

Title: Towards a general description of derived self-intersections

Date: Apr 20, 2016 11:00 AM

URL: <http://pirsa.org/16040080>

Abstract: Thanks to a result of Arinkin and Căldărău, the derived self-intersection of a closed smooth subscheme of an ambient scheme (over a field of characteristic zero) is a formal object if and only if the conormal bundle of the subscheme extends to a locally free sheaf at the first order. In this talk, we will explain a program as well as new results in order to describe these derived self-intersections in the non-formal case.

Go

schemes

$$\text{char}(k) = 0$$

$$X \subset Y$$

$$S = (X, \mathcal{O}_Y / \mathcal{I}_X^2)$$

$$j: X \hookrightarrow S$$

→ Understand the derived scheme

$$X \times_Y X$$

$$(Y = X^2)$$

$$X \hookrightarrow X^2$$

$$X \times_{X \times Y X} X = LX$$

→ Today complete description of $X \times_S X$

$$= \text{spec}(\text{Sym} \Omega_X^1)$$

Con. seq $0 \rightarrow N_{X/Y}^* \rightarrow \Omega_{Y/X}^1 \rightarrow \Omega_X^1 \rightarrow 0$

$$\eta \in \text{Ext}_{\mathcal{O}_X}^1(\Omega_X^1, N_{X/Y}^*)$$

1) Def th of perfect complexes

$K \in D^{perf}(X)$ Q when is K in $\text{Im}(\mathbb{L}_f^* : D^{perf}(S) \rightarrow D^{perf}(X))$?

Th Huybrechts-Thomas

$$\begin{array}{ccc}
 K & \xrightarrow{\theta_K} & N_{X/Y}^* \otimes K[2] \\
 \downarrow \text{at}_K & & \uparrow \cong \\
 \Omega_X^1 \otimes K[1] & &
 \end{array}$$

$$\begin{array}{l}
 K \cong \mathbb{L}_f^* \mathcal{L} \quad \mathcal{L} \in D^{perf}(S) \\
 \Leftrightarrow \theta_K = 0
 \end{array}$$

For sheaves on S

\mathcal{F} complex of sheaves on S .

$$\theta_{\mathcal{F}} \quad \mathcal{F}^* \rightarrow \mathrm{Tor}_{\mathcal{O}_S}^1(\mathcal{F}, \mathcal{O}_x)[2]$$

$$\mathcal{E} \otimes \mathcal{F} \rightarrow \mathcal{P}_S^1(\mathcal{F})$$

Thm TFAE

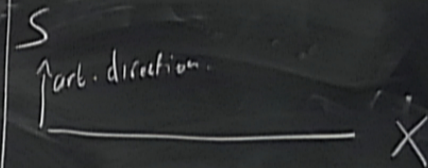
1) $\theta_{\mathcal{F}} = 0$

2) $\mathbb{L}_j^* \mathcal{F} \rightarrow j^* \mathcal{F}$ has a right inverse

3) \exists "admissible" complex \mathcal{L} and $\mathcal{L} \rightarrow \mathcal{F}$ s.t.
bounded

$$\mathbb{L}_j^* \mathcal{L} \rightarrow \mathbb{L}_j^* \mathcal{F} \rightarrow j^* \mathcal{F} \text{ is an iso}$$

admissible
 \Leftrightarrow flat in the art. direction



$$X \times_S X = \text{Spec} \left(\underbrace{\mathbb{L}_X \oplus \mathcal{O}_X}_{\geq -1} \right)$$

$$\mathcal{O}_X \oplus \underbrace{\mathbb{L}_X^* \otimes N_{X/Y}^*}_{\geq -1}$$

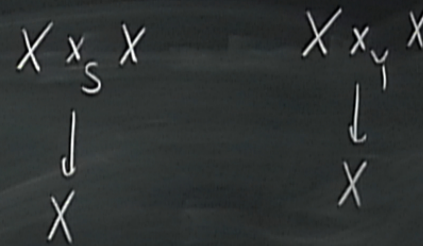
$(N_{X/Y}^*)^{\otimes 2} \oplus N^* \rightsquigarrow$ class in $\text{Ext}^2(N^*, N^{\otimes 2})$
Th (AC) this class is $\Theta_{N_{X/Y}^*}$.

