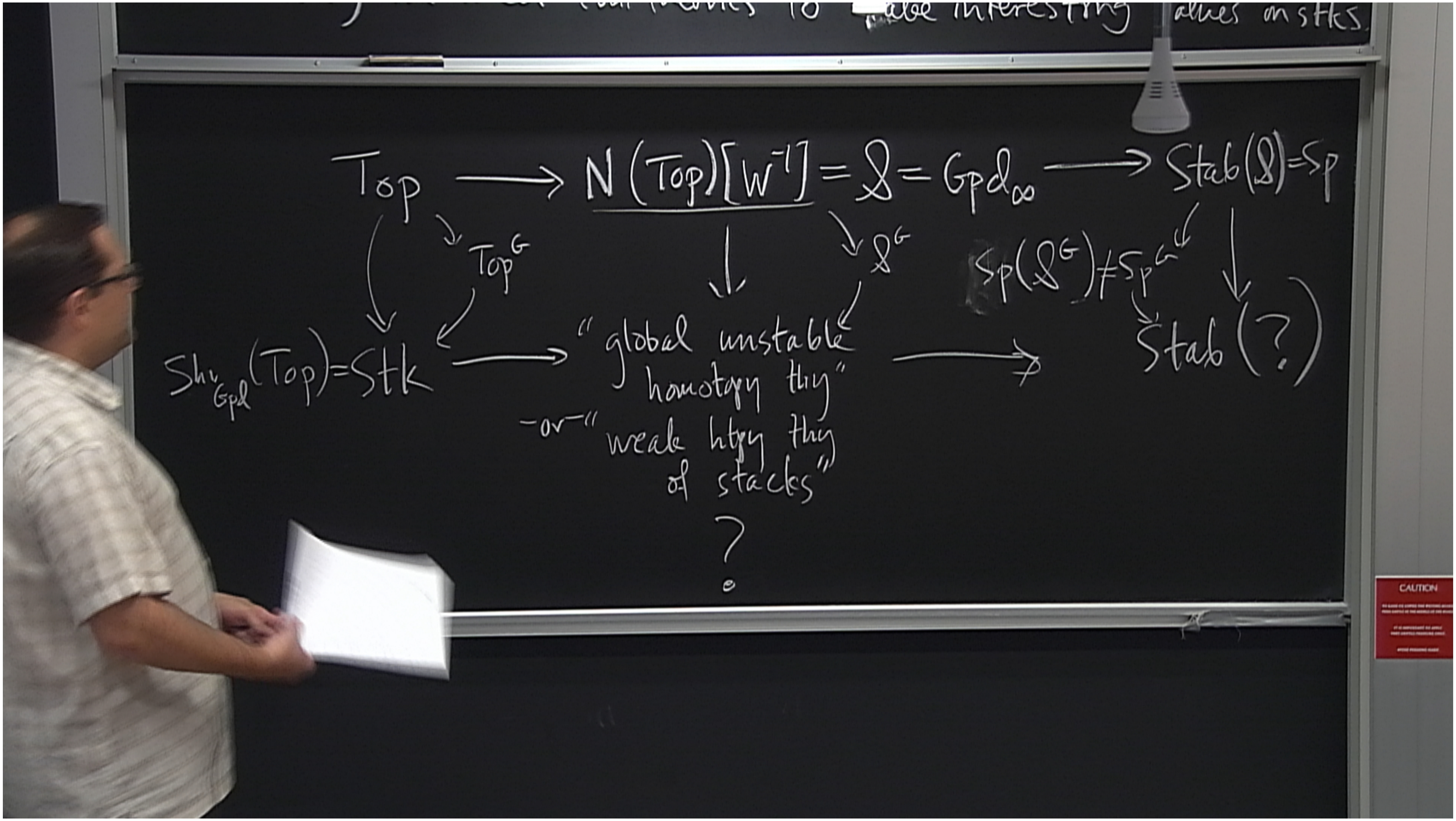
Top
N(Top)[W^-1] = S = Gpd∞
Stab(S)=Sp
Top^G
S^G
Sp(S^G) ≠ Sp^G
Sh_Gpd(Top)=Stk
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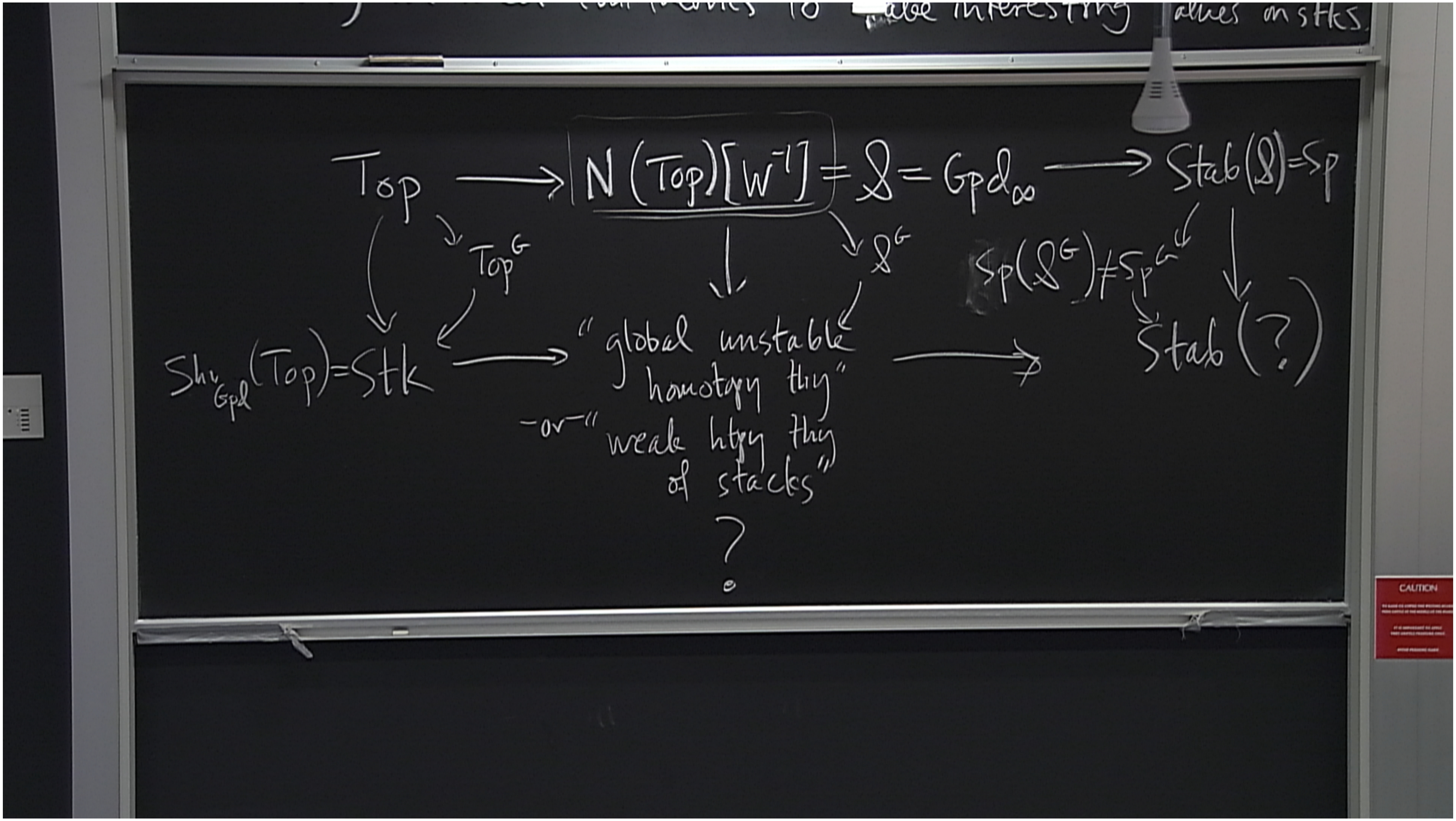
Stab(?)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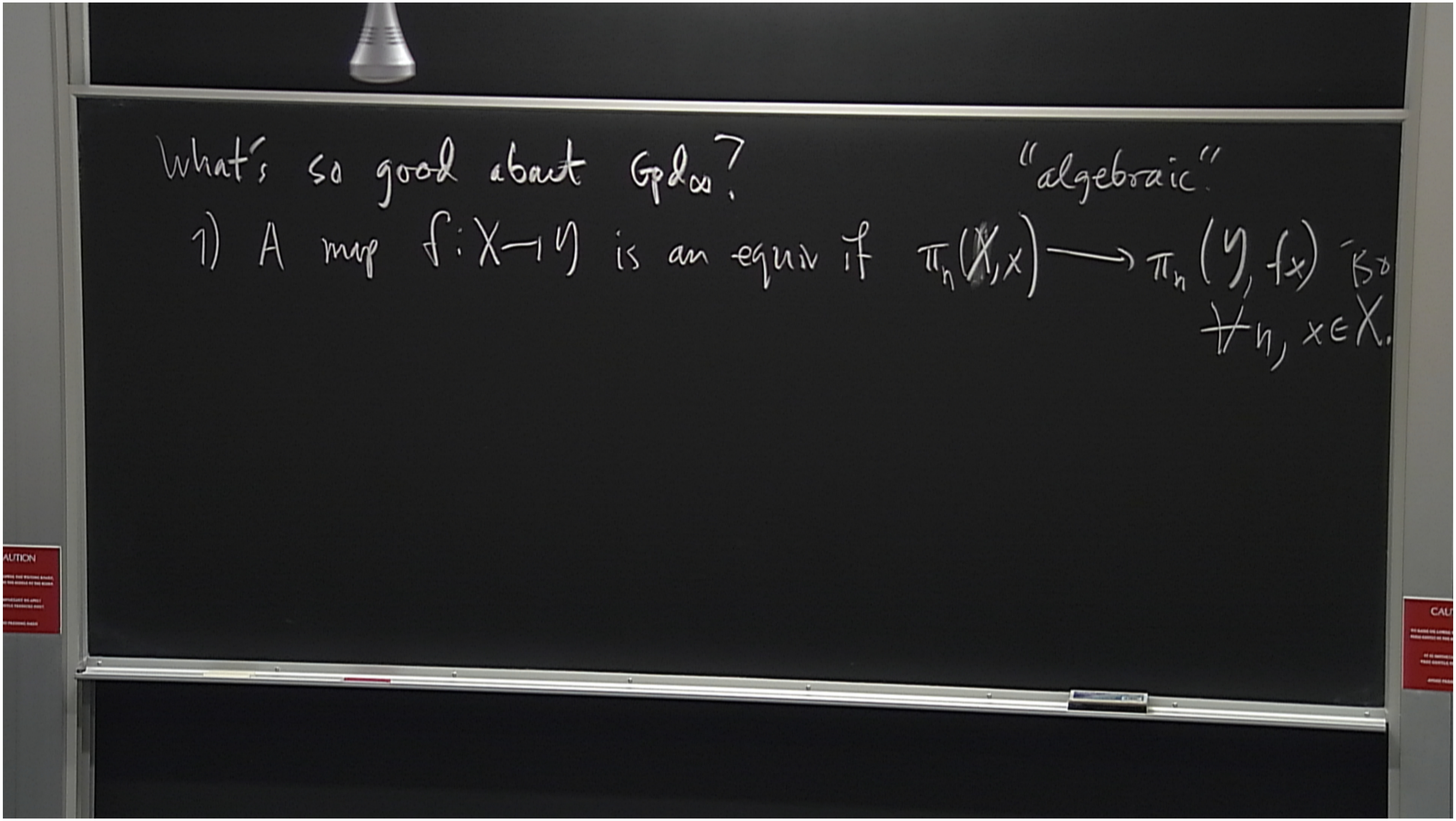
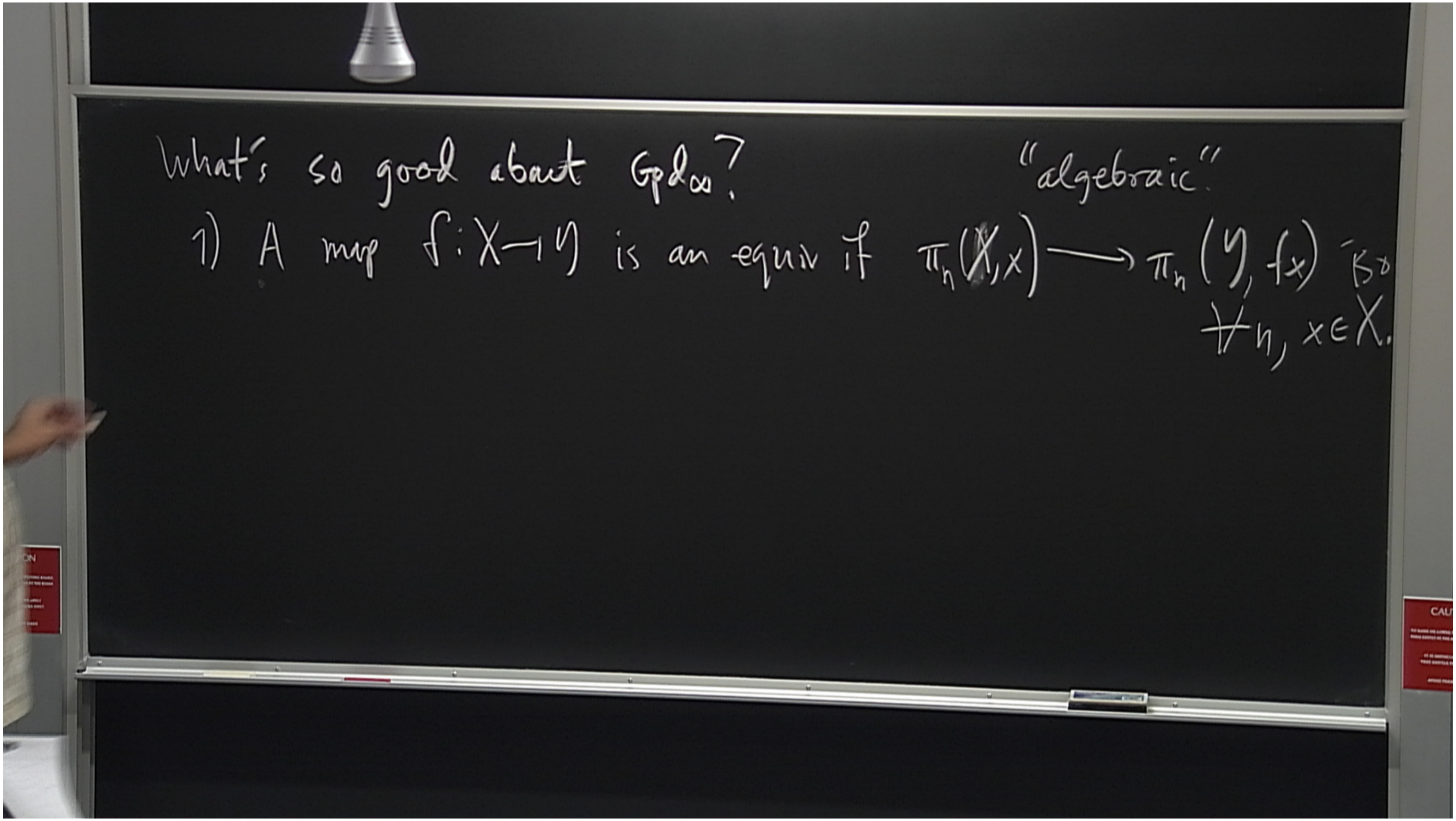
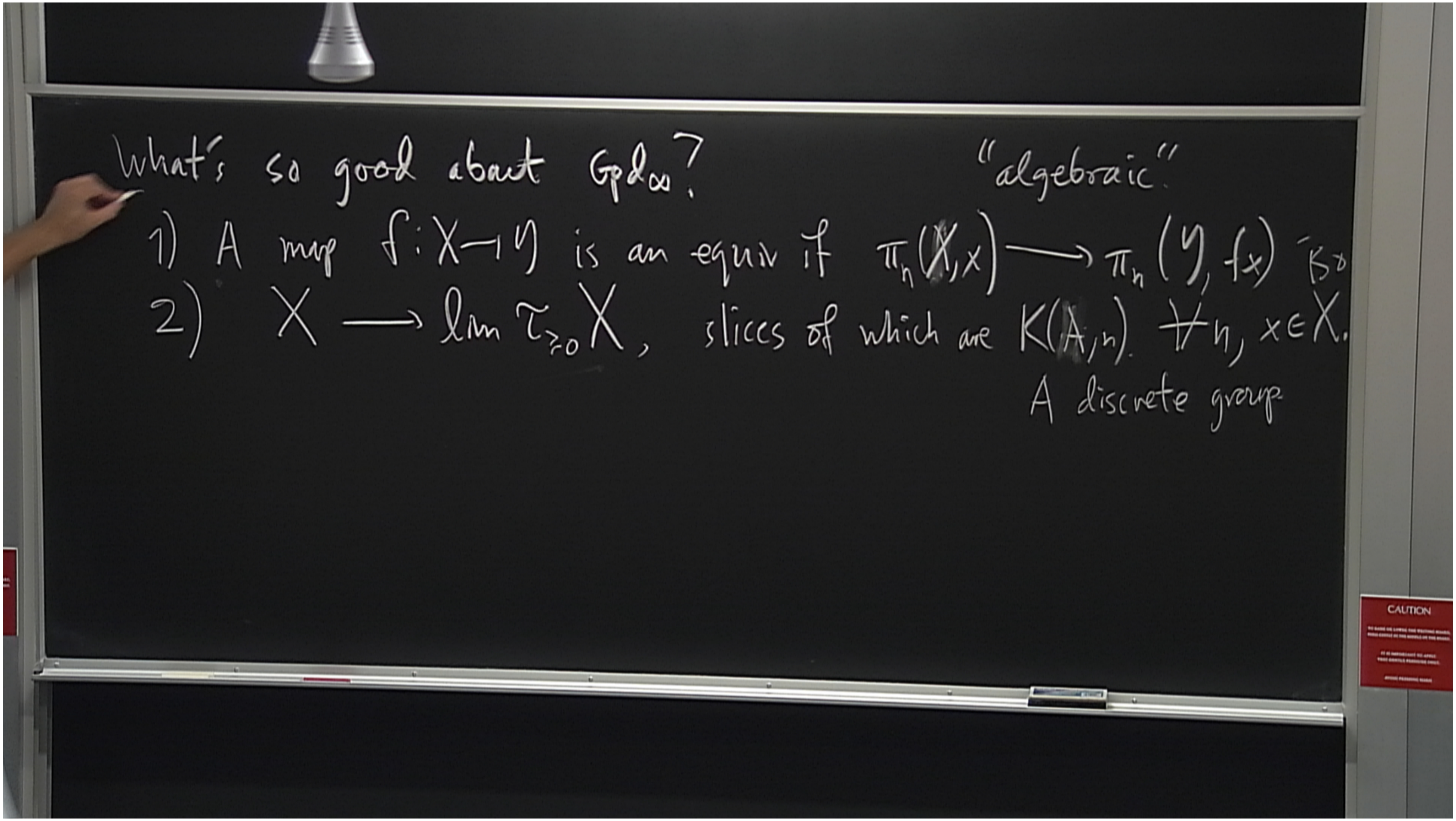
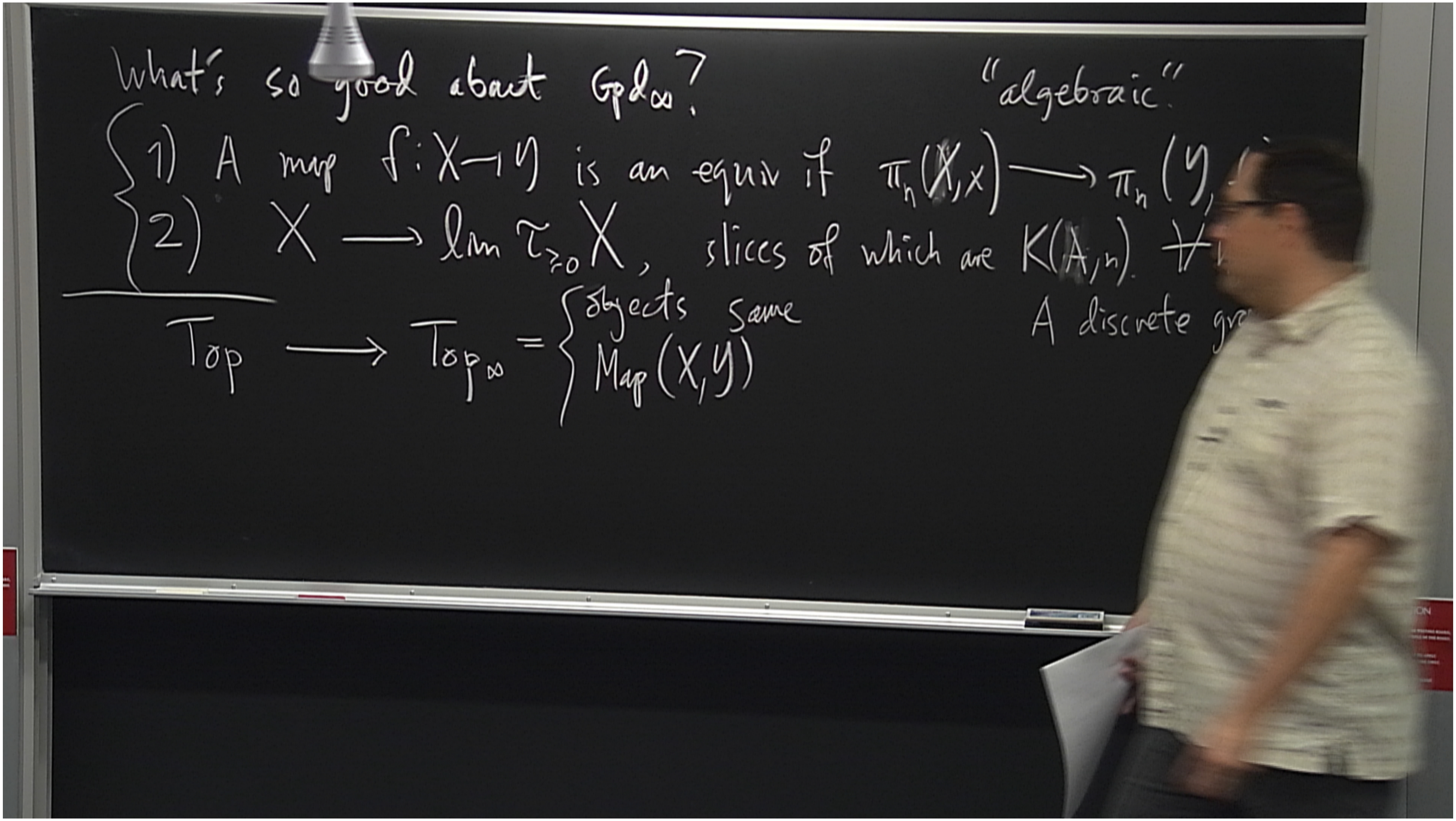

