Title: Towards a general description of derived self-intersections

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Abstract: Thanks to a result of Arinkin and CÄ•ldÄ•ru, the derived self-intersection of a closed smooth subscheme of an ambiant scheme (over a field of characteristic zero) is a formal object if and only if the conormal bundle of the subscheme extends to a locally free sheaf at the first order. In this talk, we will explain a program as well as new results in order to describe these derived self-intersections in the non-formal case.

Go[illegible] schemes $\mathrm{car}(k)=0$

$X \subset Y$

$S = \left(X, \mathcal{O}_Y / \mathcal{J}_X^2\right)$ $\quad j: X \hookrightarrow S$

→ Understand the derived scheme $X \times_Y X$. $\left(Y = X^2 \quad X \underset{\delta}{\hookrightarrow} X^2 \quad X \times_{X \times X} X = LX\right.$

→ Today complete description of $X \times_S X$ $\quad = \mathrm{spec}\left(\mathrm{Sym}\, \Omega^1_X [\right.$

Con. seq $\quad 0 \to N^*_{X/Y} \to \Omega^1_{Y|X} \to \Omega^1_X \to 0$

$\eta \in \mathrm{Ext}^1_{\mathcal{O}_X}\left(\Omega^1_X, N^*_{X/Y}\right)$

1) Def th of perfect complexes

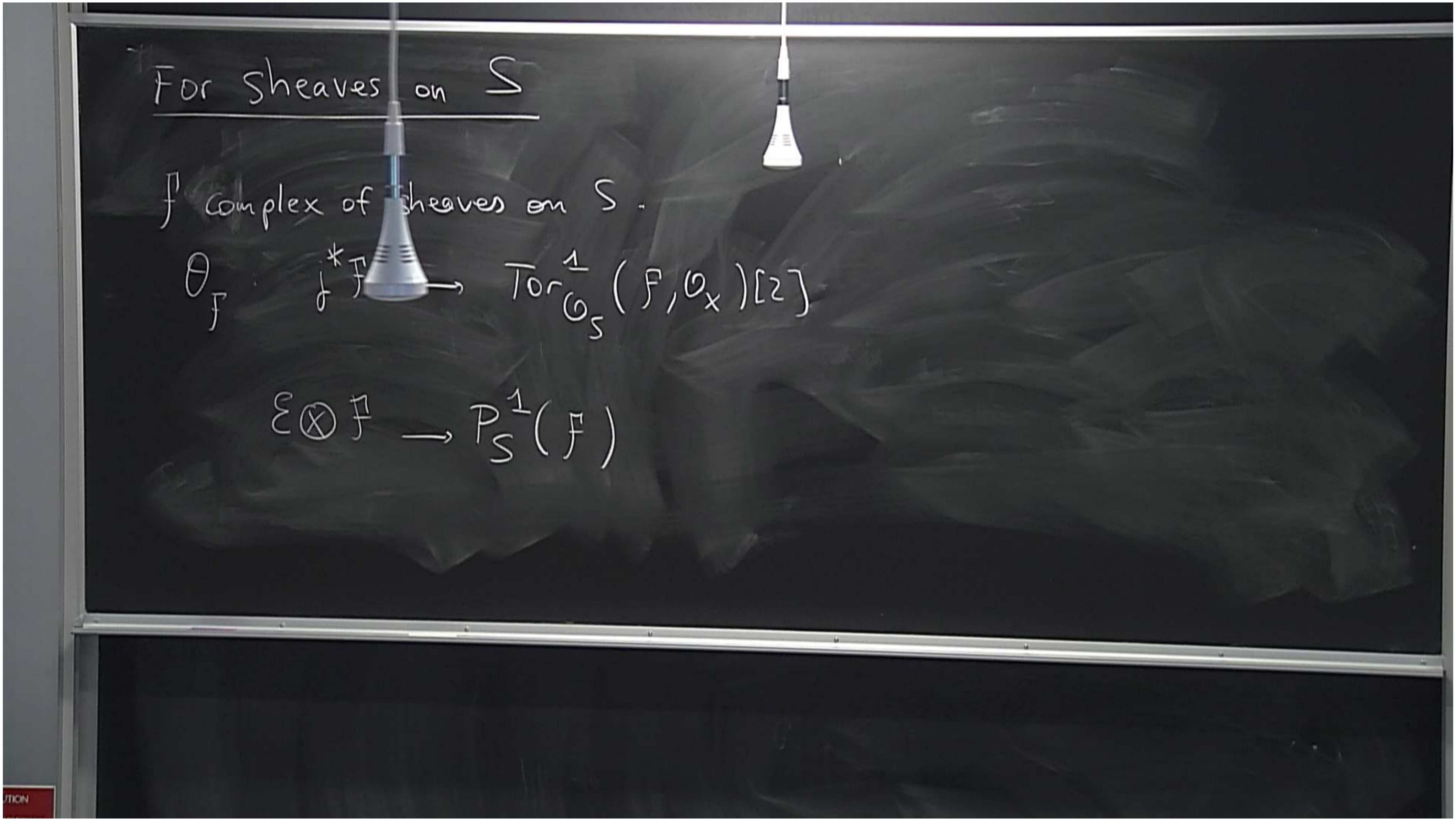
$K \in D^{perf}(X)$ $\quad$ **Q** when is K in $\mathrm{Im}(\mathbb{L}j^* : D^{perf}(S) \to D^{perf}(X))$?

