

Title: Symplectic and Lagrangian structures on mapping stacks

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Abstract: An important result in shifted symplectic geometry is the existence of shifted symplectic forms on mapping spaces with symplectic target and oriented source. I provide several examples of more complicated situations where stacks of maps shifted symplectic structures, or maps between them have Lagrangian structures. These include spaces of framed maps, pushforwards of perfect complexes, and perfect complexes on open varieties.

Symplectic and Lagrangian Structures on Mapping Stacks

Original motivation
Thm (PTVV). If X is a derived stack

$\text{Map}(X, -)$ sends n -shifted symplectic stacks
to $(n-d)$ -shifted " " " " " "

and sends Lagrangian morphisms
to L
Example: If X

CAUTION
DO NOT TOUCH THE BOARD SURFACE
AS IT IS COVERED BY A THIN LAYER OF POLYURETHANE
AND COULD BE DAMAGED

Original motivation
Thm (PTVV). If X is a derived stack
(compact, oriented in dim d)
 $\text{Map}(X, -)$ sends n -shifted symplectic stacks
to $(n-d)$ -shifted " " " "

ptv, if X

CAUTION
DO NOT TOUCH THE BOARD
OR THE BOARDER
AS IT IS HEAVY AND MAY
FALL OFF THE WALL
PLEASE BE CAREFUL

$\text{Map}(X, -)$ sends n -shifted symplectic stacks
to $(n-d)$ -shifted " " " " " "

and

Example. If X CYd then everything works

Goal: Find shifted symplectic structures on other
Stacks of maps



Ex: $X = \mathbb{P}^n$, $D = L$, $Y = BG$, $g = \mathcal{D} = G \times L$

(Donaldson), $\text{Map}(\mathbb{P}^n, L, BG, \mathcal{D})$ has a symplectic structure

$X = \mathbb{P}^1$, $D = p$, $Y = G/B$, $g = \mathfrak{g}$

(FKMM); $\text{Map}(\mathbb{P}^1, p, G/B, \mathfrak{g})$ has a symplectic structure

CAUTION

Ex: $X = \mathbb{P}^n$, $D = L$, $Y = BG$, $g = \mathcal{H} = G \times L$

(Donaldson), $\text{Map}(\mathbb{P}^n, L, BG, \mathcal{H})$ has a symplectic structure

• $X = \mathbb{P}^1$, $D = p$, $Y = G/B$, $g = \mathfrak{g}$

(FKMM); $\text{Map}(\mathbb{P}^1, p, G/B, \mathfrak{g})$ has a symplectic structure

CAUTION

Thm (-): $\mathbb{P}^1 \times X$ is a Fano
 \tilde{D} is ^{smooth} effective anticanonical

D, E are effective divisors

$$\tilde{D} = 2D + E$$

Y is n -shifted symplectic

$g: D \rightarrow Y$ is a map such that

$$\text{Map}(D+E, Y) \rightarrow \text{Map}(D, Y)$$

is étale over g

γ is a map such that

$$\text{Map}(D+E, Y) \rightarrow \text{Map}(D, Y)$$

is étale over γ

Then $\text{Map}(X, D, Y, g)$ has a $(n-d)$ -shifted structure

If $Z \rightarrow Y$ has a Lagrangian structure

$\text{Map}(X, D, Z, g) \rightarrow \text{Map}(X, D, Y, g)$ has a Lagrangian structure

$$X = \mathbb{P}^2$$

$$D = E$$

γ is a map such that

$$\text{Map}(D+E, Y) \rightarrow \text{Map}(D, Y)$$

is étale over g

Then $\text{Map}(X, D, Y, g)$ has a $(n-d)$ -shifted structure

If $Z \rightarrow Y$ has a Lagrangian structure

$\text{Map}(X, D, Z, g) \rightarrow \text{Map}(X, D, Y, g)$ has a Lagrangian structure

Ex, $X = \mathbb{P}^2$
 $D = E = L$

CAUTION

Small text warning on the left side of the chalkboard.