What's so good about Gpd∞?
"algebraic"
1) A map f: X → Y is an equiv if πn(X, x) → πn(Y, f(x)) iso
∀n, x∈X.


What's so good about Gpd∞?
"algebraic"
1) A map f : X → Y is an equiv if π_n(X, x) ⟶ π_n(Y, f(x)) iso
∀ n, x ∈ X.


What's so good about Gpd∞?
"algebraic"
1) A map f: X → Y is an equiv if π_n(X, x) → π_n(Y, fx) iso
2) X → lim τ_{≥0} X, slices of which are K(A, n) ∀n, x ∈ X.
A discrete group
CAUTION


What's so good about Gpd∞?
"algebraic"
1) A map f : X → Y is an equiv if πn(X,x) → πn(Y,
2) X → lim τ≥0 X, slices of which are K(A,n)
Top → Top∞ = { objects same
Map(X,Y)
A discrete

What's so good about Gpd_∞? "algebraic"

1) A map $f: X \to Y$ is an equiv if $\pi_n(X,x) \longrightarrow \pi_n(Y, fx)$ iso

2) $X \longrightarrow \lim \tau_{\geq 0} X$, slices of which are $K(A,n)$ $\forall n, x \in X$.

A discrete group

$$\mathrm{Top} \longrightarrow \mathrm{Top}_\infty = \begin{cases} \text{objects same} \\ \mathrm{Map}(X,Y) = \left| n \longmapsto \mathrm{Hom}(\Delta^n \times X, Y) \right| \end{cases}$$

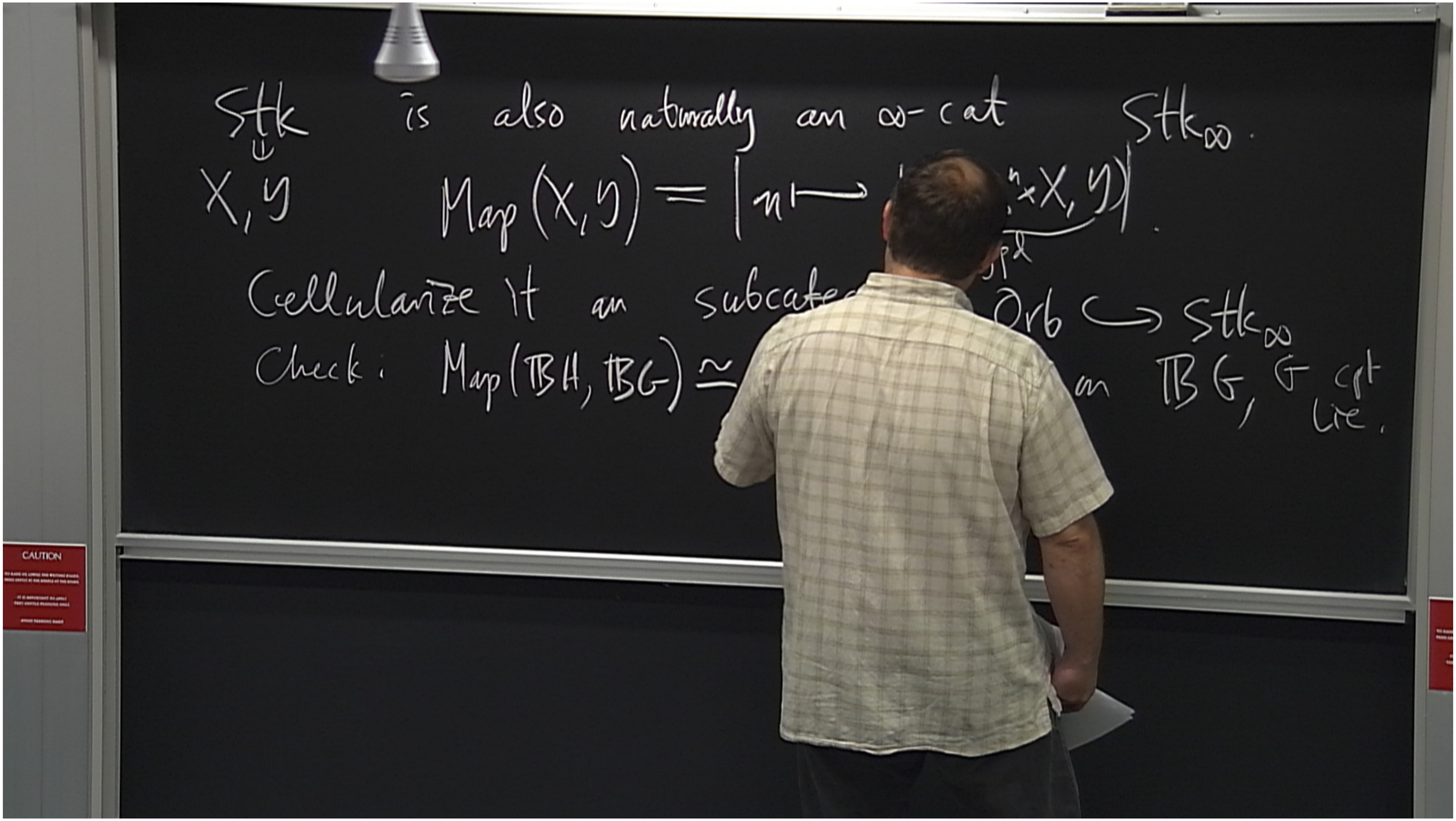
Stk is also naturally an ∞-cat Stk_∞.

$\Downarrow$

X, Y $\quad$ $\mathrm{Map}(X,Y) = \Big| n \longmapsto \underbrace{\mathrm{Hom}(\Delta^n \times X, Y)}_{\mathrm{gpd}} \Big|$.

