Title: Symplectic and Lagrangian structures on mapping stacks

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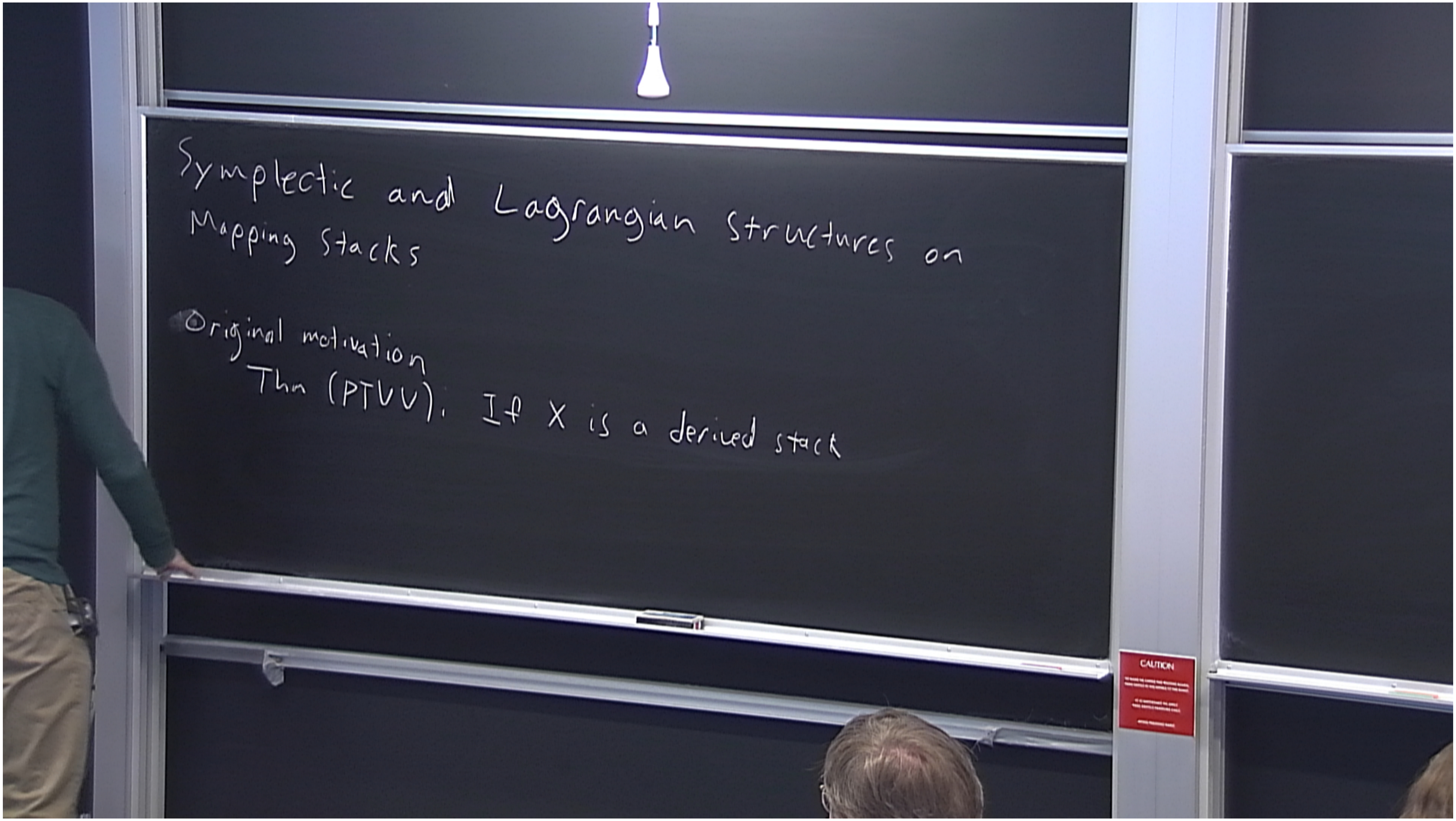
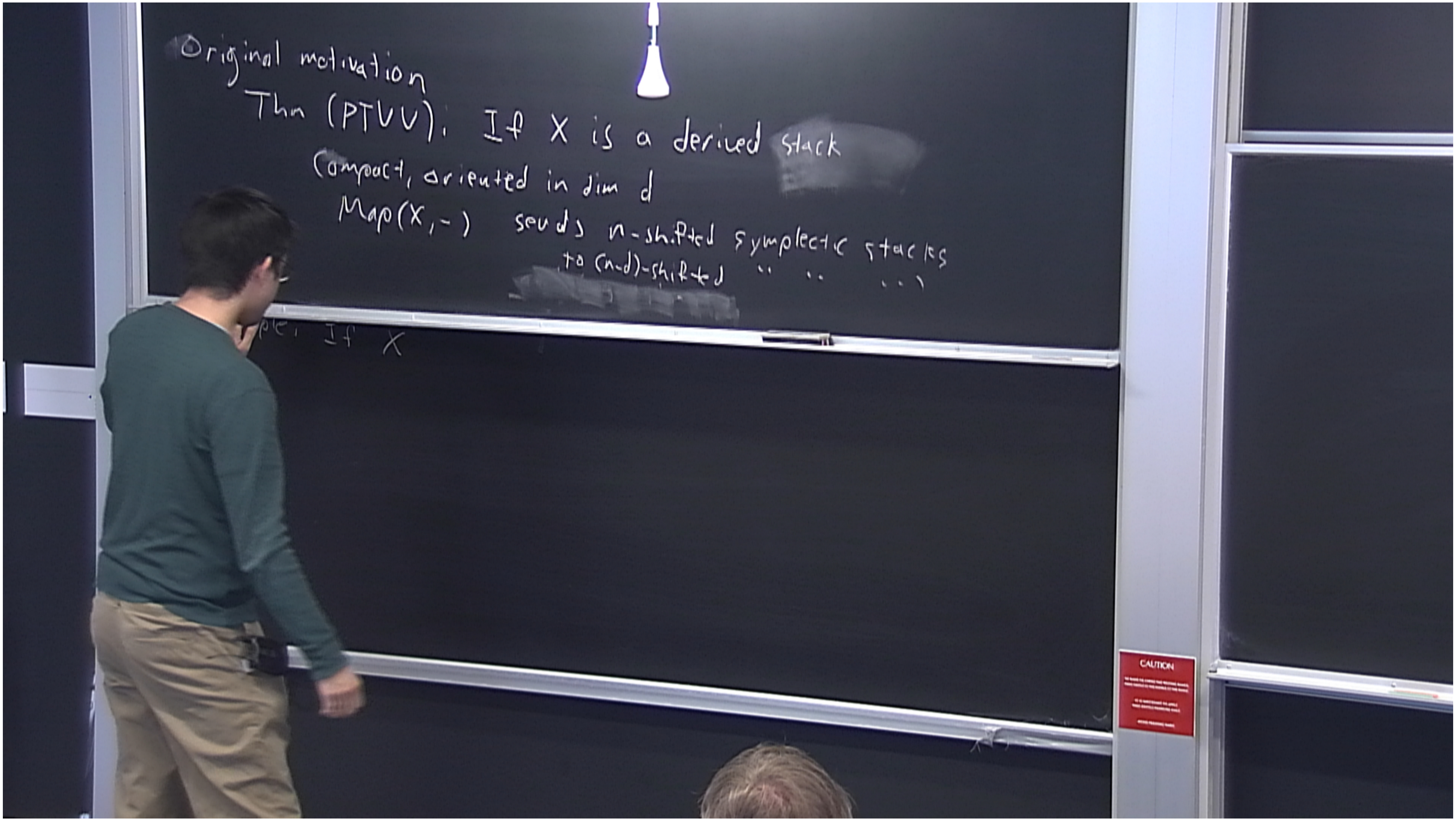
Abstract: An important result in shifted symplectic geometry is the existence of shifted symplectic forms on mapping spaces with symplectic target and oriented source. I provide several examples of more complicated situations where stacks of maps shifted symplectic structures, or maps between them have Lagrangian structures. These include spaces of framed maps, pushforwards of perfect complexes, and perfect complexes on open varieties.

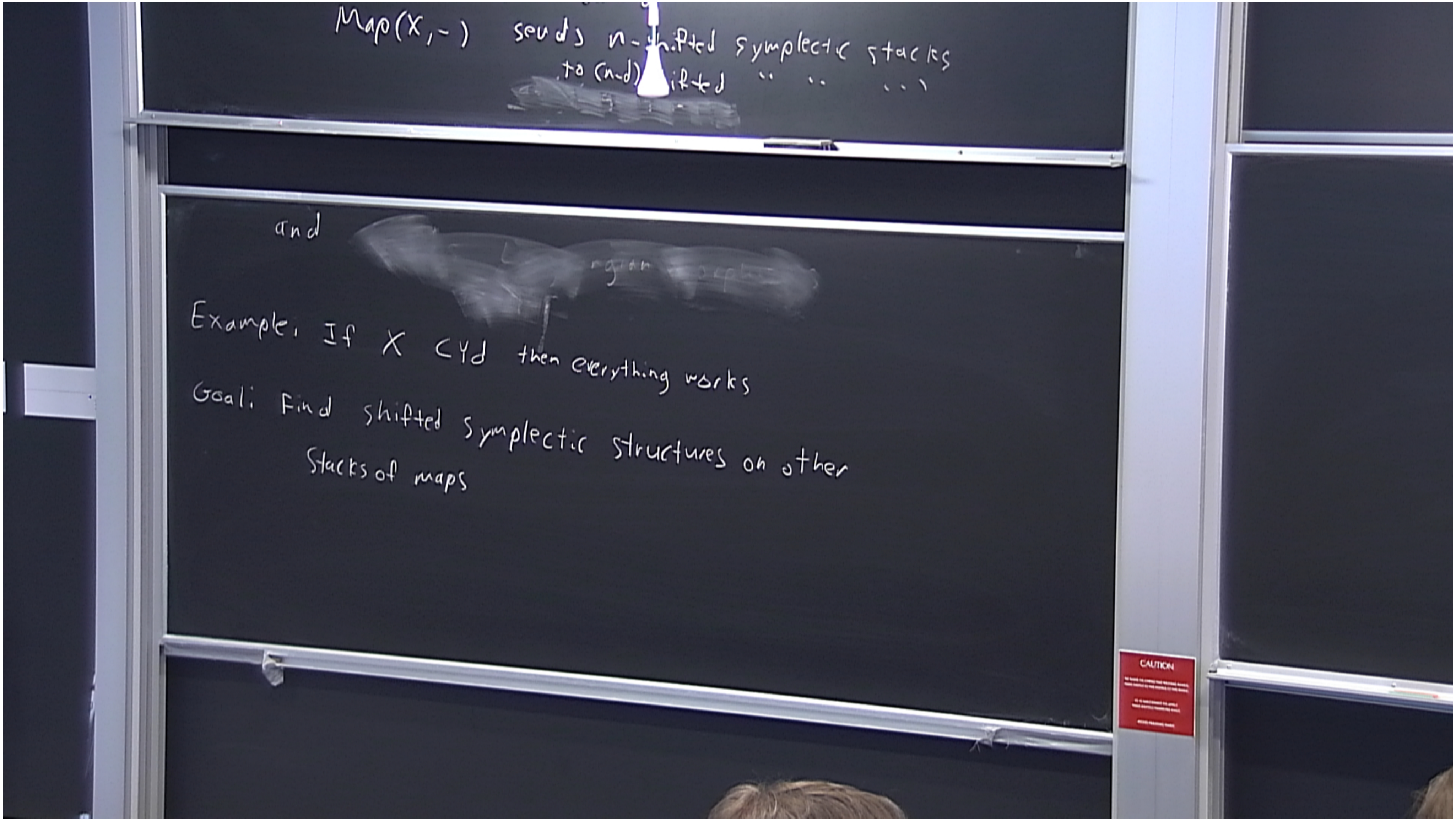

Symplectic and Lagrangian structures on
Mapping Stacks
• Original motivation
Thm (PTVV). If X is a derived stack

