Title: Derived symplectic geometry and classical Chern-Simons theory

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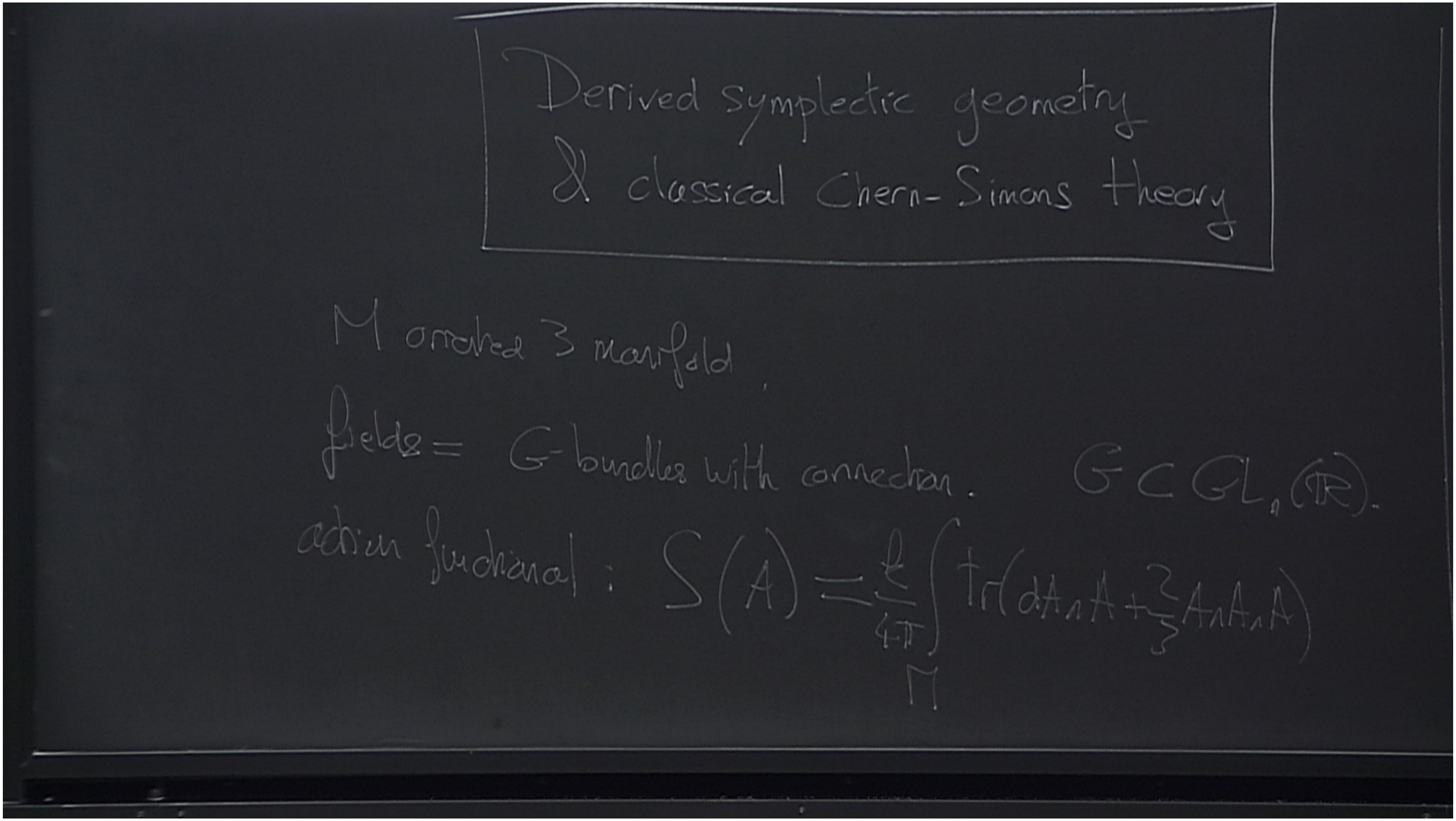
Abstract: In this talk we will review various point-of-views on classical Chern-Simons theory and moduli of flat connections. We will explain how derived symplectic geomletry (after
Pantev-ToÃ«n-VaquiÃ©-Vezzosi) somehow reconciles all of these. If time permits, we will discuss a bit the quantization problem.

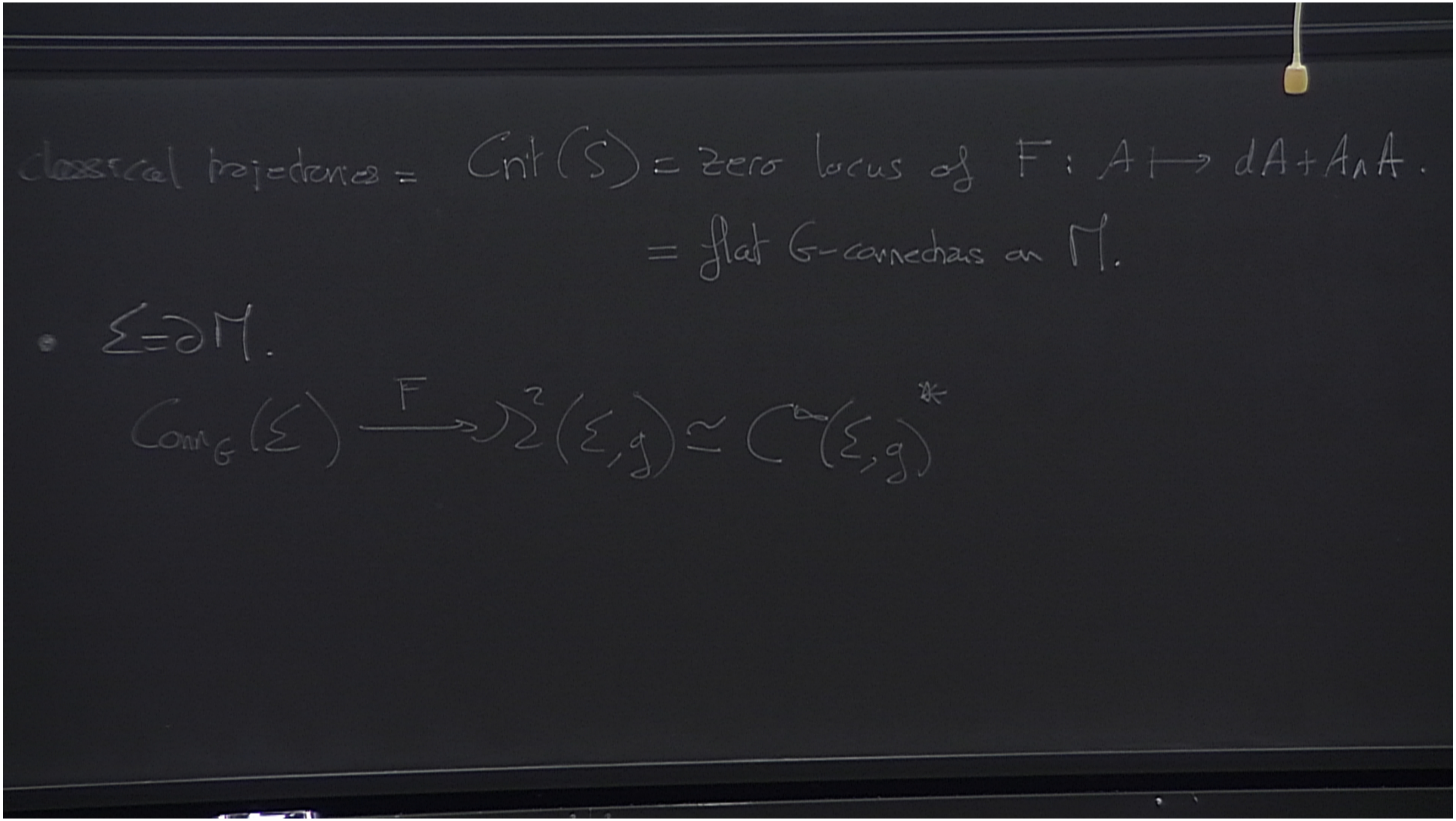


Derived symplectic geometry & classical Chern-Simons theory

M oriented 3 manifold

fields = G-bundles with connection. $G \subset GL_n(\mathbb{R})$.

action functional: $S(A) = \frac{k}{4\pi}\int_M \mathrm{tr}(dA \wedge A + \frac{2}{3} A \wedge A \wedge A)$

