Title: Relative non-commutative Calabi-Yau structures and shifted Lagrangians

Date: Apr 20, 2016 09:30 AM

URL: http://pirsa.org/16040079

Abstract: We give a definition of relative Calabi-Yau structure on a dg functor f: A --> B, discussing a examples coming from algebraic geometry, homotopy theory, and representation theory. When A=0, this returns the usual definition of Calabi-Yau structure on a smooth dg category B. When A itself is endowed with a Calabi-Yau structure and relative Calabi-Yau structure on f is compatible with the absolute structure on A, then we sketch the construction of a shifted symplectic structure on the derived moduli space M_A of pseudo-perfect A-modules, as well as the construction of a Lagrangian structure on the induced map f* : M_B --> M_A of derived moduli. This is joint work with Tobias Dyckerhoff.

There is an adjunction

$$\mathrm{Mod}^k \underset{\mathrm{RHom}_{S^e}(S,-)}{\overset{S\otimes_k -}{\rightleftarrows}} \mathrm{Mod}^{S^e}$$

$$\mathrm{RHom}_{S^e}(S,-) \simeq S^! \otimes_{S^e} - \qquad S^! = \mathrm{RHom}_{S^e}(S, S^e)$$

pre[illegible] when $S \otimes_k -$ takes perfects to perfects

$$\iff S \in \mathrm{Perf}^{S^e}$$

Such a dg category is called 'smooth'.

There is an adjunction

$$\mathrm{Mod}^k \underset{\mathrm{RHom}_{S^e}(S,-)}{\overset{S\otimes_k -}{\rightleftarrows}} \mathrm{Mod}^{S^e}$$

$$S^! \otimes_{S^e} - \simeq \qquad S^! = \mathrm{RHom}_{S^e}(S, S^e)$$

pre[illegible] when
$S\otimes_k -$ takes perfects
to perfects

$\Leftrightarrow \quad S \in \mathrm{Perf}^{S^e}$

Such a dg category
is called 'smooth'.

If S smooth, get

$$\mathrm{Mod}_k \underset{-\otimes_{S^e} S}{\overset{-\otimes_k S^!}{\rightleftarrows}} \mathrm{Mod}_{S^e}$$

$$CC_*(S) \simeq S \otimes_{S^e} S \simeq \mathrm{RHom}_{S^e}(S^!, S)$$

There is an adjunction

$$\mathrm{Mod}^k \underset{\mathrm{RHom}_{S^e}(S,-)}{\overset{S\otimes_k -}{\rightleftarrows}} \mathrm{Mod}^{S^e}$$

$$S^! \otimes_{S^e} - \simeq$$

$$S^! = \mathrm{RHom}_{S^e}(S, S^e)$$

prec[illegible] when

$S\otimes_k -$ takes perfects to perfects

$\Leftrightarrow \; S \in \mathrm{Perf}^{S^e}$

Such a dg category is called 'smooth'.

If S smooth, get

$$\mathrm{Mod}_k \underset{-\otimes_{S^e} S}{\overset{-\otimes_k S^!}{\rightleftarrows}} \mathrm{Mod}_{S^e}$$

$$CC_*(S) \simeq S \otimes_{S^e} S \simeq \mathrm{RHom}_{S^e}(S^!, S)$$

Hochschild chains

has extra symmetry making it a S^1/mixed complex b, B

differential

action of $[S^1]$.

Negative cyclic chains

$$CC_*(S)^{S^1} \simeq CC^-_*(S) = \left(CC_*(S)[[u]],\ b+Bu \right)$$

If coho grading, then

$$|b| = 1, \quad |B| = -1, \quad |u| = 2.$$

(Kontsevich, Ginzburg)

Def'n A Calabi-Yau structure of dim'n d on a (smooth) dg category S is

$$k[d] \xrightarrow{\theta} CC^-_*(S)$$

$$[\theta] \in HC^-_d(S)$$

s.t. under the map

$$CC^-_*(S) \to CC_*(S) \simeq R\mathrm{Hom}_{S^e}(S^!, S)$$

$$\theta \longmapsto \left(S^![d] \to S \right)$$

$\uparrow$ an equivalence.