Map(X, -) sends n-shifted symplectic stacks
to (n-d) shifted " " ")

and sends Lagrangian morphisms
to L

Example: If X


Original motivation
Thm (PTVV): If X is a derived stack
compact, oriented in dim d
Map(X,-) sends n-shifted symplectic stacks
to (n-d)-shifted " " ")
If X
CAUTION

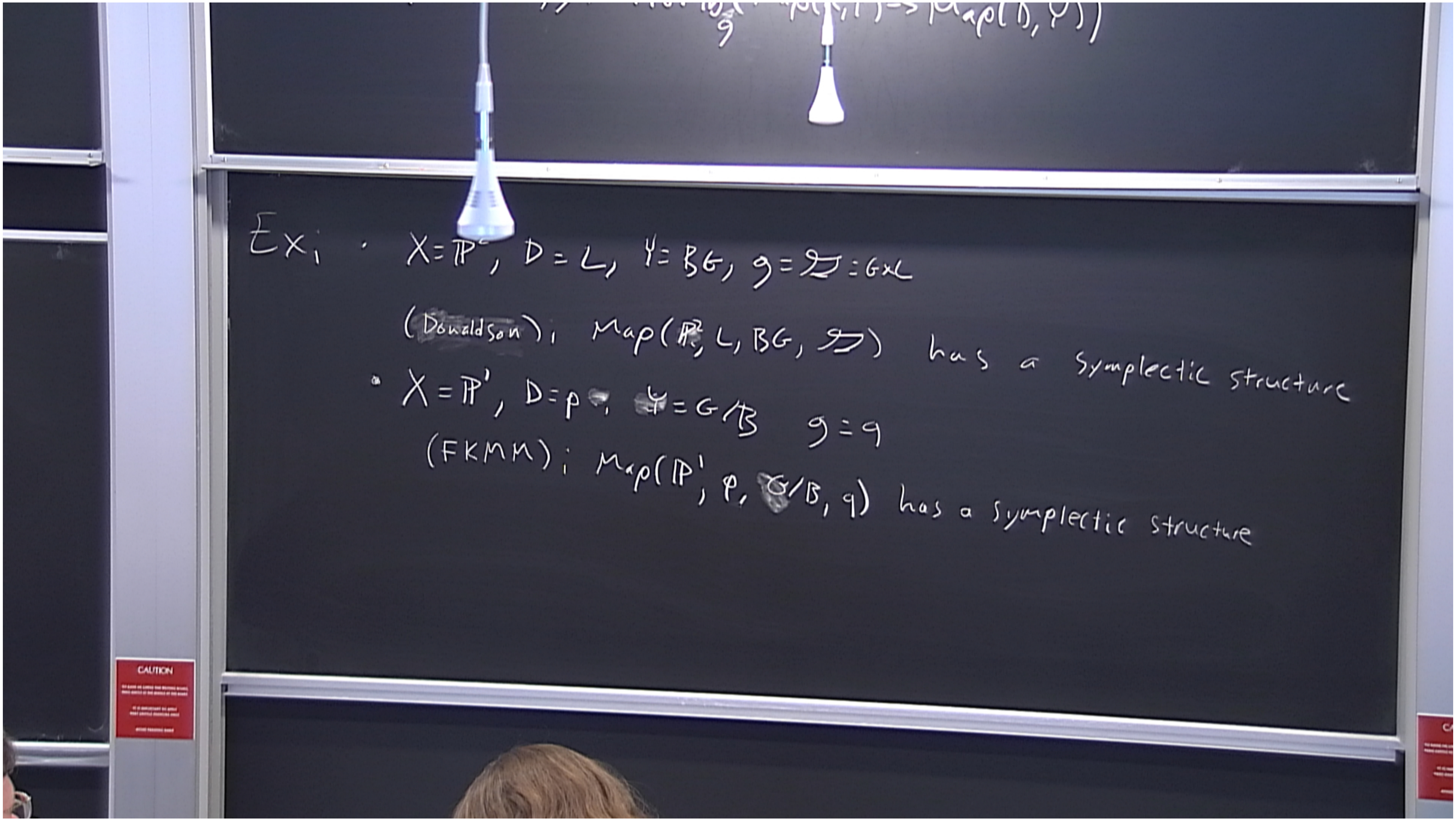

Map(X, -) sends n-shifted symplectic stacks
to (n-d) shifted
and
Example. If X CYd then everything works
Goal: Find shifted symplectic structures on other
stacks of maps

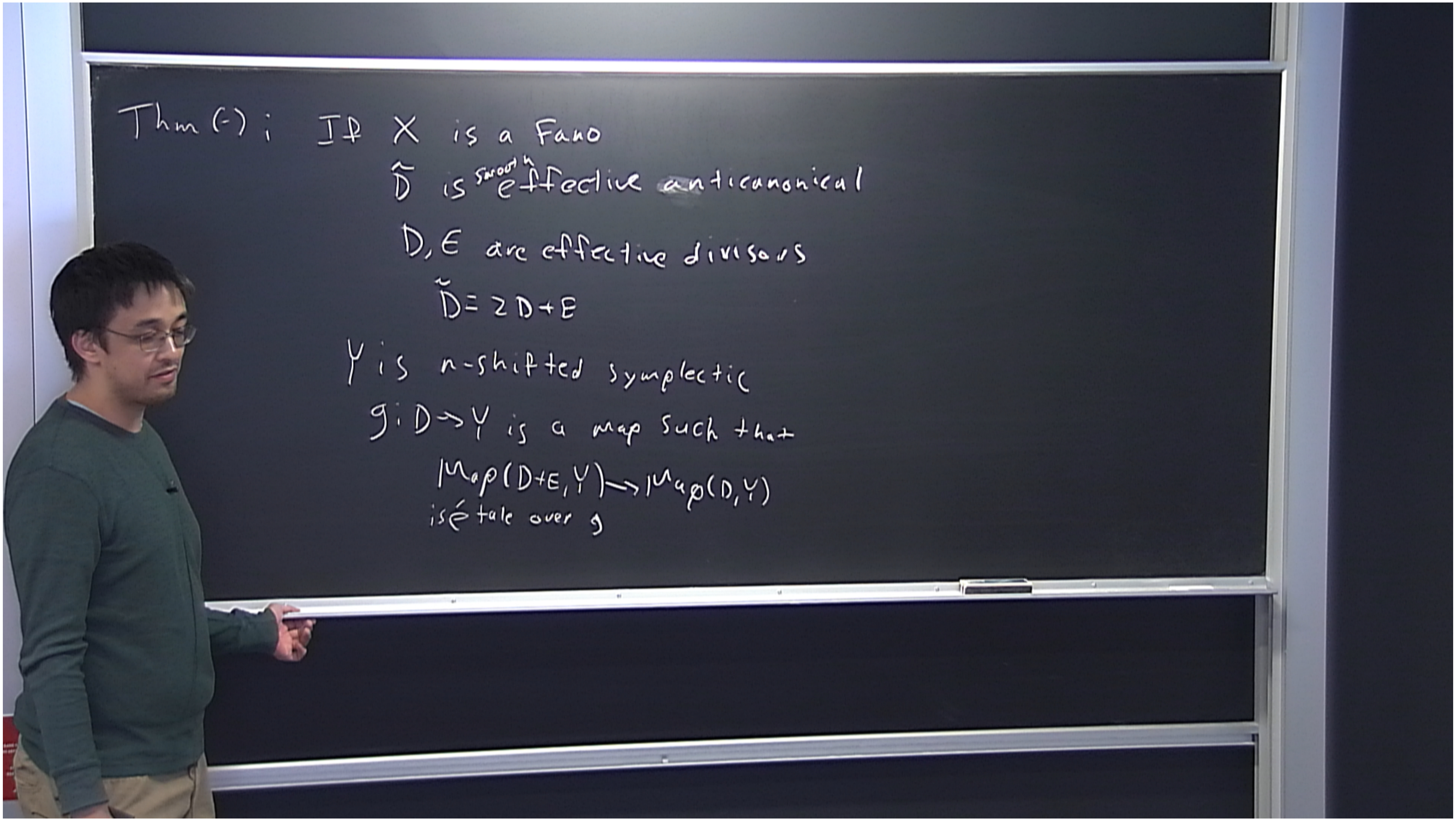
Ex: • $X = \mathbb{P}^2$, $D = L$, $Y = BG$, $g = \mathfrak{G} := G \times L$

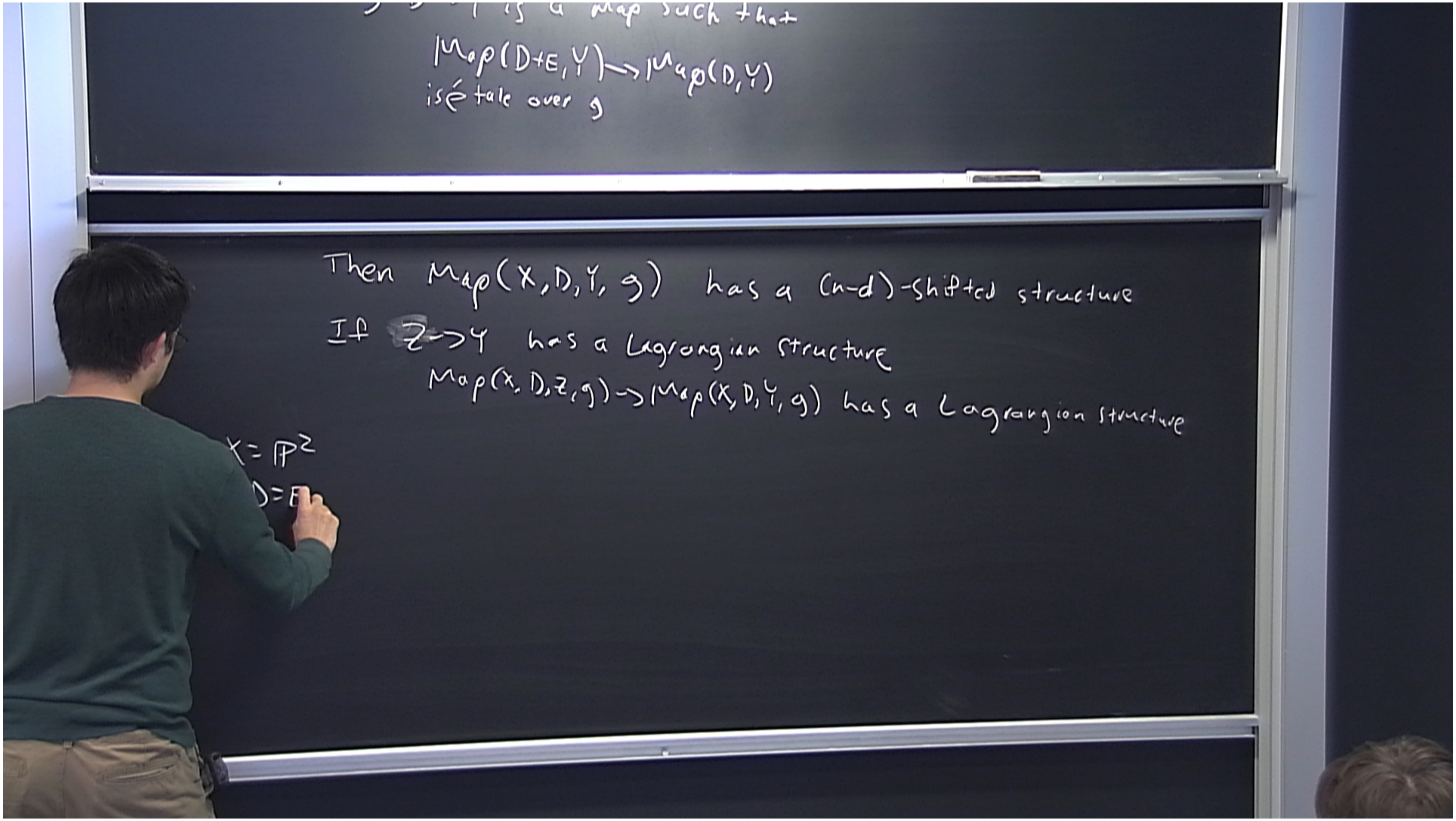
(Donaldson): $\mathrm{Map}(\mathbb{P}^2, L, BG, \mathfrak{G})$ has a symplectic structure