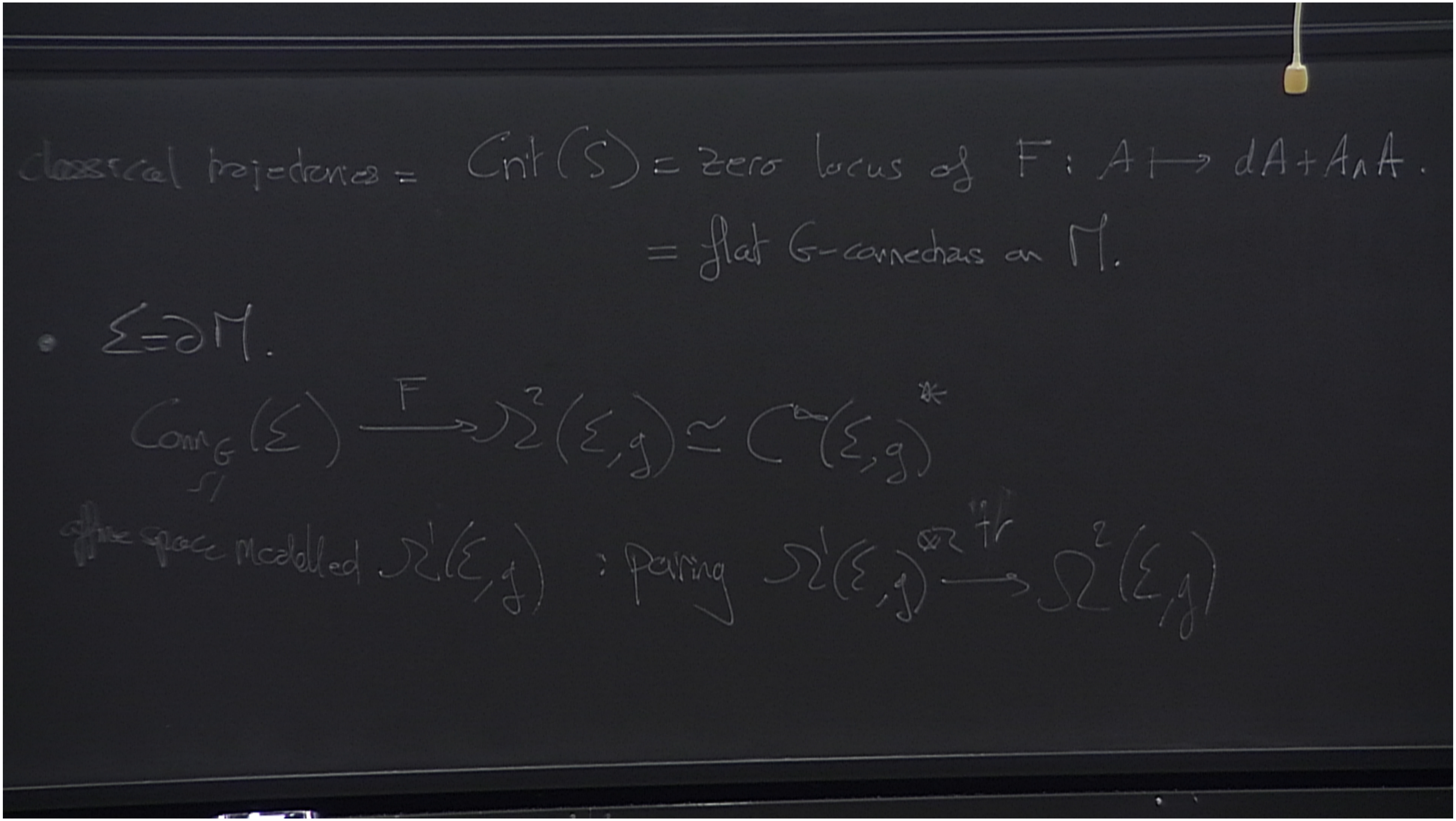


classical trajectories = $\mathrm{Crit}(S)$ = zero locus of $F: A \mapsto dA + A \wedge A$.

= flat G-connections on M.

- $\Sigma = \partial M$.

$$\mathrm{Conn}_G(\Sigma) \xrightarrow{F} \Omega^2(\Sigma, \mathfrak{g}) \simeq C^\infty(\Sigma, \mathfrak{g})^*$$



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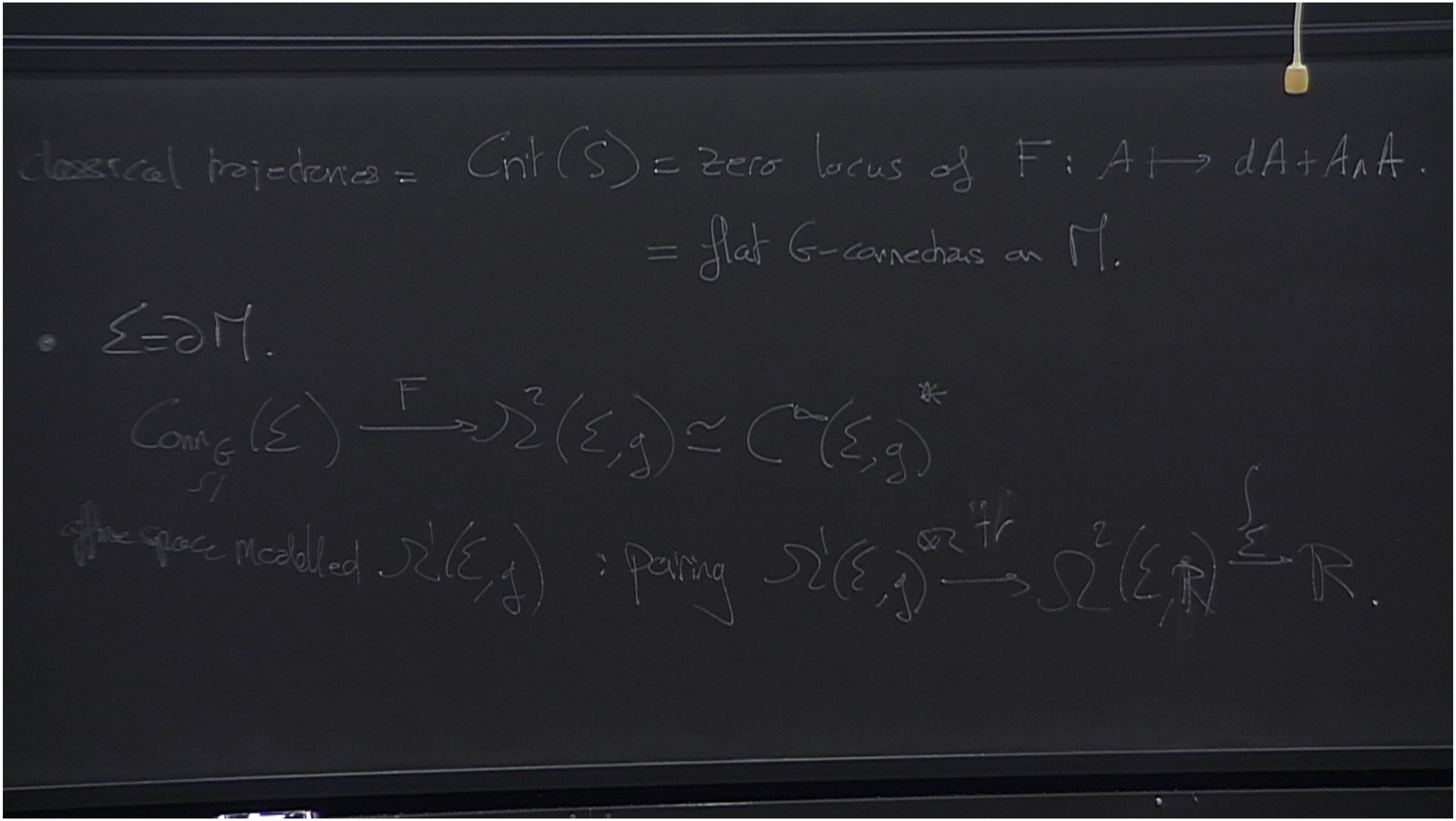
$\wr$

affine space modelled $\Omega^1(\Sigma, \mathfrak{g})$: pairing $\Omega^1(\Sigma, \mathfrak{g})^{\otimes 2} \xrightarrow{\text{"tr"}} \Omega^2(\Sigma, \mathfrak{g})$



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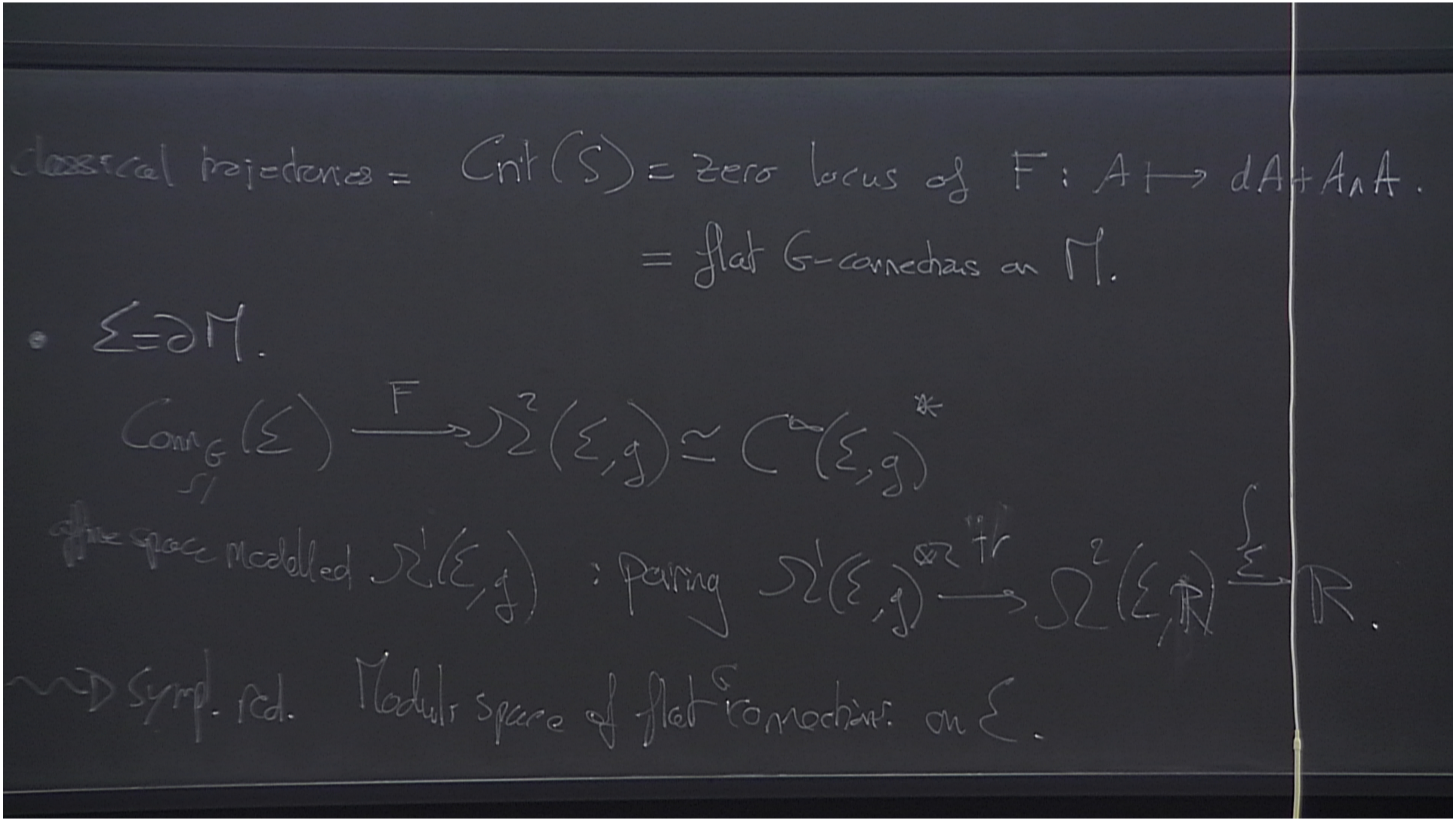