Th Huybrechts - Thomas

$$\begin{tikzcd} K \arrow[r, "\theta_K"] \arrow[d, "at_K"'] & N^*_{X/Y} \otimes K[2] \\ \Omega^1_X \otimes K[1] \arrow[ur, bend right, "\xi \otimes id_K"'] & \end{tikzcd}$$

$K \cong \mathbb{L}j^* \mathcal{L} \quad \mathcal{L} \in D^{perf}(S)$

$\iff \theta_K = 0$


For sheaves on S
$\mathcal{F}^{\cdot}$ complex of sheaves on S.
$\theta_{\mathcal{F}} \quad j^{*}\mathcal{F} \longrightarrow \mathrm{Tor}^{1}_{\mathcal{O}_S}(\mathcal{F}, \mathcal{O}_X)[2]$
$\mathcal{E} \otimes \mathcal{F} \longrightarrow P^{1}_{S}(\mathcal{F})$

Thm TFAE

1) $\Theta_{\mathcal{F}} = 0$

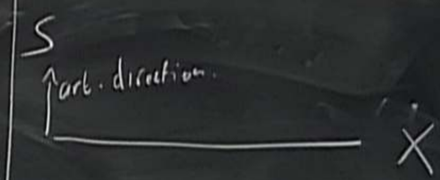
2) $\mathbb{L}j^*\mathcal{F} \longrightarrow j^*\mathcal{F}$ has a right inverse

3) $\exists$ "admissible" bounded complex $\mathcal{L}$ and $\mathcal{L} \to \mathcal{F}$ s.t.

$$\mathbb{L}j^*\mathcal{L} \longrightarrow \mathbb{L}j^*\mathcal{F} \longrightarrow j^*\mathcal{F} \text{ is an iso}$$

admissible

$\Leftrightarrow$ flat in the art. direction



$X \times_S X = \mathrm{Spec}\,(\underbrace{\mathbb{L}j^{*}\, \mathcal{O}_X}$

$\mathcal{O}_X \oplus \underbrace{\mathbb{L}j^{*} N^{*}_{X/Y}}$

$\tau^{\geq -1}(\quad) = \underset{(N^{*}_{X/Y})^{\otimes 2}}{+}\!\!\longrightarrow\!\!\underset{N^{*}}{+} \rightsquigarrow$ class in $\mathrm{Ext}^2(N^{*}, N^{*\otimes 2})$

$\underline{\mathrm{Th}}$ (AC) this class is $\Theta_{N^{*}_{X/Y}}$.