• $X = \mathbb{P}^1$, $D = p$, $Y = G/B$ $g = g$

(FKMM): $\mathrm{Map}(\mathbb{P}^1, p, G/B, g)$ has a symplectic structure


Ex: • $X = \mathbb{P}^2$, $D = L$, $Y = BG$, $\mathfrak{g} = \mathfrak{g} : G \times L$
(Donaldson): $\mathrm{Map}(\mathbb{P}^2, L, BG, \mathfrak{g})$ has a symplectic structure
• $X = \mathbb{P}^1$, $D = p$, $Y = G/B$, $\mathfrak{g} = \mathfrak{g}$
(FKMM): $\mathrm{Map}(\mathbb{P}^1, p, G/B, \mathfrak{g})$ has a symplectic structure


Thm (-): If X is a Fano
D̃ is smooth effective anticanonical
D, E are effective divisors
D̃ = 2D + E
Y is n-shifted symplectic
g: D → Y is a map such that
Map(D+E, Y) → Map(D, Y)
is étale over g


is a map such that
Map(D+E, Y) → Map(D, Y)
is étale over g
Then Map(X, D, Y, g) has a (n-d)-shifted structure
If Z → Y has a Lagrangian structure
Map(X, D, Z, g) → Map(X, D, Y, g) has a Lagrangian structure
X = P^2
D = E

... is a map such that

$$\mathrm{Map}(D+E, Y) \to \mathrm{Map}(D, Y)$$

is étale over g

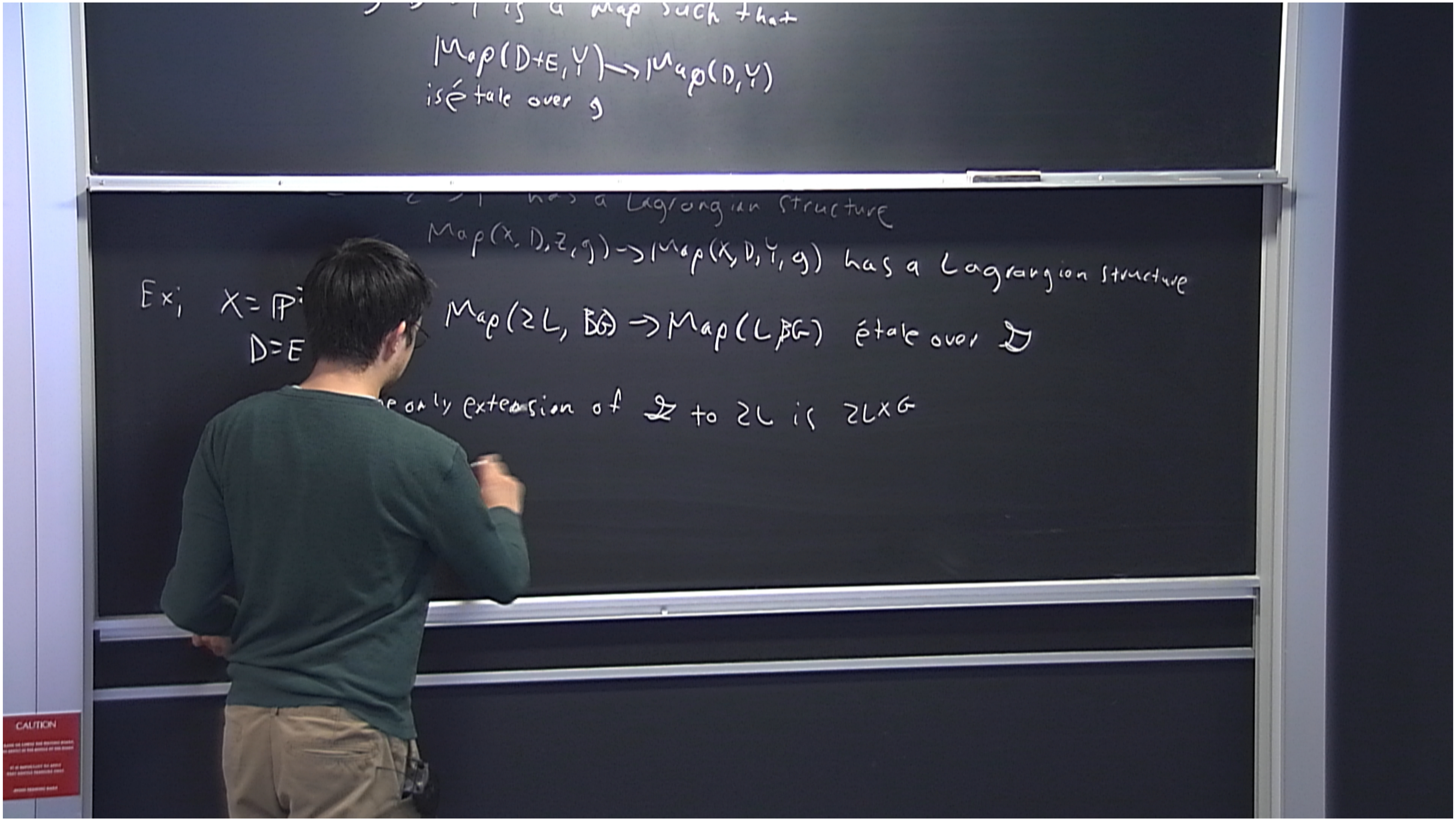
Then $\mathrm{Map}(X, D, Y, g)$ has a $(n-d)$-shifted structure

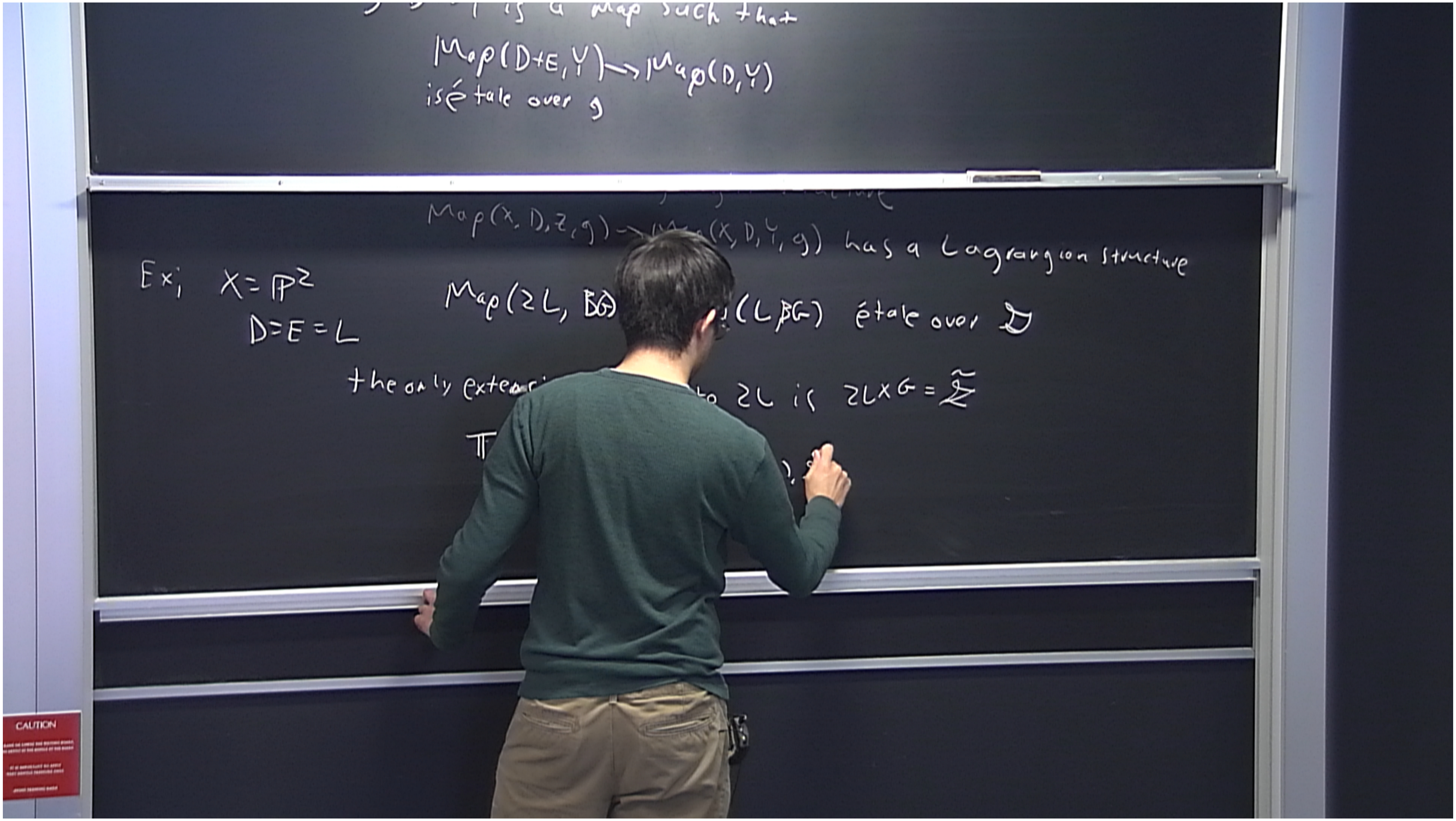
If $Z \to Y$ has a Lagrangian structure

$\mathrm{Map}(X, D, Z, g) \to \mathrm{Map}(X, D, Y, g)$ has a Lagrangian structure

Ex; $X = \mathbb{P}^2$

$D = E = L$


is a map such that
Map(D+E, Y) → Map(D, Y)
is étale over g
has a Lagrangian structure
Map(X, D, Z, g) → Map(X, D, Y, g) has a Lagrangian structure
Ex; X = P
D=E
Map(2L, BG) → Map(L, BG) étale over D
e only extension of D to 2L is 2L×G
CAUTION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Ex; X = P2
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CAUTION

... is a map such that

$$\mathrm{Map}(D+E, Y) \to \mathrm{Map}(D, Y)$$

is étale over g

Then $\mathrm{Map}(X, D, Y, g)$ has a $(n-d)$-shifted structure

If $Z \to Y$ has a Lagrangian structure

$\mathrm{Map}(X, D, Z, g) \to \mathrm{Map}(X, D, Y, g)$ has a Lagrangian structure