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$$\underset{\Omega^1}{\mathrm{Conn}_G(\Sigma)} \xrightarrow{F} \Omega^2(\Sigma, \mathfrak{g}) \simeq C^\infty(\Sigma, \mathfrak{g})^*$$

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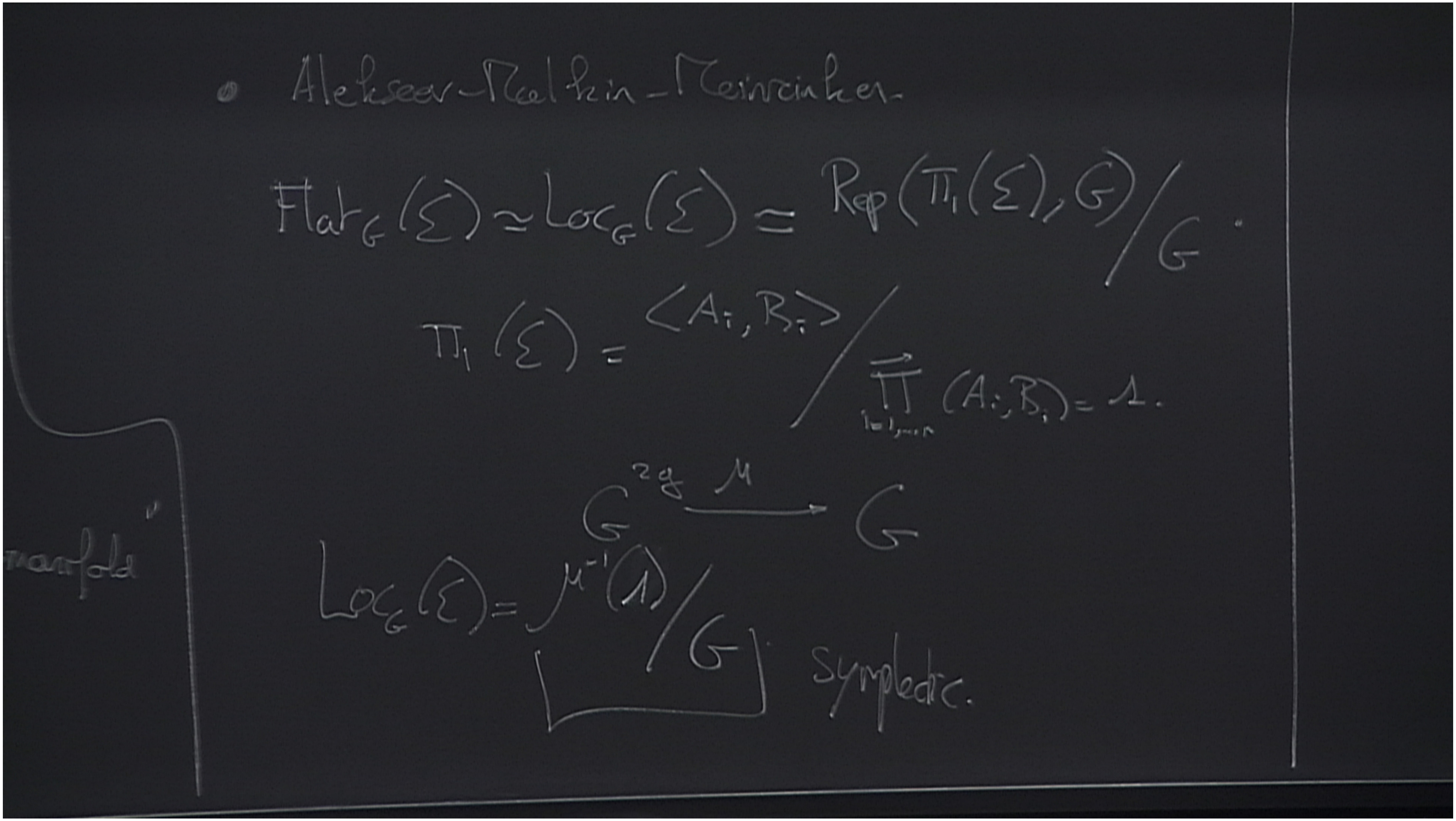
⇝ Symp. red. Moduli space of flat G connections on Σ.

- $T_{[\nabla]} \mathrm{Flat}_G(\Sigma) \simeq H^1(\Sigma, \mathfrak{g})$.

$$H^1(\Sigma,\mathfrak{g})^{\otimes 2} \xrightarrow{tr} H^2(\Sigma,\mathbb{R}) \simeq \mathbb{R}.$$

Can prove that the 2-form is closed.

<u>Moreover</u>: $\mathrm{Flat}_G(M) \longrightarrow \mathrm{Flat}_G(\Sigma)$ "Lagr. submanifold"



- Alekseev–Malkin–Meinrenken

$$\mathrm{Flat}_G(\Sigma) \simeq \mathrm{Loc}_G(\Sigma) \simeq \mathrm{Rep}(\pi_1(\Sigma), G)/G.$$

$$\pi_1(\Sigma) = \langle A_i, B_i \rangle \Big/ \overrightarrow{\prod_{i=1,\dots,n}} (A_i, B_i) = 1.$$

$$G^{2g} \xrightarrow{\mu} G$$

$$\mathrm{Loc}_G(\Sigma) = \underbrace{\mu^{-1}(1)/G}_{} \quad \text{symplectic.}$$

manifold"

$\Sigma), G)/G$.

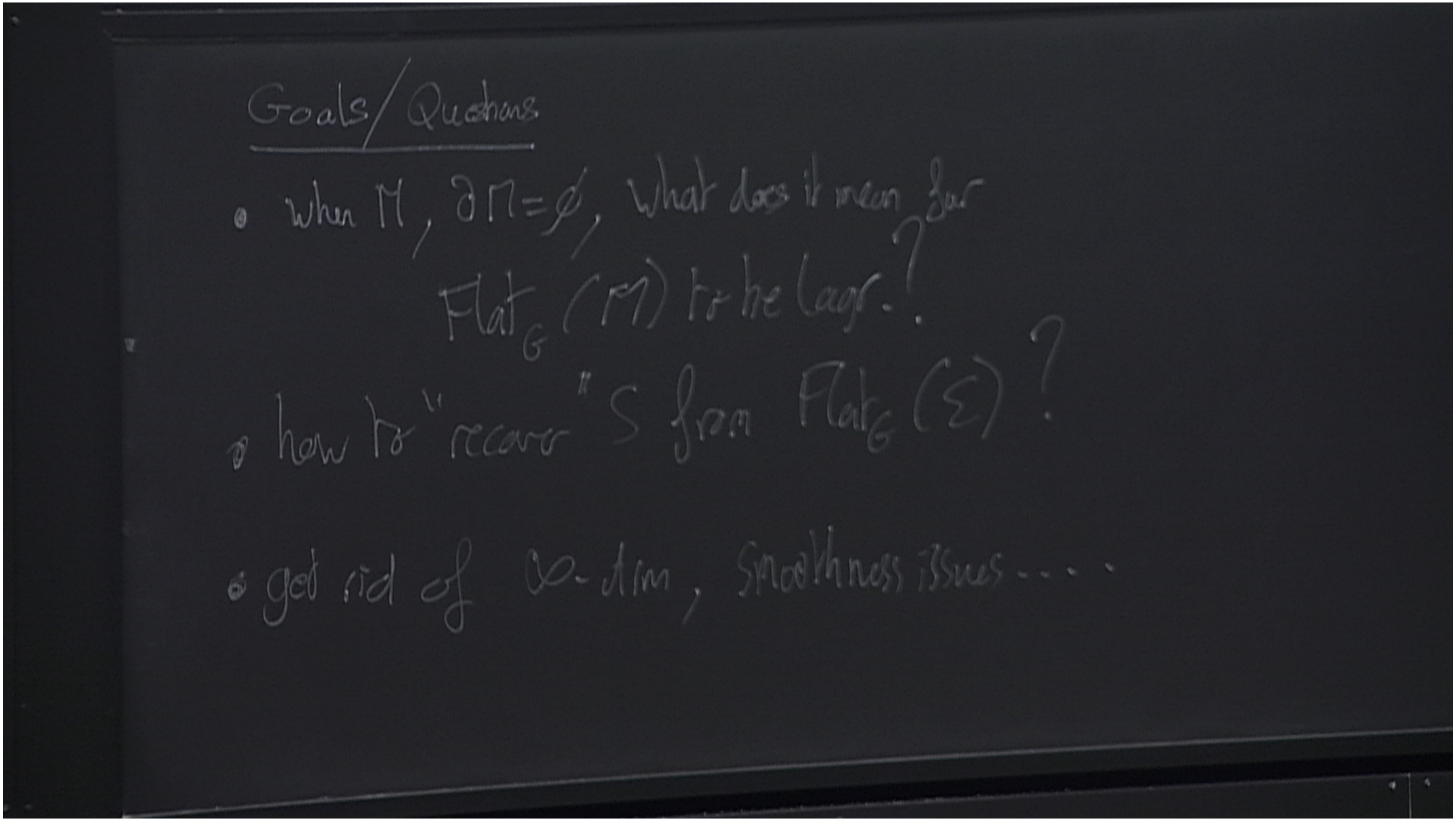
$, B_i) = 1$.

$$\begin{tikzcd}
\Omega^1(S^1, g) \arrow[r, "\mathrm{Hol}"] & G \\
LG \arrow[u, phantom, "\circlearrowright"] \arrow[r] & G \arrow[u, phantom, "\circlearrowright"] \\
\gamma \arrow[r, mapsto] & \gamma(1)
\end{tikzcd}$$