Cellularize it an subcategory $\mathrm{Orb} \hookrightarrow \mathrm{Stk}_\infty$

full on $\mathbb{B}G$,


Stk is also naturally an ∞-cat Stk∞.
X, Y
Map(X, Y) = |n ↦
Cellularize it an subcate
Orb ↪ Stk∞
Check: Map(BH, BG) ≃
on BG, G cpt Lie.
CAUTION

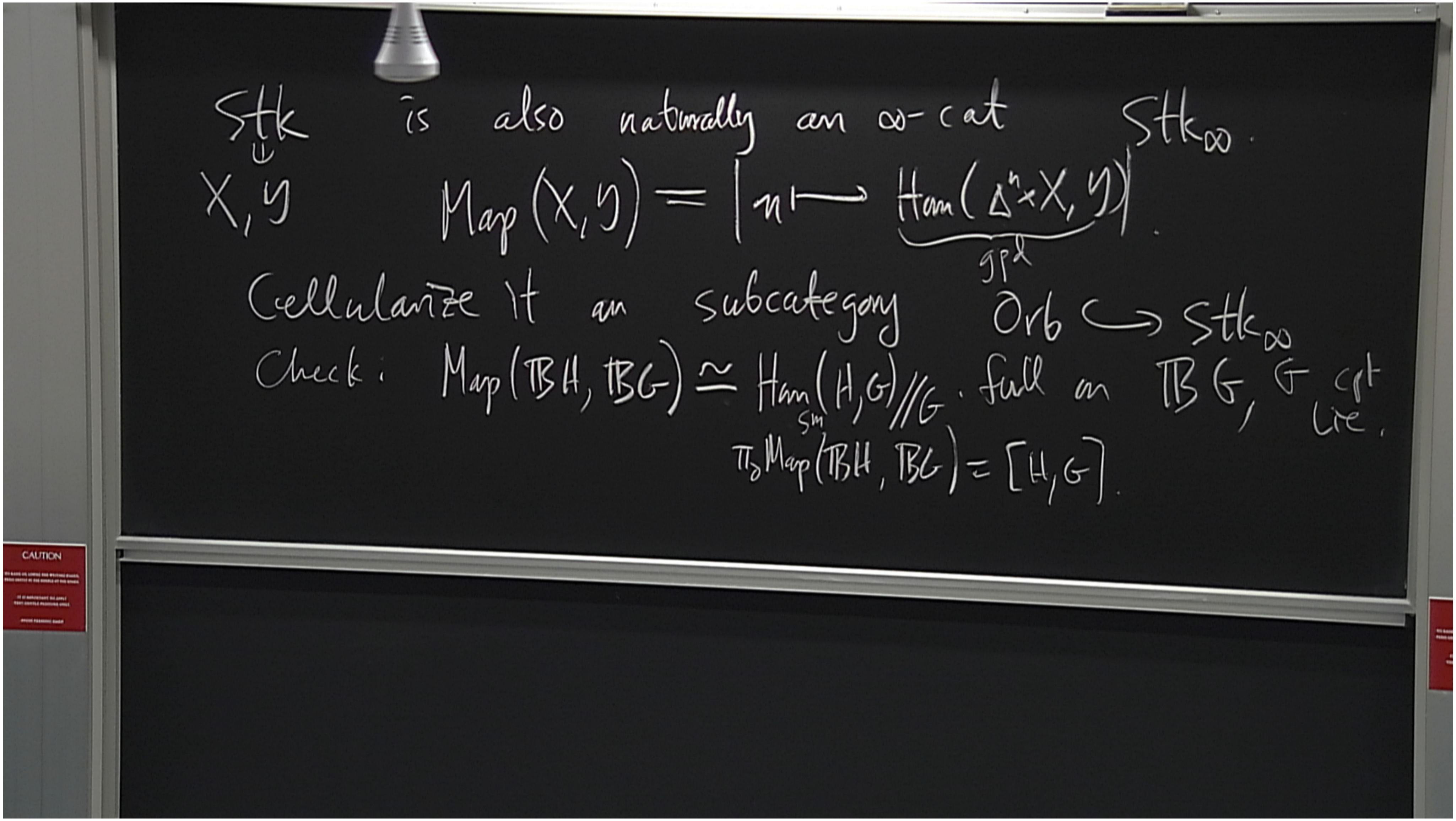
Stk is also naturally an ∞-cat Stk_∞.

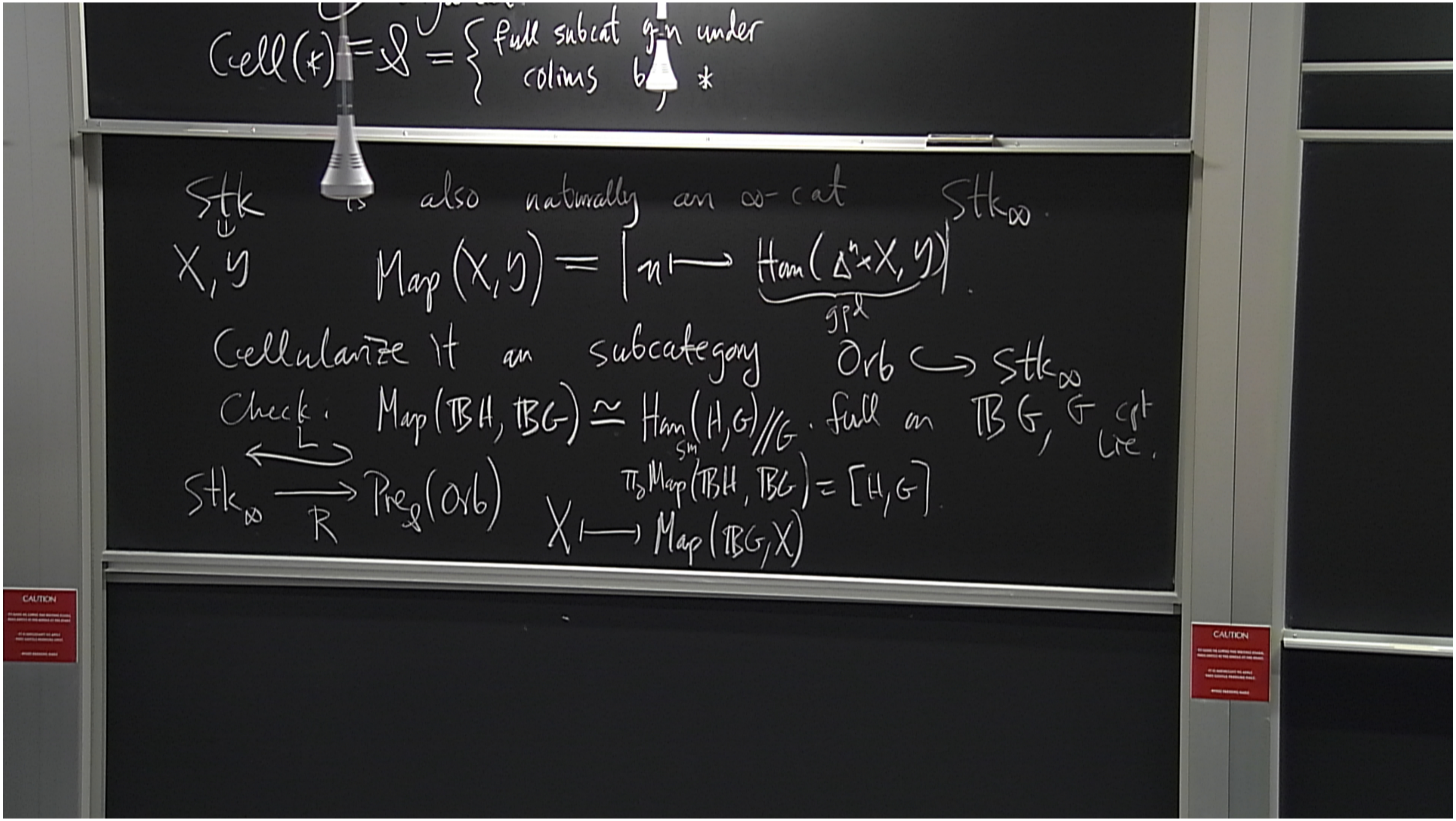
$X, Y \in \mathrm{Stk}$ $\quad \mathrm{Map}(X,Y) = \left| n \mapsto \underbrace{\mathrm{Hom}(\Delta^n \times X, Y)}_{\text{gpd}} \right|$.

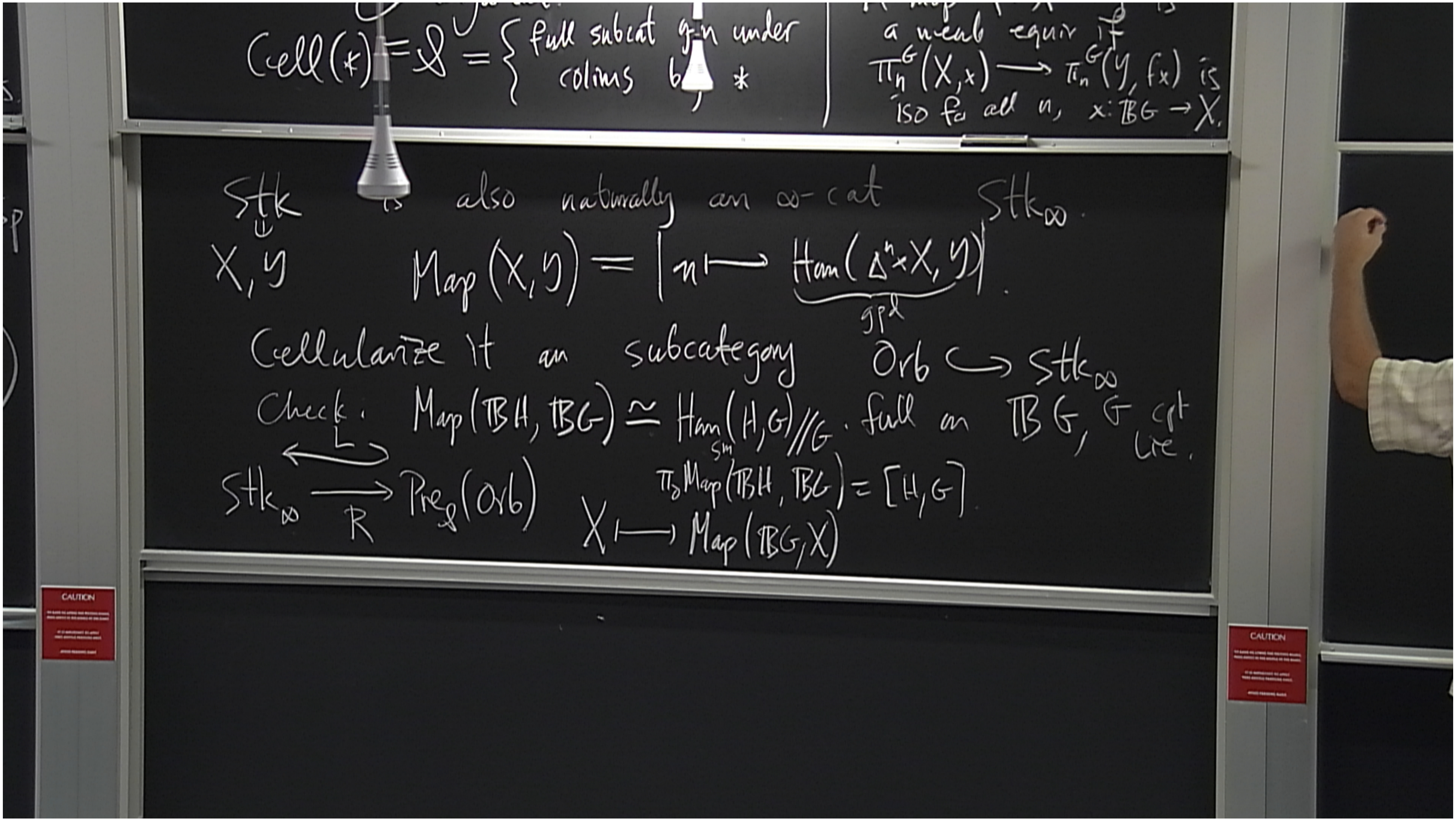
Cellularize it an subcategory $\mathrm{Orb} \hookrightarrow \mathrm{Stk}_\infty$

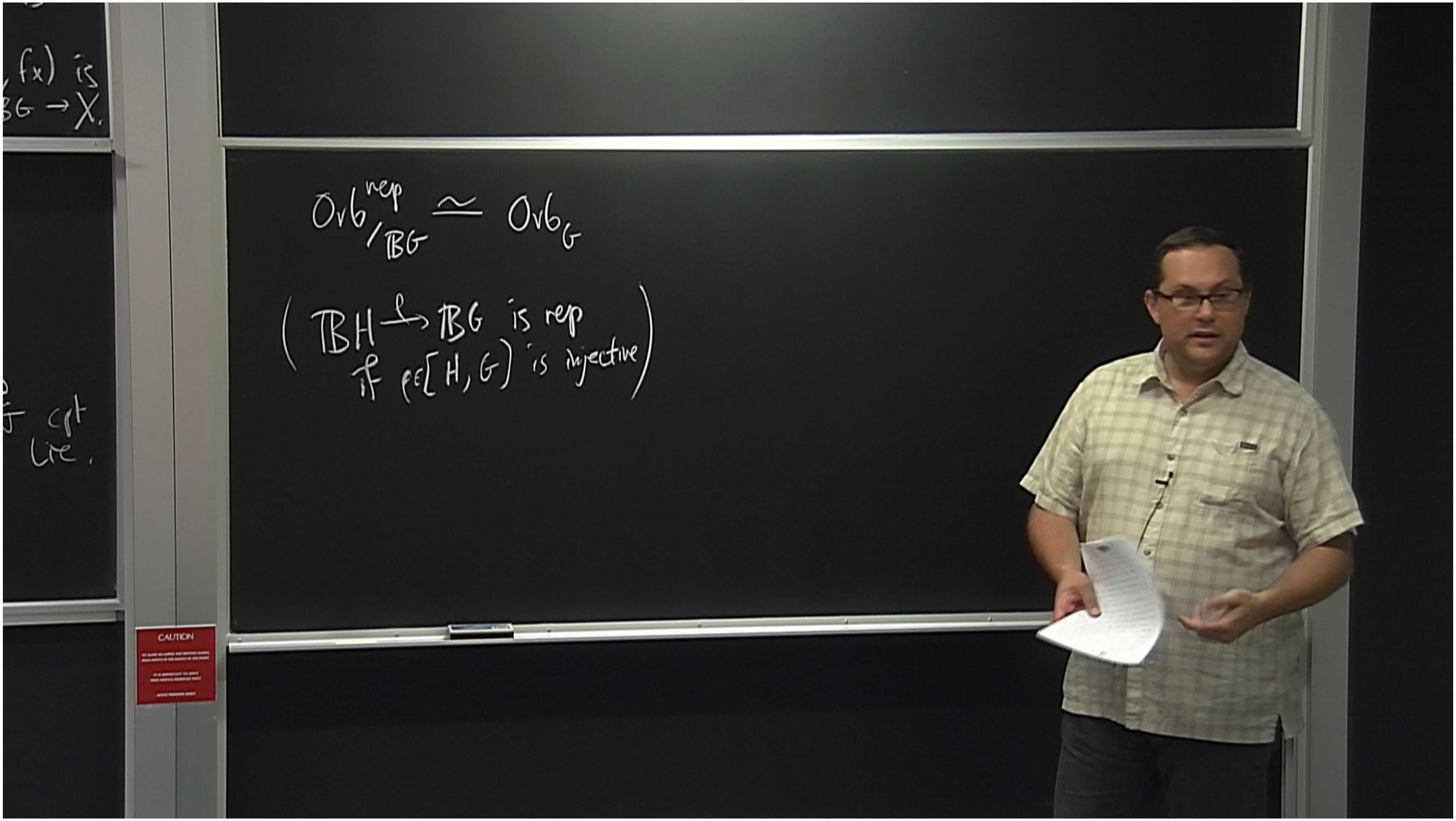
Check: $\mathrm{Map}(\mathbb{B}H, \mathbb{B}G) \simeq \mathrm{Hom}_{sm}(H,G)/\!/G$. full on $\mathbb{B}G$, G cpt Lie.

$\pi_0 \mathrm{Map}(\mathbb{B}H, \mathbb{B}G) = [H, G]$.