Ex; $X = \mathbb{P}^2$

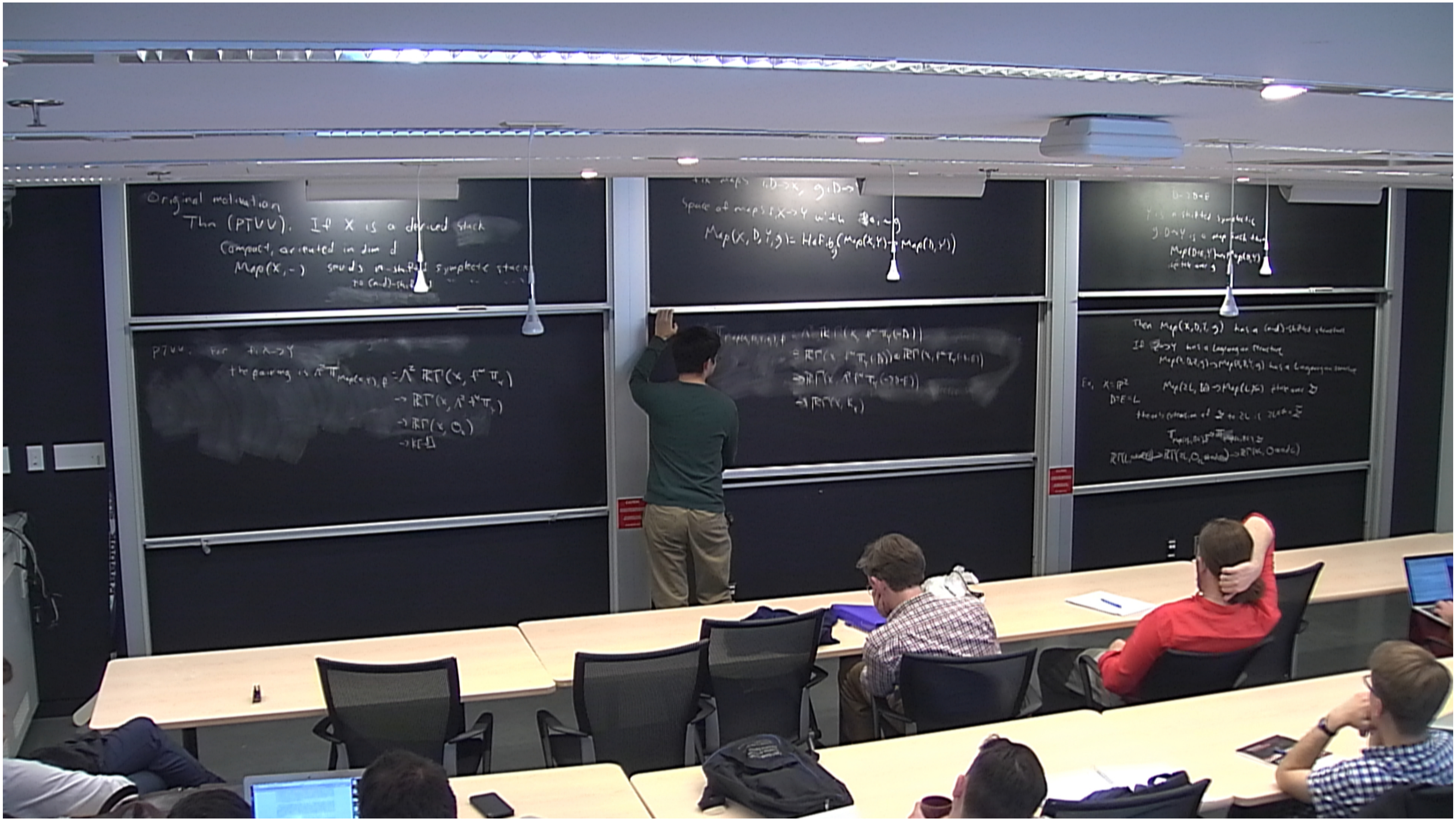
$D = E = L$

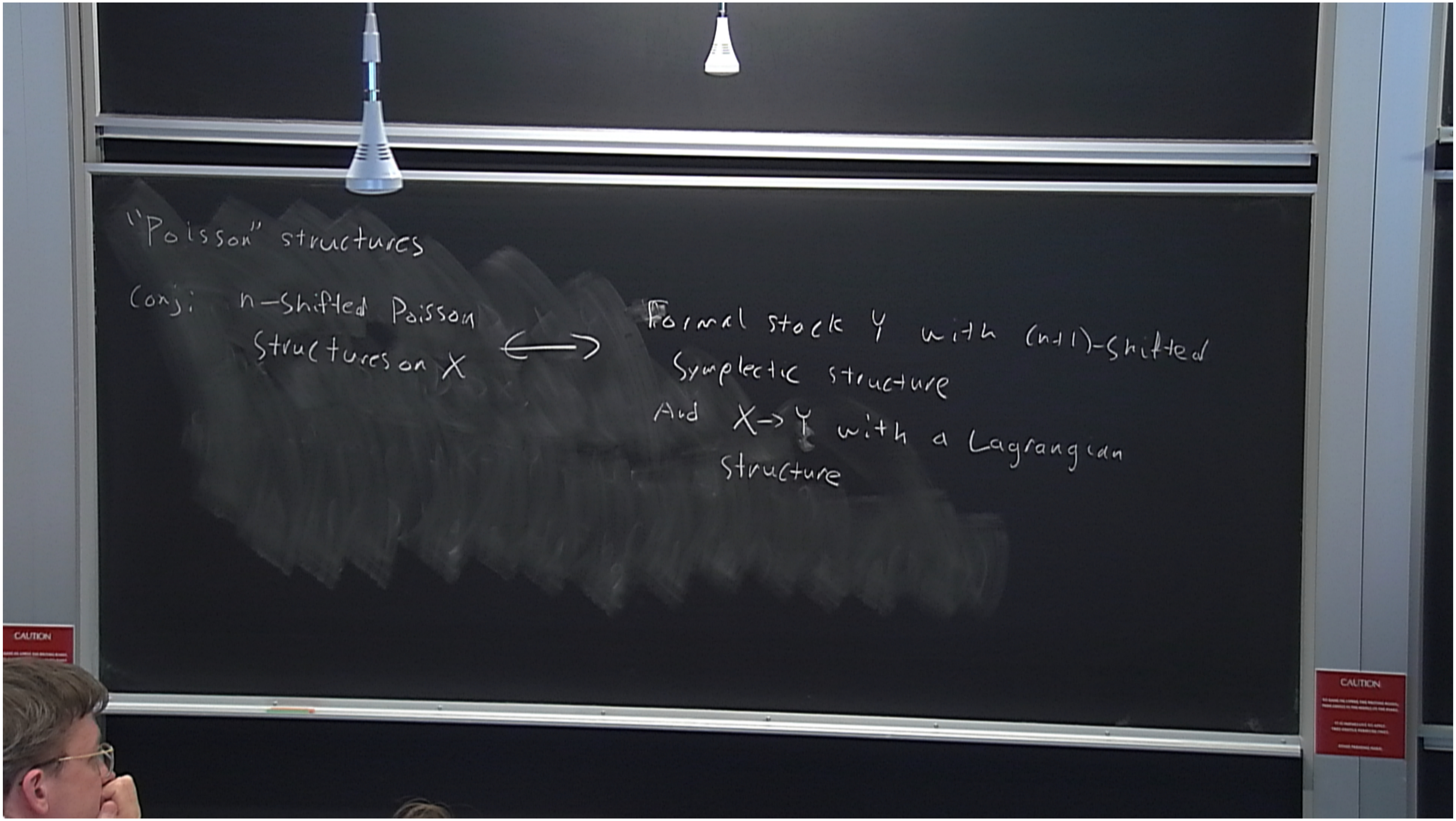
$$\mathrm{Map}(2L, BG) \to \mathrm{Map}(L, BG) \text{ étale over } \mathcal{G}$$

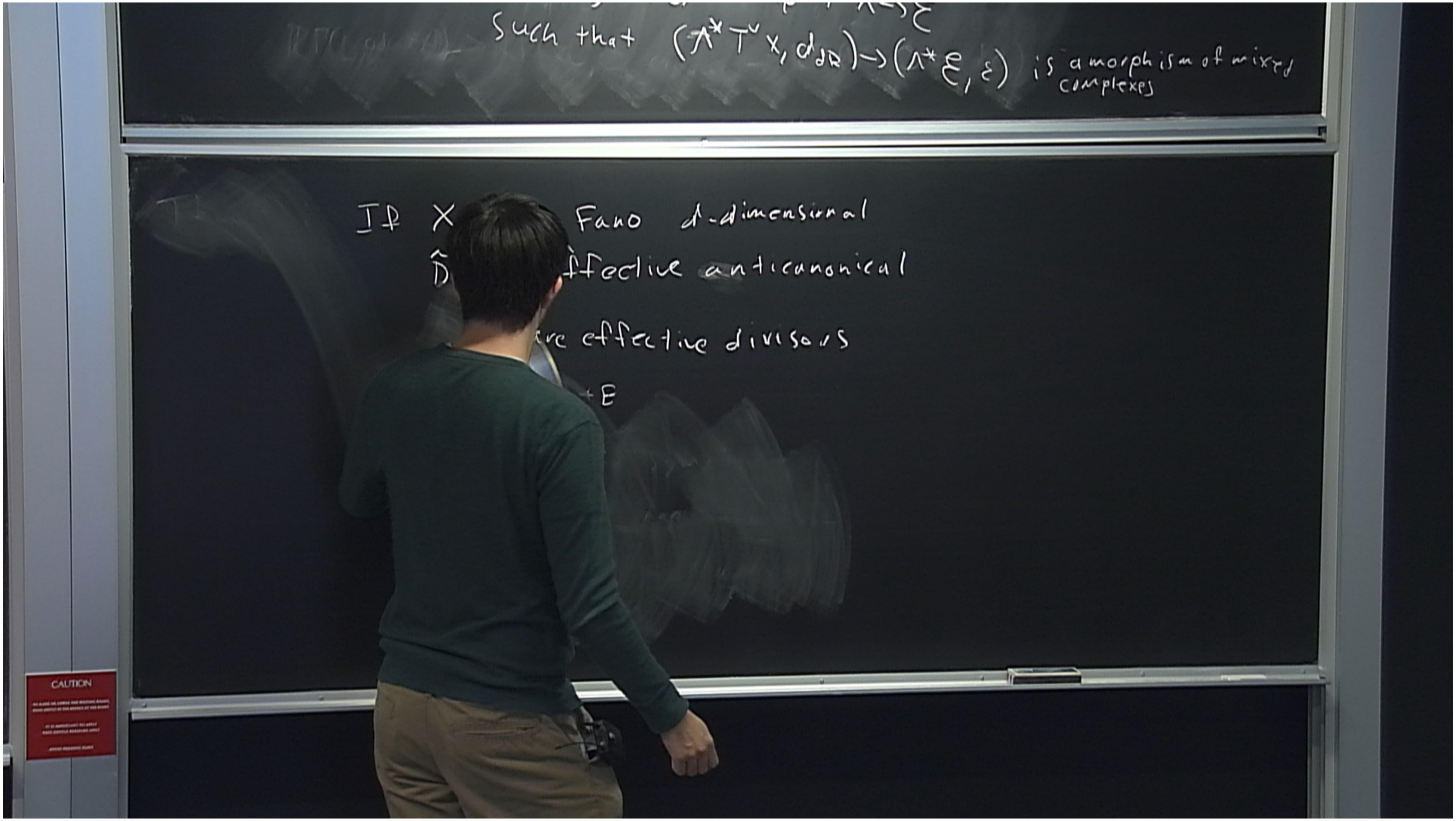
the only extension of $\mathcal{G}$ to $2L$ is $2L \times G = \tilde{\mathcal{G}}$

$$T_{\mathrm{Map}(2L, BG), \tilde{\mathcal{G}}} \to T_{\mathrm{Map}(L, BG), \mathcal{G}}$$

$$\mathbb{R}\Gamma(L, \mathrm{ad}\, G \otimes \mathcal{O}(-1)) \to \mathbb{R}\Gamma(2L, \mathcal{O}_{2L} \otimes \mathrm{ad}\, G) \to \mathbb{R}\Gamma(L, \mathcal{O} \otimes \mathrm{ad}\, G)$$


Original motivation
Thm (PTVV). If X is a derived stack
compact, oriented in dim d
Map(X, -) sends n-shifted symplectic stacks
Space of maps f: X→Y with
Then Map(X, D, Y, g) has a (n-d)-shifted structure