$\Omega^1(S^1,g)/LG \longrightarrow G/G$

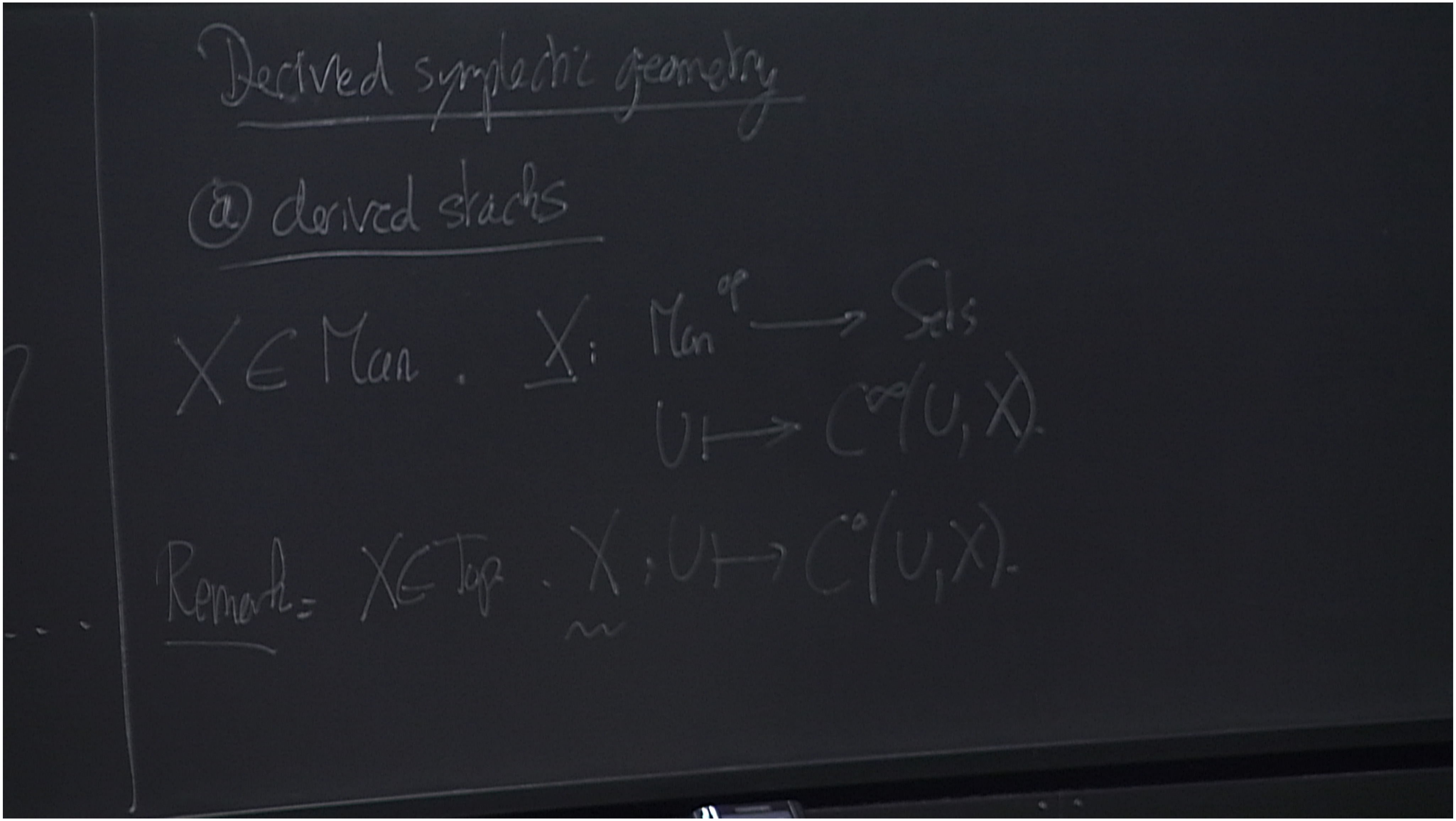
equivalence.

$$\mathrm{Flat}_G(\Sigma \setminus D) \longrightarrow \Omega^1(S^1, g)$$



Goals/Questions

- when M, $\partial M = \emptyset$, what does it mean for $\mathrm{Flat}_G(M)$ to be Lagr.?
- how to "recover" S from $\mathrm{Flat}_G(\Sigma)$?
- get rid of ∞-dim, smoothness issues . . . .



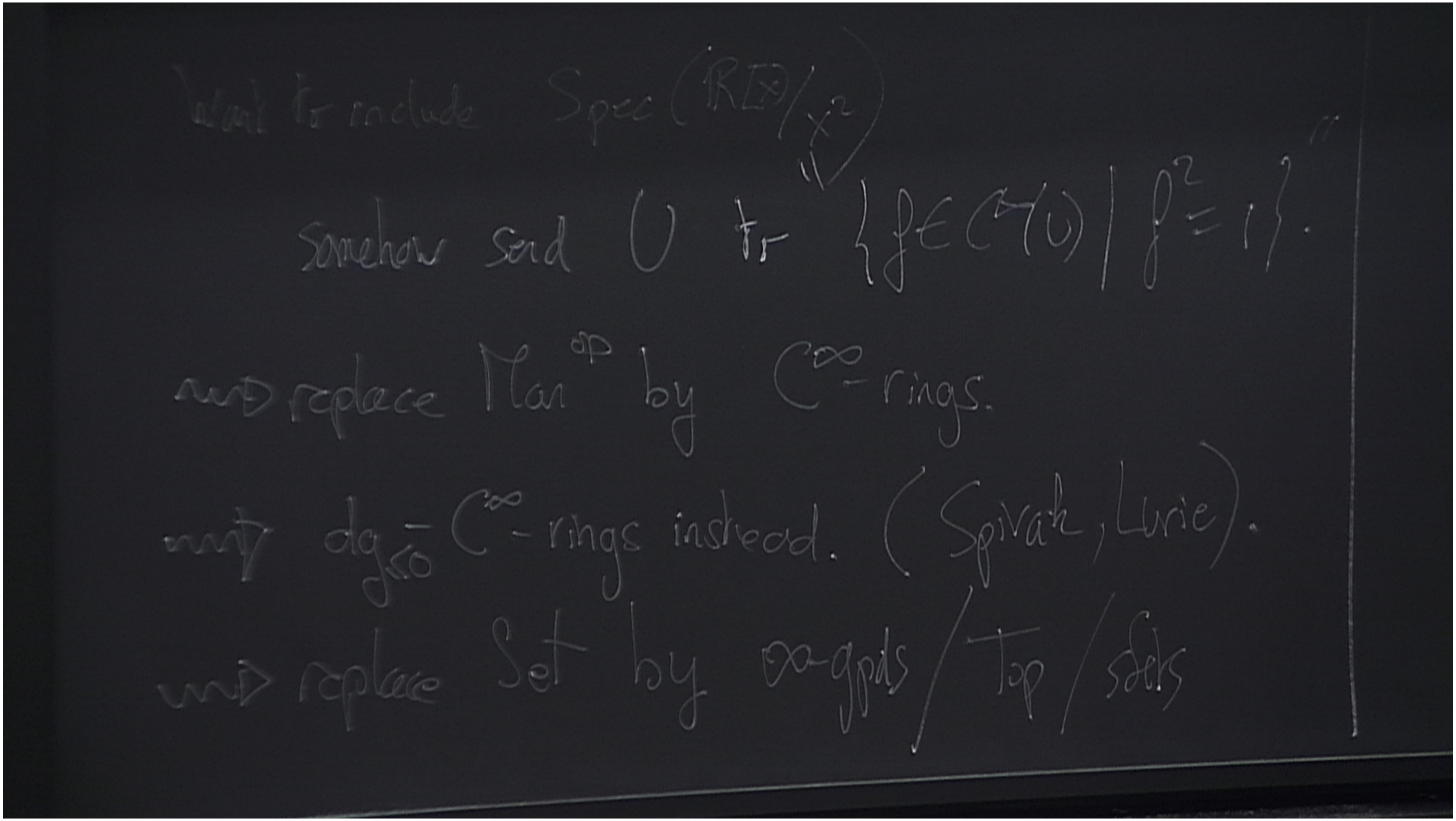
Derived symplectic geometry

ⓐ derived stacks

$X \in \mathrm{Man}$. $\underline{X} : \mathrm{Man}^{op} \longrightarrow \mathrm{Sets}$

$U \longmapsto C^\infty(U, X)$.

Remark: $X \in \mathrm{Top}$. $\underset{\sim}{X} : U \longmapsto C^0(U, X)$.