Finally, project to wt 2 to get map

This part $X = \text{Proj } k[\epsilon]$
 and $Y = \text{Proj } k[\epsilon]$

$\mathcal{L}(AC)$ • \mathcal{V} v.b. on X
 then $\mathbb{Z} \rightarrow \mathbb{Z}^{-1}(\mathbb{L}_{\mathcal{L}}^* \mathcal{V}) = \text{cone } \theta_{\mathcal{V}} [1]$

- $\mathbb{L}_{\mathcal{L}}^* \mathcal{V}$ is formal in $\overline{\mathcal{D}}(X)$
- $\Leftrightarrow \mathbb{L}_{i(i)^*}^* \mathcal{V}$ is formal in $\overline{\mathcal{D}}(X)$
- $\Leftrightarrow \theta_{\mathcal{V}} = \theta_{N_{X/Y}^*} = 0$

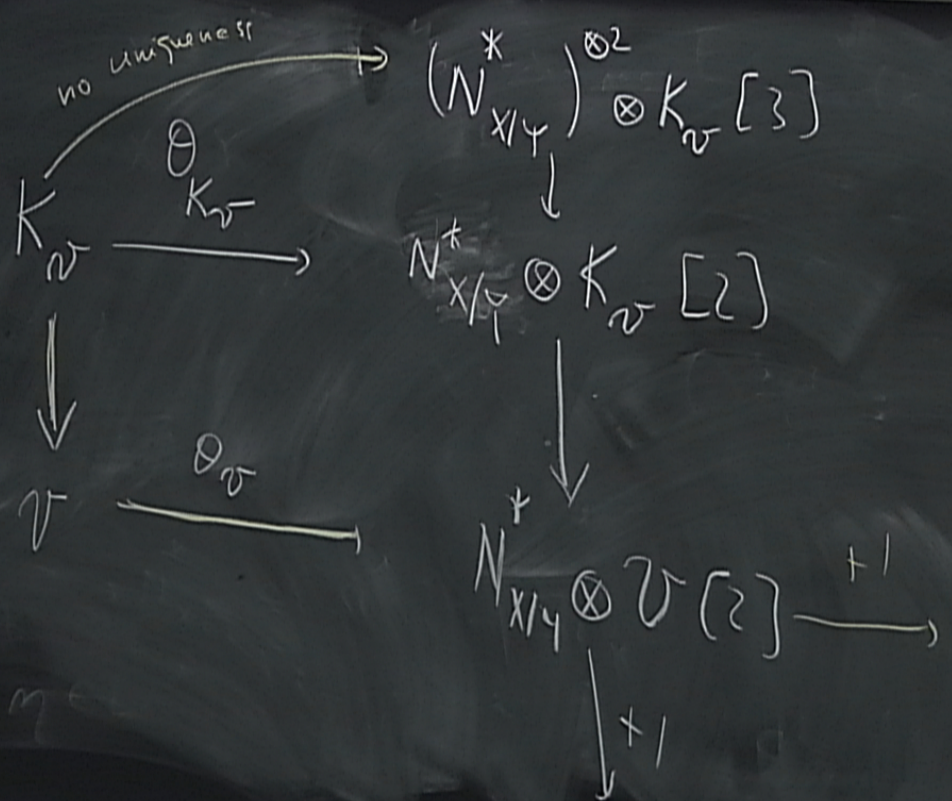


$X \times_S X$ formal / X
 $\Leftrightarrow X \times_Y X$ formal / X
 $\Leftrightarrow \theta_{N_{X/Y}^*} = 0$

Question What happens if $\Theta_{N^*_{x/y}} \neq 0$?