Stk is also naturally an ∞-cat Stk∞.
X, Y
Map(X, Y) = |n ↦ Hom(Δⁿ × X, Y)|
gpd
Cellularize it an subcategory Orb ↪ Stk∞
Check: Map(BH, BG) ≃ Hom(H, G)//G . full on BG, G cpt Lie.
sm
π0 Map(BH, BG) = [H, G].
CAUTION


$\mathrm{Cell}(*) = \mathcal{S} = \{$ full subcat g-n under colims by $*$
Stk is also naturally an ∞-cat Stk_∞.
X, Y
$\mathrm{Map}(X,Y) = |n \mapsto \underbrace{\mathrm{Hom}(\Delta^n \times X, Y)}_{\mathrm{gpd}}|$.
Cellularize it an subcategory $\mathrm{Orb} \hookrightarrow \mathrm{Stk}_\infty$
Check: $\mathrm{Map}(\mathbb{B}H, \mathbb{B}G) \simeq \mathrm{Hom}_{\mathrm{sm}}(H,G)/\!/G$. full on $\mathbb{B}G$, G cpt Lie.
$\pi_0 \mathrm{Map}(\mathbb{B}H, \mathbb{B}G) = [H,G]$.
$\mathrm{Stk}_\infty \underset{R}{\overset{L}{\rightleftarrows}} \mathrm{Pre}_{\mathcal{S}}(\mathrm{Orb})$
$X \longmapsto \mathrm{Map}(\mathbb{B}G, X)$
CAUTION
CAUTION


Cell(*) = S = { full subcat gen under colims by *
a weak equiv if
π_n^G(X,x) ⟶ π_n^G(Y, fx) is
iso for all n, x: BG → X.
Stk is also naturally an ∞-cat Stk_∞.
X, Y   Map(X,Y) = | n ↦ Hom(Δ^n × X, Y) |
gpd
Cellularize it an subcategory Orb ↪ Stk_∞
Check: Map(BH, BG) ≃ Hom_sm(H,G)//G. full on BG, G cpt Lie.
L
Stk_∞ ⟶ Pre_S(Orb)
R
π_0 Map(BH, BG) = [H,G].
X ↦ Map(BG, X)


Orb^rep /BG ≃ Orb_G
(BH —f→ BG is rep
if f∈[H, G] is injective)
fx) is
BG → X.
cpt
Lie.

, fx) is
BG → X.

cpt
Lie.

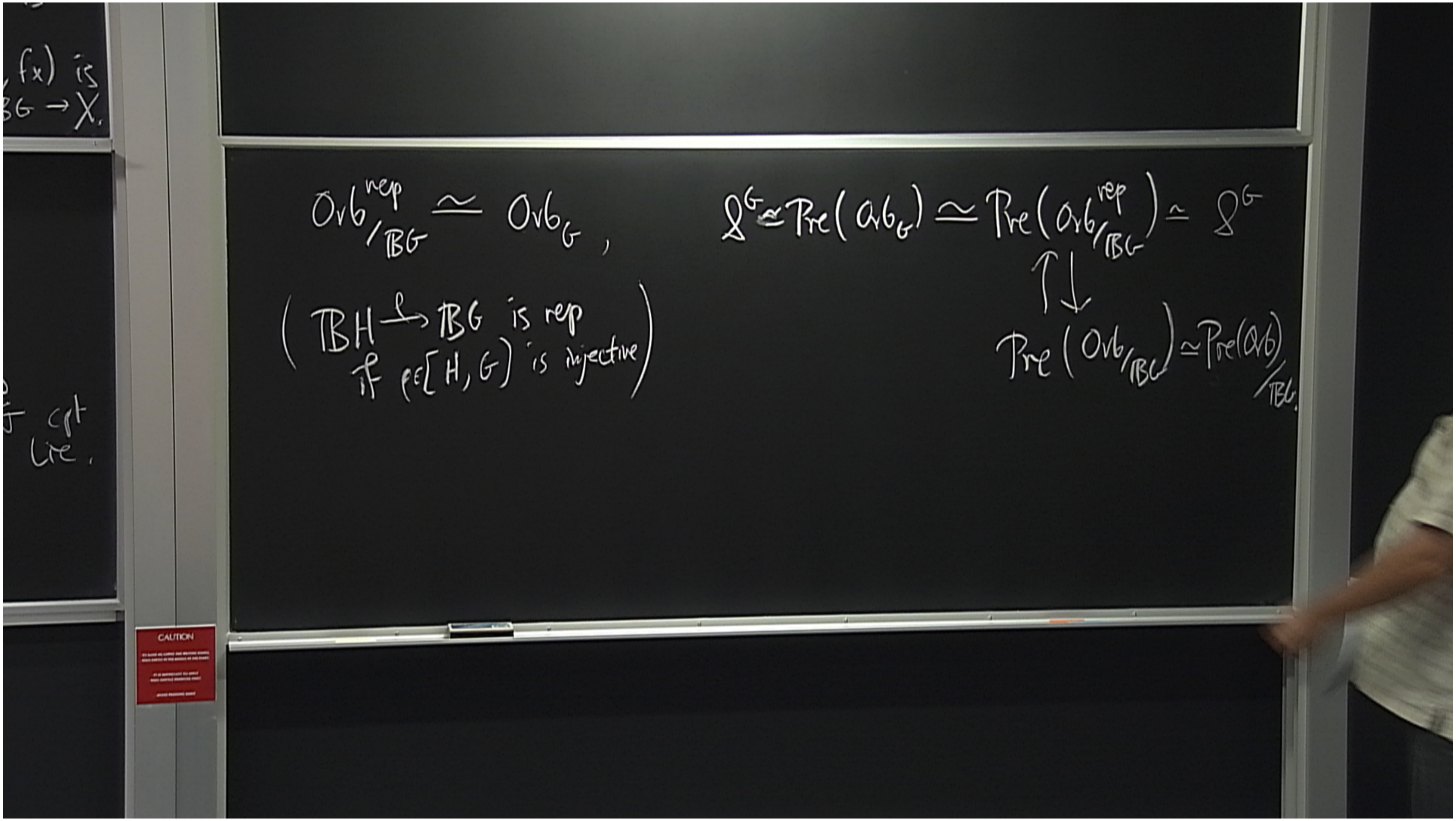
$$\mathrm{Orb}^{rep}_{/\mathbb{B}G} \simeq \mathrm{Orb}_G ,$$

($\mathbb{B}H \xrightarrow{f} \mathbb{B}G$ is rep if $f \in [H, G]$ is injective)

$$\mathcal{S}^G \simeq \mathrm{Pre}(\mathrm{Orb}_G) \simeq \mathrm{Pre}(\mathrm{Orb}^{rep}_{/\mathbb{B}G})$$

$$\uparrow\downarrow$$

$$\mathrm{Pre}(\mathrm{Orb}_{/\mathbb{B}G}) \simeq \mathrm{Pre}(\mathrm{Orb})_{/\mathbb{B}G} .$$



$\mathrm{Orb}^{rep}_{/\mathbb{B}G} \simeq \mathrm{Orb}_G$,
($\mathbb{B}H \xrightarrow{f} \mathbb{B}G$ is rep if $f \in [H, G]$ is injective)
$\mathcal{S}^G \simeq \mathrm{Pre}(\mathrm{Orb}_G) \simeq \mathrm{Pre}(\mathrm{Orb}^{rep}_{/\mathbb{B}G}) \simeq \mathcal{S}^G$
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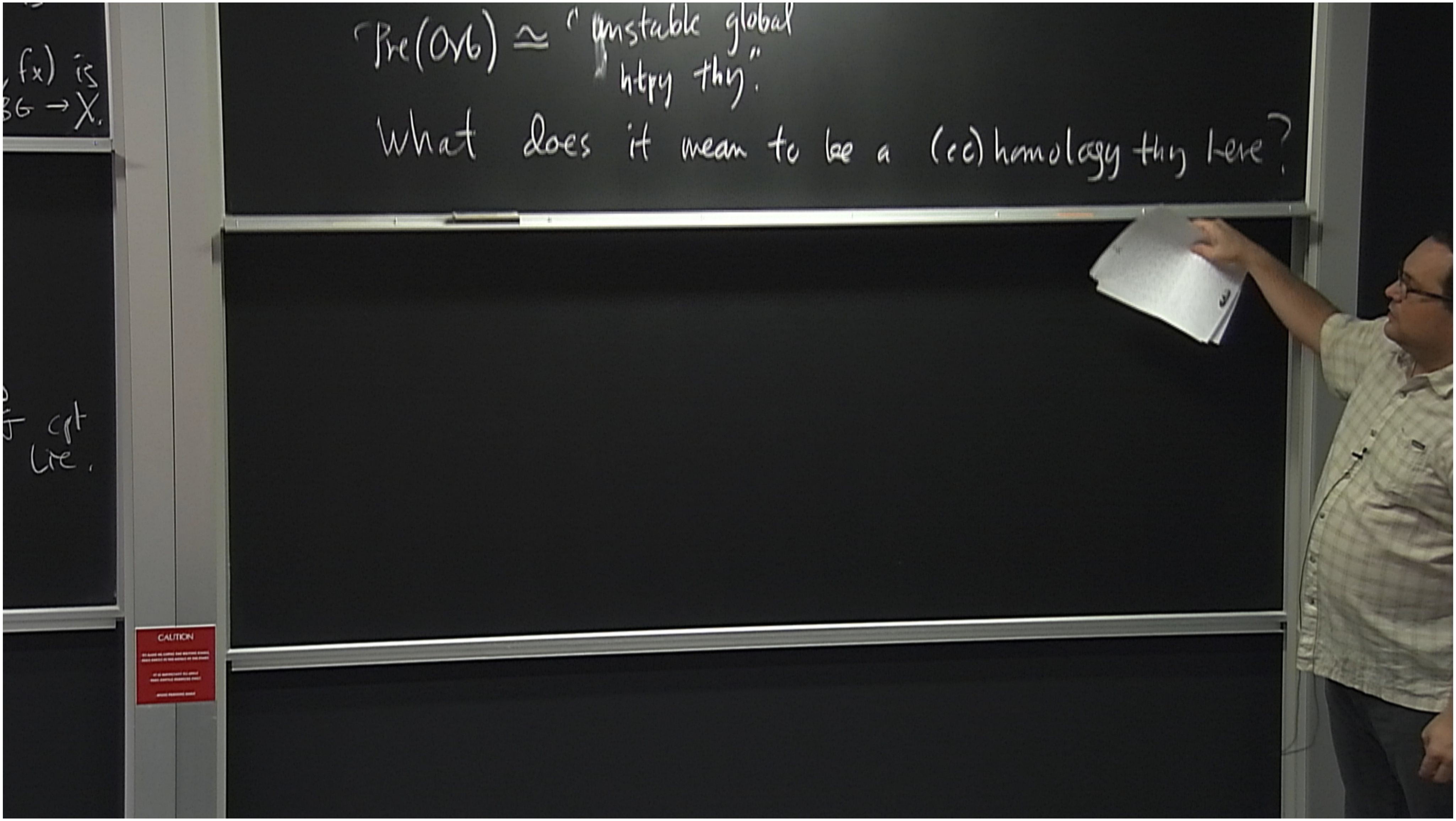
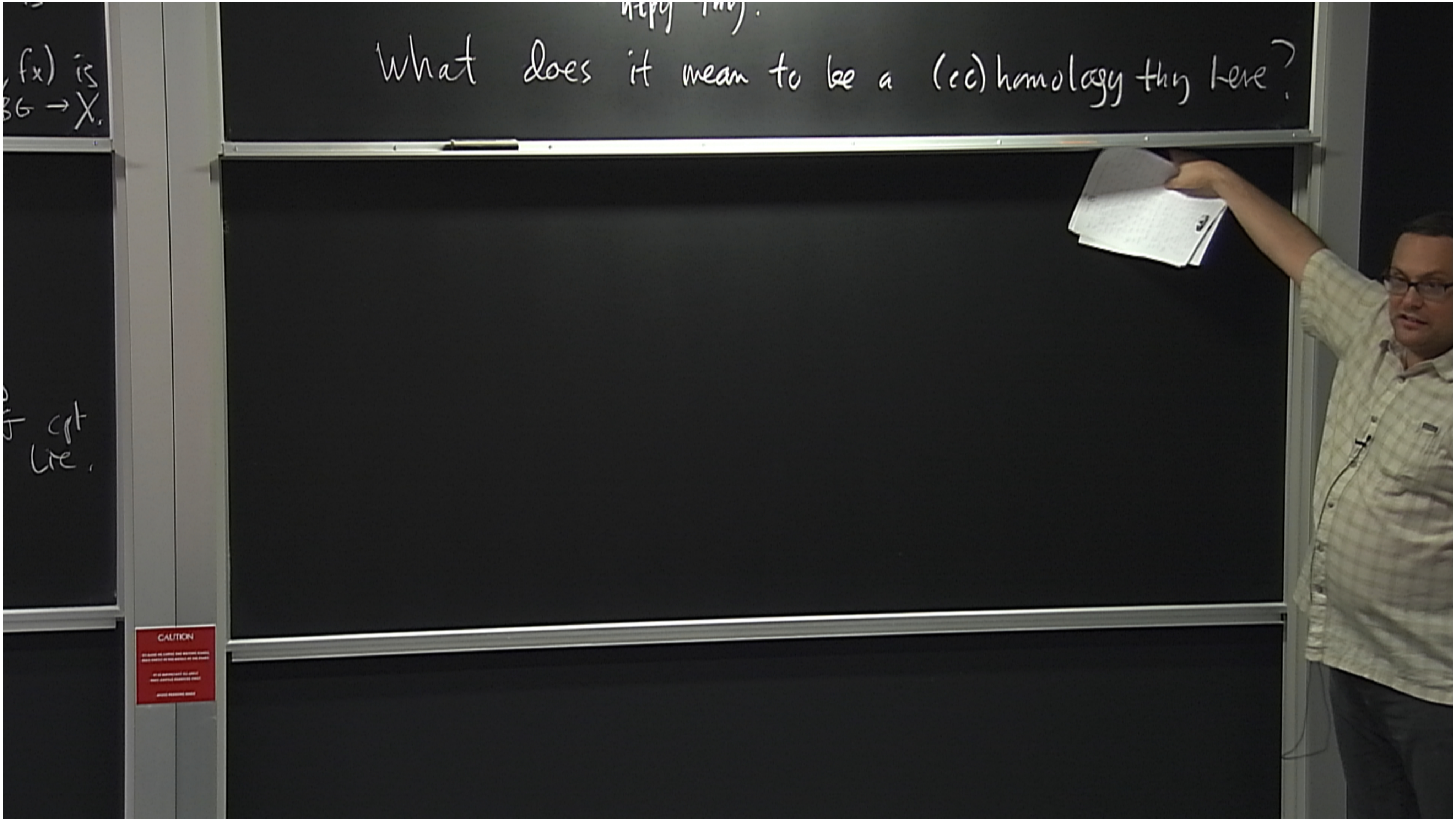
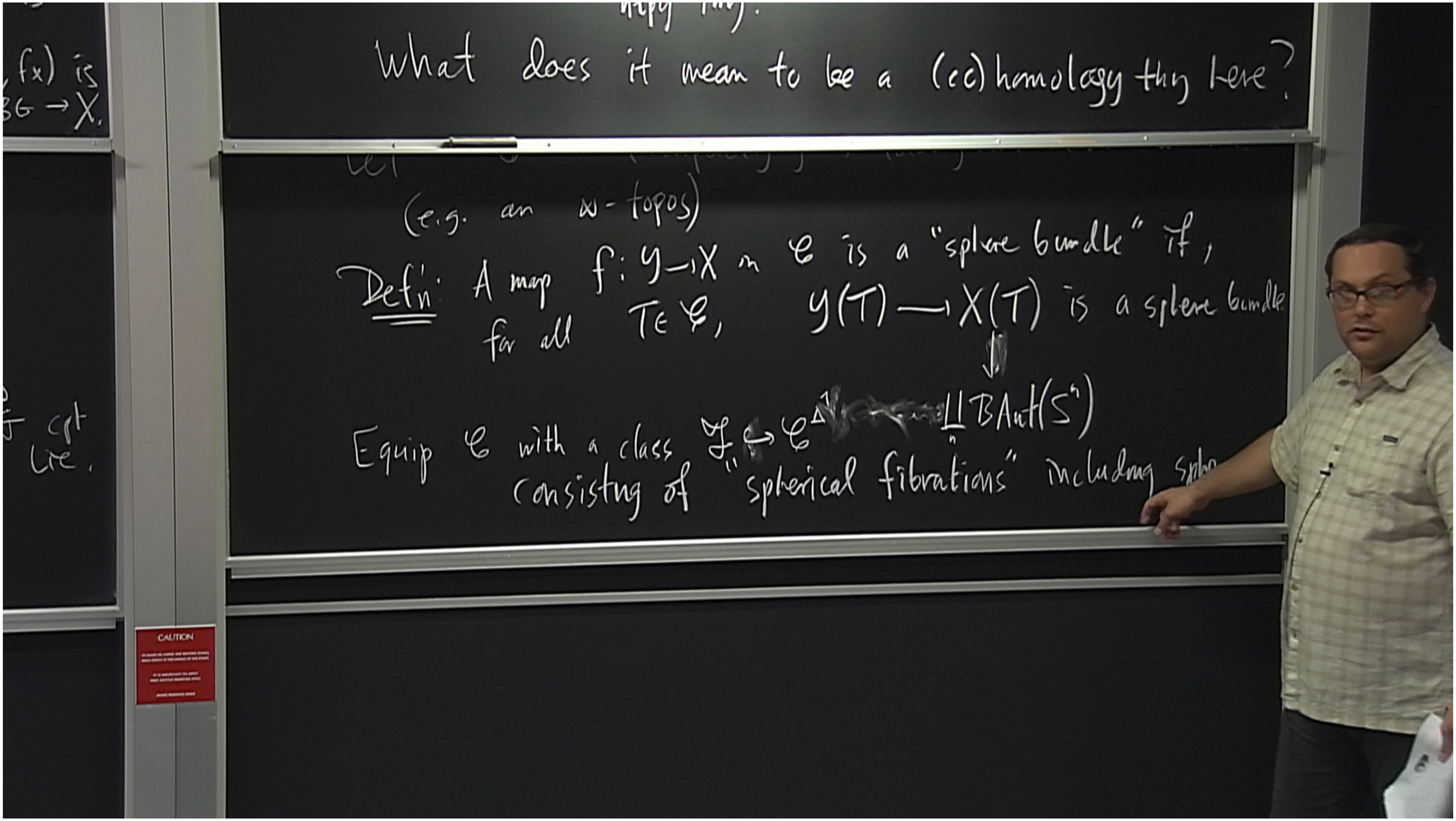
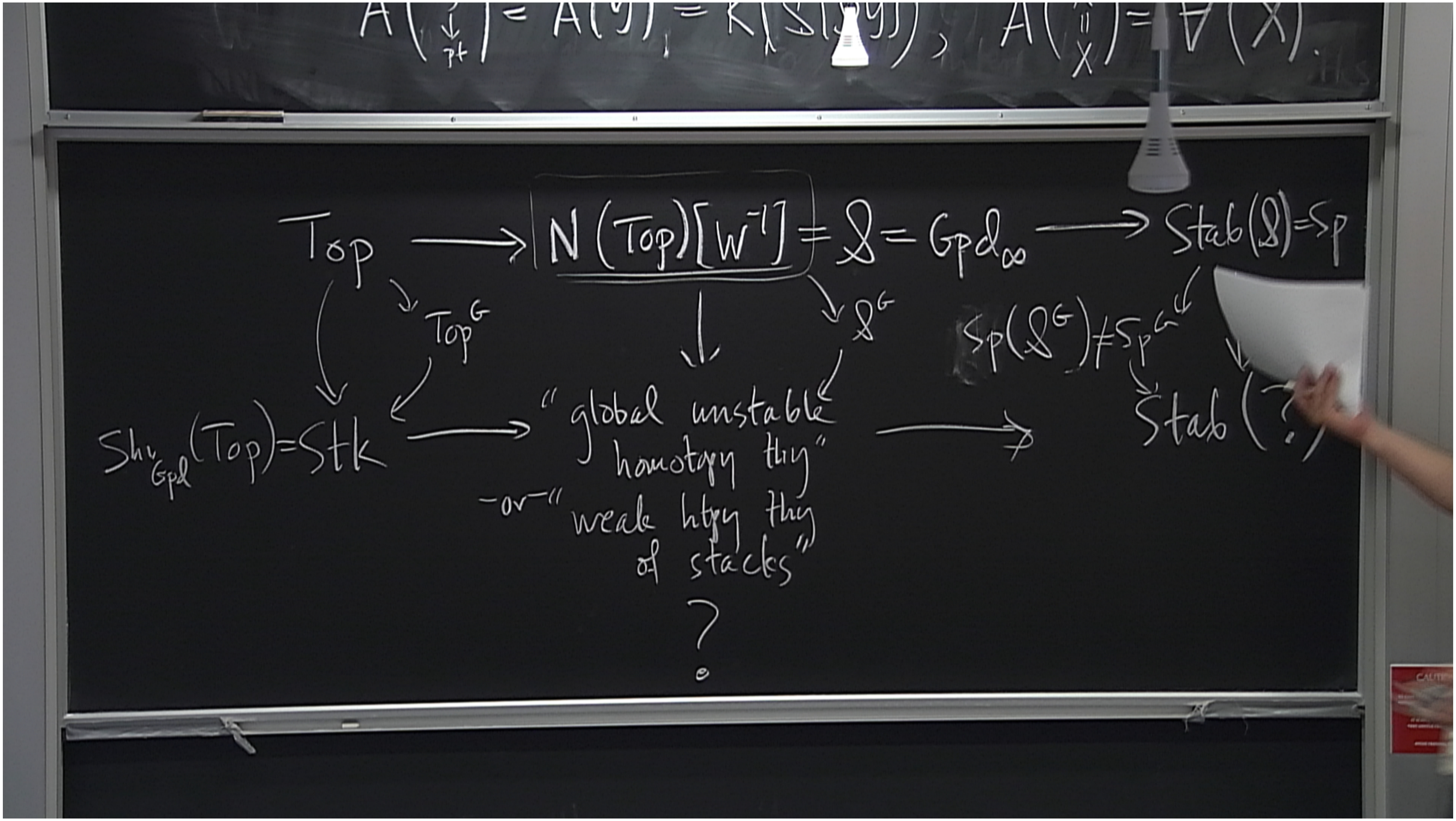
CAUTION


