Title: Singular support of categories

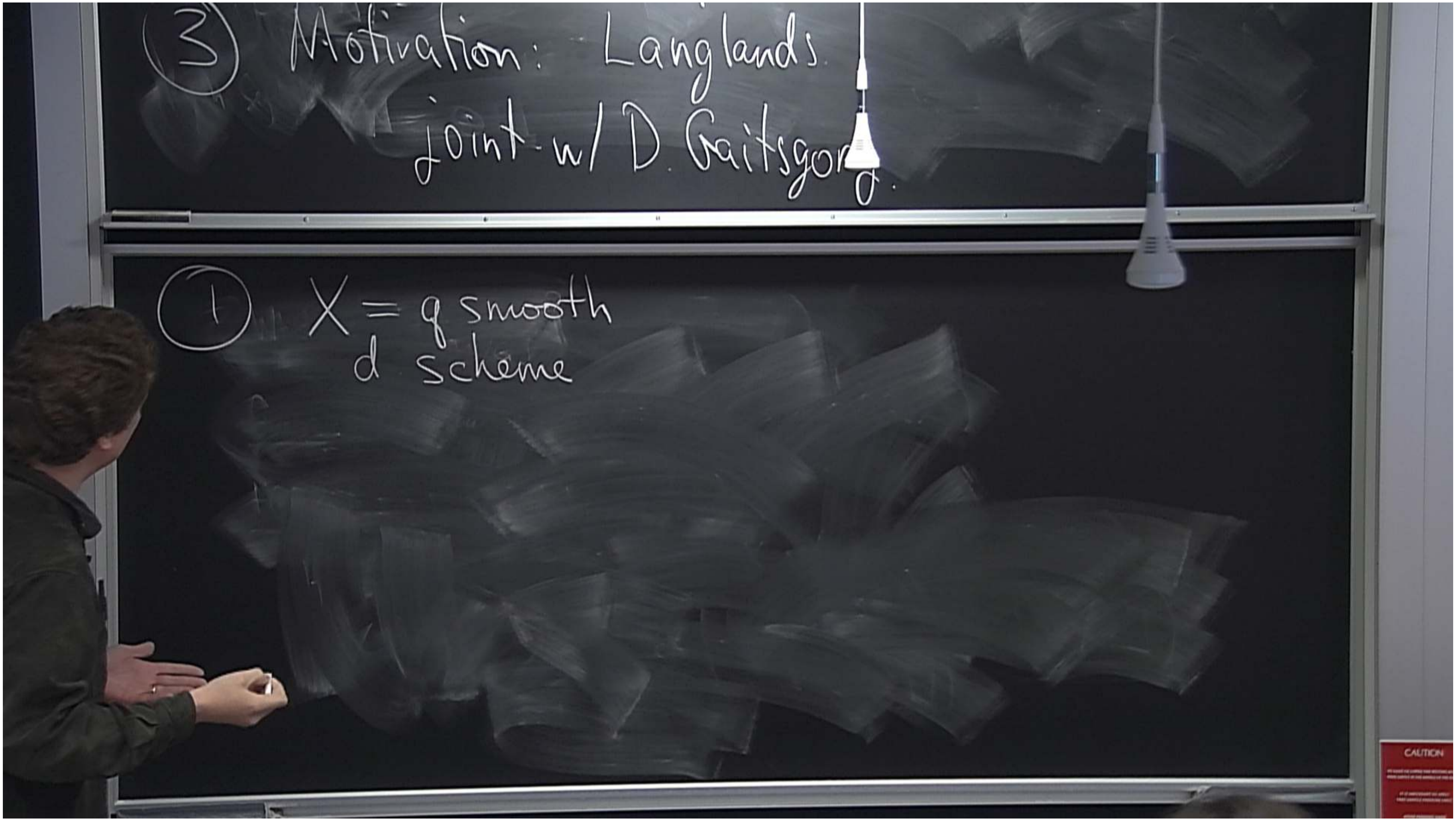
Date: Apr 19, 2016 11:00 AM

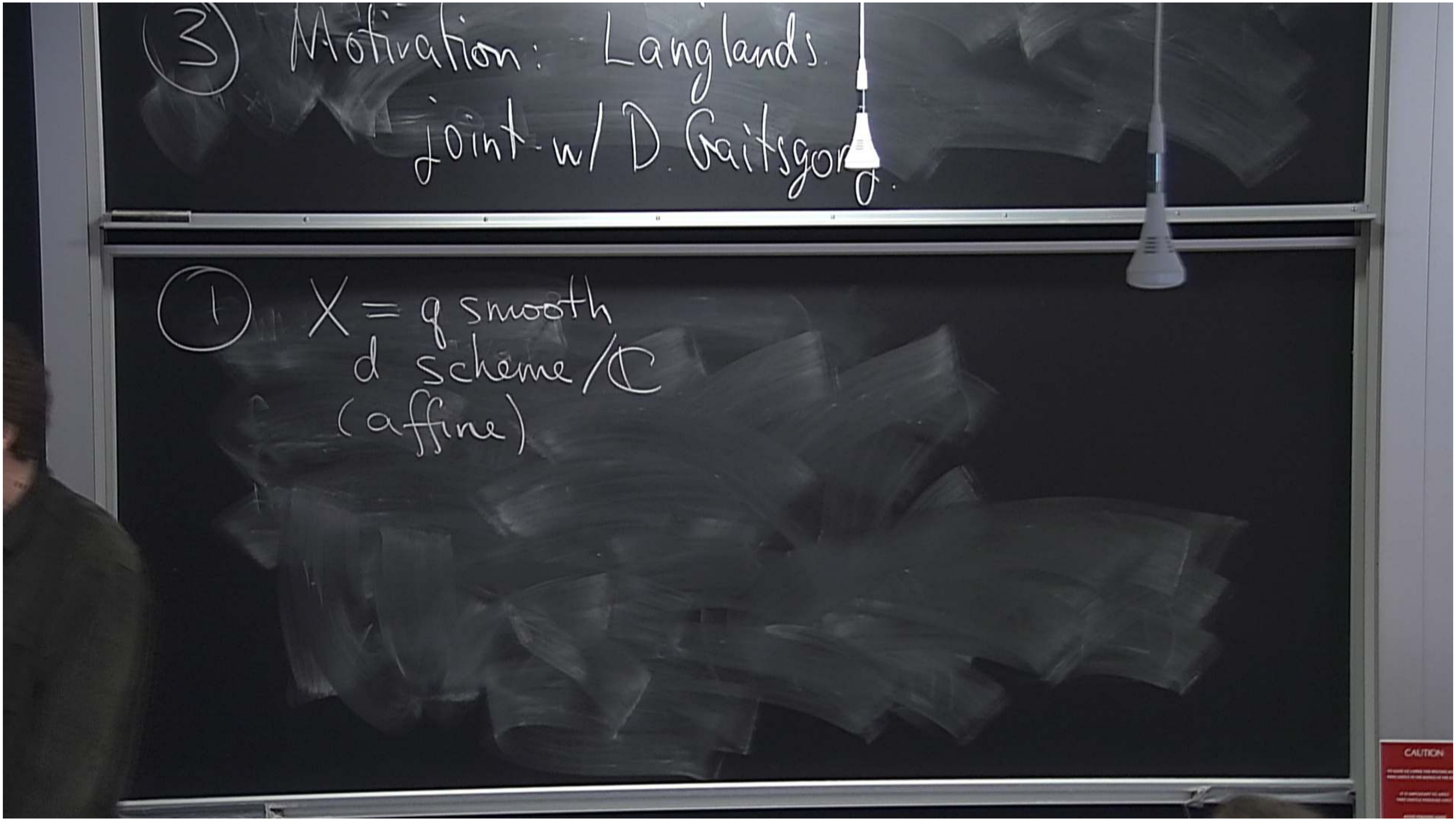
URL: http://pirsa.org/16040075

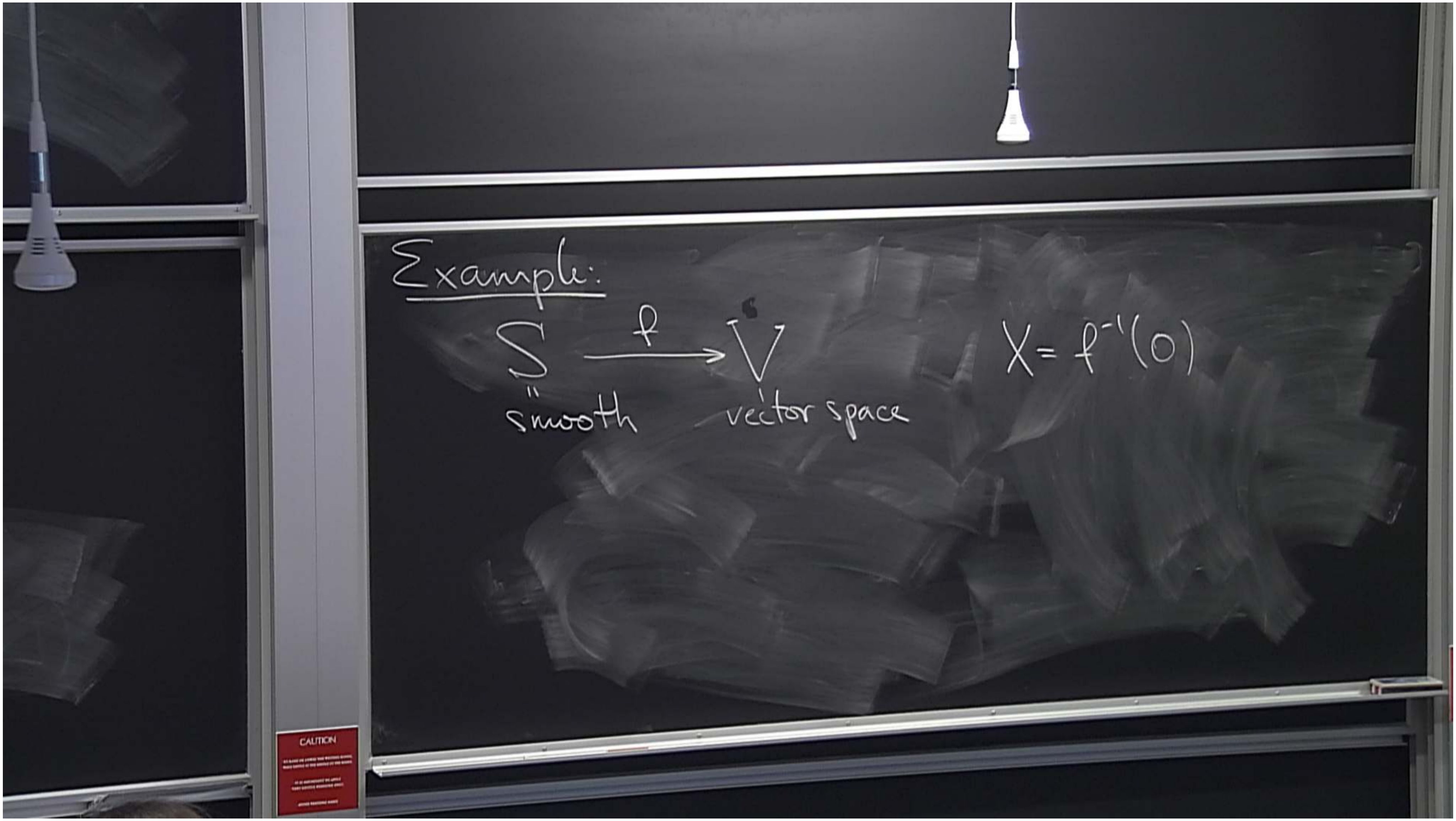
Abstract: In many situations, geometric objects on a space have some kind of singular support, which refines the usual support. For instance, for smooth X, the singular support of a D-module (or a perverse sheaf) on X is as a conical subset of the cotangent bundle; similarly, for quasi-smooth X, the singular support of a coherent sheaf on X is a conical subset of the cohomologically shifted cotangent bundle. I would like to describe a higher categorical version of this notion.