"Poisson" structures
Conj: n-shifted Poisson structures on X ⟷ Formal stack Y with (n+1)-shifted symplectic structure
And X → Y with a Lagrangian structure



Such that $(\Lambda^* T^\vee X, d_{dR}) \to (\Lambda^* \mathcal{E}, \varepsilon)$ is a morphism of mixed complexes
If X Fano d-dimensional
$\hat{D}$ ffective anticanonical
re effective divisors
+E

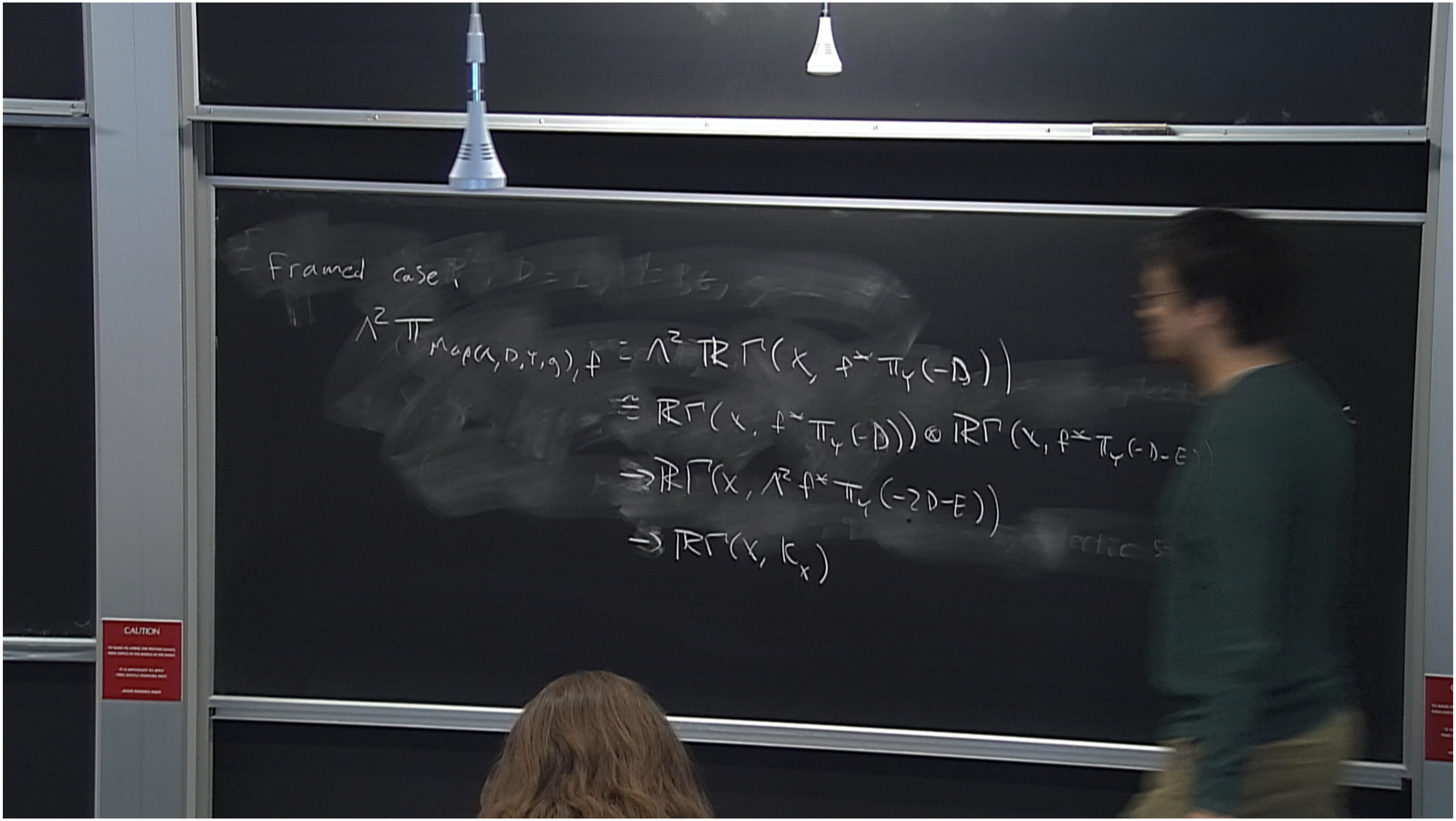
such that $(\Lambda^* T^\vee X, d_{dR}) \to (\Lambda^* \mathcal{E}, \varepsilon)$ is a morphism of mixed complexes

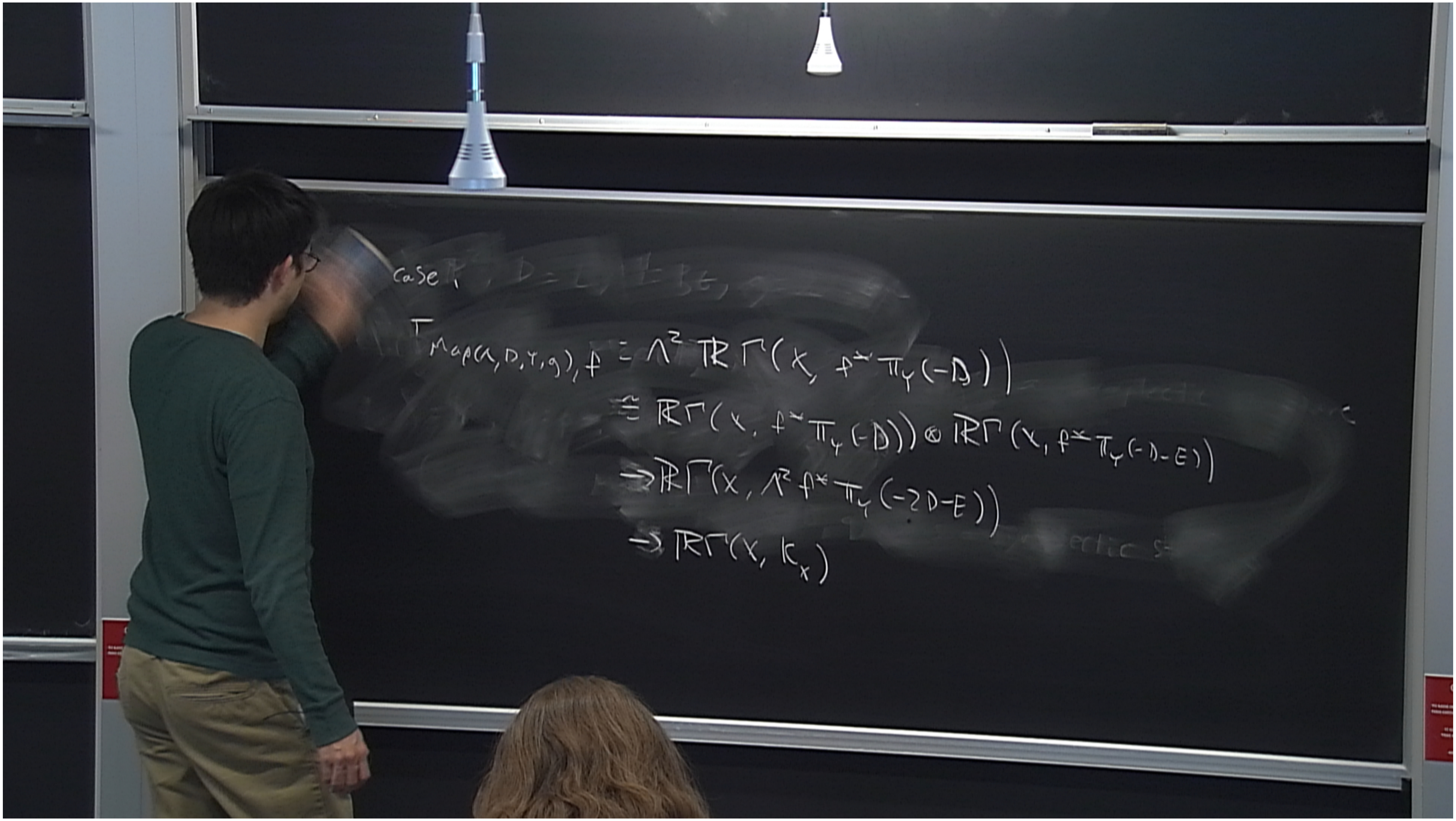
$(\Lambda^* T^\vee X, d_{dR}) \to (\wedge T X, [\pi, -])$ is a foliation

So we can form $[X/\pi^b]$

$$\mathbb{L}_{[X/\pi^b]} = \left\{ \underset{0}{T^\vee X} \xrightarrow{\pi^b} \underset{1}{TX} \right\}$$

$$\Lambda^2 \mathbb{L}_{[X/\pi^b]} = \left\{ \Lambda^2 T^\vee X \to \underset{\substack{\oplus \\ id}}{T^\vee X \otimes TX} \to Sym^2 TX \right\}$$


Framed case:
$\Lambda^2 T_{Map(X,D,Y,g),f} \cong \Lambda^2 \mathbb{R}\Gamma(X, f^* T_Y(-D))$
$\cong \mathbb{R}\Gamma(X, f^* T_Y(-D)) \otimes \mathbb{R}\Gamma(X, f^* T_Y(-D-E))$
$\to \mathbb{R}\Gamma(X, \Lambda^2 f^* T_Y(-2D-E))$
$\to \mathbb{R}\Gamma(X, K_X)$