Finally, project to wt 2 to get map

$\mathrm{Perf}\, X = \cancel{\mathrm{Perf}}\; k[\varepsilon]$

[illegible]

Th (AC) • $\mathcal{V}$ v.b. on X.

then $\tau^{\geq -1}(\mathbb{L}j^{*}\mathcal{V}) = \mathrm{cone}\ \Theta_{\mathcal{V}}[-1]$

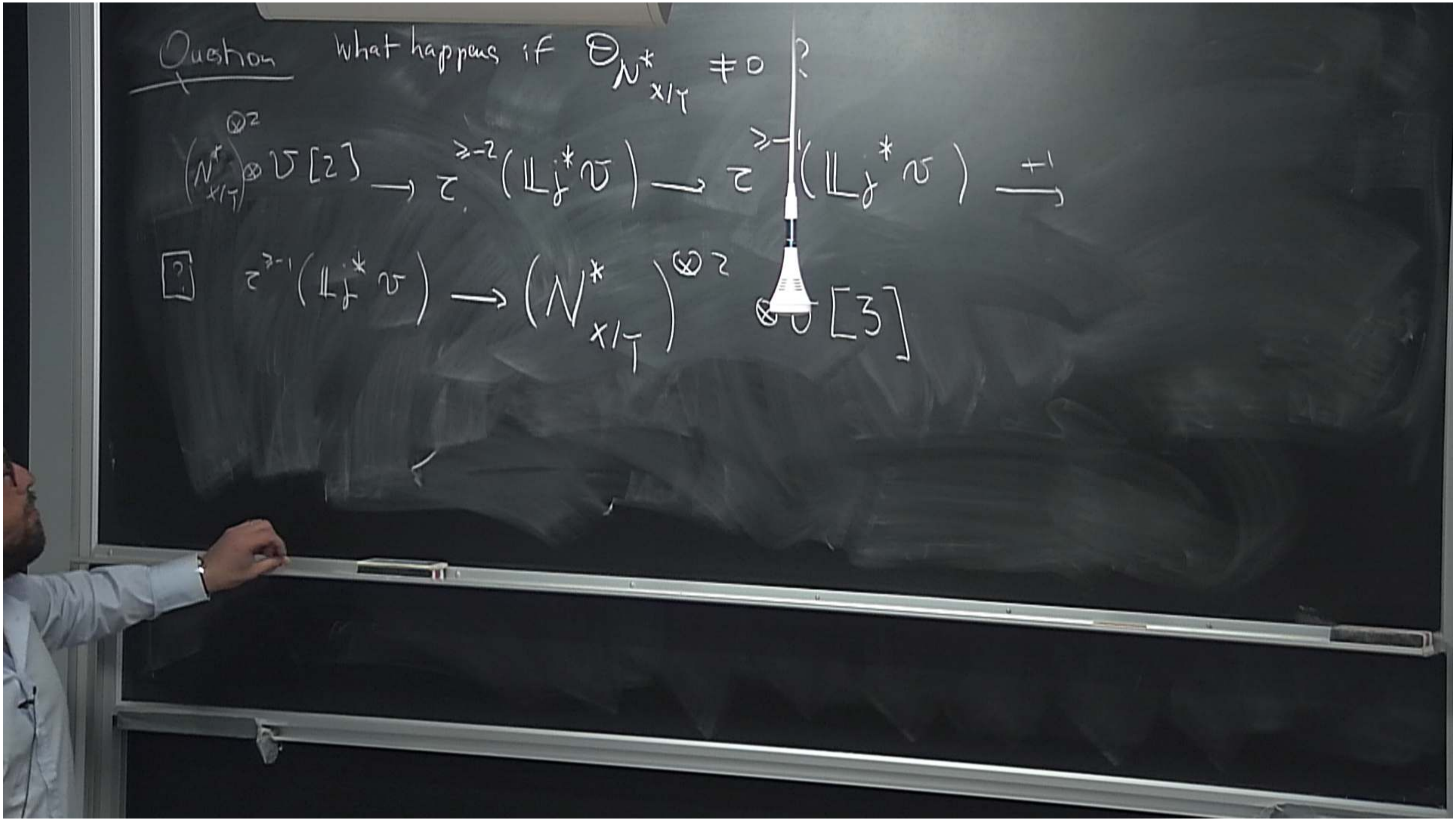
• $\mathbb{L}j^{*}(j_{*}\mathcal{V})$ is formal in $D^{-}(X)$

$\iff \mathbb{L}i^{*}(i_{*}\mathcal{V})$ is formal in $D^{-}(X)$

$\iff \Theta_{\mathcal{V}} = \Theta_{N^{*}_{X/Y}} = 0$

$$\begin{array}{cc} X \times_S X & X \times_Y X \\ \downarrow & \downarrow \\ X & X \end{array}$$

$$\boxed{\begin{array}{l} X \times_S X \text{ formal } /X \\ \iff X \times_Y X \text{ formal } /X \\ \iff \Theta_{N^{*}_{X/Y}} = 0 \end{array}}$$