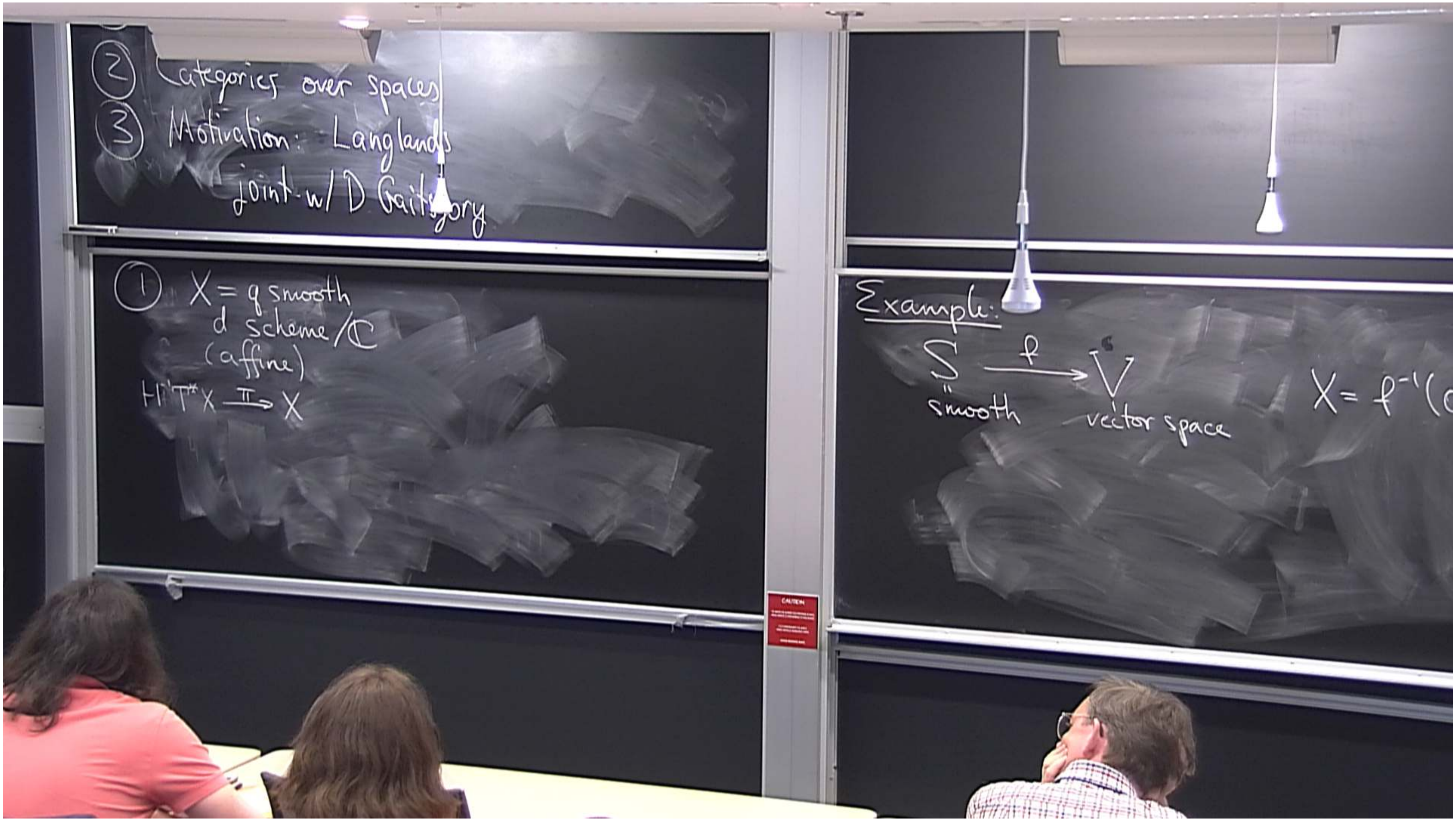
Let X be a smooth variety, and let Z be a closed conical isotropic subset of the cotangent bundle of X. I will define a 2-category associated with Z; its objects may be viewed as `categories over X with singular support in Z'. In particular, if Z is the zero section, we simply consider categories over Z in the usual sense.

This talk is based on a joint project with D.Gaitsgory. The project is motivated by the local geometric Langlands correspondence; I plan to sketch the relation with the Langlands correspondence in the talk.


(3) Motivation: Langlands.
joint w/ D. Gaitsgory.
(1) X = q smooth
d scheme
CAUTION


③ Motivation: Langlands
joint w/ D. Gaitsgory.
① X = q smooth
d scheme/ℂ
(affine)


Example:
S → V
f
smooth
vector space
X = f⁻¹(0)


2 Categories over spaces
3 Motivation: Langlands
joint w/ D Gaitsgory
1 X = q smooth
d scheme /ℂ
(affine)
H'T*X →π X
Example:
S →f V
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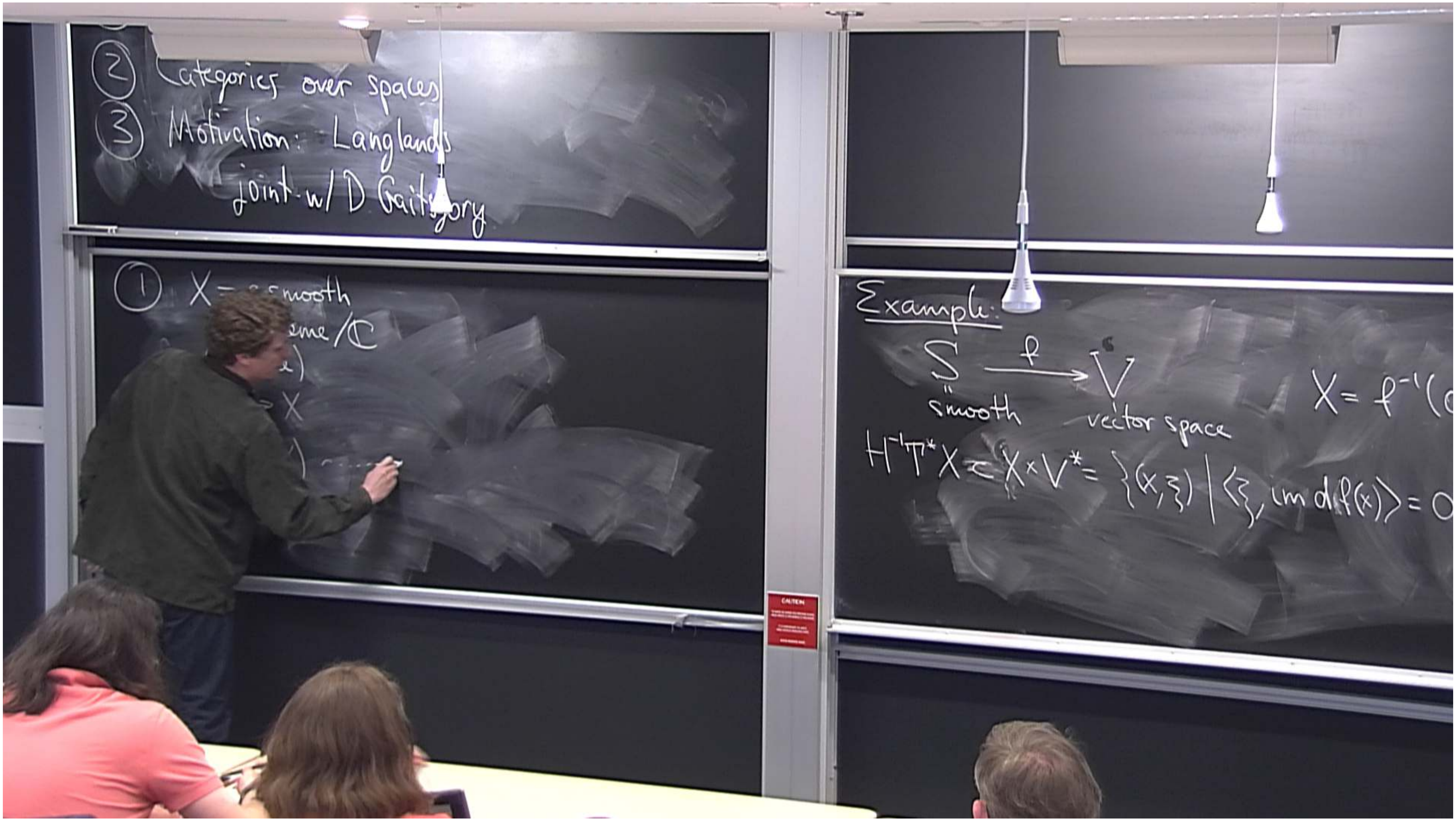
vector space
X = f⁻¹(