γ is a map such that

$$\text{Map}(D+E, Y) \rightarrow \text{Map}(D, Y)$$

is étale over \mathfrak{g}

γ has a Lagrangian structure

$$\text{Map}(X, D, Z, \mathfrak{g}) \rightarrow \text{Map}(X, D, Y, \mathfrak{g}) \text{ has a Lagrangian structure}$$

Ex; $X = \mathbb{P}^1$
 $D = E$

$$\text{Map}(ZL, BG) \rightarrow \text{Map}(L, BG) \text{ étale over } \mathfrak{D}$$

the only extension of Z to ZL is $ZL \times G$

CAUTION
DO NOT CLIMB OR SWING FROM THE CHALKBOARD
IF YOU DO YOU MAY BE INJURED
OR EVEN KILLED

f is a map such that

$$\text{Map}(D+E, Y) \rightarrow \text{Map}(D, Y)$$

is étale over g

$\text{Map}(X, D, Z, g) \rightarrow \text{Map}(X, D, Y, g)$ has a Lagrangian structure

Ex; $X = \mathbb{P}^2$
 $D = E = L$

$$\text{Map}(ZL, BG) \rightarrow \text{Map}(L, BG) \text{ étale over } D$$

the only extension to ZL is $ZL \times G = \tilde{Z}$

$$\pi$$

CAUTION
DO NOT CLIMB ON BOARDING BRACKETS
OR STAND ON THE BRACKETS OF THE BOARD
ALL INFORMATION IS UNCLASSIFIED
DATE 01/20/2010 BY 60322 UCBAW/STP

γ is a map such that

$$\text{Map}(D+E, Y) \rightarrow \text{Map}(D, Y)$$

is étale over g

Then $\text{Map}(X, D, Y, g)$ has a $(n-d)$ -shifted structure

If $Z \rightarrow Y$ has a Lagrangian structure

$\text{Map}(X, D, Z, g) \rightarrow \text{Map}(X, D, Y, g)$ has a Lagrangian structure

Ex: $X = \mathbb{P}^2$
 $D = E = L$

$$\text{Map}(ZL, BG) \rightarrow \text{Map}(L, BG) \text{ étale over } D$$

the only extension of Z to ZL is $ZL \times G = \tilde{Z}$

$$\pi_{\text{Map}(ZL, BG)} \rightarrow \pi_{\text{Map}(L, BG)}$$

$$\mathbb{R}\Gamma(L, \text{ad}_G) \rightarrow \mathbb{R}\Gamma(ZL, \mathcal{O}_{ZL} \otimes \text{ad}_G) \rightarrow \mathbb{R}\Gamma(L, \mathcal{O} \otimes \text{ad}_G)$$



Original motivation
 Thin (PTVV). If X is a derived stack
 (compact, oriented in dim d)
 $\text{Map}(X, -)$ sends n -shifted symplectic stacks
 to $(n-d)$ -shifted

PTVV. For $f: X \rightarrow Y$
 the pairing is $\mathbb{R}^T \text{Map}(X, f) = \mathbb{R}^T \text{Map}(X, f^* \pi_*)$
 $\rightarrow \mathbb{R}^T(X, f^* \pi_*)$
 $\rightarrow \mathbb{R}^T(X, \mathcal{O}_Y)$
 $\rightarrow \mathbb{R}^T(X, \mathcal{O}_X)$
 $\rightarrow \mathbb{R}^T(X, \mathcal{O}_X)$

for maps $(D, \omega, g) \rightarrow (D', \omega', g')$
 Space of maps $(X, \omega, g) \rightarrow (Y, \omega', g')$ with $\mathbb{R}^T \omega = \mathbb{R}^T \omega'$
 $\text{Map}(X, D, \omega, g) = \text{Hom}_{\mathbb{R}^T}(\text{Map}(X, Y), \text{Map}(D, D'))$

$\mathbb{R}^T \text{Map}(X, D, \omega, g) = \mathbb{R}^T \text{Map}(X, f^* \pi_*)$
 $= \mathbb{R}^T(X, f^* \pi_*) = \mathbb{R}^T(Y, \pi_* f_*)$
 $\rightarrow \mathbb{R}^T(Y, \pi_* f_* \mathcal{O}_X)$
 $\rightarrow \mathbb{R}^T(Y, \mathcal{O}_Y)$

$D = D' = 0$
 Y is a shifted symplectic
 \mathcal{O}_Y is a map back along
 $\text{Map}(D, D') = \text{Map}(0, 0) = \text{pt}$
 is the same g

Then $\text{Map}(X, D, \omega, g)$ has a $(n-d)$ -shifted structure
 If \mathcal{O}_Y has a Lagrangian structure
 $\text{Map}(D, D') \rightarrow \text{Map}(D, D')$ has a Lagrangian structure
 For $X = \mathbb{P}^2$, $D = \mathbb{A}^1$, $\omega = 0$, $g = 0$
 $D' = \mathbb{A}^1$, $\omega' = 0$, $g' = 0$
 then the structure of \mathbb{Z} on \mathbb{Z} is $\mathbb{Z} \oplus \mathbb{Z}$
 $\mathbb{R}^T \text{Map}(X, D, \omega, g) = \mathbb{R}^T \text{Map}(X, \mathbb{A}^1) = \mathbb{R}^T(X, \mathcal{O}_Y)$
 $\mathbb{R}^T(X, \mathcal{O}_Y) = \mathbb{R}^T(X, \mathcal{O}_X) = \mathbb{R}^T(X, \mathcal{O}_X)$