Negative cyclic chains

$$CC_*(S)^{S^1} \simeq CC_*^-(S) = \left(CC_*(S)[[u]], b+Bu \right)$$

If coho grading, then

$|b|=1$, $|B|=-1$, $|u|=2$.

(Kontsevich, Ginzburg)

<u>Def'n</u> A Calabi-Yau structure of dim'n d on a (smooth) dg category S is

$$k[d] \xrightarrow{\theta} CC_*^-(S)$$

$$[\theta] \in HC^-_d(S)$$

s.t. under the map

$$CC_*^-(S) \to CC_*(S) \simeq R\mathrm{Hom}_{S^e}(S^!, S)$$

$$\theta \longmapsto \left(S^![d] \to S \right)$$

an equivalence.

Ganatra's thesis for A_∞-formulas / + Perutz, Sheridan.

dual notion involving S-coinvariants.

$\Rightarrow$ cyclically invariant non-degenerate pairings in Hom-spaces... strong CY category.

Show If S smooth, CY can show $S^{pr} = \{ s \in S \mid S(-,s) : S^{op} \to \mathrm{Perf}_k \}$ is CY in proper sense.

so that $\omega_X^\circ[-d]$ is a line bundle.

$S = \mathrm{Coh}(X)$,

$CC_*(S) \simeq \mathrm{RHom}_{X \times X}(\Delta_* \mathcal{O}, \Delta_* \omega_X^\circ)$

in particular

$HH_d(\mathrm{Coh}\, X) \simeq H^0(X, \omega_X^\circ[-d])$

The non-degeneracy cond'n of def'n $\Longleftrightarrow$ $\theta : \mathcal{O}_X \xrightarrow{\sim} \omega_X^\circ[-d]$

So $\mathrm{Coh}(X)$ being CY of dim'n d happens when X is CY.

Need a lift from HH to HC^-

$HH \Rightarrow HC^-$ spectral sequence

see $HC^-_d(\mathrm{Coh}\,X) \xrightarrow{\sim} HH_d(\mathrm{Coh}\,X)$

<u>Ex.</u> M connected compact oriented manifold w/ $[M] \in H_d(M;k)$

$\mathcal{S} = \mathrm{Loc}\,M \simeq \mathrm{Perf}\; C_*\Omega M$

<u>Goodwillie</u> $CC_*(\mathrm{Loc}\,M) \simeq C_*(LM)$

S^1 equiv.

Check non-degeneracy $[M] \in C_*(M)$ in terms of 'Spivak stable normal bundle'

$\Rightarrow$ Poincaré duality w/ local coefficients. <u>Malm.</u>

• 'Moduli of objects'

d'après Töen-Vaquie.

$$dSt_k \underset{\mathcal{M}}{\overset{\mathrm{Perf}}{\rightleftarrows}} dgcat^{op}$$

quasi-equiv.

Given $S \in dgcat$, $\mathcal{M}_S$ is ...

$$\mathrm{Map}_{dSt}(\mathrm{Spec}\,A, \mathcal{M}_S) \simeq \mathrm{Map}_{dgcat^{op}}(\mathrm{Perf}\,A, S)$$

$$\simeq \mathrm{Map}_{dgcat}(S^{op}, \mathrm{Perf}_A)$$

Thm (Toën-Vaquie) If S is of finite type, then $\mathcal{M}_S$ is locally of finite presentation & locally geometric. And moreover given k-pt $s \in \mathcal{M}_S$

$$T_{\mathcal{M}_S|s} \simeq \mathrm{RHom}(s,s)[1].$$

Ex. X f.t. scheme, $\mathcal{S} = \mathrm{Coh}(X)$

Then k-pts of $\mathcal{M}_{\mathcal{S}}$ are given by $\mathrm{Coh}(X)^{op} \longrightarrow \mathrm{Perf}_k$

$$\mathrm{RHom}(-, \mathcal{F})$$

Restrict $\mathcal{F}$ to be perfect and have compact support.

can show $\mathcal{S}^{pr} = \{ s \in \mathcal{S} \mid \mathcal{S}(-,s) : \mathcal{S}^{op} \to \mathrm{Perf}_k \}$ in particular

is CY in proper sense.