② Categories over spaces
③ Motivation: Langlands
joint w/ D Gaitsgory
① X = smooth
Example:
S → V
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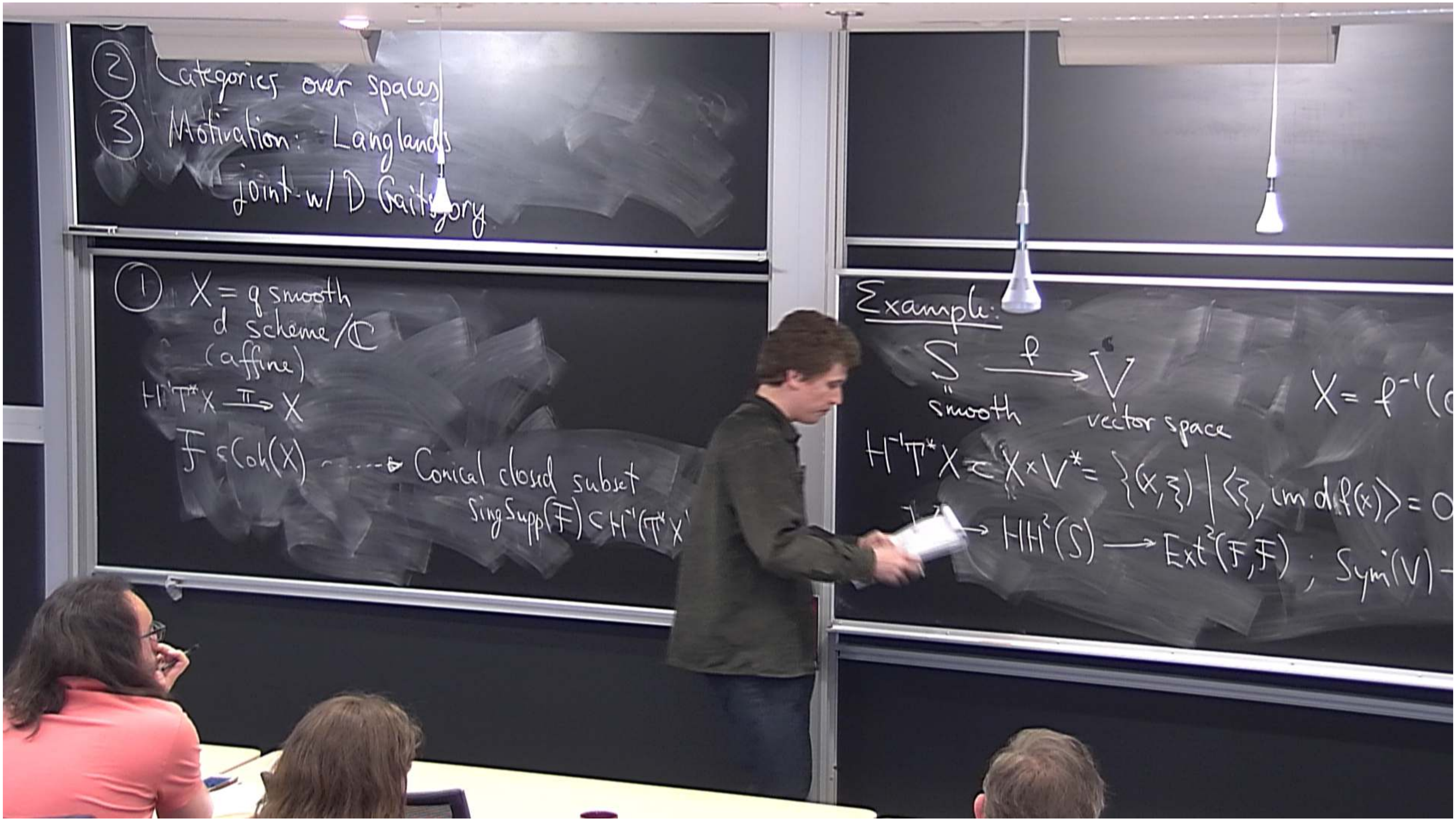