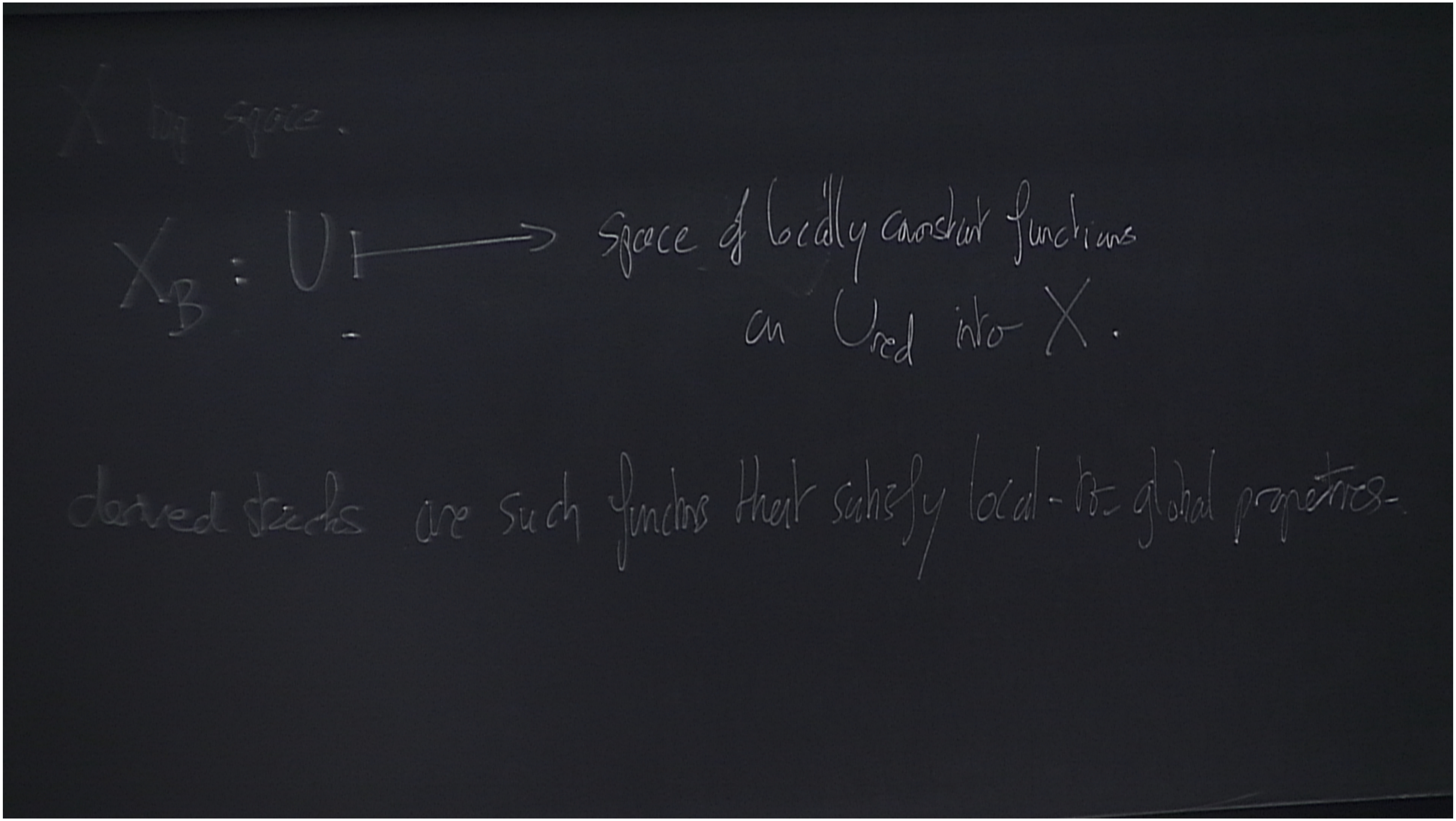
Want to include $\mathrm{Spec}\left(\mathbb{R}[x]/x^2\right)$

"somehow send U to $\{f \in C^\infty(U) \mid f^2 = 1\}$."

⇝ replace Man^{op} by C^∞-rings.

⇝ $dg_{\leq 0}$ C^∞-rings instead. (Spivak, Lurie).

⇝ replace Set by ∞-gpds / Top / sSets



X top space.

$X_B : U \longmapsto$ space of locally constant functions on U_{red} into X.

derived stacks are such functions that satisfy local-to-global properties.

ver: $\mathrm{Flat}_G(M) \longrightarrow \mathrm{Flat}_G(\Sigma)$ "Lagr. submanifold"

$G \longrightarrow G$

$\mathrm{Loc}_G(\Sigma) = \underbrace{\mu^{-1}(1)/G}$ symplectic.

$\mathrm{Flat}_G(\Sigma \setminus$

ⓑ Shifted sympl structures [PTVV]

to any X, one can associate $DR(X)$ ⇝ 2 gradings

internal/coh. degree $\quad d_{int}$ $(1,0)$

$+ \quad = d_{tot}$.

weight (degree of forms) $\quad d_{dR}$ $(1,1)$

Example: $X = [M/G]$. M manifold.

G compact simpl.

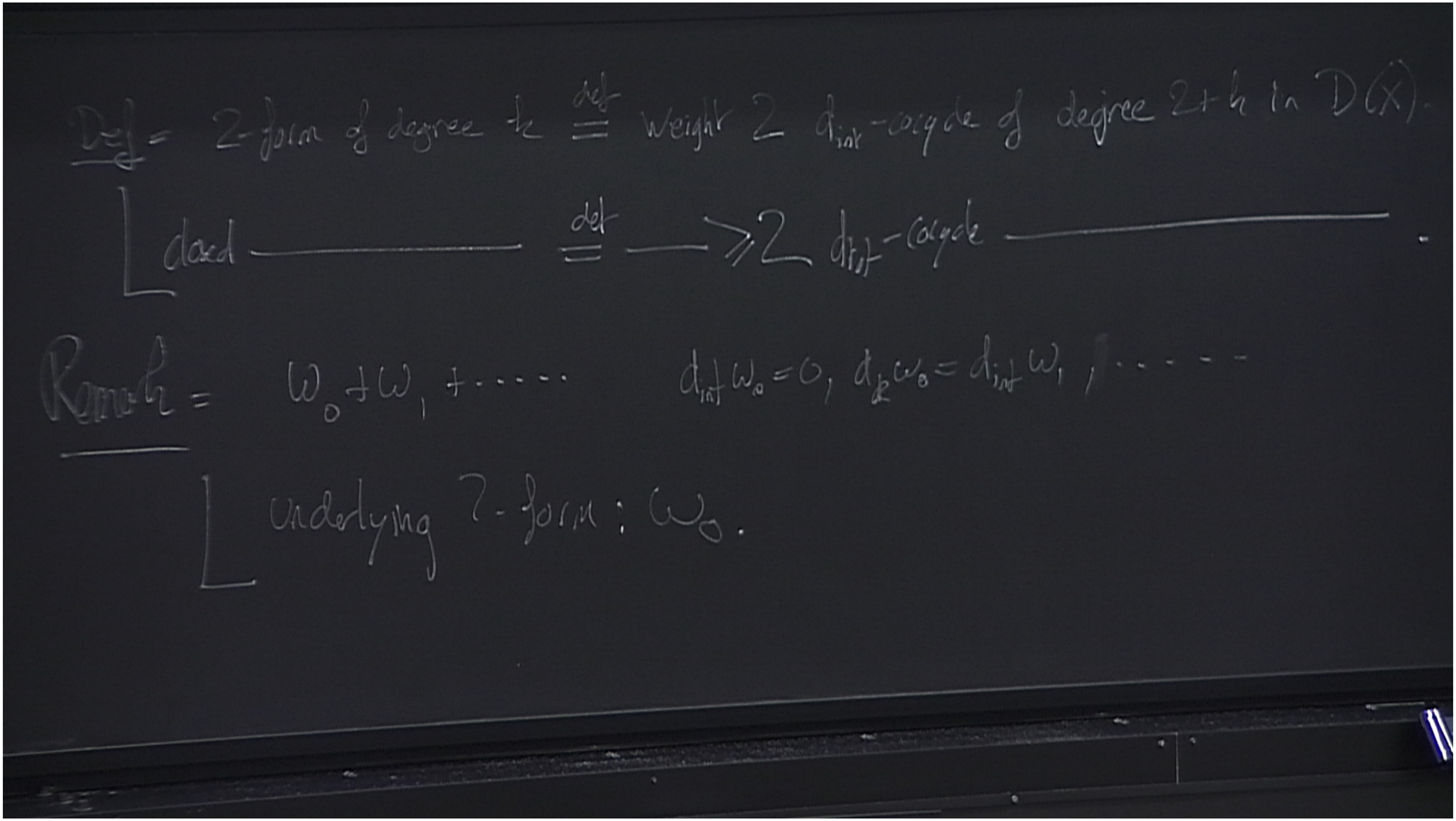
$$DR(X) = \left(\Omega^\bullet(M) \otimes S(\mathfrak{g}^*[-2])\right)^G$$