"Poisson" structures

Conj: n -shifted Poisson
Structures on X



Formal stack \mathcal{Y} with $(n+1)$ -shifted
Symplectic structure

And $X \rightarrow \mathcal{Y}$ with a Lagrangian
structure

Such that $(\Lambda^* T^* X, d_{\mathbb{R}}) \rightarrow (\Lambda^* E, \varepsilon)$ is a morphism of mixed complexes

$\mathbb{P}^1 \times$ Fano d -dimensional
 \hat{D} effective anticanonical
effective divisors
 $+E$

CAUTION
DO NOT TOUCH THE BOARD
OR THE SURFACE OF THE BOARD
OR THE SURFACE OF THE BOARD
OR THE SURFACE OF THE BOARD

Such that $(\Lambda^k T^u X, d_{dR}) \rightarrow (\Lambda^k \mathcal{E}, \mathcal{E})$ is a morphism of mixed complexes

$(\Lambda^k T^u X, d_{dR}) \rightarrow (\Lambda^k T X, [\pi, -])$ is a foliation

So we can form $[X/\pi^b]$

$$\mathbb{L}_{[X/\pi^b]} = \left\{ \begin{array}{c} T^u X \\ \downarrow \pi^b \\ T X \end{array} \right\}$$

$$\Lambda^2 \mathbb{L}_{[X/\pi^b]} = \left\{ \begin{array}{c} \Lambda^2 T^u X \rightarrow T^u X \otimes T X \rightarrow \text{Sym}^2 T X \\ \downarrow \pi^b \\ \text{Sym}^2 T X \end{array} \right\}$$



Framed case,

$$D = L, E = 3E,$$

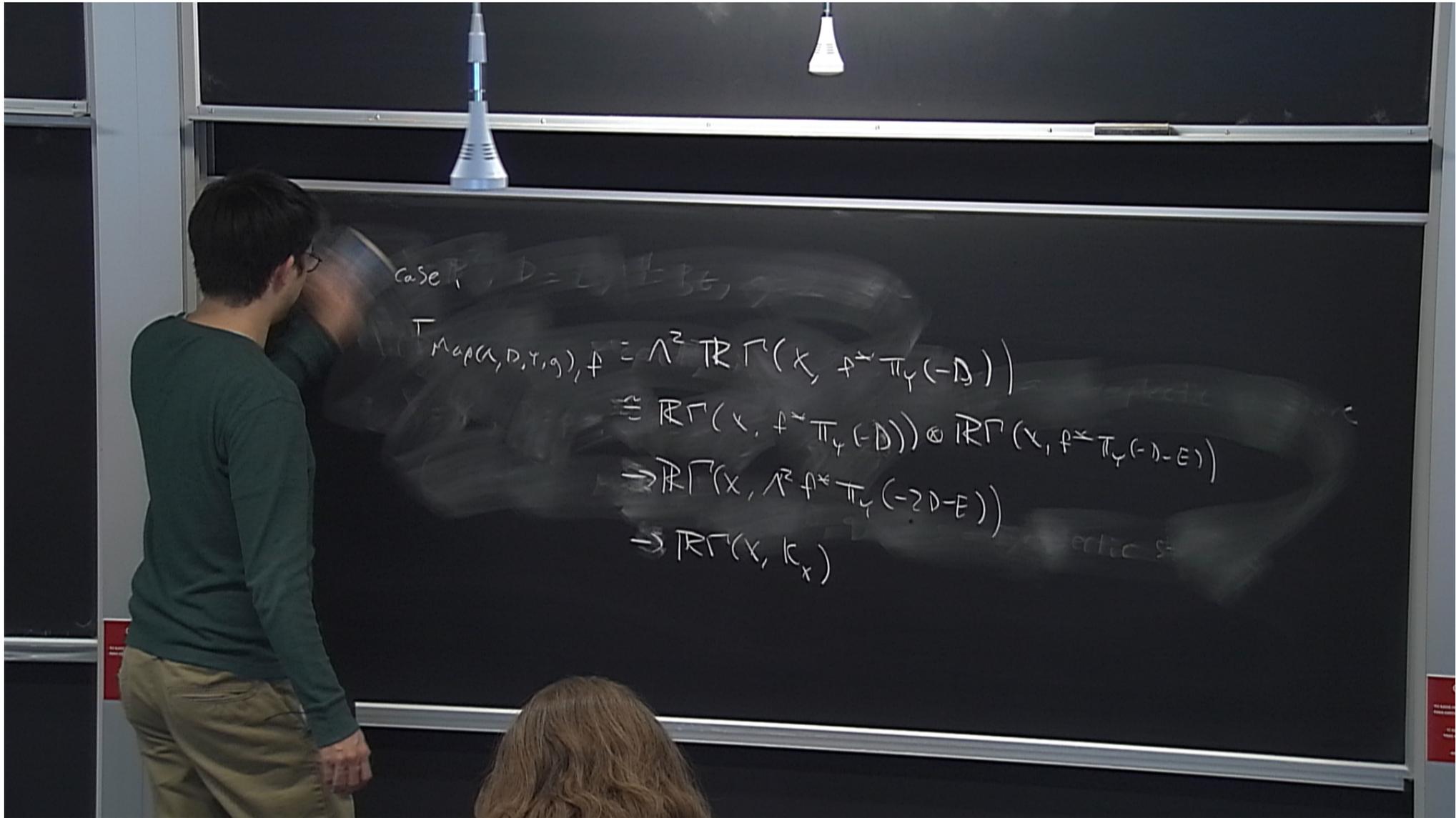
$$\Lambda^2 \pi_{\text{Map}(X, D, Y, g), f} = \Lambda^2 \text{TR} \Gamma(X, f^* \pi_Y(-D))$$