Pre(Orb) ≃ "unstable global
htpy thy".
What does it mean to be a (co)homology thy here?


What does it mean to be a (co)homology thy here?
CAUTION


What does it mean to be a (co)homology thy here?
(e.g. an ∞-topos)
Def'n: A map f: Y → X in 𝒞 is a "sphere bundle" if,
for all T ∈ 𝒞, Y(T) → X(T) is a sphere bundle
∐ BAut(S^n)
Equip 𝒞 with a class ℱ ↪ 𝒞^Δ¹
consisting of "spherical fibrations" including sph
f x) is
BG → X.
cpt
Lie.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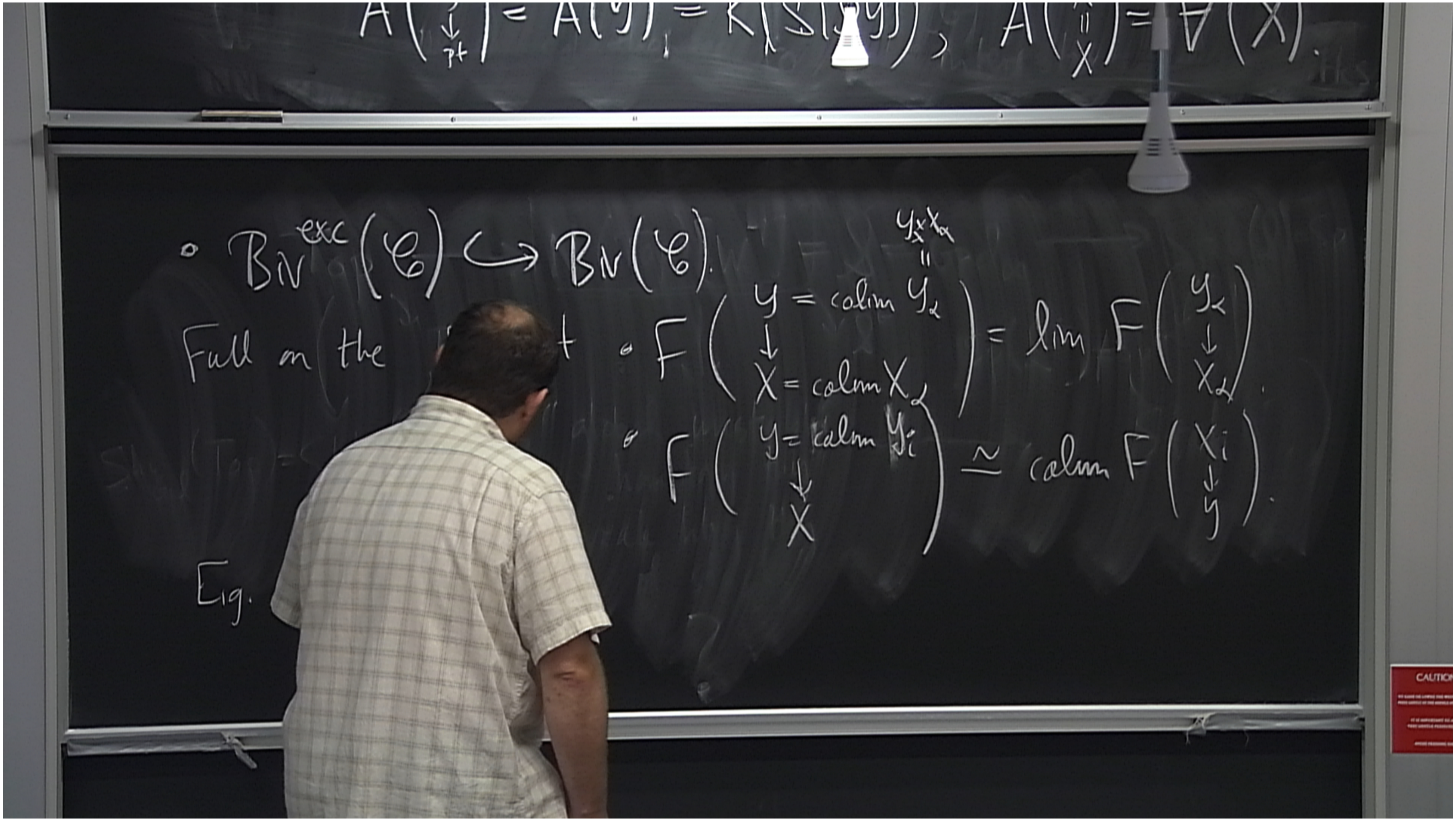
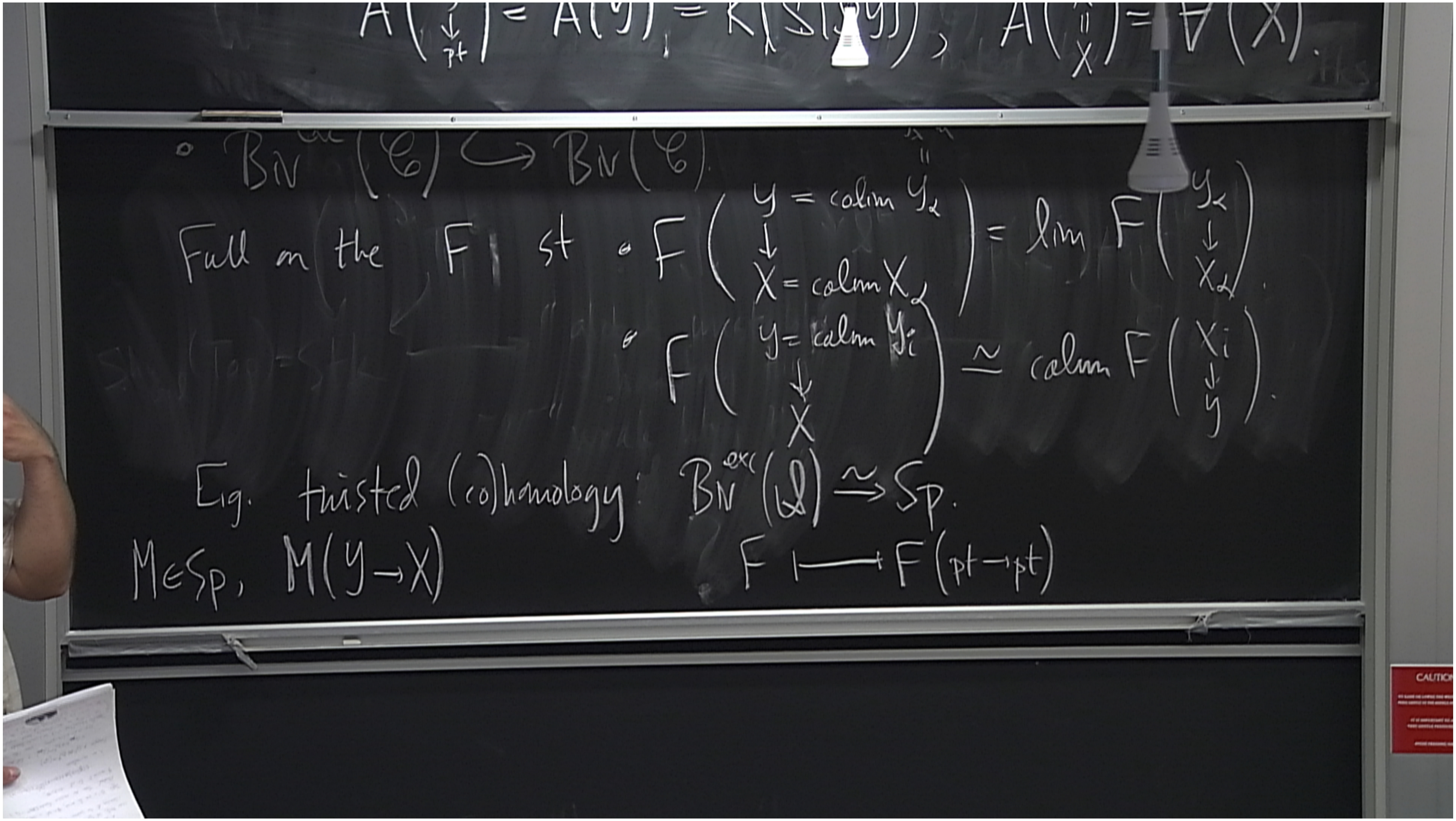
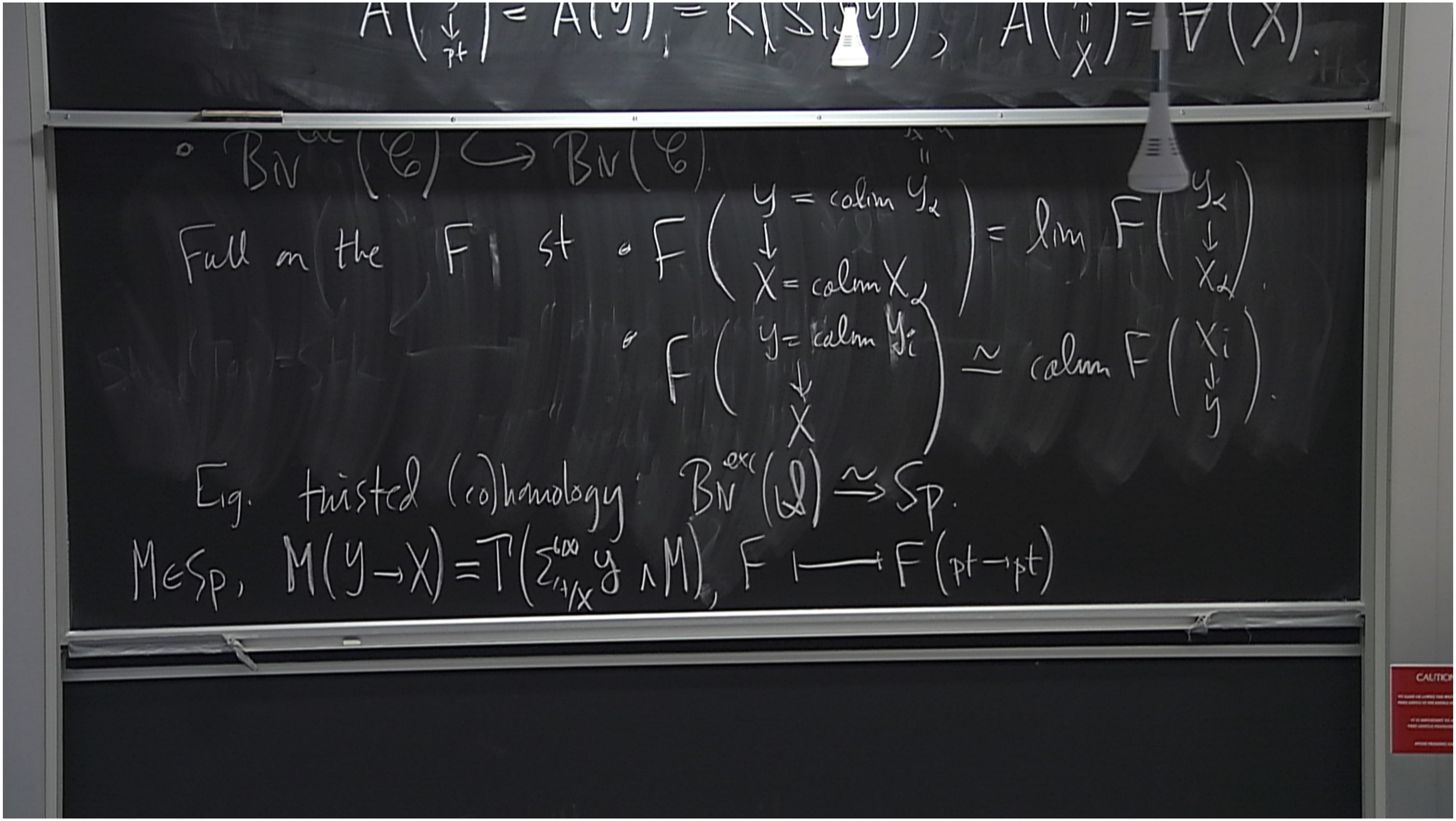
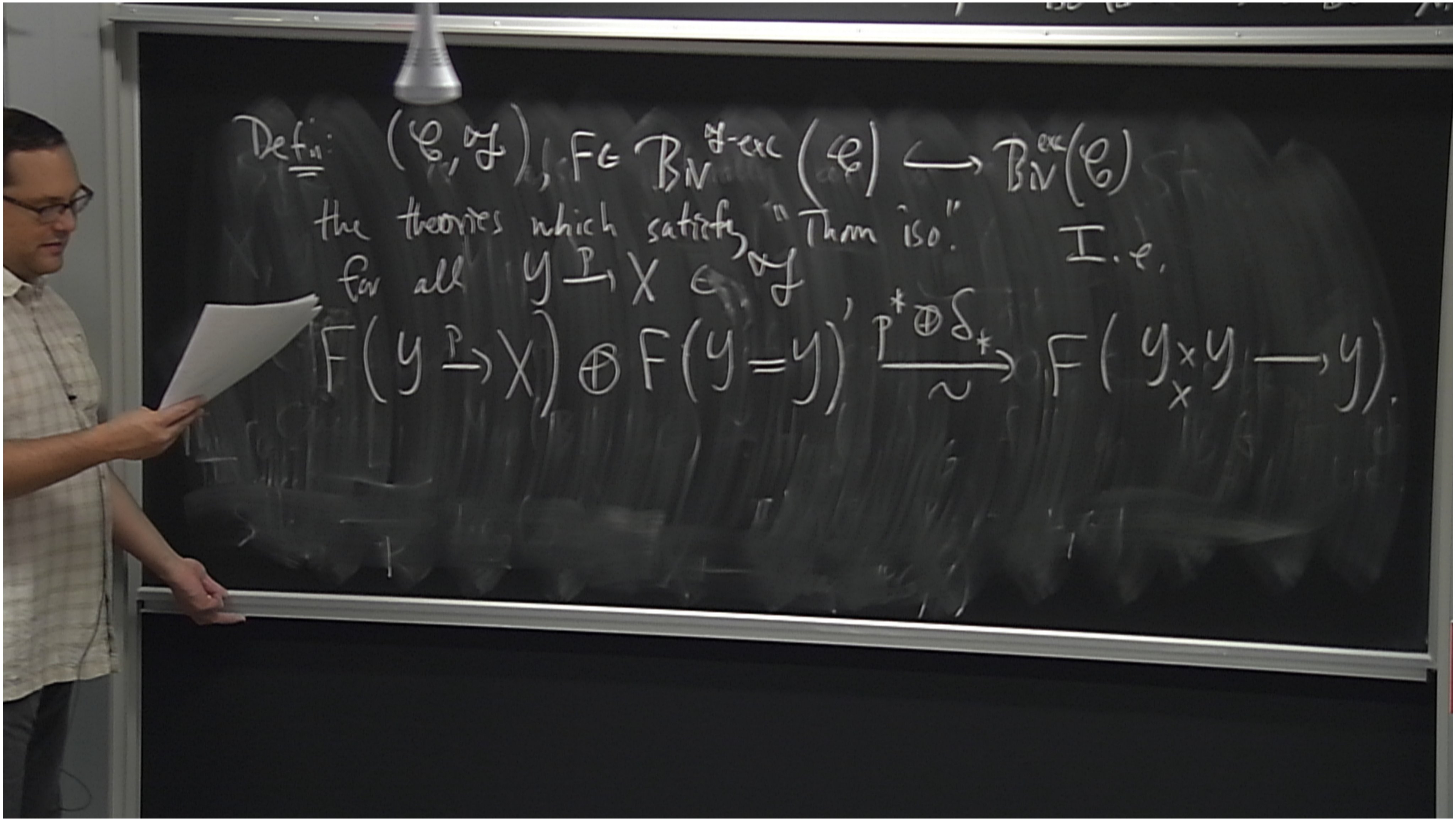
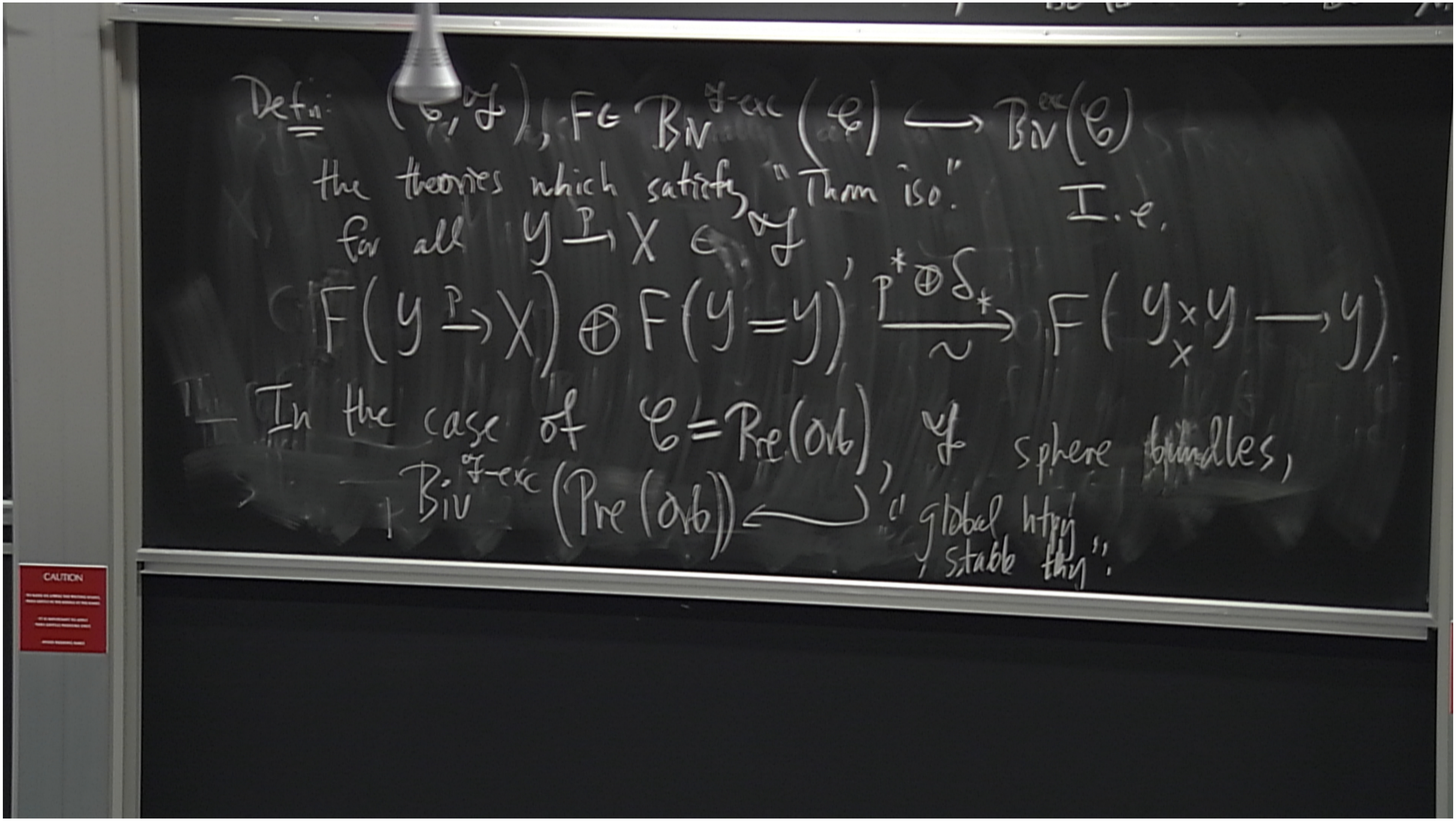

Bv^exc(𝒞) ↪ Bv(𝒞).
Full on the
F(Y = colim Y_α → X = colim X_α) = lim F(Y_α → X_α).
F(Y = colim Y_i → X) ≃ colim F(X_i → Y).
E.g.


A(Y ↓ pt) = A(Y) = K(...), A(X = X) = A(X).
∘ Bv^exc(C) ↪ Bv(C).
Full on the F st ∘ F(Y = colim Y_α ↓ X = colim X_α) = lim F(Y_α ↓ X_α).