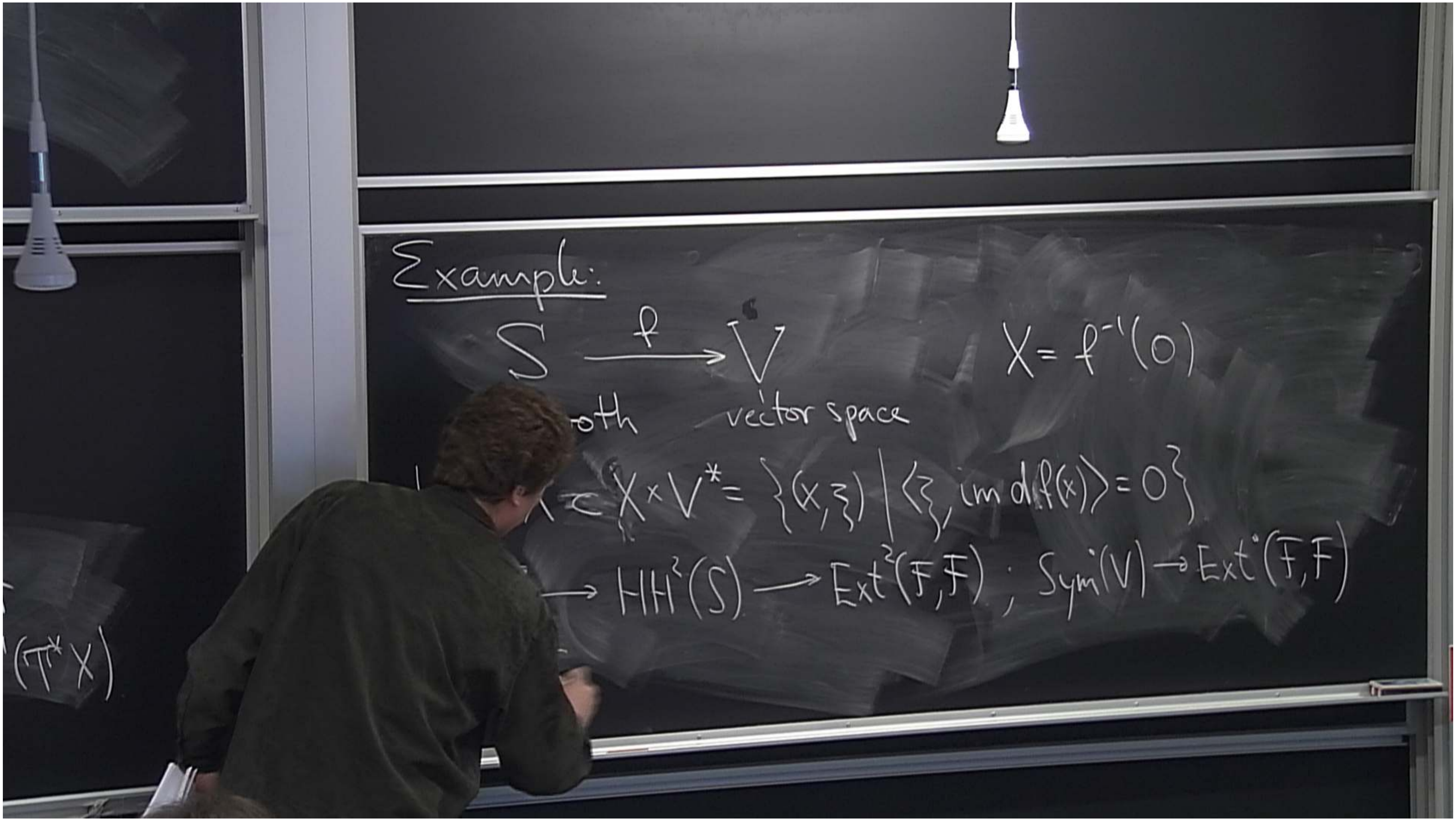
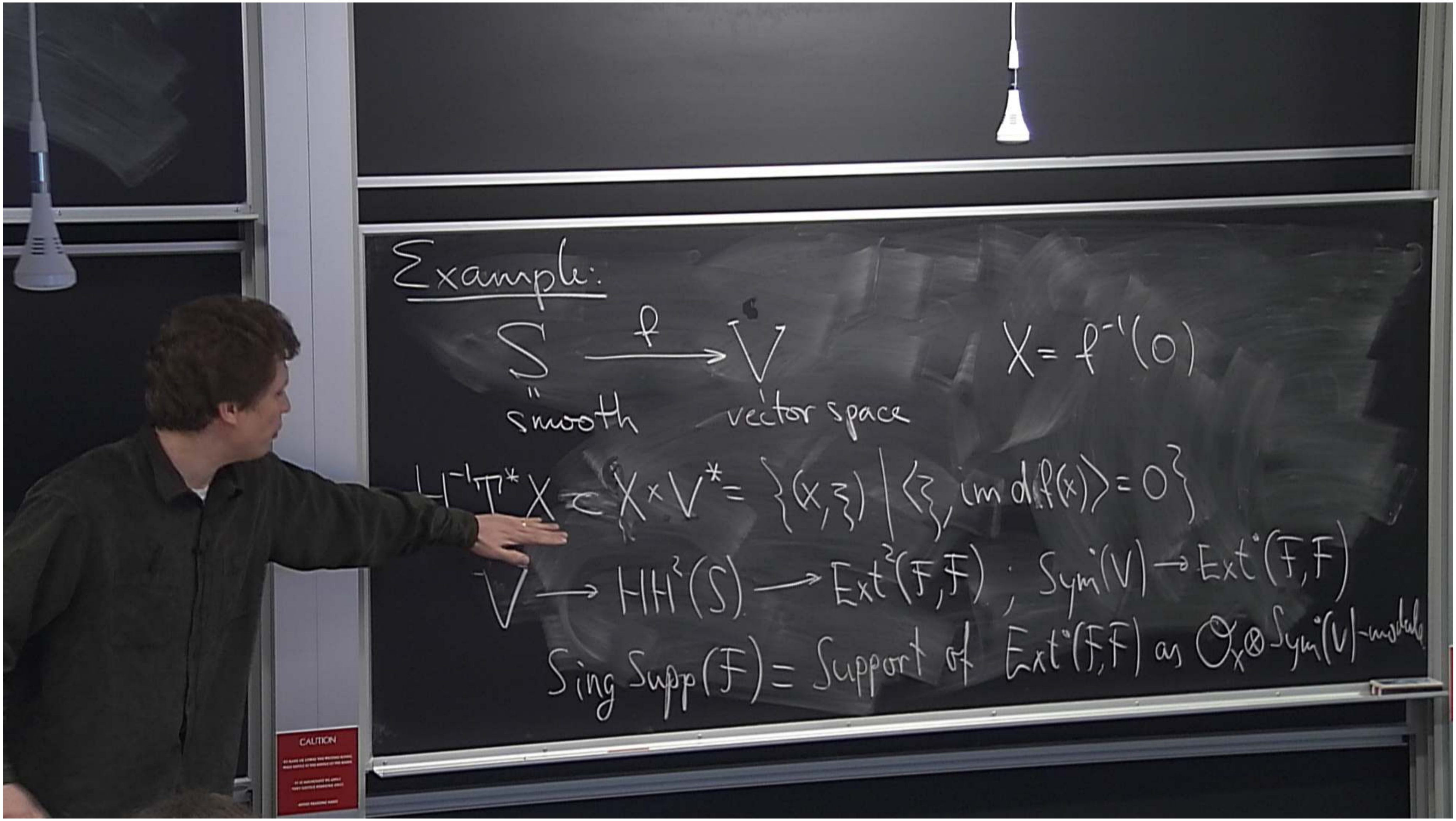
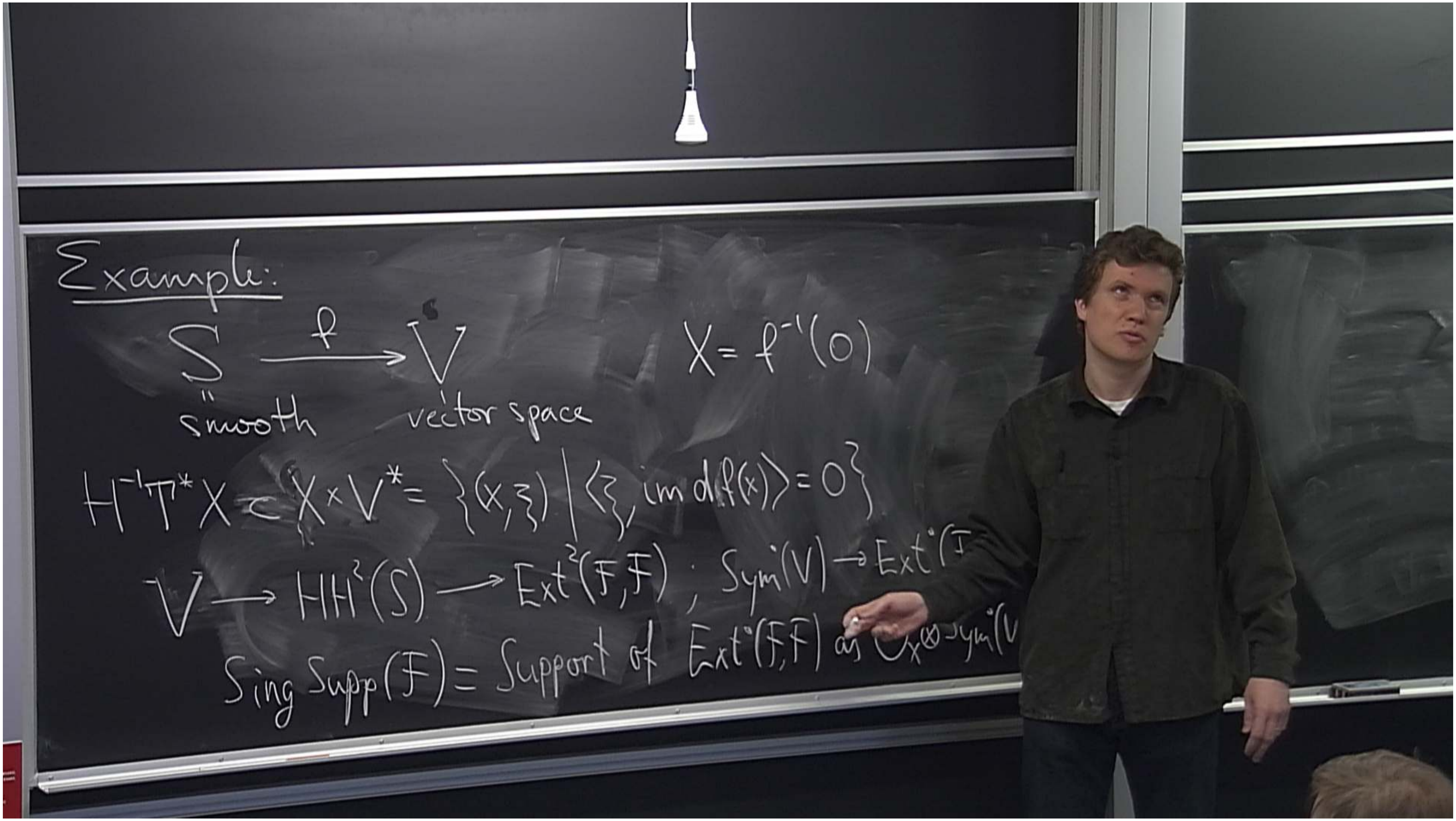
X = f⁻¹(
H⁻¹T*X = X × V* = {(x, ξ) | ⟨ξ, im df(x)⟩ = 0