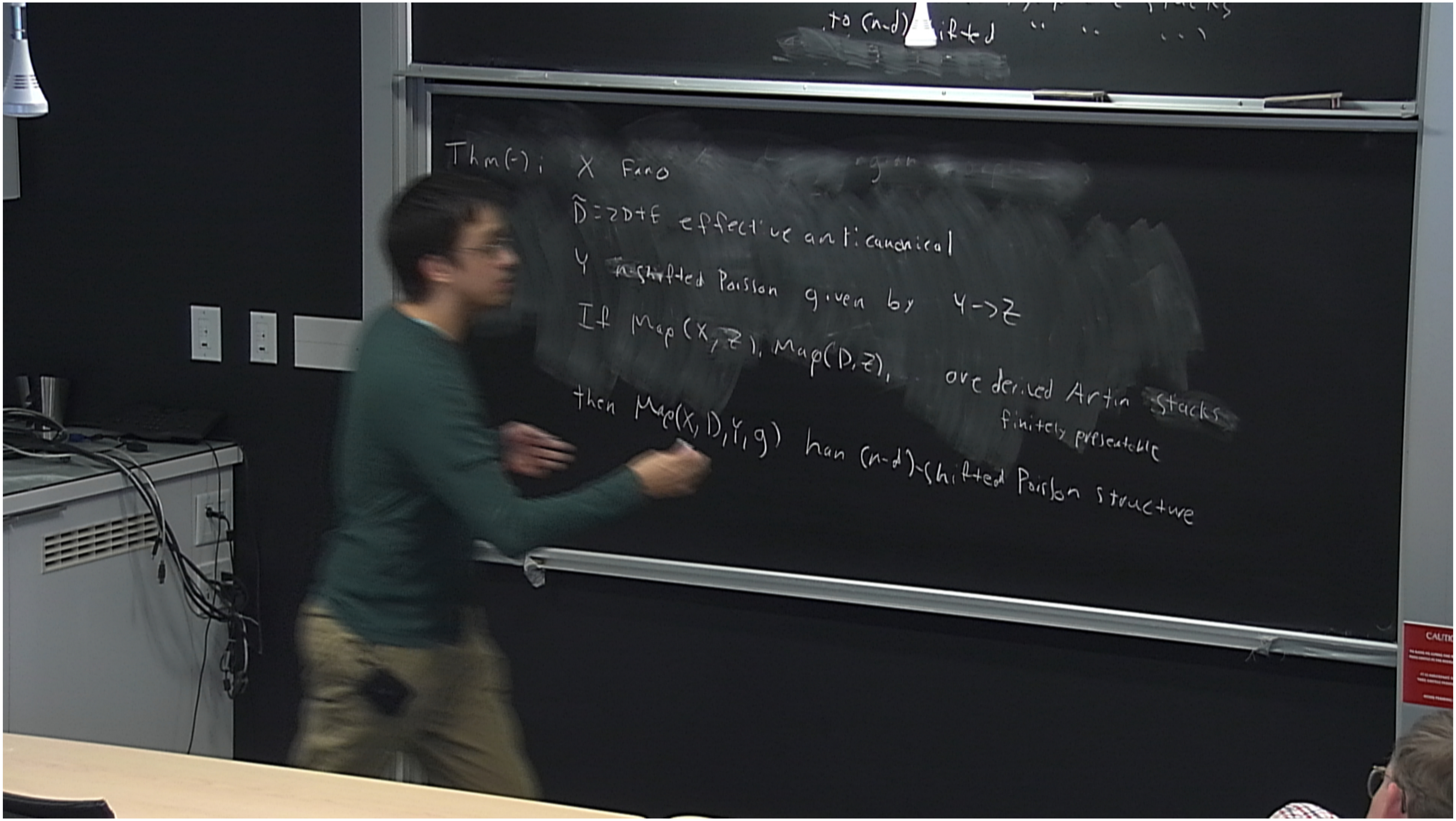

case
$T_{Map(X,D,Y,g),f} \cong \Lambda^2 \mathbb{R}\Gamma(X, f^* T_Y(-D))$
$\cong \mathbb{R}\Gamma(X, f^* T_Y(-D)) \otimes \mathbb{R}\Gamma(X, f^* T_Y(-D-E))$
$\to \mathbb{R}\Gamma(X, \Lambda^2 f^* T_Y(-2D-E))$
$\to \mathbb{R}\Gamma(X, K_X)$

Def. n-shifted Poisson structure on X is $(n+1)$-shifted symplectic Y

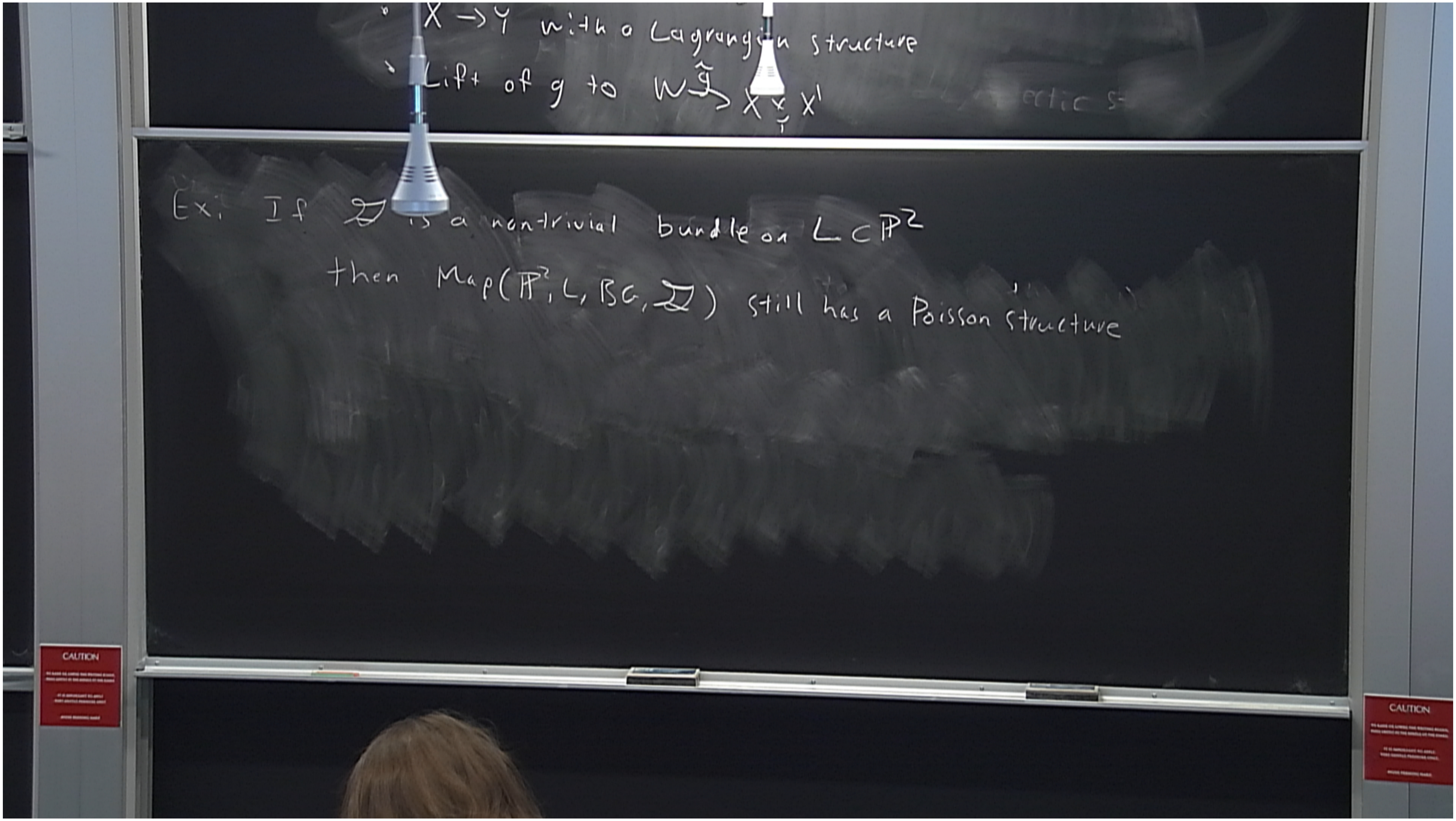
$+$ $X \to Y$ with Lagrangian structure

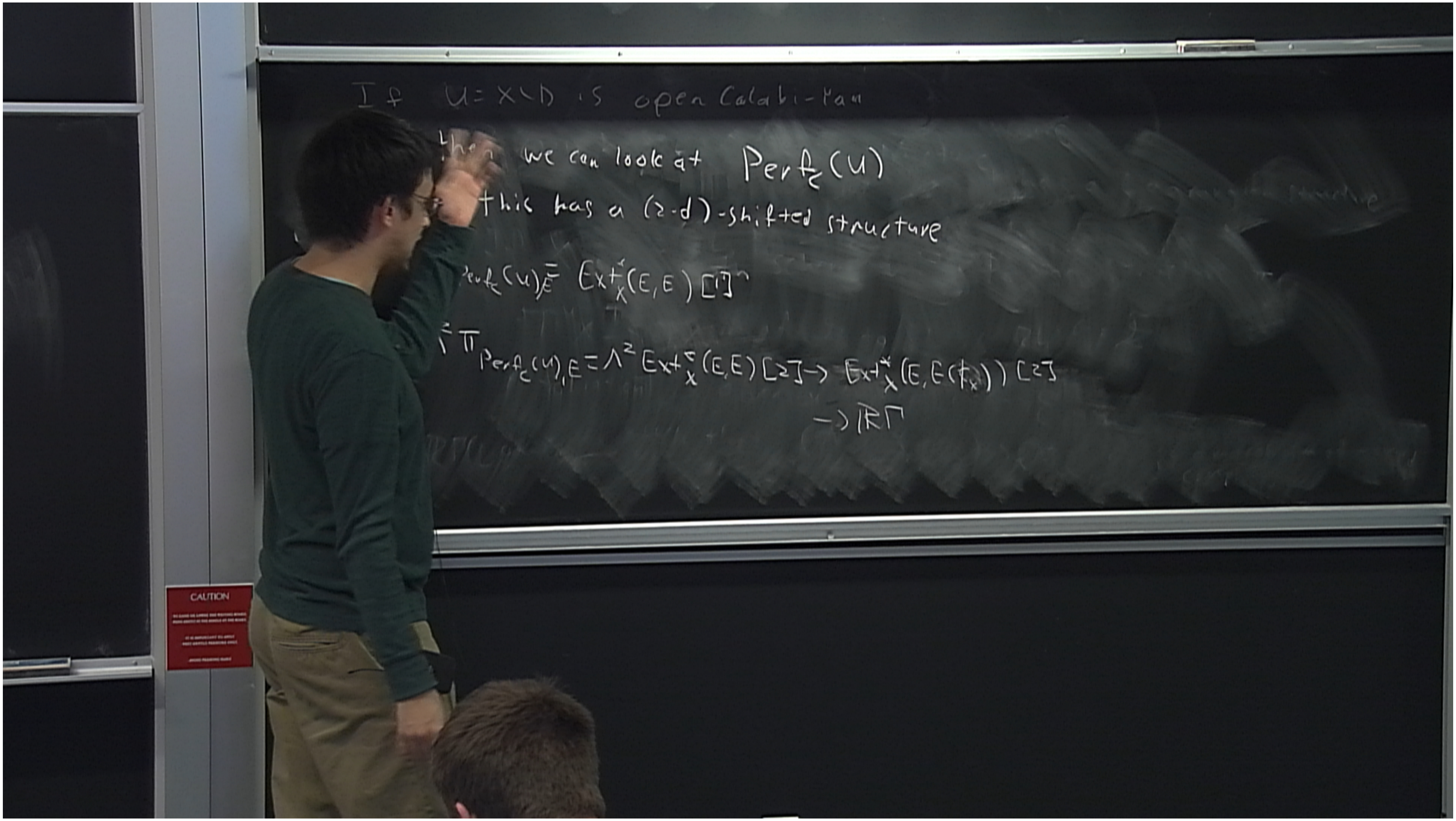
Coisotropic structure on $W \xrightarrow{g} X$ is

- $X' \to Y$ with a Lagrangian structure
- Lift of g to $W \xrightarrow{\hat{g}} X \times_Y X'$
- Lagrangian structure on $\hat{g}$