∘ F(Y = colim Y_i ↓ X) ≃ colim F(X_i ↓ Y).
E.g. twisted (co)homology: Bv^exc(Δ) ⥲ Sp.
M ∈ Sp, M(Y → X)
F ⟼ F(pt → pt)


A(Y → pt) = A(Y) = K(Σ(Y)) , A(X = X) = A(X).
∘ Bv^exc(𝒞) ↪ Bv(𝒞).
Full on the F st ∘ F(Y = colim Y_α → X = colim X_α) = lim F(Y_α → X_α)
∘ F(Y = colim Y_i → X) ≃ colim F(X_i → Y).
E.g. twisted (co)homology: Bv^exc(𝒰) ≃ Sp.
M ∈ Sp, M(Y → X) = Γ(Σ^∞_{+/X} Y ∧ M), F ⟼ F(pt → pt)


Defn: (𝒞, 𝒥), F ∈ BV^{𝒥-exc}(𝒞) ⟶ BV^{exc}(𝒞)
the theories which satisfy "Thom iso". I.e.
for all Y →p X ∈ 𝒥,
F(Y →p X) ⊕ F(Y = Y) —p* ⊕ δ* ~→ F(Y ×X Y ⟶ Y).


Def'n: (𝒞, 𝒴), F ∈ Biv^{𝒴-exc}(𝒞) ⟶ Biv^{exc}(𝒞)
the theories which satisfy "Thom iso". I.e.
for all Y →p X ∈ 𝒴,
F(Y →p X) ⊕ F(Y = Y) —p*⊕δ*, ~→ F(Y ×_X Y → Y).
In the case of 𝒞 = Pre(Orb), 𝒴 sphere bundles,
Biv^{𝒴-exc}(Pre(Orb)) ⟵ "global htpy stable thy".
CAUTION

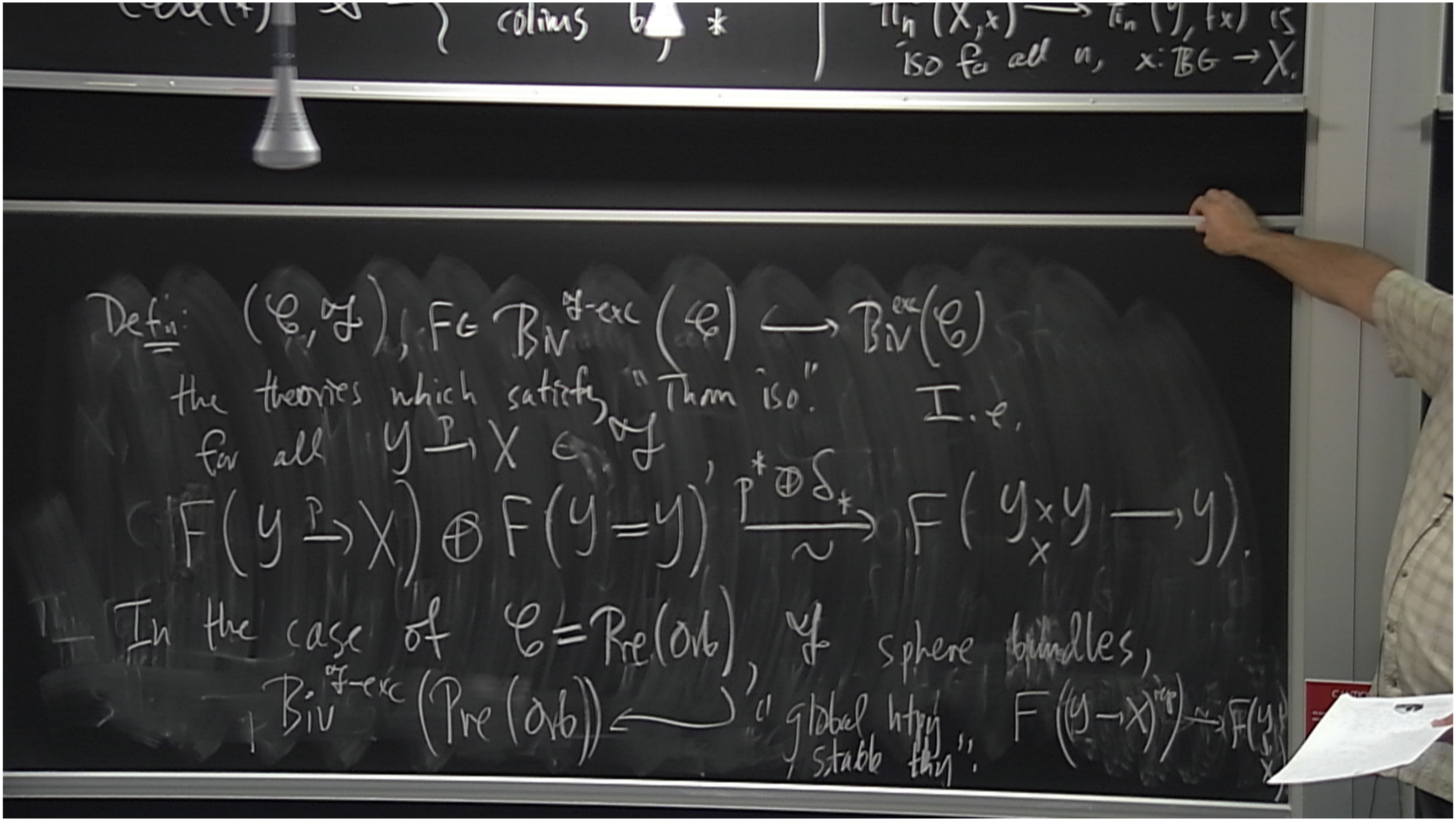
colims b/ $*$ | $\pi_n(X,x) \to \pi_n(Y,fx)$ is iso for all n, $x: \mathbb{B}G \to X$.