$$\cong \text{TR} \Gamma(X, f^* \pi_Y(-D)) \otimes \text{TR} \Gamma(X, f^* \pi_Y(-D-E))$$

$$\rightarrow \text{TR} \Gamma(X, \Lambda^2 f^* \pi_Y(-2D-E))$$

$$\rightarrow \text{TR} \Gamma(X, K_X)$$

CAUTION
Do not touch the board
Do not touch the board
Do not touch the board



case 1, $D = L, E = B, g = \dots$

$$T \text{Map}(X, D, Y, g), f = \Lambda^2 TR \Gamma(X, f^* \pi_Y(-D))$$

$$\cong R\Gamma(X, f^* \pi_Y(-D)) \otimes R\Gamma(X, f^* \pi_Y(-D-E))$$

$$\rightarrow R\Gamma(X, \Lambda^2 f^* \pi_Y(-2D-E))$$

$$\rightarrow R\Gamma(X, K_X)$$

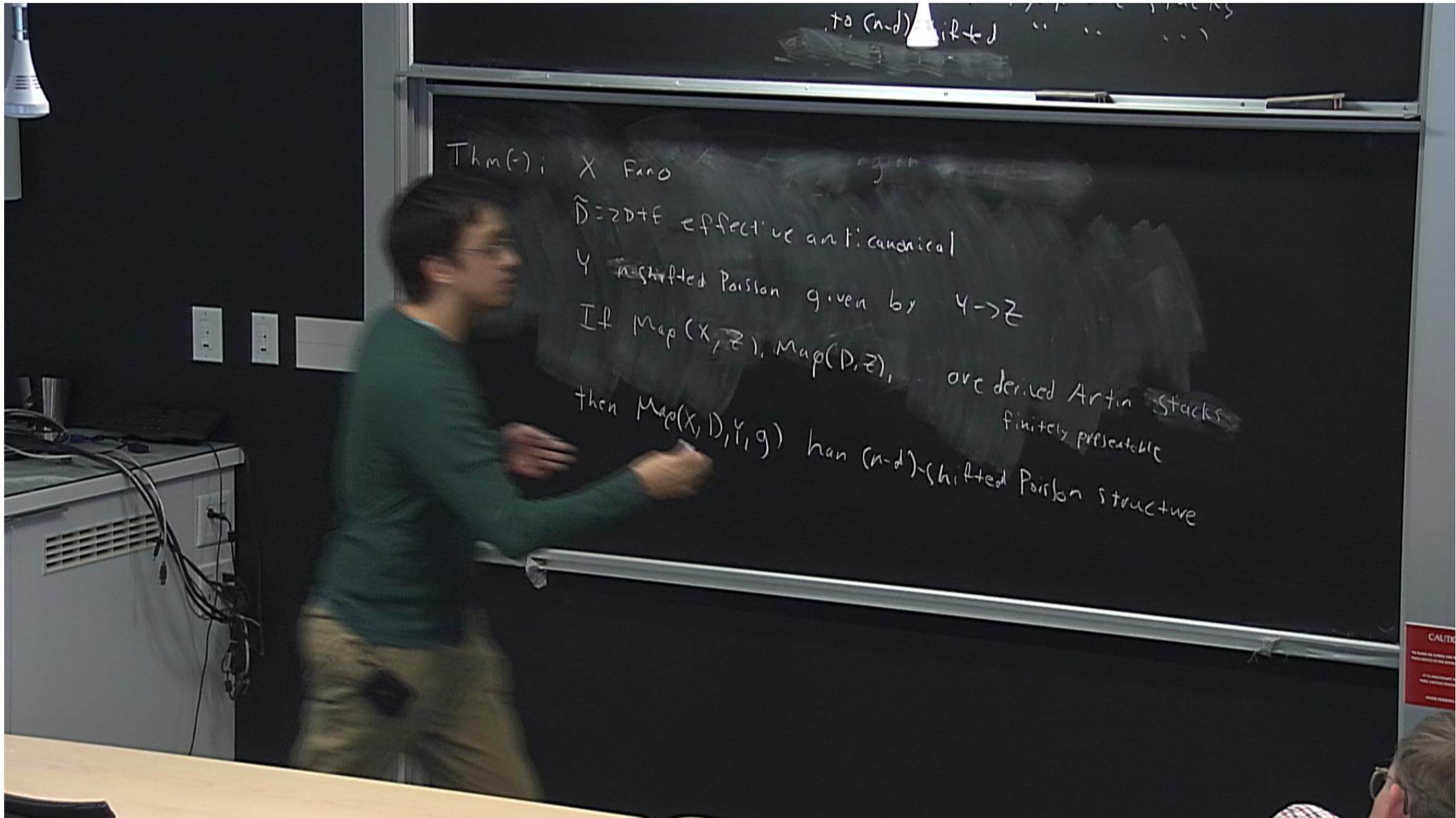
Def. ^{n-shifted} Poisson structure on X is $(n+1)$ -shifted symplectic Y
 $+ X \rightarrow Y$ with Lagrangian structure

coisotropic structure on $W \xrightarrow{g} X$ is

- $X' \rightarrow Y$ with a Lagrangian structure

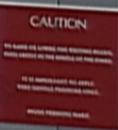
- Lift of g to $W \xrightarrow{\hat{g}} X \times_{\mathbb{A}^1} X'$