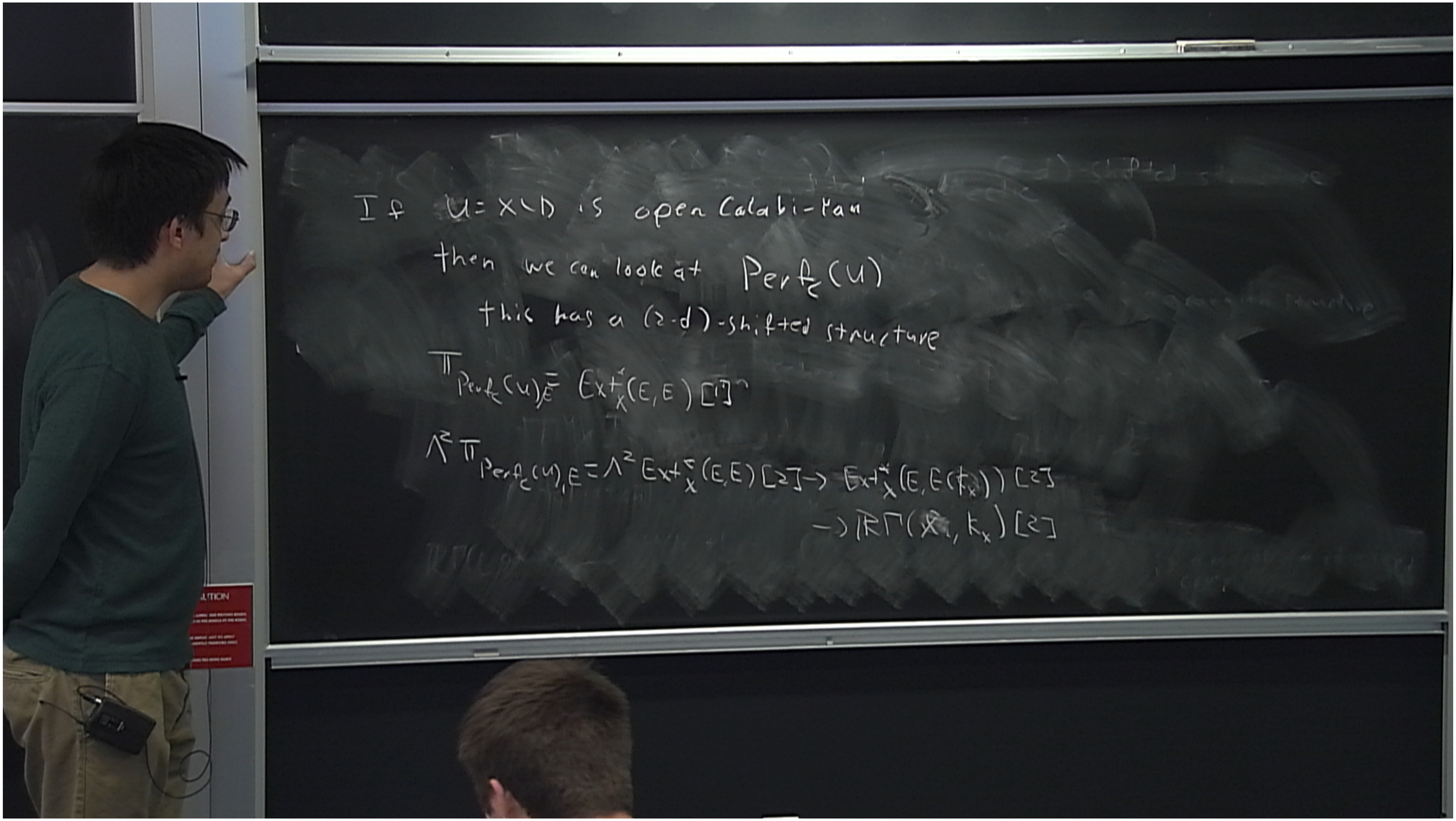


to (n-d)-shifted
Thm(-): X Fano
D̃ = 2D+E effective anticanonical
Y n-shifted Poisson given by Y -> Z
If Map(X,Z), Map(D,Z), are derived Artin stacks
finitely presentable
then Map(X,D,Y,g) has (n-d)-shifted Poisson structure


X → Y with a Lagrangian structure
Lift of g to W → X ×_Y X'
Ex: If 𝔓 is a nontrivial bundle on L ⊂ ℙ²
then Map(ℙ², L, BG, 𝔓) still has a Poisson structure



If U=X\D is open Calabi-Yau
we can look at Perf_c(U)
this has a (2-d)-shifted structure


If U = X∖D is open Calabi-Yau
then we can look at Perf_c(U)
this has a (2-d)-shifted structure
T_Perf_c(U),E = Ext*_X(E,E)[1]
Λ² T_Perf_c(U),E = Λ² Ext*_X(E,E)[2] → Ext*_X(E,E(K_X))[2]
→ RΓ(X, K_X)[2]

If $U = X \setminus D$ is open Calabi-Yau

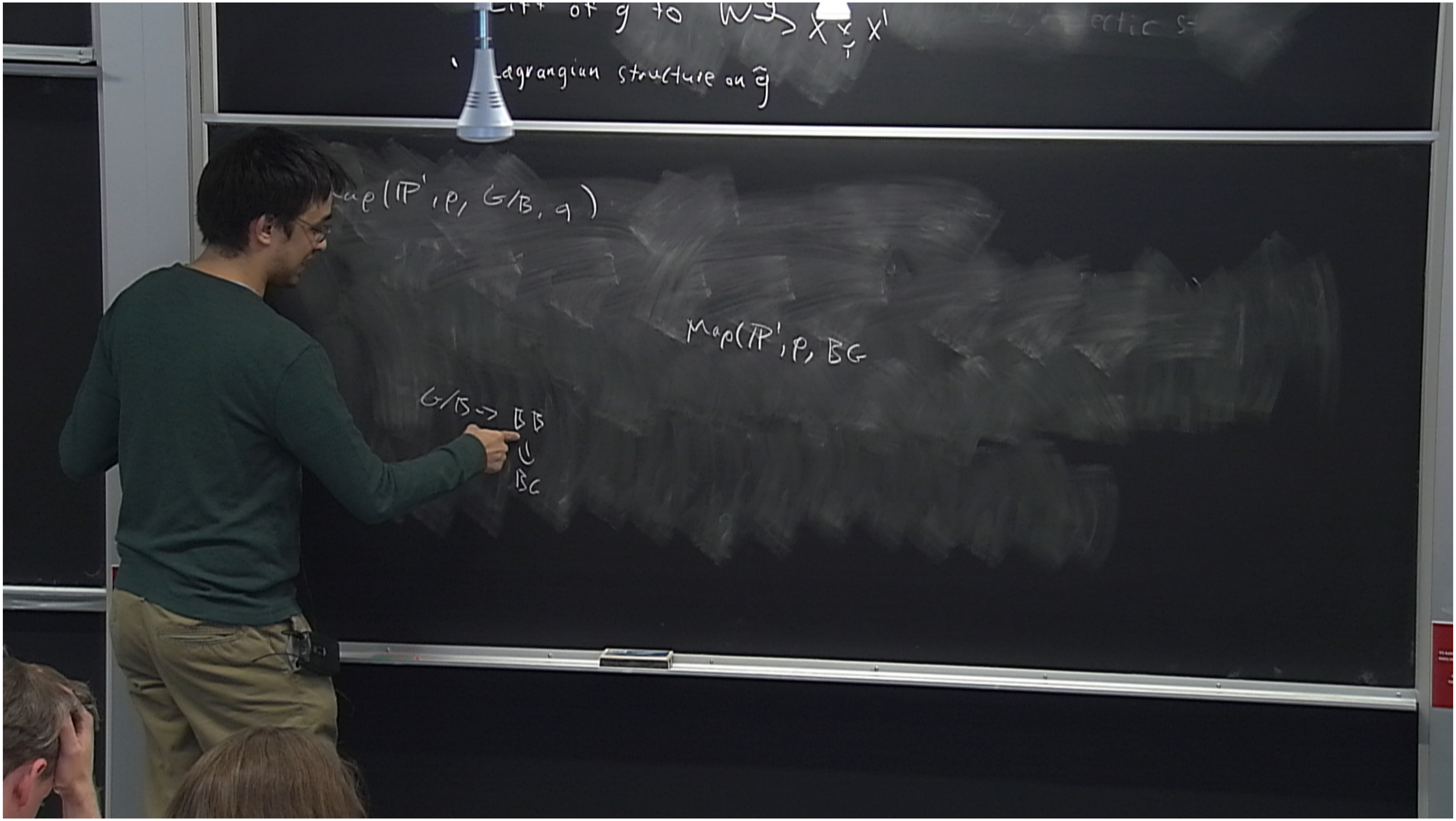
then we can look at $\mathrm{Perf}_c(U)$