for $Y \xrightarrow{p} X \in \mathcal{Y}$,

$$F(Y \xrightarrow{p} X) \oplus F(Y = Y) \xrightarrow[\sim]{p^* \oplus \delta_*} F(Y \times_X Y \longrightarrow Y).$$

Thm In the case of $\mathcal{C} = \mathrm{Pre}(\mathrm{Orb})$, $\mathcal{Y}$ sphere bundles,

$\mathrm{Biv}^{\mathcal{Y}\text{-exc}}(\mathrm{Pre}(\mathrm{Orb})) \hookleftarrow$ "global htpy stable thy": $F(\mathbb{B}G \to pt) \to F(\mathbb{B}G \to pt$

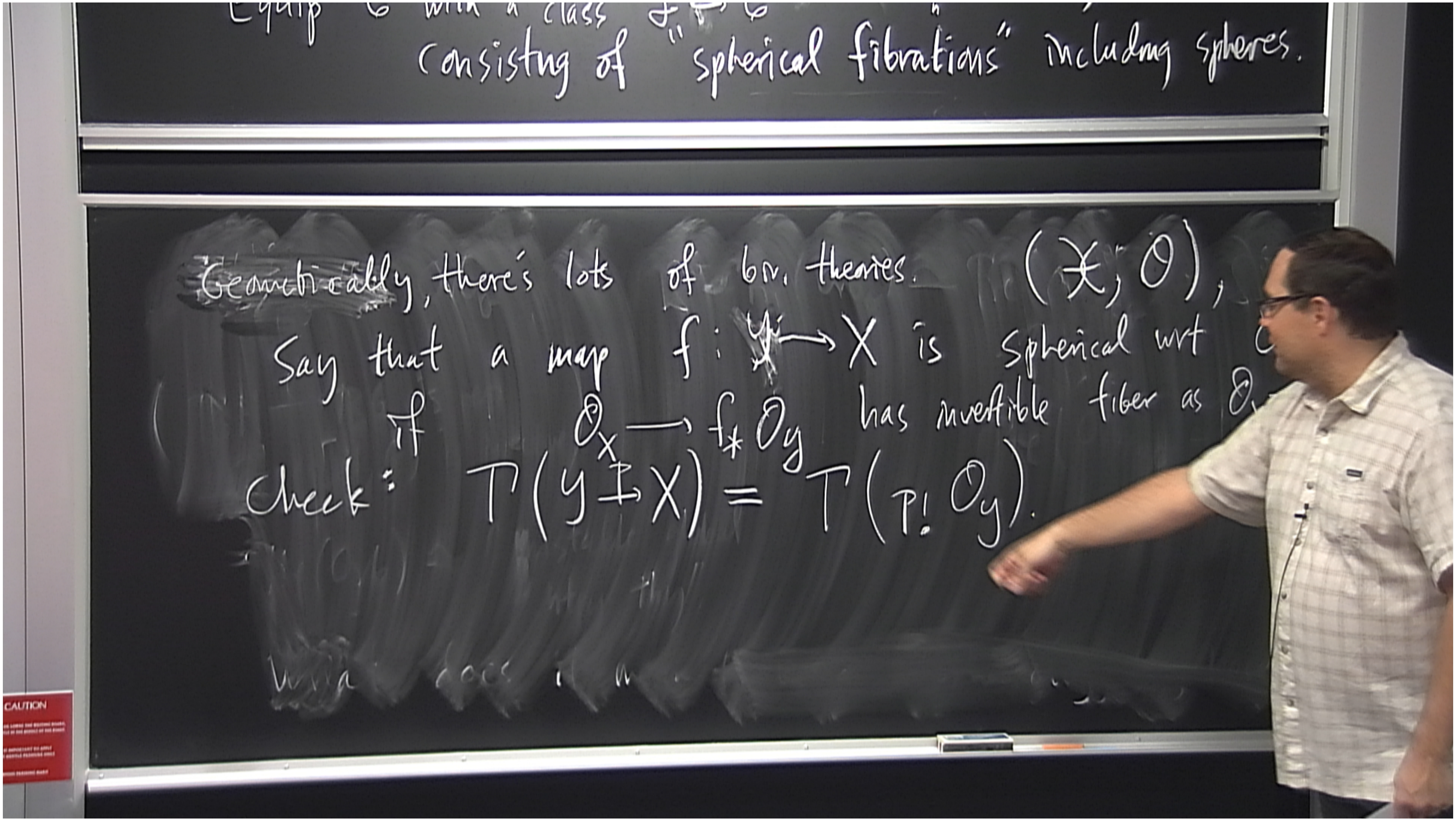

colims b *
π_n(X,x) → π_n(Y,fx) is iso for all n, x: BG → X.
Defn: (C, Y), F ∈ Biv^{Y-exc}(C) ↪ Biv^{exc}(C)
the theories which satisfy "Thom iso" wrt Y. I.e.
for all Y →p X ∈ Y,
F(Y →p X) ⊕ F(Y = Y) → p* ⊕ δ* ~ F(Y ×X Y → Y).
In the case of C = Pre(Orb), Y sphere bundles,
Biv^{Y-exc}(Pre(Orb)) ⟵ "global htpy stable thy": F((Y → X)^rep) ~→ F(Y ×X

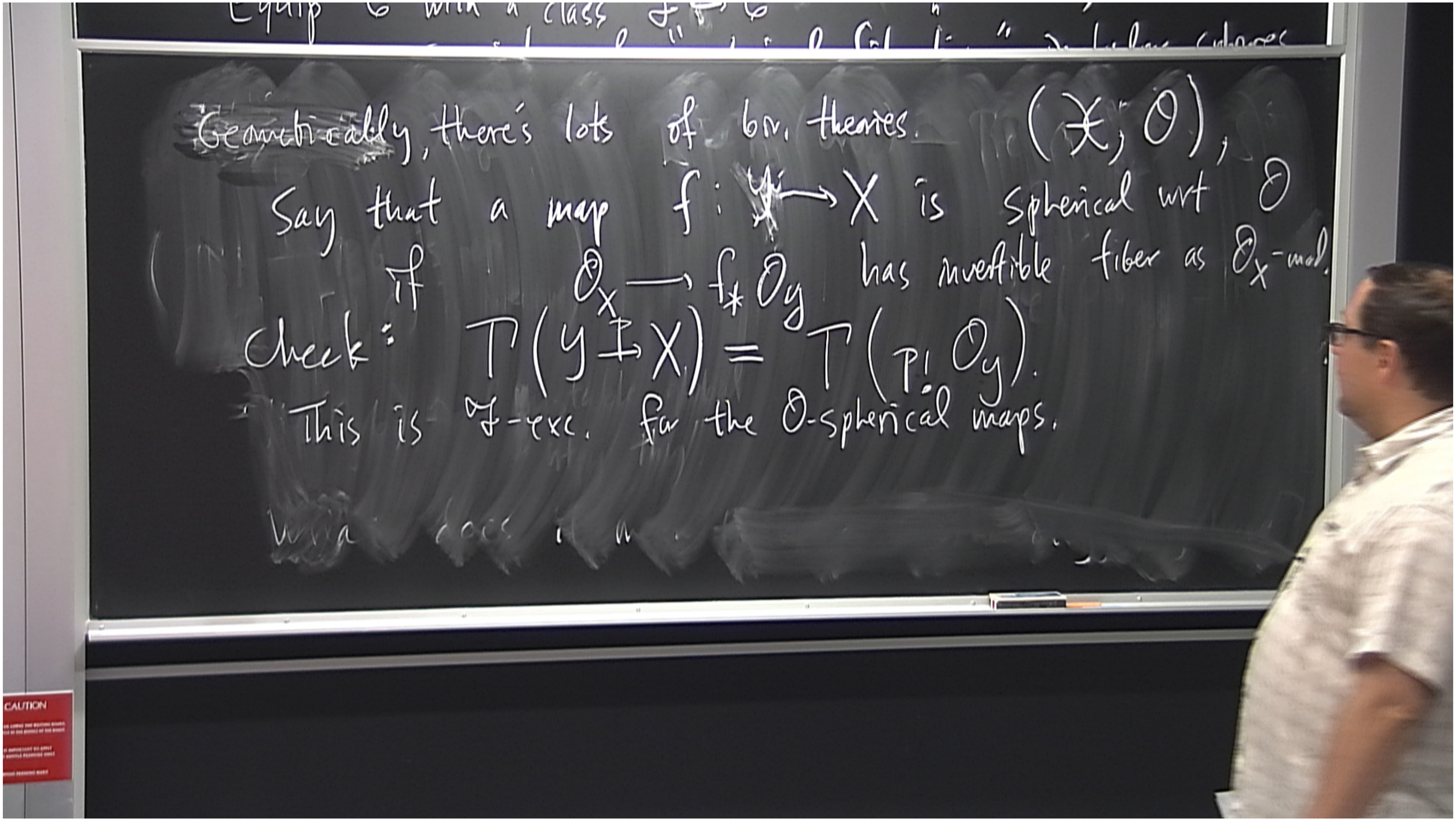
consisting of "spherical fibrations" including spheres.

Geometrically, there's lots of br. theories $(\mathcal{X}, \mathcal{O})$,

Say that a map $f : Y \to X$ is spherical wrt $\mathcal{O}$

if $\mathcal{O}_X \longrightarrow f_* \mathcal{O}_Y$ has invertible fiber as $\mathcal{O}_X$-mod.


consisting of "spherical fibrations" including spheres.
Geometrically, there's lots of 6r. theories (X; O),
Say that a map f: Y → X is spherical wrt O
if O_X → f_* O_Y has invertible fiber as O_X
Check: T(Y → X) = T(p_! O_Y).

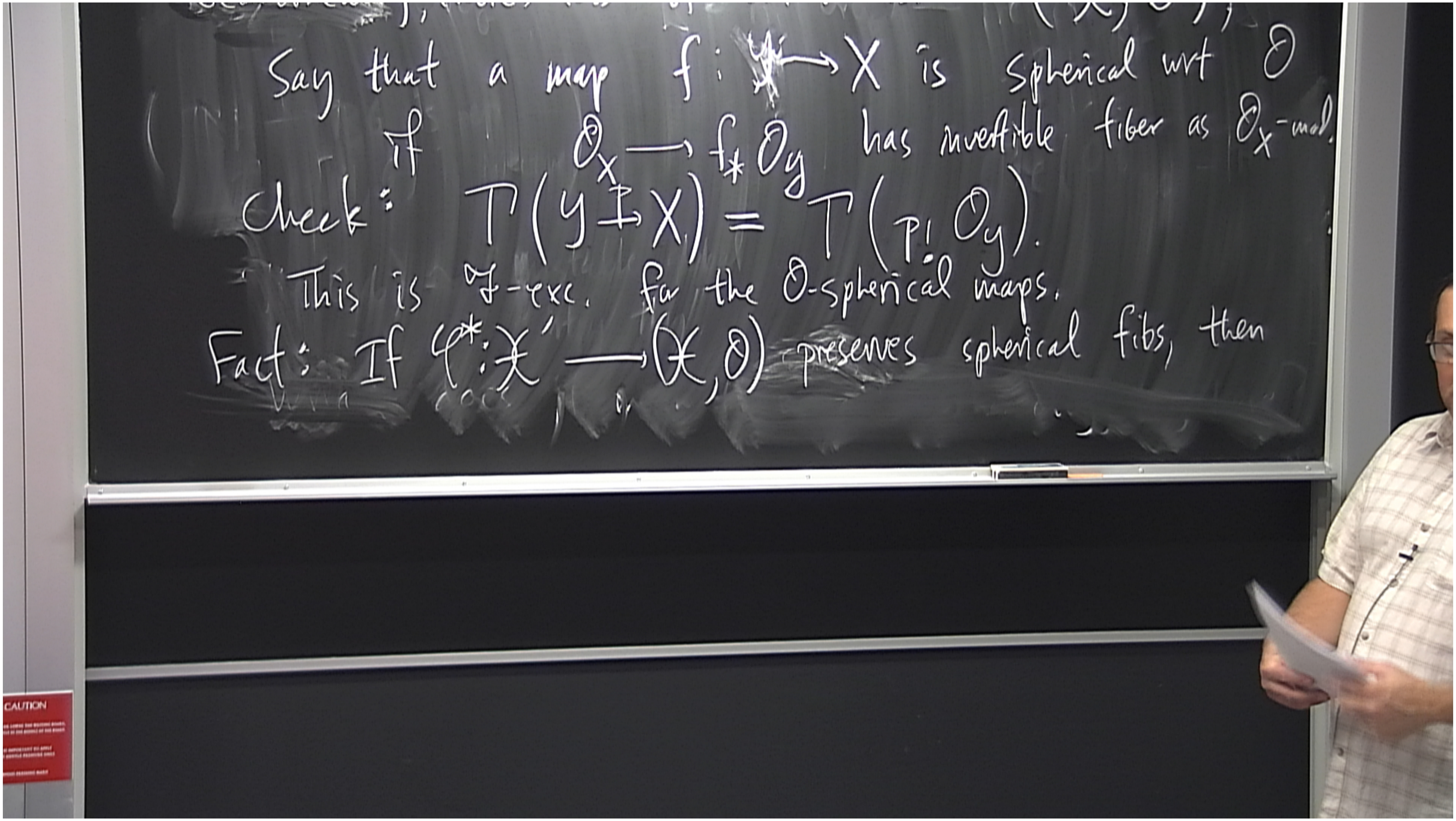

Geometrically, there's lots of br. theories (𝔛, 𝒪),
Say that a map f: Y → X is spherical wrt 𝒪
if 𝒪_X → f_* 𝒪_Y has invertible fiber as 𝒪_X-mod.
Check: T(Y →f X) = T(p_! 𝒪_Y).
This is 𝒯-exc. for the 𝒪-spherical maps.

Geometrically, there's lots of br. theories. $(\mathfrak{X}, \mathcal{O})$,

Say that a map $f: Y \to X$ is spherical wrt $\mathcal{O}$ if $\mathcal{O}_X \to f_* \mathcal{O}_Y$ has invertible fiber as $\mathcal{O}_X$-mod.