$$d_{dR} = d_X \otimes id$$

$$d_{int} = \sum_i \iota_{\vec{X}_i} \otimes \xi^i$$

$(X_i)_i$ basis of $\mathfrak{g}$ $\quad \vec{X}_i$

$(\xi^i)_i$ dual basis

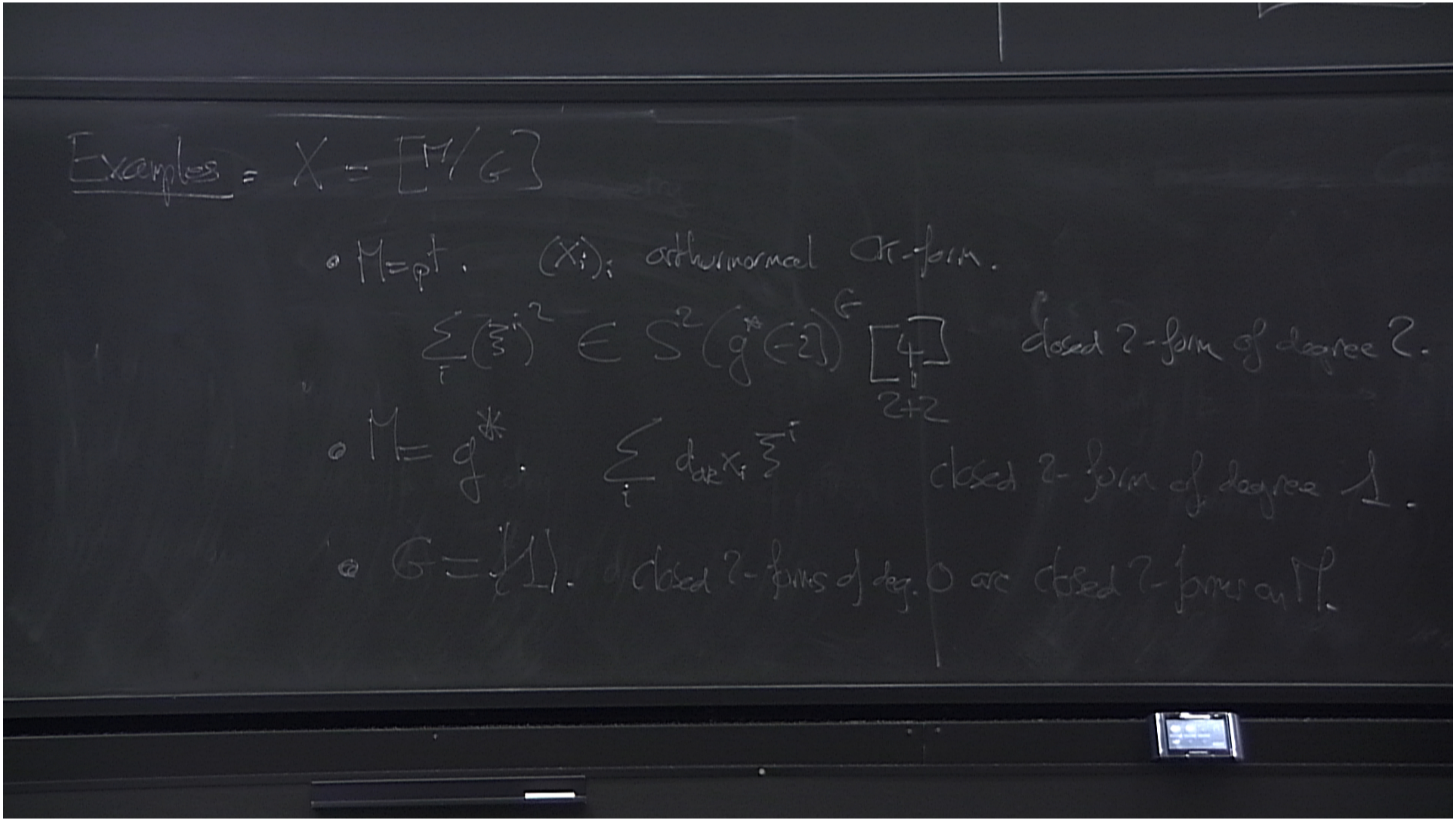


Def = 2-form of degree k $\overset{\text{def}}{=}$ weight 2 d_{int}-cocycle of degree $2+k$ in $D(X)$.

└ closed ______ $\overset{\text{def}}{=}$ ___ ≥ 2 d_{int}-cocycle ______.

Remark = $\omega_0 + \omega_1 + \cdots\cdots$ $\quad d_{int}\,\omega_0 = 0,\ d_{dR}\,\omega_0 = d_{int}\,\omega_1,\ \cdots\cdots$

└ underlying 2-form : ω_0.



Examples: $X = [M/G]$

- $M = pt$. (X_i): orthonormal CK-form.

  $\sum_i (\xi^i)^2 \in S^2(\mathfrak{g}^*(-2))^G \underbrace{[4]}_{2+2}$ closed 2-form of degree 2.

- $M = \mathfrak{g}^*$. $\sum_i d_{dR} x_i \xi^i$ closed 2-form of degree 1.

- $G = \{1\}$. closed 2-forms of deg. 0 are closed 2-forms on M.

e ?.

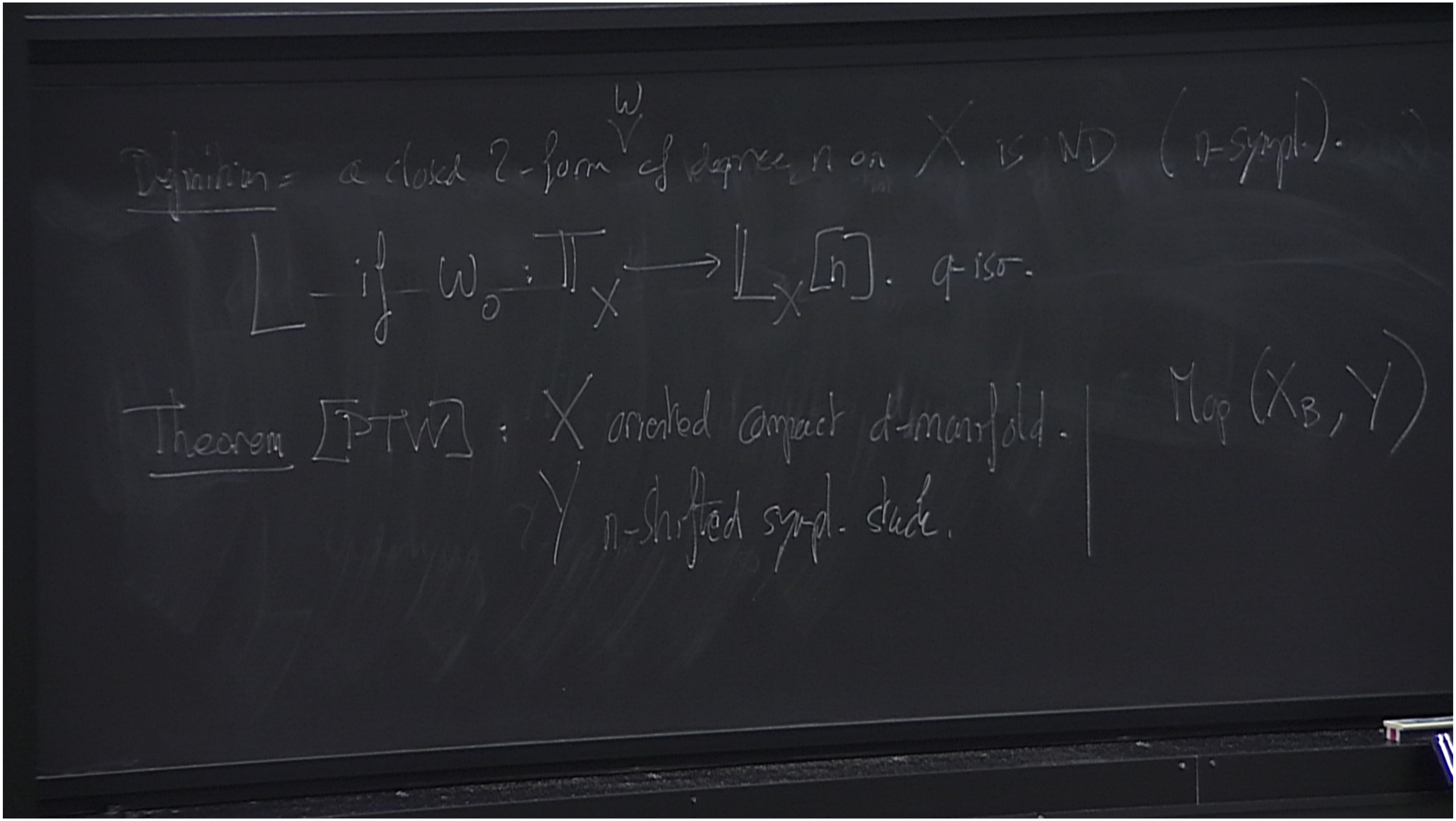
1.

- $M = G$ acted by conj. $\theta_R = g^{-1}dg$ $\theta_L = dgg^{-1}$.

$$\omega_0 = \frac{1}{2}(\theta_R + \theta_L) = \left(\Omega^1(G)\otimes\mathfrak{g}\right)^G \simeq \left(\Omega^1(G)\otimes\mathfrak{g}^*\right)^G$$

$$\omega_1 = \frac{1}{12}\langle \theta_R, [\theta_R, \theta_R]\rangle \in \Omega^3(G)^G.$$

$\omega_0 + \omega_1$ is the closed 2-form of degree 1 on $\left[G/G^{ad}\right]$.

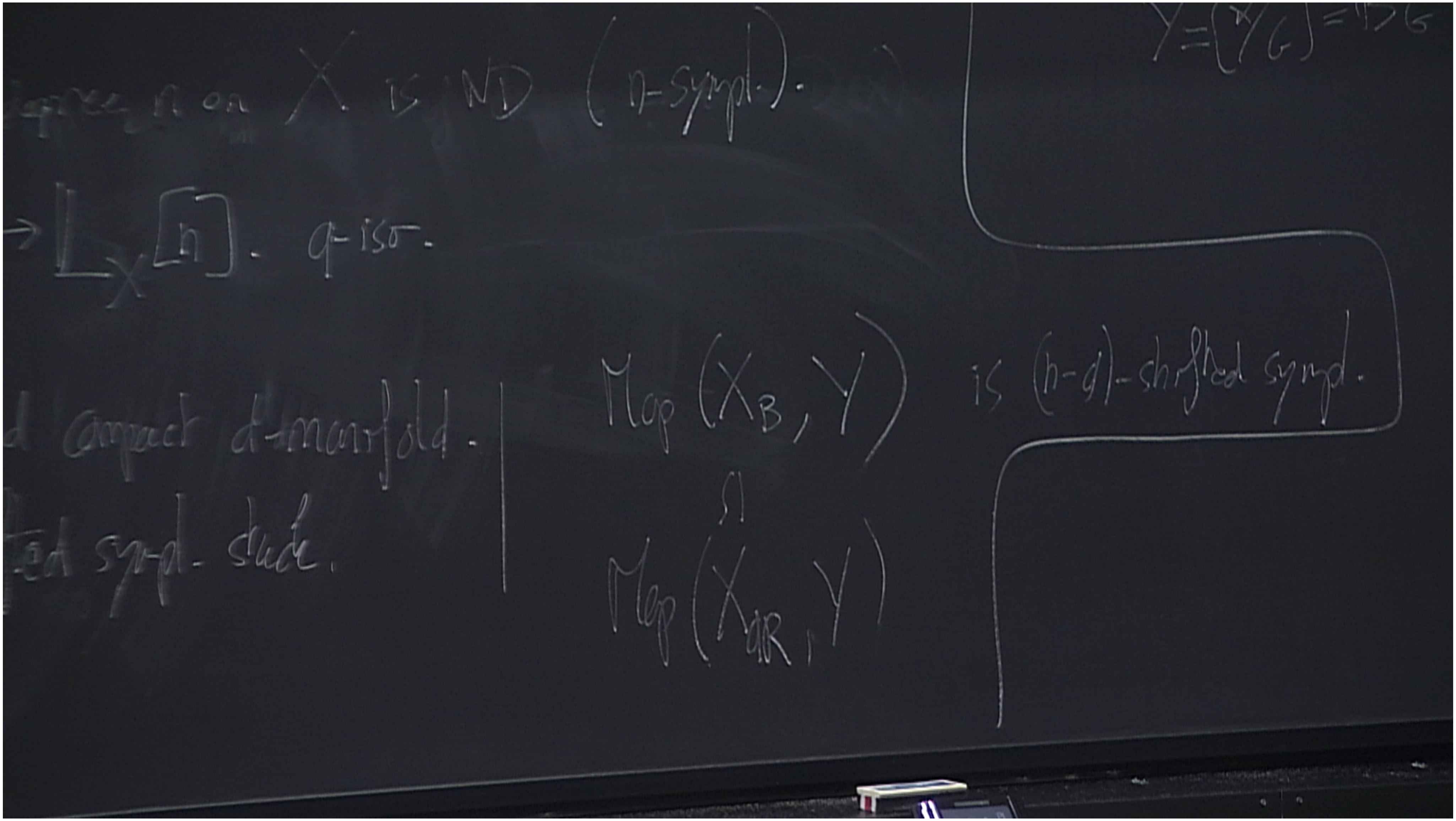


Definition: a closed 2-form $\overset{\omega}{\vee}$ of degree n on X is ND (n-sympl.)