$HH_d(\mathrm{Coh}\, X) \simeq H^0(X, \omega_X^{\circ}[-d])$

Ex. X f.t. scheme, $\mathcal{S} = \mathrm{Coh}(X)$

Then k-pts of $\mathcal{M}_{\mathcal{S}}$ are given by $\mathrm{Coh}(X)^{op} \xrightarrow{} \mathrm{Perf}_k$

$R\mathrm{Hom}(-,\mathcal{F})$

Restrict $\mathcal{F}$ to be perfect and have compact support.

Thm (- Dyckerhoff)

Let $\mathcal{S}$ be a finite type dg category w/ CY structure $\theta \in HC^{-}_{d}(\mathcal{S})$ of dim'n d.

Then $\mathcal{M}_{\mathcal{S}}$ has an induced symplectic structure of deg $2-d$.

is called 'smooth'. differential of $[S^1]$

Sketch of construction

The universal property of $\mathcal{M}_S$ gives a universal module

$$S^{op} \longrightarrow \mathrm{Perf}\,\mathcal{M}_S$$

adjoint to $\mathcal{M}_S = \mathcal{M}_S$.

Apply CC_*:

$$CC_*(S) \simeq CC_*(S^{op}) \xrightarrow{S^1\text{-equiv.}} CC_*(\mathrm{Perf}\,\mathcal{M}_S)$$

Need to compose w/ the map

$$CC_*(\mathrm{Perf}\,\mathcal{M}_S) \longrightarrow CC_*(\mathcal{M}_S)$$

(Rmk. If $\mathcal{M}_S$ were quasi-compact, this would be an equiv.)

is called 'smooth'.

differential of $[S^1]$

Sketch of construction

The universal property of $\mathcal{M}_S$ gives a universal module

$$S^{op} \longrightarrow \mathrm{Perf}\, \mathcal{M}_S$$

adjoint to $\mathcal{M}_S = \mathcal{M}_S$.

Apply CC_*:

$$CC_*(S) \simeq CC_*(S^{op}) \xrightarrow{S^1\text{-equiv.}} CC_*(\mathrm{Perf}\, \mathcal{M}_S)$$

Need to compose w/ the map

$$CC_*(\mathrm{Perf}\, \mathcal{M}_S) \longrightarrow CC_*(\mathcal{M}_S)$$

(Rmk. If $\mathcal{M}_S$ were quasi-compact, this would be an equiv.)

Using Hochschild-Kostant-Rosenberg get an equiv of mixed complexes

$$CC_*(\mathcal{M}_S) \xrightarrow{\sim} DR(\mathcal{M}_S)$$

$$k[d] \xrightarrow{\theta} CC_*^-(S)$$

$$[\theta] \in HC^-_d(S)$$

Finally, project to wt 2 to get map

$$CC^-_*(S) \longrightarrow DR(M_S)(2)$$

$$\theta \longmapsto \omega.$$

$$k[d] \xrightarrow{\theta} CC_*^-(S)$$

$$[\theta] \in HC_d^-(S)$$

Finally, project to wt 2 to get map

$$CC_*^-(S) \longrightarrow DR(M_S)(2)$$

$$\theta \longmapsto \omega.$$

Get closed 2-form.

Need to check non-degenerate.

$$k[d] \xrightarrow{\theta} CC_*^-(S)$$

$$[\theta] \in HC^-_d(S)$$

Finally, project to wt 2 to get map

$$CC_*^-(S) \longrightarrow DR(M_S)(2)$$
$$\theta \longmapsto \omega.$$

Get closed 2-form.

Need to check non-degenerate.

This involves comparing two a priori different maps

$$\mathbb{T}_{M_S}[-1] \longrightarrow \mathbb{T}^*_{M_S}[1-d]$$

Compatibility is a generalised compatibility between Chern/Atiyah.