Check: $T(Y \xrightarrow{p} X) = T(p_! \mathcal{O}_Y)$.

This is $\mathcal{J}$-exc. for the $\mathcal{O}$-spherical maps.

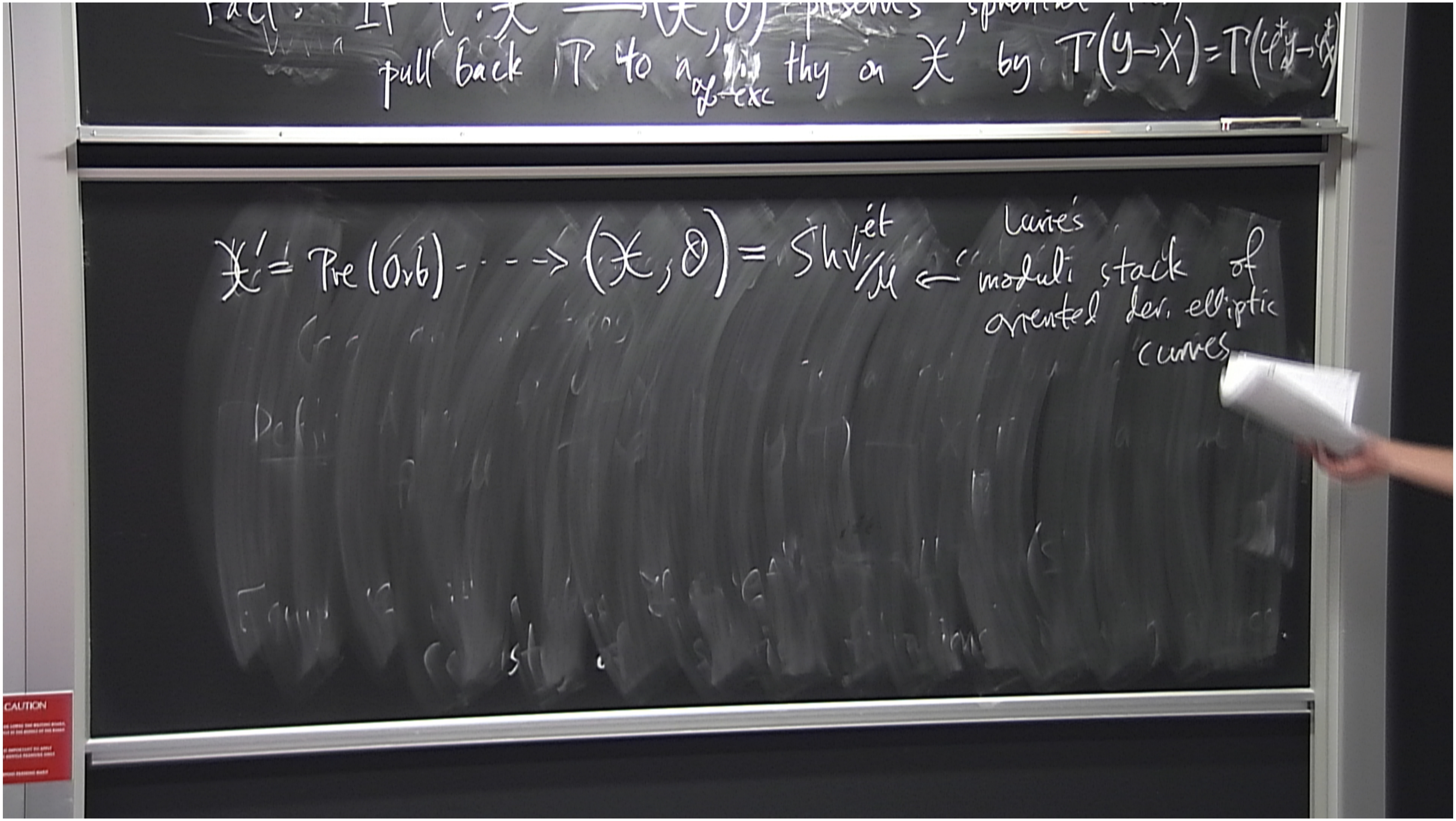

Say that a map f : Y → X is spherical wrt O
if O_X → f_* O_Y has invertible fiber as O_X-mod.
Check: T(Y →f X) = T(p_! O_Y).
This is T-exc. for the O-spherical maps.
Fact: If φ*: X' → (X, O) preserves spherical fibs, then
CAUTION

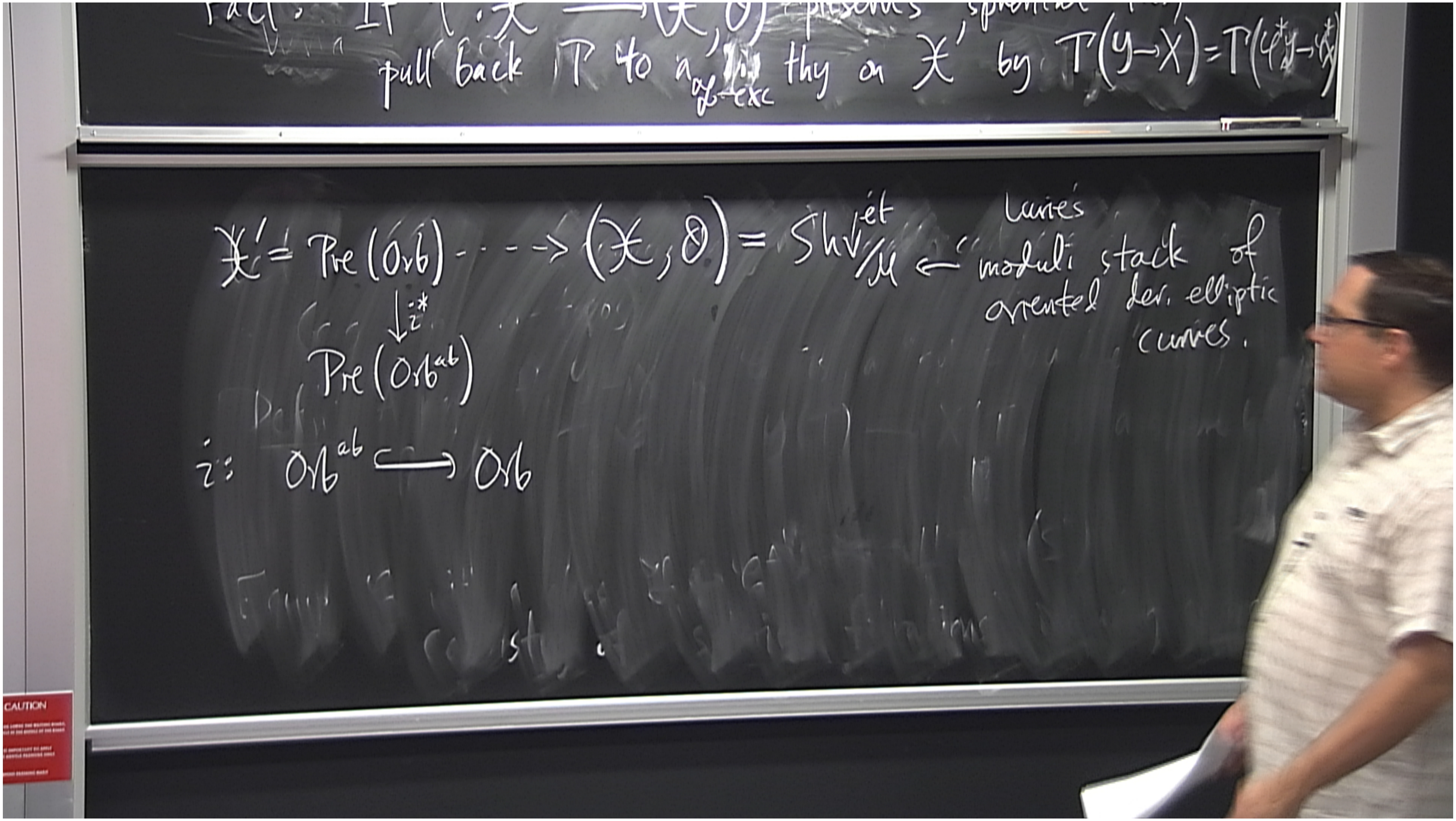
Say that a map $f: Y \to X$ is spherical wrt $\mathcal{O}$ if $\mathcal{O}_X \longrightarrow f_* \mathcal{O}_Y$ has invertible fiber as $\mathcal{O}_X$-mod.

Check: $T(Y \xrightarrow{p} X) = T(p_! \mathcal{O}_Y)$.

This is $\mathcal{Y}$-exc. for the $\mathcal{O}$-spherical maps.

Fact: If $\varphi^*: \mathcal{X}' \longrightarrow (\mathcal{X}, \mathcal{O})$ preserves spherical fibs, then pull back T to a $\mathcal{Y}$-exc thy on $\mathcal{X}'$ by $T(Y \to X) = T(\varphi^* Y \to \varphi^* X)$


pull back P to a X-exc thy on X' by P(Y→X) = P(φ*Y→X')
X' = Pre(Orb) - - -> (X, O) = Shv^ét /M ← "moduli stack of oriented der. elliptic curves"
Lurie's


pull back Γ to a 𝒢-exc thy on 𝒳' by Γ(Y→X)=Γ(φ*Y→φ*X)
𝒳' = Pre(Orb) - - - -> (𝒳, 𝒪) = Shv^ét/M ← Lurie's moduli stack of oriented der. elliptic curves.
↓ i*
Pre(Orb^ab)
i: Orb^ab ↪ Orb

$\mathrm{Ell}(\mathbb{BT} \longrightarrow \mathbb{BT}) = \Gamma(\mathcal{O}_C)$, $\mathrm{Ell}(pt \to pt) = \Gamma(\mathcal{O}_{\mathcal{M}})$.

(G., N., Meier) $\quad TMF \oplus \Sigma^{?} TMF$. $\quad TMF$

Geometrically, there's lots of bv. theories. $(\mathfrak{X}, \mathcal{O})$,

Say that a map $f: \mathcal{Y} \to X$ is spherical wrt $\mathcal{O}$

if $\mathcal{O}_X \longrightarrow f_* \mathcal{O}_Y$ has invertible fiber as $\mathcal{O}_X$-mod.

Check: $\Gamma(Y \xrightarrow{p} X) = \Gamma(p_! \mathcal{O}_Y)$.

This is $\mathcal{J}$-exc. for the $\mathcal{O}$-spherical maps.

Fact: If $\varphi^*: \mathfrak{X}' \longrightarrow (\mathfrak{X}, \mathcal{O})$ preserves spherical fibs, then pull back Γ to a $\mathcal{J}$-exc thy on $\mathfrak{X}'$ by $\Gamma(Y \to X) = \Gamma(\varphi^* Y \to \varphi^* X)$

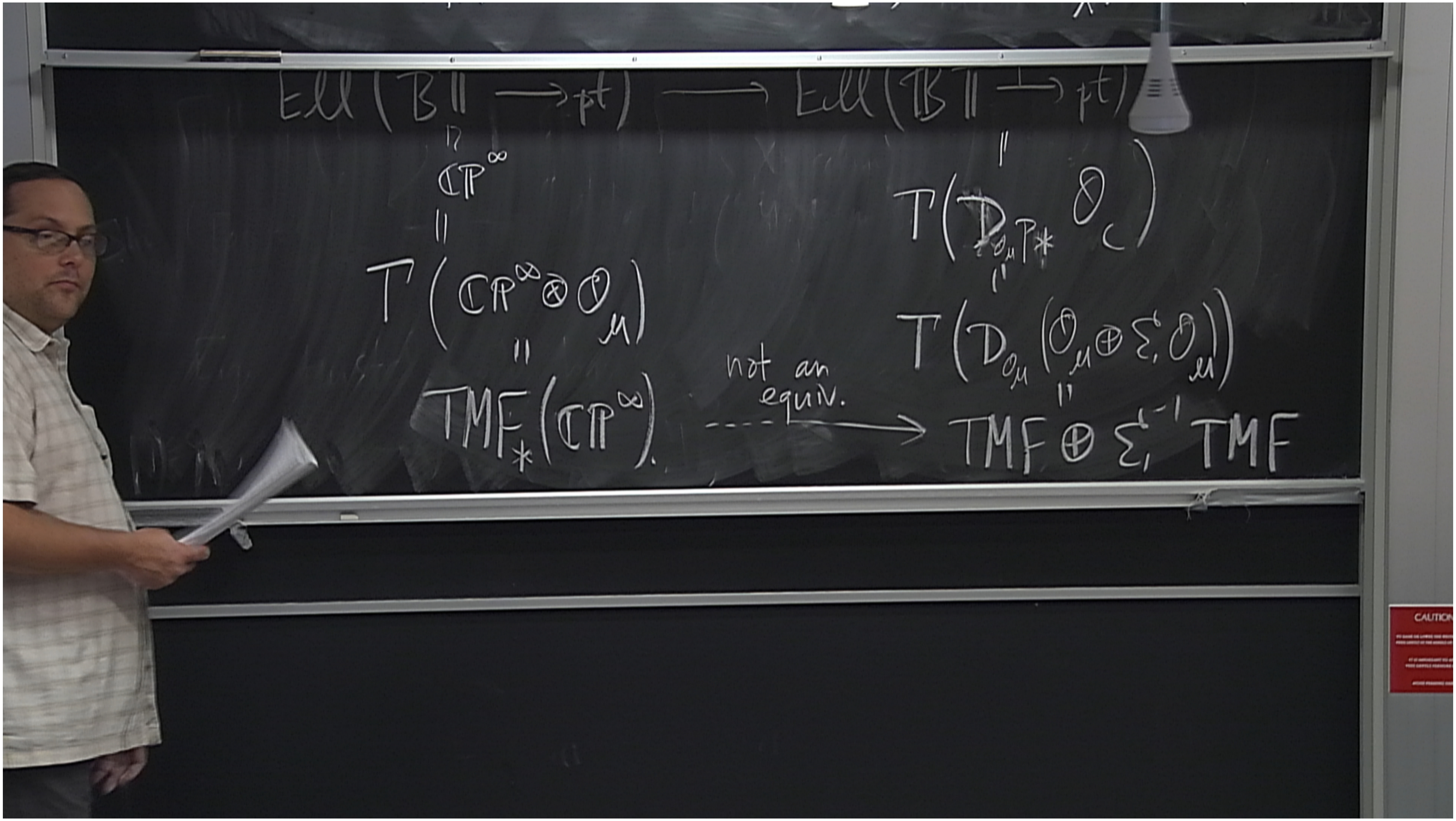
We want the homological cocompletion (?) map

$$\mathrm{Ell}(B\mathbb{T} \longrightarrow pt) \longrightarrow \mathrm{Ell}(B\mathbb{T} \xrightarrow{p} pt)$$

$$B\mathbb{T} \cong \mathbb{CP}^\infty$$

$$\mathrm{Ell}(B\mathbb{T} \xrightarrow{p} pt) = \Gamma(\mathbb{D}\, p_* \mathcal{O}_C)$$

$$\mathrm{Ell}(\mathbb{CP}^\infty) = \Gamma(\mathbb{CP}^\infty \otimes \mathcal{O}_M) = TMF_*(\mathbb{CP}^\infty).$$



not an equiv.
TMF ⊕ Σ⁻¹ TMF