$$\left(N^*_{x/y} \right)^{\otimes 2} \otimes \mathcal{U}[2] \xrightarrow{\tau^{-2}} \left(\mathcal{L}_f^* \mathcal{U} \right)^{\otimes 2} \xrightarrow{\tau^{-1}} \left(\mathcal{L}_f^* \mathcal{U} \right)^{\otimes 2} \xrightarrow{+1}$$

$$\left[? \right] \tau^{-2} \left(\mathcal{L}_f^* \mathcal{U} \right)^{\otimes 2} \rightarrow \left(N^*_{x/y} \right)^{\otimes 2} \otimes \mathcal{U}[3]$$



Sketch of construction

Need to compose w/ the map

Thm $\Gamma_b(X, \mathcal{F})$ of sheaves of \mathcal{O}_X -modules

$$\rightarrow \varinjlim^* (j_* \mathcal{F}) \simeq \varinjlim H^{[m]}(\mathcal{F})$$

\rightarrow If \mathcal{F} is concentrated in deg. 0

$$\varinjlim_{n \leq -2}^* (j_* \mathcal{F}) \simeq H^{[n]}(\mathcal{F})$$

$H^{[n]}$

= equalizer of

$$H^n \begin{array}{c} \rightrightarrows \\ \longrightarrow \\ \longrightarrow \end{array} H^{n-1}$$

$H \rightarrow \text{id}$

$$\begin{array}{l}
 \text{loc} \\
 \hline
 \mathbb{L} i^* (i_* \mathcal{O}_X) \rightarrow \mathbb{L} \tilde{f}^* j_* \\
 \oplus \Lambda^p (N_{X/Y}^*)^{\otimes p} [p] \xrightarrow{\text{Symmetrization}} \oplus T^p N_{X/Y}^* [p] \xrightarrow{\text{anti-symmetrization}}
 \end{array}$$

Sketch of construction

Need to compose w/ the map
 $\rho: \mathbb{C}(M) \rightarrow \mathbb{C}(M)$

Conjectures

1) $X \times_Y X$ is formal over Y

\Downarrow

2) weaker version

$X \times_Y X$ is formal over $\text{Spec}(k)$

! $X \times_S X$ is not formal over S

Conjectures

1) $X \times_Y X$ if formal over Y

\Leftrightarrow

2) weaker version

$X \times_Y X$ is formal over $\text{Spec}(k)$

$\triangle!$ $X \times_S X$ is not formal over S

Thm $\tau^{[p+1, p]}$ $\prod_{i \geq 1} (i \circ \theta_X)$ are formal in $D(Y)$

Conjectures

1) $X \times_Y X$ is formal over Y

\Downarrow

2) weaker version

$X \times_Y X$ is formal over $\text{Spec}(k)$

\rightarrow $\triangle!$ $X \times_S X$ is not formal over S

Thm $\tau^{[p+1, p]}$ $\prod_i^* (i_* \mathcal{O}_X)$ are formal in $D(Y)$

$$\mathcal{O}_X \oplus^u_{\mathcal{O}_Y} \mathcal{O}_X \simeq \bigoplus_i \Lambda^i N_{X/Y}^* [i]$$

$D(Y)$

\mathcal{O}_V is given by the ex. seq

$$0 \rightarrow N_{X/Y}^* \otimes \mathcal{O}_V \rightarrow \left[\Sigma \otimes \mathcal{O}_V \rightarrow P_X^1(\mathcal{O}_V) \right] \rightarrow \mathcal{O}_V \rightarrow 0$$

Def $C^b(X) \xrightarrow{H} C^b(X)$ dg functor
 $V \rightarrow \text{cone}(\Sigma \otimes \mathcal{O}_V \rightarrow P_X^1(\mathcal{O}_V))$

$X \subset Y$ divisor $V \subset \mathcal{O}_V \neq 0$

$$\begin{array}{c} \mathcal{O}_X \otimes \mathcal{O}_Y \mathcal{O}_V \end{array}$$