② Categories over spaces
③ Motivation: Langlands
joint w/ D Gaitsgory
① X = a smooth
d scheme/ℂ
(affine)
H⁻¹T*X → X
F ∈ Coh(X) ----→ Conical closed subset
SingSupp(F) ⊂ H⁻¹(T*X)
Example:
S → V
smooth
vector space
X = f⁻¹(
H⁻¹T*X = X × V* = {(x, ξ) | ⟨ξ, im df(x)⟩ = 0
HH²(S) → Ext²(F,F) ; Sym(V)


Example:
S —f→ V
X = f⁻¹(0)
oth
vector space
⊂ X × V* = {(x,ξ) | ⟨ξ, im df(x)⟩ = 0}
→ HH²(S) → Ext²(F,F) ; Symⁱ(V) → Extⁱ(F,F)
(T*X)


Example:
$S \xrightarrow{f} V$ $X = f^{-1}(0)$
smooth vector space
$H^{-1}T^*X \subset X \times V^* = \{(x,\xi) \mid \langle \xi, \mathrm{im}\, df(x) \rangle = 0\}$
$V \to HH^2(S) \to Ext^2(F,F)$; $Sym^\bullet(V) \to Ext^\bullet(F,F)$
$Sing\,Supp(F) =$ Support of $Ext^\bullet(F,F)$ as $\mathcal{O}_X \otimes Sym^\bullet(V)$-module


Example:
S —f→ V   X = f⁻¹(0)
smooth   vector space
H⁻¹T*X ⊂ X × V* = {(x, ξ) | ⟨ξ, im df(x)⟩ = 0}
V → HH²(S) → Ext²(F,F) ; Sym(V) → Ext(F
Sing Supp(F) = Support of Ext(F,F) as O_X ⊗ Sym(V

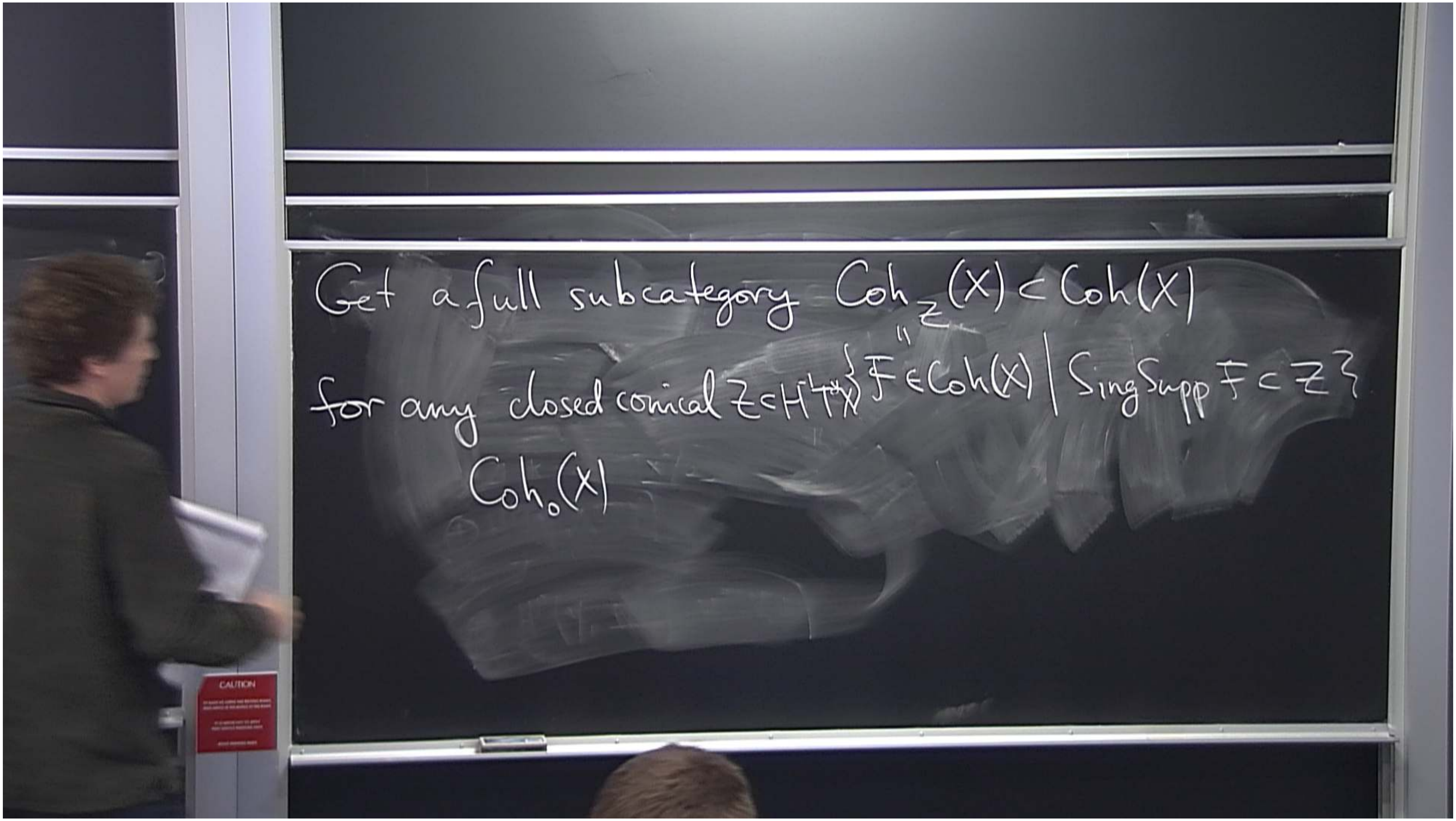
Example:

$$\underset{\text{smooth}}{S} \xrightarrow{f} \underset{\text{vector space}}{V} \qquad X = f^{-1}(0)$$

$$H^{-1}T^*X = \{(x,\xi) \mid \langle \xi, \operatorname{im} df(x)\rangle = 0\} \subset X \times V^*$$

$$V \longrightarrow HH^2(X) \longrightarrow \operatorname{Ext}^2(\mathcal{F},\mathcal{F}) \; ; \; \operatorname{Sym}^\bullet(V) \to \operatorname{Ext}^\bullet(\mathcal{F},\mathcal{F})$$

$$\operatorname{Sing\,Supp}(\mathcal{F}) = \text{Support of } \operatorname{Ext}^\bullet(\mathcal{F},\mathcal{F}) \text{ as } \mathcal{O}_X \otimes \operatorname{Sym}^\bullet(V)\text{-module}$$