- Lagrangian structure on \hat{g}



$X \rightarrow Y$ with a Lagrangian structure
Lift of g to $W \xrightarrow{\hat{g}} X \times X'$

Ex: If \mathcal{D} is a non-trivial bundle on $L\mathbb{C}P^2$
then $\text{Map}(\mathbb{P}^2, L, \mathcal{B}\mathcal{G}, \mathcal{D})$ still has a Poisson structure



If $U = X \setminus D$ is open Calabi-Yau

then we can look at $\text{Perf}_Z(U)$

this has a $(2-d)$ -shifted structure

$$\text{Perf}_Z(U)_E = \text{Ext}_X^*(E, E)[1]$$

$$\begin{aligned} \cong \Pi \text{Perf}_Z(U)_E &= \Lambda^2 \text{Ext}_X^*(E, E)[2] \rightarrow \text{Ext}_X^*(E, E(\mathcal{O}_X)) [2] \\ &\rightarrow \mathbb{R}\Gamma \end{aligned}$$

CAUTION

DO NOT TOUCH THE BOARD WITH YOUR HANDS
OR FEET. IT IS A PUBLIC RESOURCE.
IT IS PROHIBITED TO WRITE
ON THE BOARD WITH YOUR HANDS.
PLEASE RESPECT THE BOARD.

If $U = X \setminus D$ is open Calabi-Yau
 then we can look at $\text{Perf}_E(U)$
 this has a $(2-d)$ -shifted structure