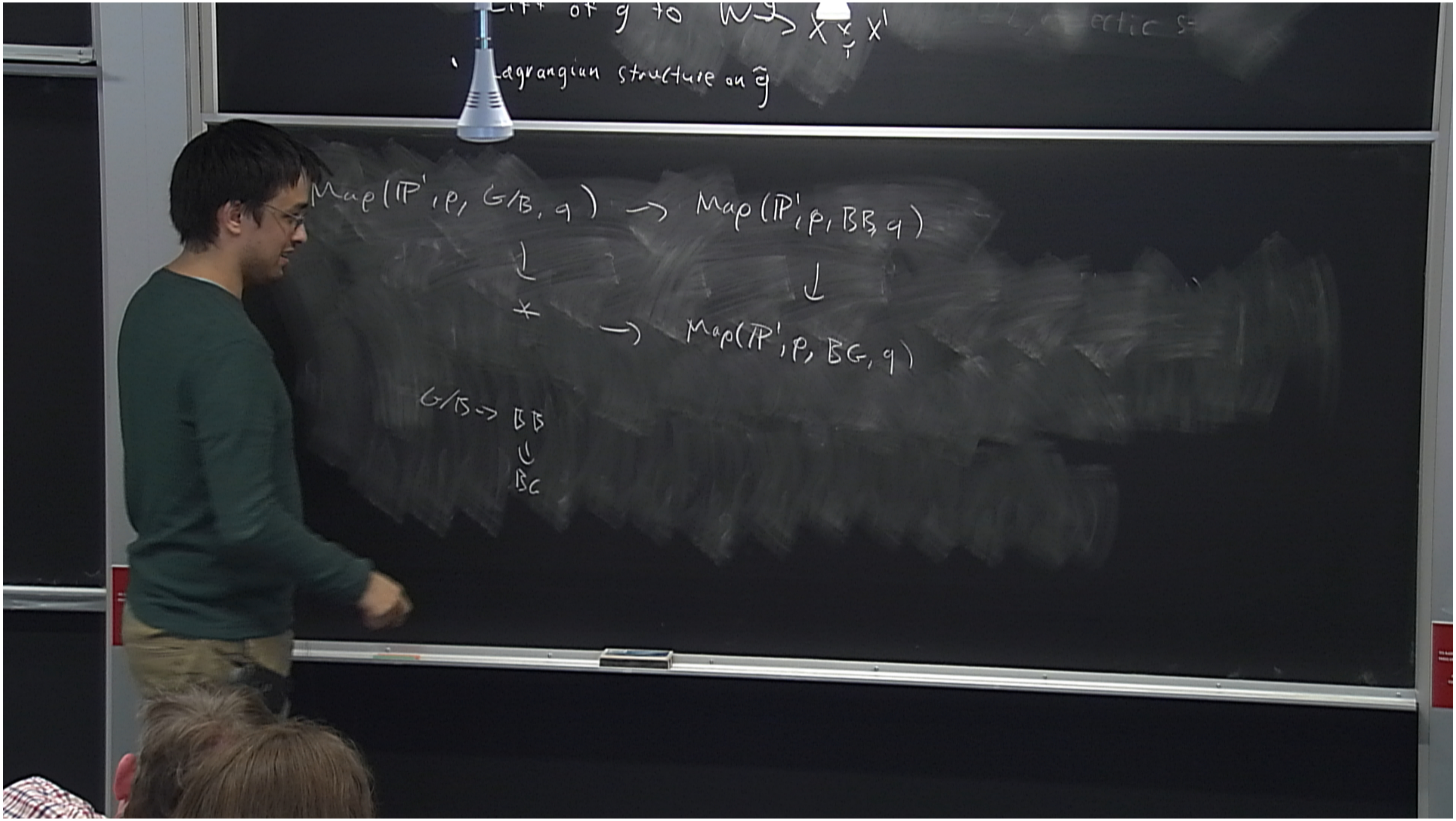
this has a $(2-d)$-shifted structure

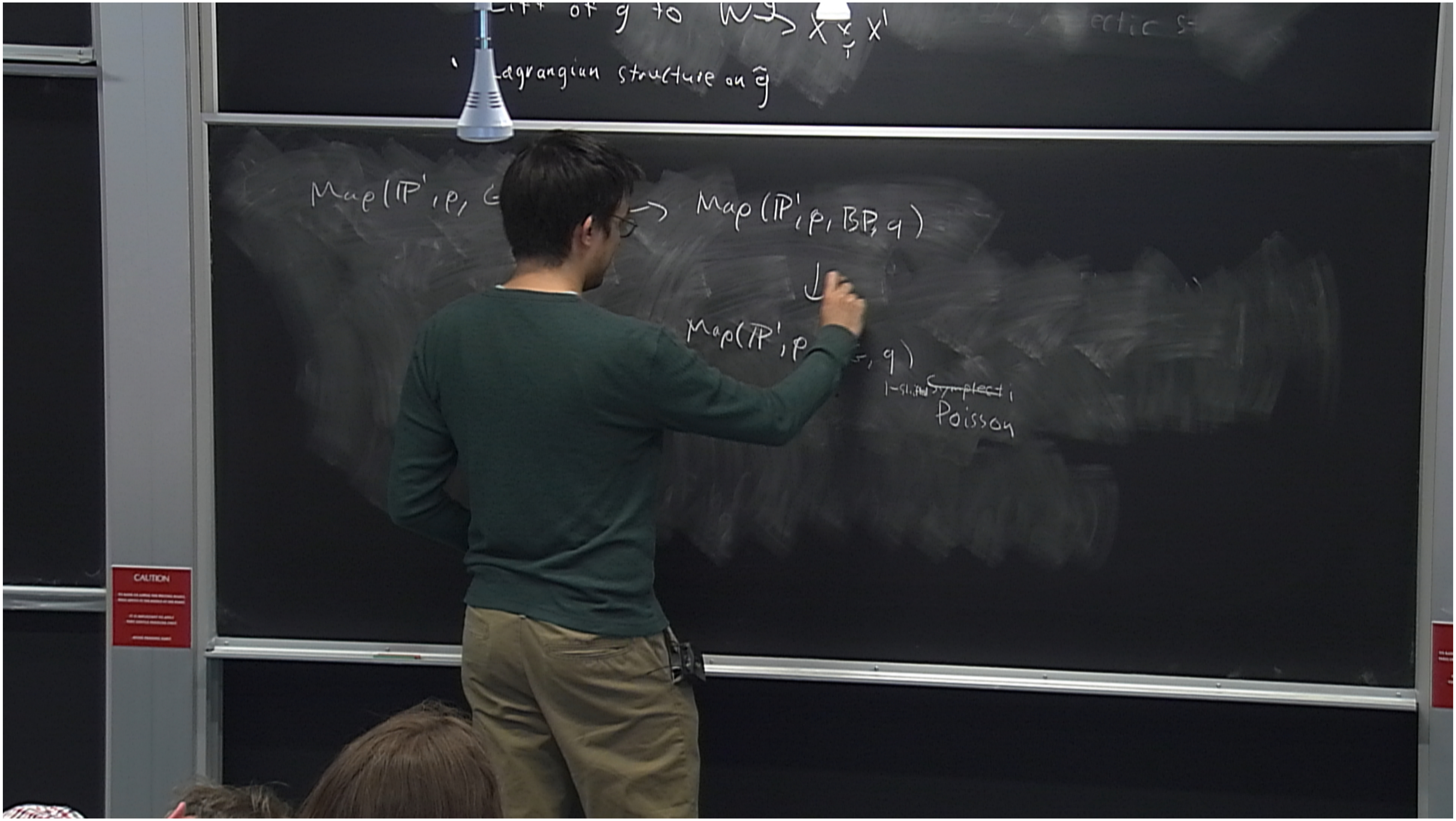
$$\mathbb{T}_{\mathrm{Perf}_c(U),E} = \mathrm{Ext}^*_X(E,E)[1]$$

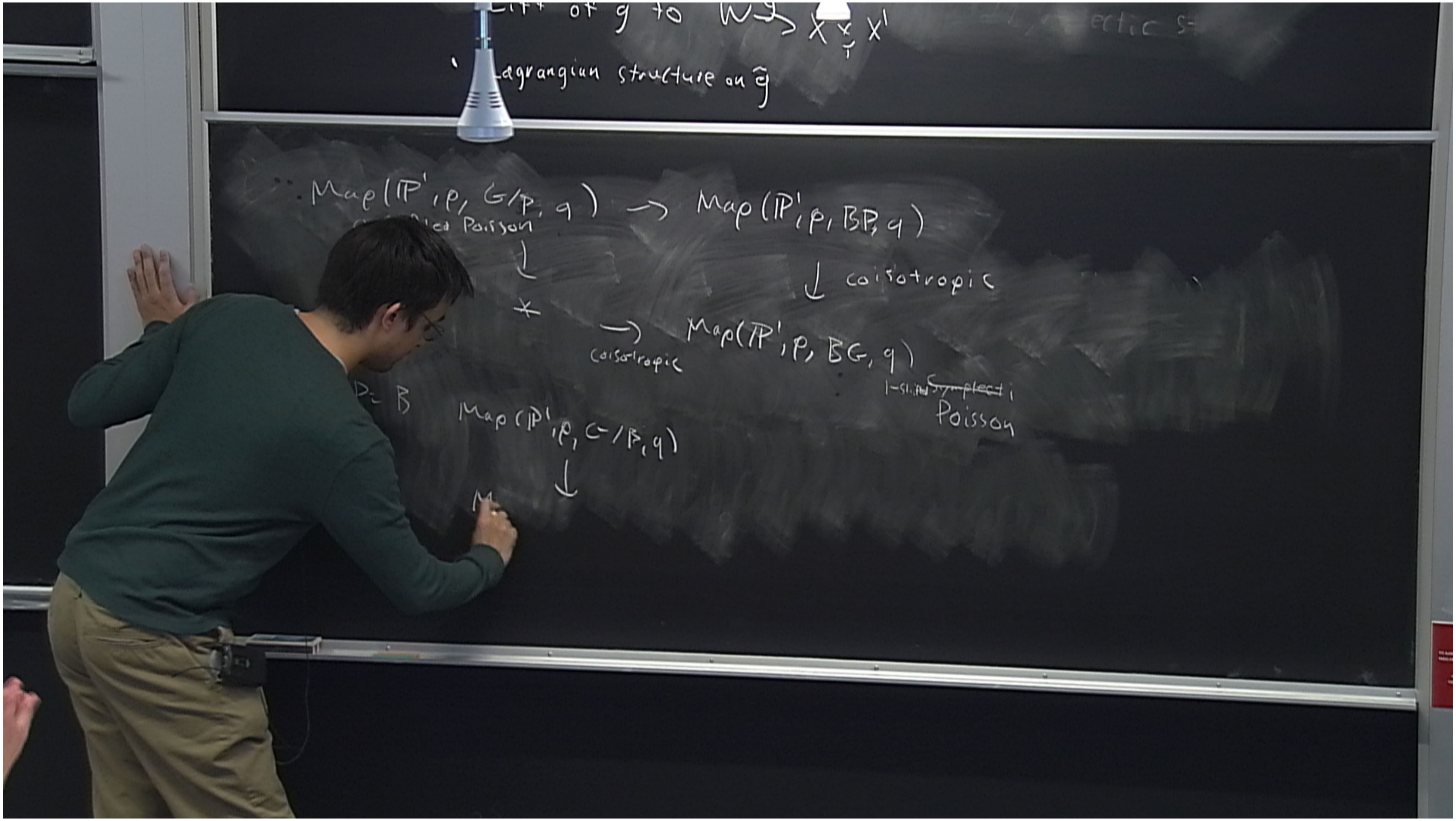
$$\Lambda^2 \mathbb{T}_{\mathrm{Perf}_c(U),E} = \Lambda^2 \mathrm{Ext}^*_X(E,E)[2] \to \mathrm{Ext}^*_X(E,E(K_X))[2]$$

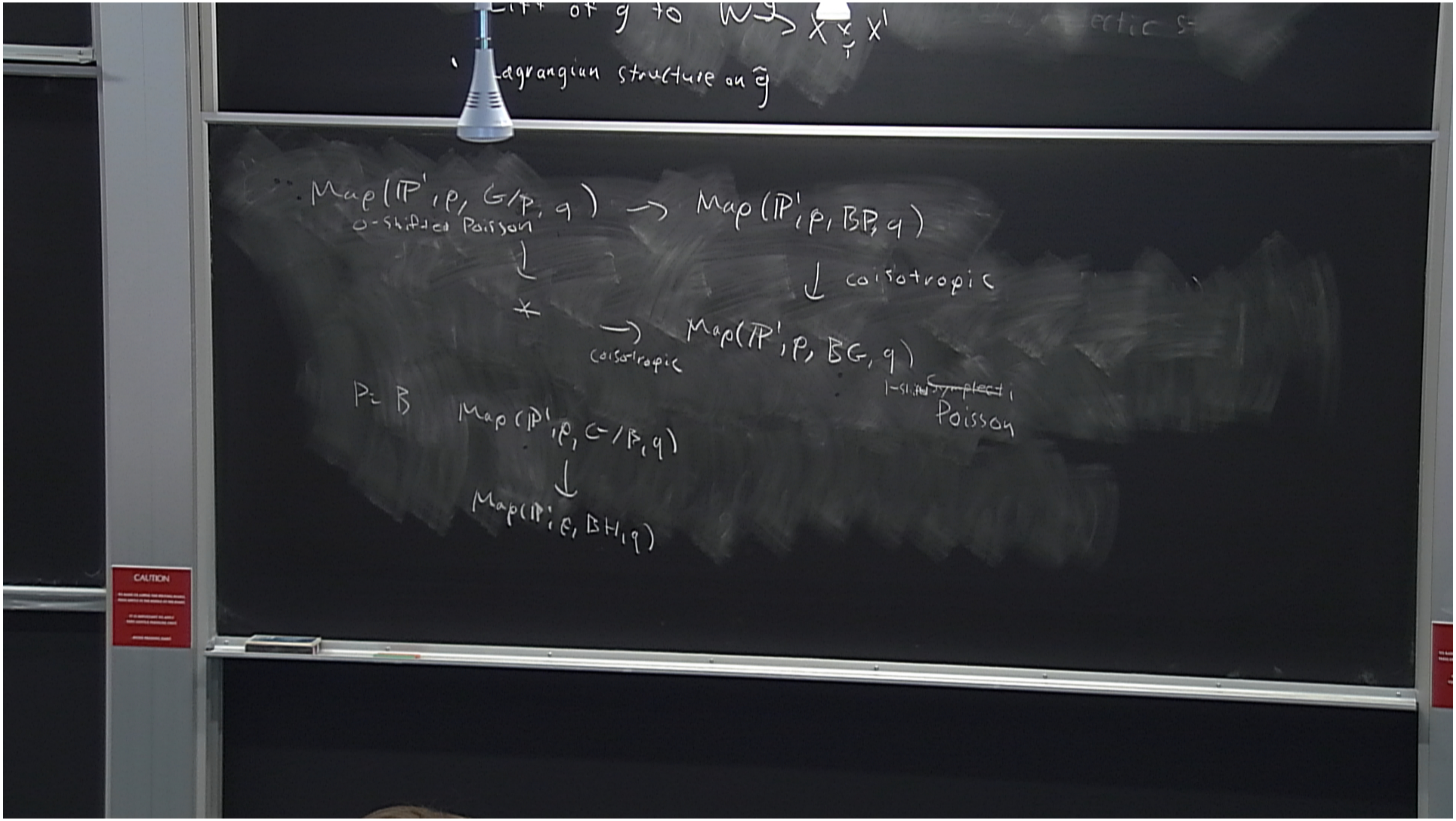
$$\to \mathbb{R}\Gamma(X, K_X)[2]$$


structure on
Map(P', p, BG
G/B -> BB


of g to W → X ×_Y X'
Lagrangian structure on g̃
Map(P^1, p, G/B, q) → Map(P^1, p, BB, q)
* → Map(P^1, p, BG, q)
G/B → BB
BG


structure on
Map(ℙ¹, p, BP, q)
Poisson


of g to
Lagrangian structure on g̃
Map(P', p, G/P, q) → Map(P', p, BP, q)
Poisson
coisotropic
coisotropic
Map(P', p, BG, q)
1-shifted Symplectic Poisson
Map(P', p, G/B, q)


of g to
Lagrangian structure on g̃
Map(P¹, p, G/P, q) → Map(P¹, p, BP, q)
0-shifted Poisson
coisotropic
* → Map(P¹, p, BG, q)
coisotropic
1-shifted ~~symplectic~~ Poisson
P = B
Map(P¹, p, G/B, q)
Map(P¹, p, BH, q)