$\mathbb{L}$ if $\omega_0 : \mathbb{T}_X \longrightarrow \mathbb{L}_X[n]$. q-iso.

Theorem [PTVV]: X oriented compact d-manifold. | $\mathrm{Map}(X_B, Y)$

Y n-shifted sympl. stack.



$Y = [*/G] = BG$

...degree n on X is ND (n-sympl.).

$\to \mathbb{L}_X[n]$. q-iso.

...d compact d-manifold.

...ted sympl. stack.

$\mathrm{Map}(X_B, Y)$ is $(n-d)$-shifted sympl.

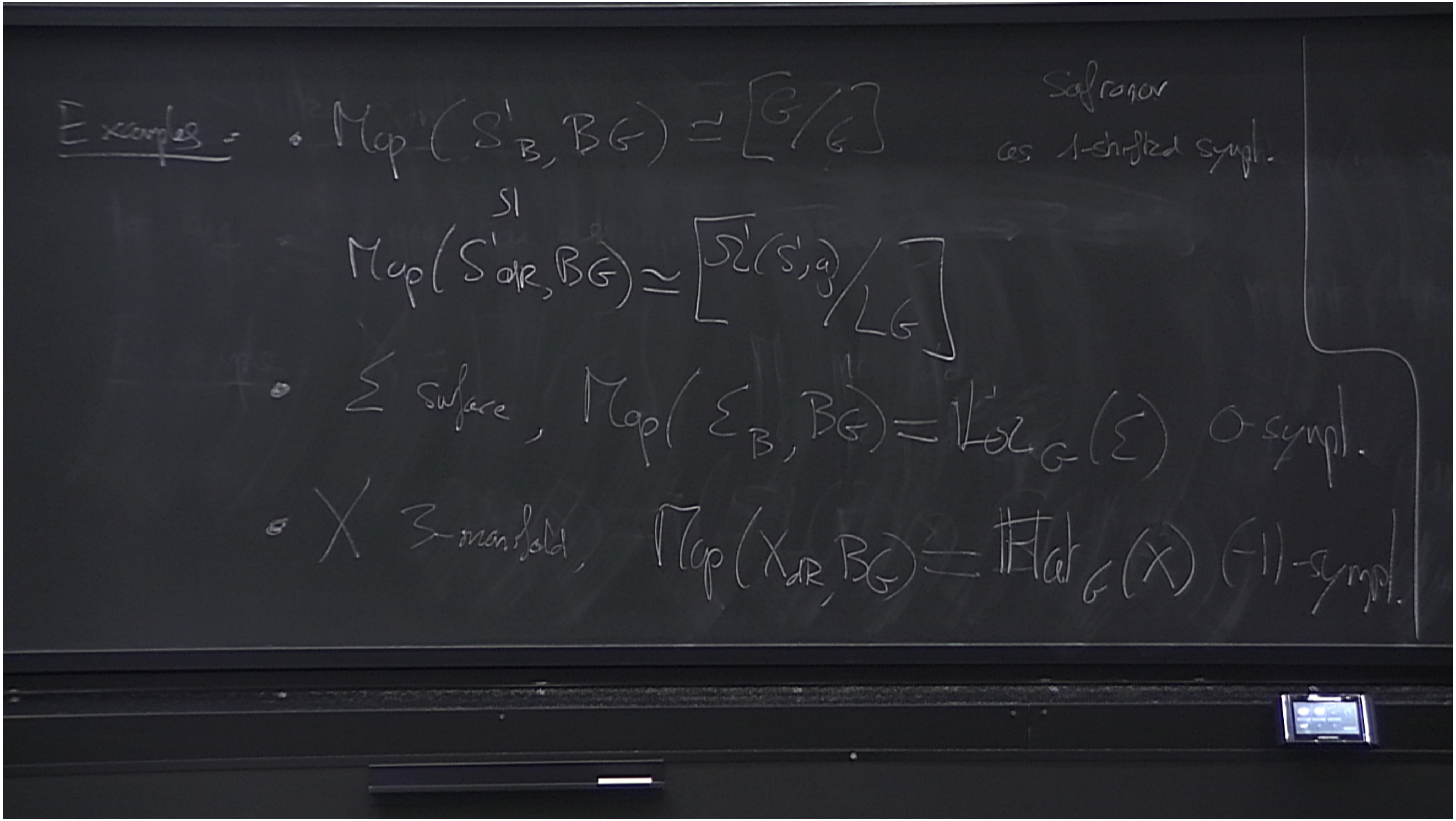
$\simeq$

$\mathrm{Map}(X_{dR}, Y)$

$Y = (*/G) = BG \quad \mathrm{Map}(X_B, Y) = \mathbb{L}\mathrm{oc}_G(X)$

$\mathrm{Map}(X_{dR}, Y) = \mathbb{F}\mathrm{lat}_G(X).$

$\boxed{\text{(2-d)-shifted sympl.}}$



Examples:

- $\mathrm{Map}(S^1_B, BG) \simeq [G/G]$

  Safronov

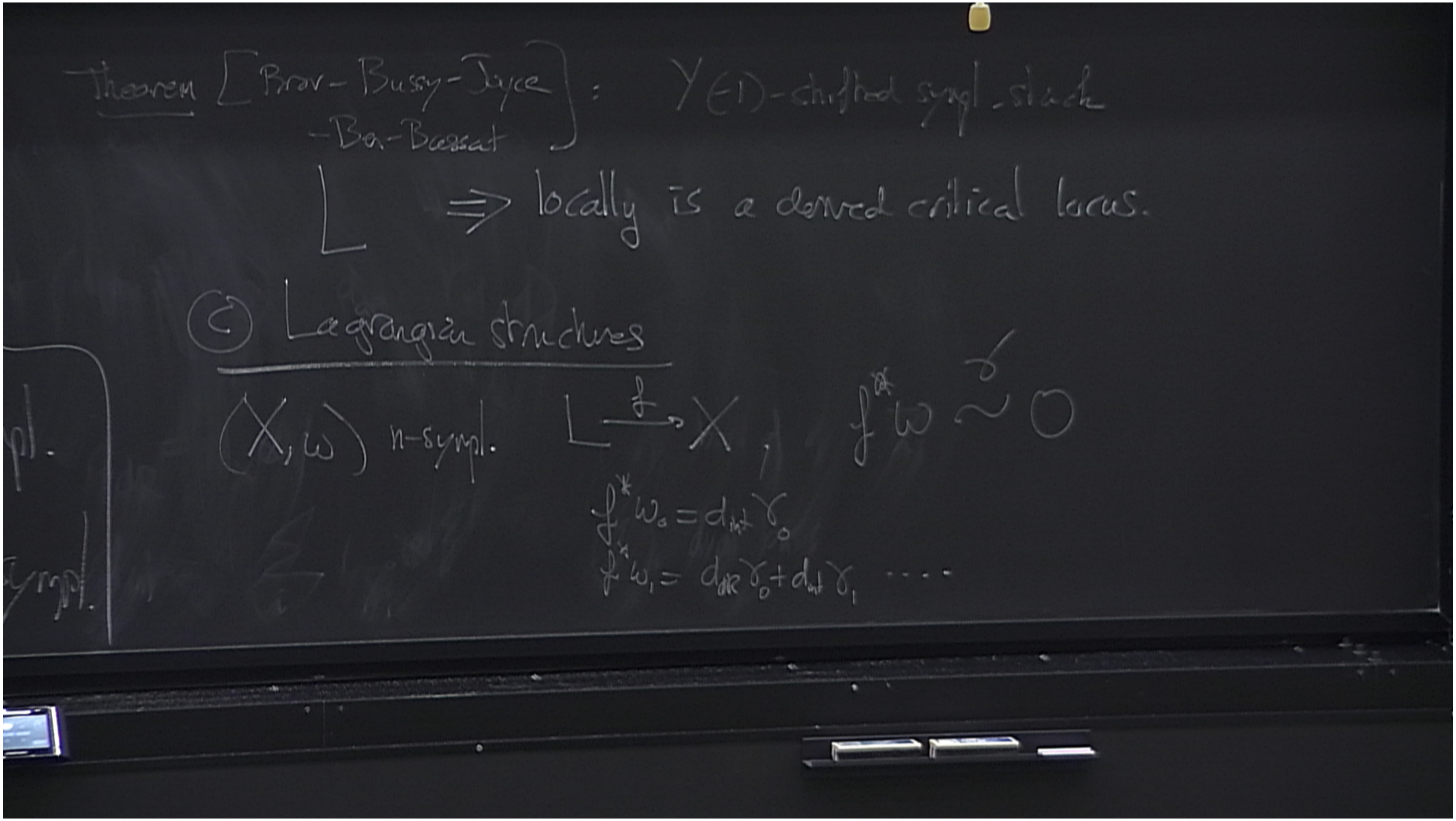
  as 1-shifted sympl.

  S1

  $\mathrm{Map}(S^1_{dR}, BG) \simeq [\Omega^1(S^1, \mathfrak{g}) / LG]$

- Σ surface, $\mathrm{Map}(\Sigma_B, BG) = \mathrm{Loc}_G(\Sigma)$ 0-sympl.