$$\pi_{\text{Perf}_E(U)} = \text{Ext}_X^d(E, E)[d]$$

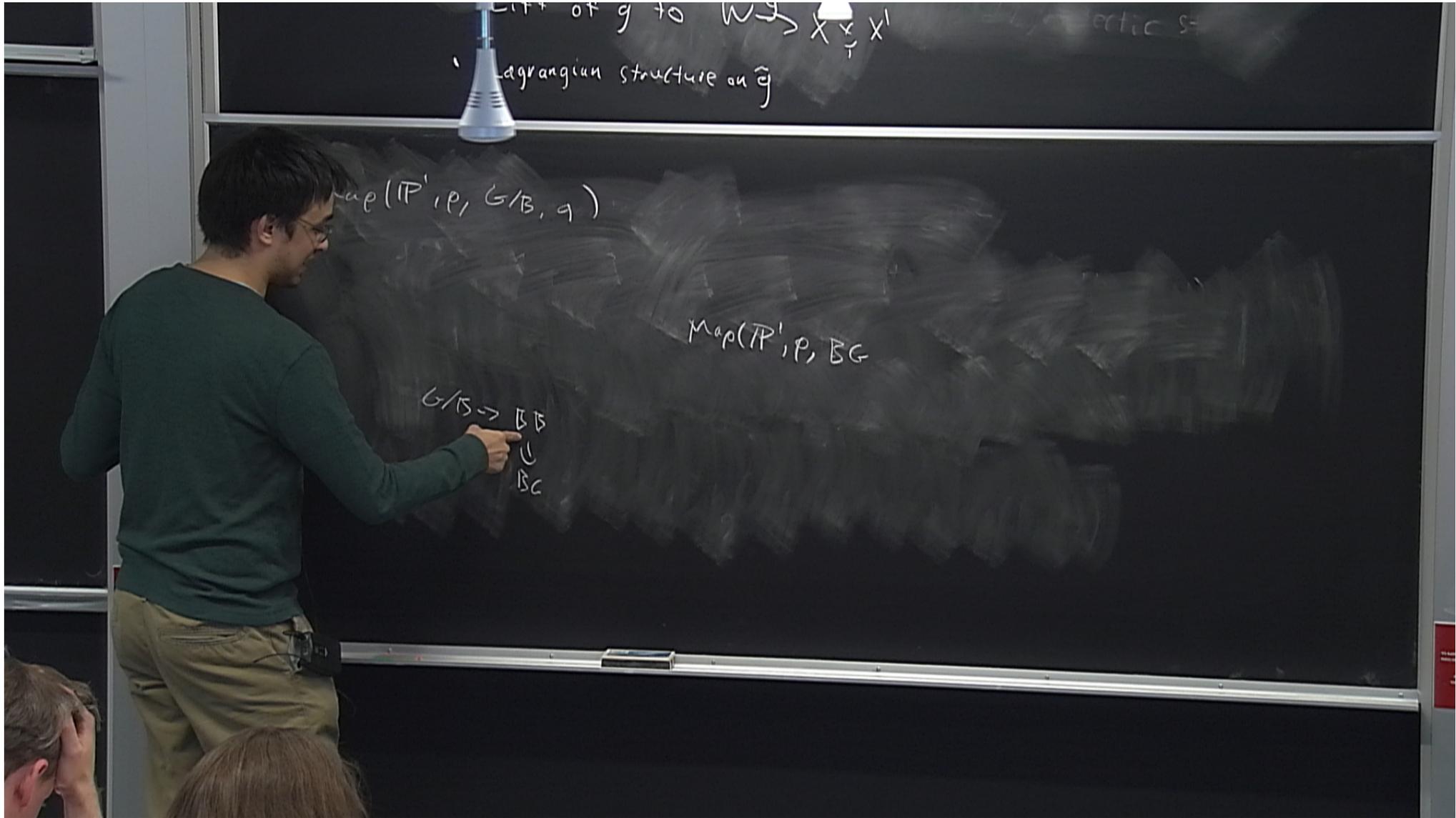
$$\begin{aligned} \wedge^2 \pi_{\text{Perf}_E(U)} = \wedge^2 \text{Ext}_X^d(E, E)[2] &\rightarrow \text{Ext}_X^d(E, E(\mathcal{O}_D)) [2] \\ &\rightarrow \text{R}\Gamma(X, \mathcal{K}_X)[2] \end{aligned}$$

If $U = X \setminus D$ is open Calabi-Yau
 then we can look at $\text{Perf}_c(U)$
 this has a $(2-d)$ -shifted structure

$$\pi_{\text{Perf}_c(U), E} = \text{Ext}_X^{\bullet}(E, E)[2]$$

$$\begin{aligned} \wedge^2 \pi_{\text{Perf}_c(U), E} &= \wedge^2 \text{Ext}_X^{\bullet}(E, E)[2] \rightarrow \text{Ext}_X^{\bullet}(E, E(\mathcal{O}_X)) [2] \\ &\rightarrow \mathbb{R}\Gamma(X, \mathcal{K}_X) [2] \end{aligned}$$