Question what happens if $\Theta_{N^*_{X/Y}} \neq 0$?
$(N^*_{X/Y})^{\otimes 2} \otimes \mathcal{O}[2] \longrightarrow \tau^{\geq -2}(\mathbb{L}j^* \mathcal{O}) \longrightarrow \tau^{\geq -1}(\mathbb{L}j^* \mathcal{O}) \xrightarrow{+1}$
$\boxed{?}$ $\tau^{\geq -1}(\mathbb{L}j^* \mathcal{O}) \longrightarrow (N^*_{X/Y})^{\otimes 2} \otimes \mathcal{O}[3]$

$$
\begin{tikzcd}
 & (N^*_{X/Y})^{\otimes 2} \otimes K_v[3] \arrow[d] & \\
K_v \arrow[ur, bend left, "\text{no uniqueness}"] \arrow[r, "\theta_{K_v}"] \arrow[d] & N^*_{X/Y} \otimes K_v[2] \arrow[d] & \\
v \arrow[r, "\theta_v"] & N^*_{X/Y} \otimes v[2] \arrow[r, "+1"] \arrow[d, "+1"] & {} \\
 & {} &
\end{tikzcd}
$$

Sketch of construction

Thm $\mathcal{F}$ b. cx of sheaves of $\mathcal{O}_X$-modules

$\to$ $\mathbb{L}j^*(j_*\mathcal{F}) \simeq \varinjlim H^{[n]}(\mathcal{F})$

$\to$ If $\mathcal{F}$ is concentrated in deg. 0

$\tau^{\geq -n}\mathbb{L}j^*(j_*\mathcal{F}) \simeq H^{[n]}(\mathcal{F})$

Need to compose w/ the map

$H^{[n]}$

= equalizer of

$H^n \rightrightarrows H^{n-1}$

$H \to id$

$$\mathbb{L} i^*(i_* \mathcal{O}_X) \longrightarrow \mathbb{L} j^* j_*$$

loc

$$\bigoplus \Lambda^p (N^*_{X/Y})^{\otimes p}[p] \longrightarrow \bigoplus T^p N^*_{X/Y}[p] \longrightarrow$$

Symetrization anti-symetrization

Sketch of c[illegible]ruction

Need to compose w/ the map

$CC(\mathrm{Perf}\ M) \to CC(M)$

Conjectures

1) $X \times_Y X$ if formal over Y.

$\Downarrow$

2) weaker version $\quad X \times_Y X$ is formal over $\mathrm{Spec}(k)$

⚠ $X \times_S X$ is not formal over S

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Thm $\tau^{[p+1,p]} \coprod i^*(i_* \mathcal{O}_X)$ are formal in $D(Y)$.

Conjectures

1) $X \times_Y X$ if formal over Y.

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2) weaker version $X \times_Y X$ is formal over $\mathrm{Spec}(k)$

$\to$ ⚠ $X \times_S X$ is not formal over S

Thm $\tau^{[p+1,p]} \mathbb{L}i^*(i_*\mathcal{O}_X)$ are formal in $D(Y)$

$$\mathcal{O}_X \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_Y} \mathcal{O}_X \simeq \bigoplus \Lambda^i N^*_{X/Y}[i]$$

$D(Y)$

$\Theta_{\mathcal{V}}$ is giv[illegible]y the ex. seq

$$0 \to N^{*}_{X/Y} \otimes \mathcal{V} \to \boxed{\mathcal{E} \otimes \mathcal{V} \to P^{1}_{X}(\mathcal{V})} \to \mathcal{V} \to 0$$

Def $\quad C^{b}(X) \xrightarrow{H} C^{b}(X) \quad$ dg functor

$$\mathcal{V} \longrightarrow \mathrm{cone}\left(\mathcal{E} \otimes \mathcal{V} \longrightarrow P^{1}_{X}(\mathcal{V}) \right)$$

$$\mathcal{V}$$

$X \subset Y$ divisor $\quad \mathcal{V} \quad \Theta_{\mathcal{V}} \neq 0$

$$\mathcal{O}_{X} \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_{Y}} \mathcal{V}$$