Get a full subcategory $Coh_Z(X) \subset Coh(X)$
$= \{ \mathcal{F} \in Coh(X) \mid SingSupp\, \mathcal{F} \subset Z \}$
for any closed conical $Z \subset H^1(T^*X)$
$Coh_0(X)$
CAUTION

$\times V^*$

$\mathcal{F}, \mathcal{F})$

$\mathrm{Sym}(V)$-module

Get a full subcategory $\mathrm{Coh}_Z(X) \subset \mathrm{Coh}(X)$

$= \{\mathcal{F} \in \mathrm{Coh}(X) \mid \mathrm{SingSupp}\,\mathcal{F} \subset Z\}$

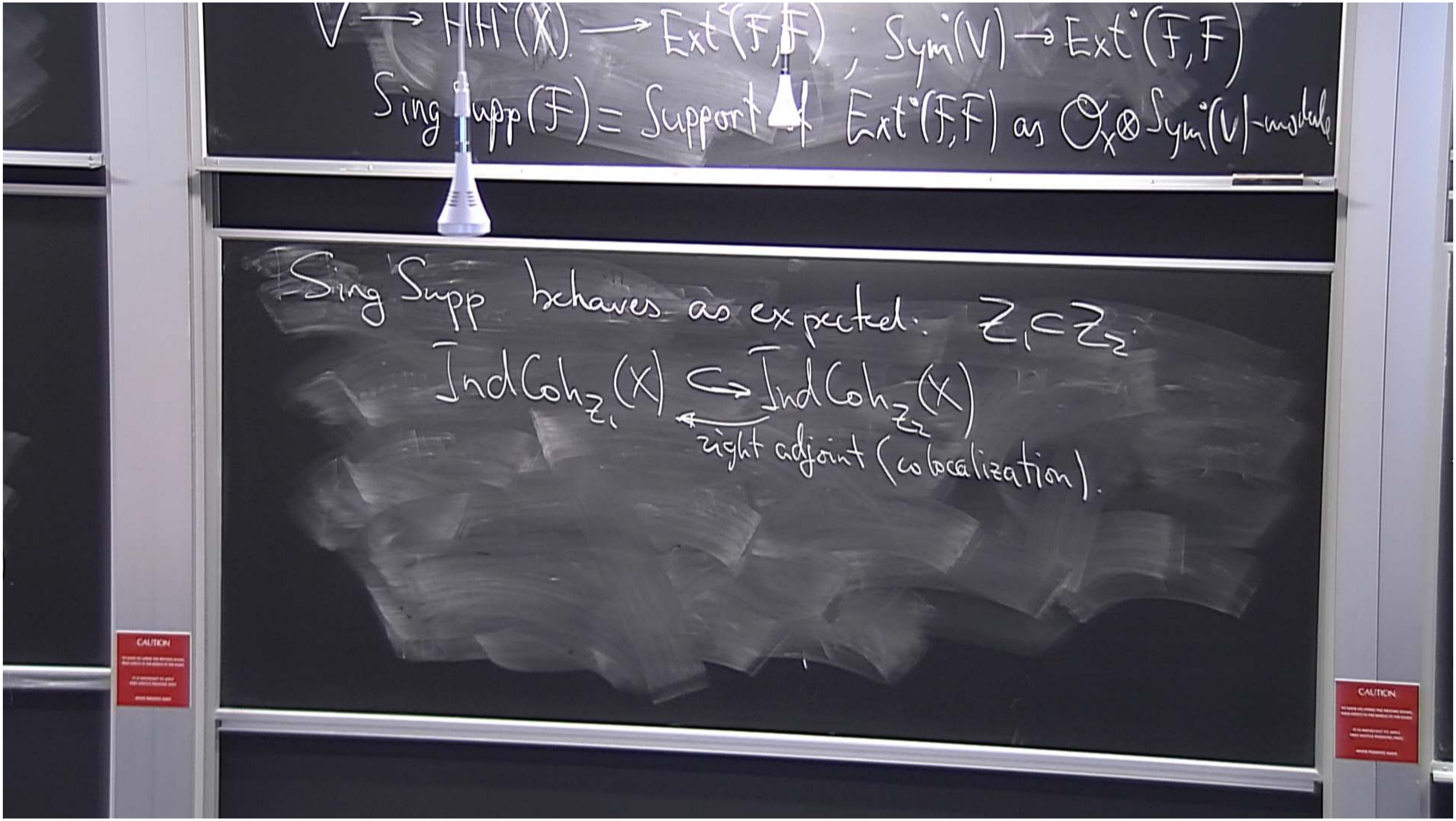
for any closed conical $Z \subset H^{-1}(T^*X)$