CAUTION
 DO NOT TOUCH THE BOARD
 IT IS PROTECTED BY A LOCK
 AND SHOULD REMAIN CLOSED



Lift of g to $W \rightarrow X, X, X'$ = Lagrangian structure on \tilde{g}

$\text{Map}(TP', P, G/B, g)$

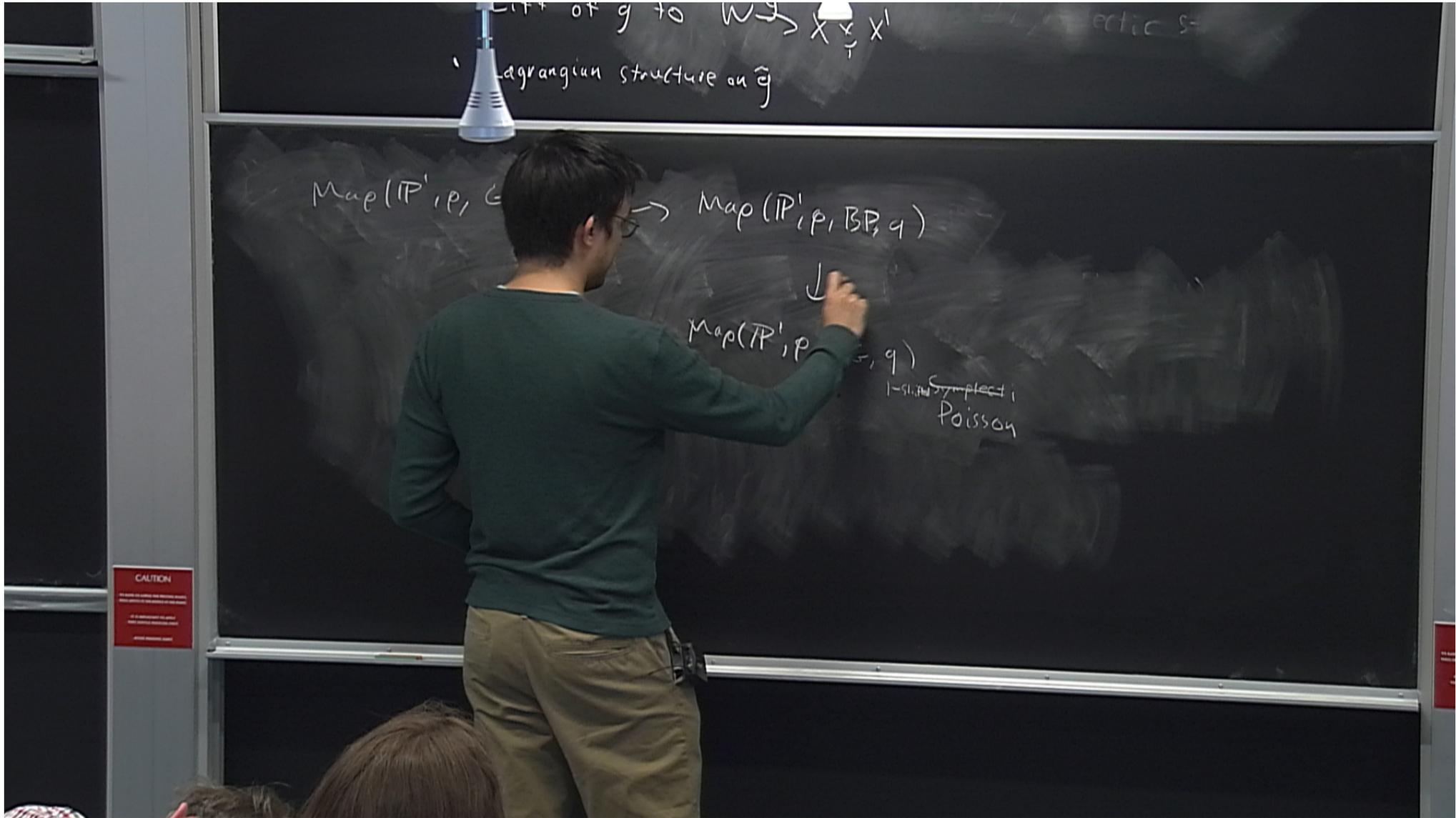
$\text{Map}(TP', P, BG)$

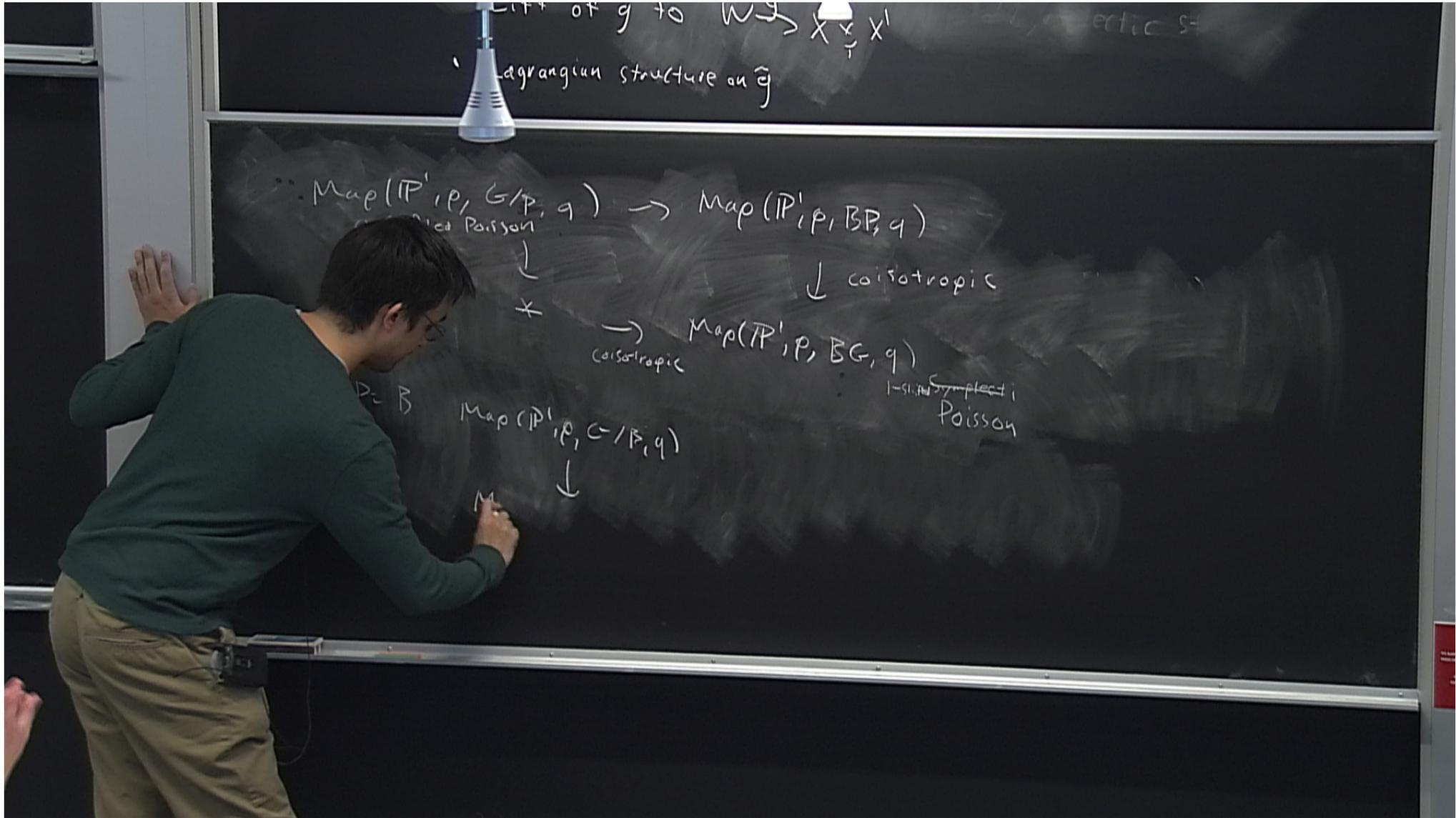
$G/B \rightarrow BB$
 \cup
 BG

Lift of g to $W \rightarrow X \times X'$ — Lagrangian structure on \tilde{g}

$$\begin{array}{ccc} \text{Map}(P', p, G/B, q) & \rightarrow & \text{Map}(P', p, BB, q) \\ \downarrow & & \downarrow \\ * & \rightarrow & \text{Map}(P', p, BG, q) \end{array}$$

$$\begin{array}{c} G/B \rightarrow BB \\ \downarrow \\ BG \end{array}$$





Lift of g to $W \rightarrow X \times X'$
 Lagrangian structure on \tilde{g}

$$\text{Map}(P', P, G/P, q) \rightarrow \text{Map}(P', P, BG, q)$$

0-shifted Poisson ↓ isotropic

$$\downarrow \times \rightarrow \text{Map}(P', P, BG, q)$$

isotropic 1-shifted symplectic Poisson

$P = B$

$$\text{Map}(P', P, G/P, q) \rightarrow \text{Map}(P', P, BG, q)$$

↓

$$\text{Map}(P', P, BT, q)$$

CAUTION