- X 3-manifold, $\mathrm{Map}(X_{dR}, BG) \simeq \mathrm{Flat}_G(X)$ (-1)-sympl.



Theorem [Brav-Bussy-Joyce -Ben-Bassat]: Y (−1)-shifted sympl. stack

L $\Rightarrow$ locally is a derived critical locus.

pl.

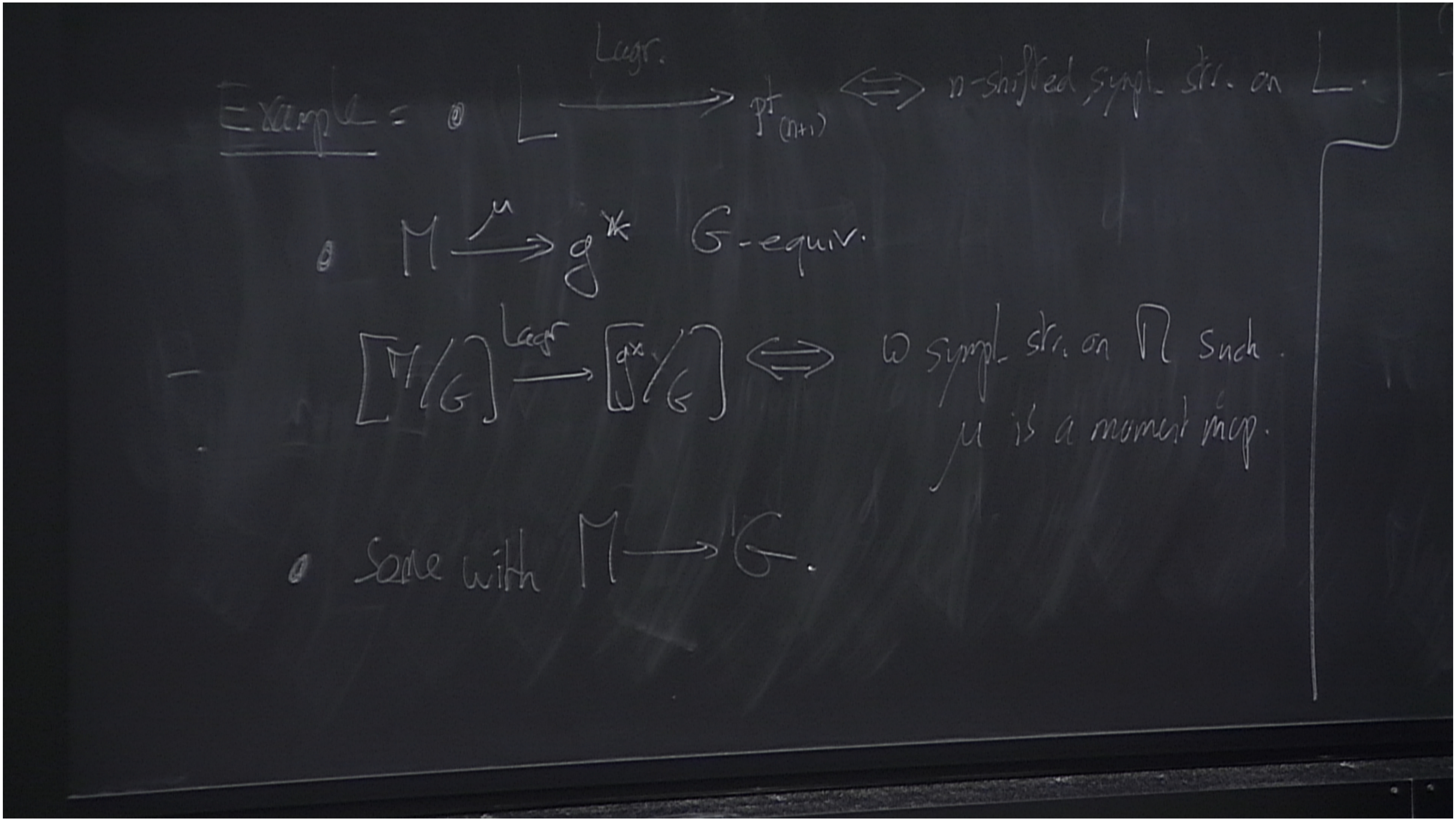
sympl.

ⓒ Lagrangian structures

(X, ω) n-sympl. $\quad L \xrightarrow{f} X, \quad f^*\omega \overset{\gamma}{\sim} 0$

$f^*\omega_0 = d_{int}\gamma_0$

$f^*\omega_1 = d_{dR}\gamma_0 + d_{int}\gamma_1 \quad \cdots$



Example: • $L \xrightarrow{\text{Lagr.}} pt_{(n+1)} \Longleftrightarrow$ n-shifted sympl. str. on L.

• $M \xrightarrow{\mu} \mathfrak{g}^*$ G-equiv.

$[M/G] \xrightarrow{\text{Lagr}} [\mathfrak{g}^*/G] \Longleftrightarrow$ ω sympl. str. on M such μ is a moment map.

• Same with $M \longrightarrow G$.

$$X_1 \qquad X_2 \qquad X_3$$

$$\begin{tikzcd} & L_{12} \times_{X_2} L_{23} \arrow[dl] \arrow[dr] & \\ X_1 & & X_3 \end{tikzcd}$$

Lagr. corr.:

$X_1 = X_3 = pt.$

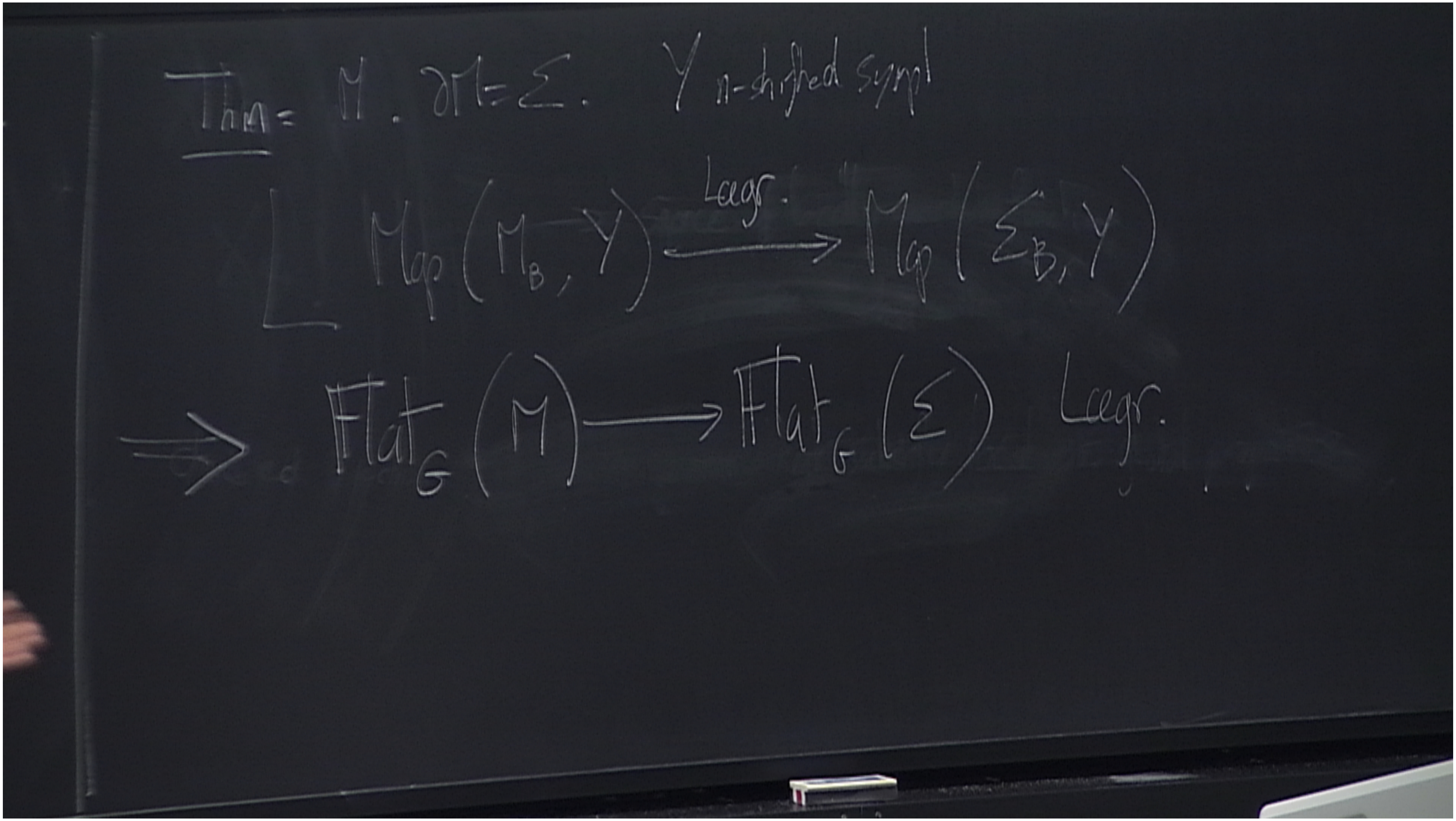
$$\begin{tikzcd} L_1 \arrow[dr] & \\ & X \\ L_2 \arrow[ur] & \end{tikzcd}$$

Lagr. X n-sympl.

$L_1 \times_X L_2$ $(n-1)$-shifted sympl.

$$\begin{tikzcd} X_{red} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & {[*/G]} \arrow[d] \\ {[M/G]} \arrow[r] & {[\mathfrak{g}^*/G]} \end{tikzcd}$$

$\Longrightarrow X_{red}$ 0-sympl.



Thm: M. $\partial M = \Sigma$. Y n-shifted sympl

$$\mathrm{Map}(M_B, Y) \xrightarrow{\text{Lagr.}} \mathrm{Map}(\Sigma_B, Y)$$

$$\Rightarrow \mathrm{Flat}_G(M) \longrightarrow \mathrm{Flat}_G(\Sigma) \quad \text{Lagr.}$$