ind completion

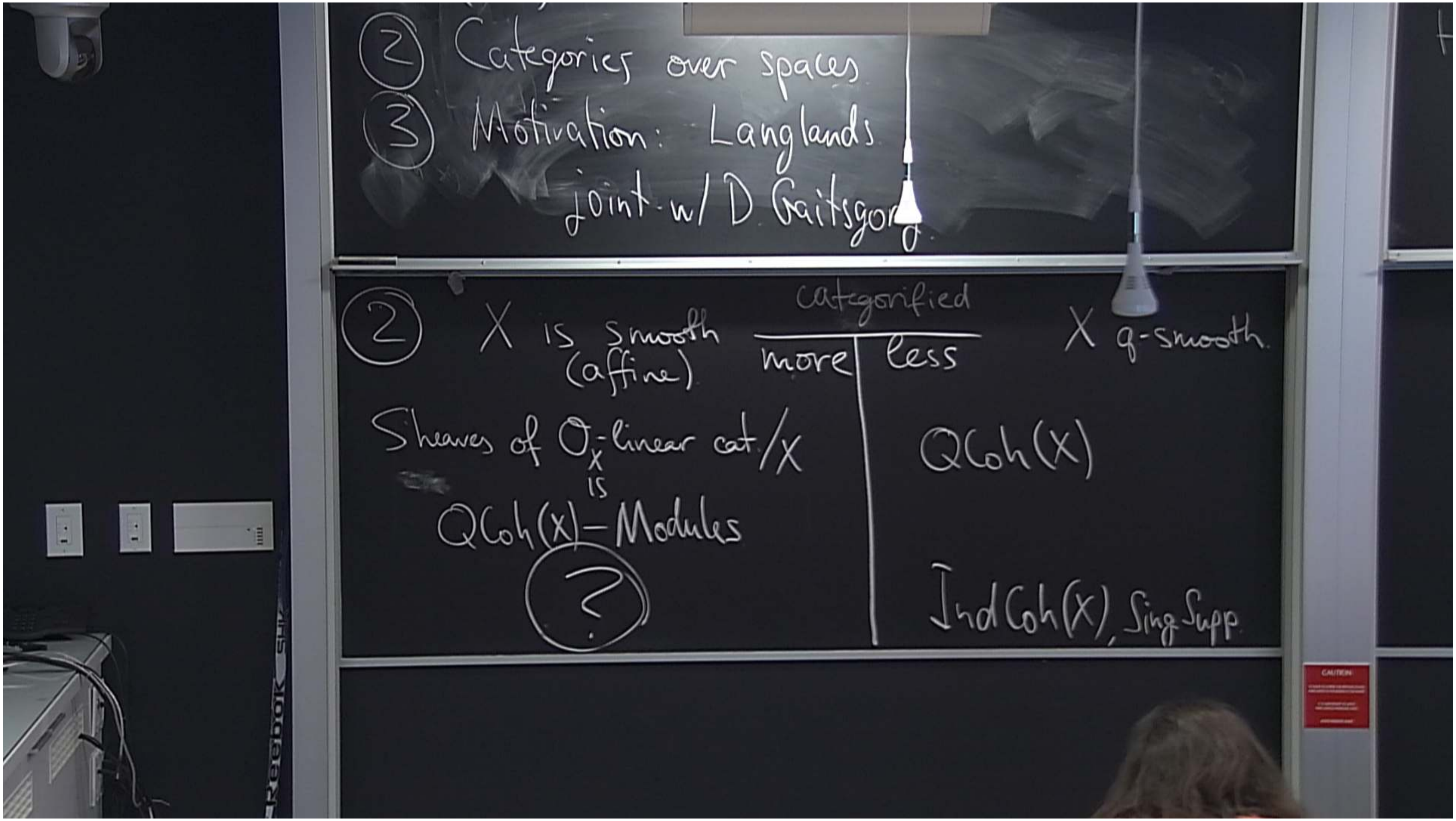
$\mathrm{Perf}(X) = \mathrm{Coh}_0(X)$ $\qquad$ $\mathrm{Ind}(\mathrm{Perf}(X)) = \mathrm{QCoh}(X)$

$\mathrm{Coh}_Z(X)$ $\qquad$ $\mathrm{Ind}(\mathrm{Coh}(X)_Z) = \mathrm{IndCoh}(X)_Z$

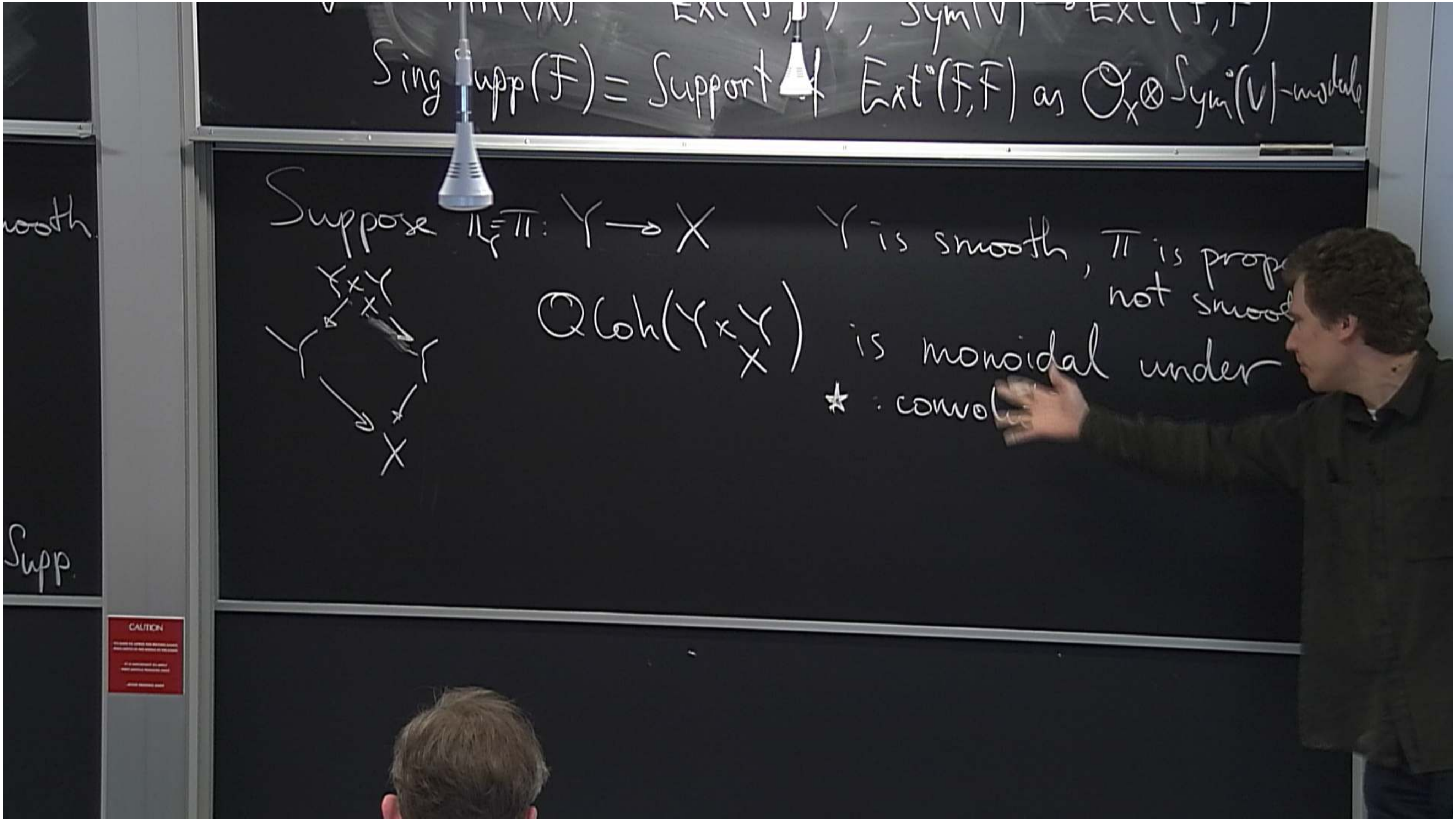
$\mathrm{Coh}(X) = \mathrm{Coh}_{H^{-1}T^*X}(X)$ $\qquad$ $\mathrm{Ind}(\mathrm{Coh}(X)) = \mathrm{IndCoh}(X)$

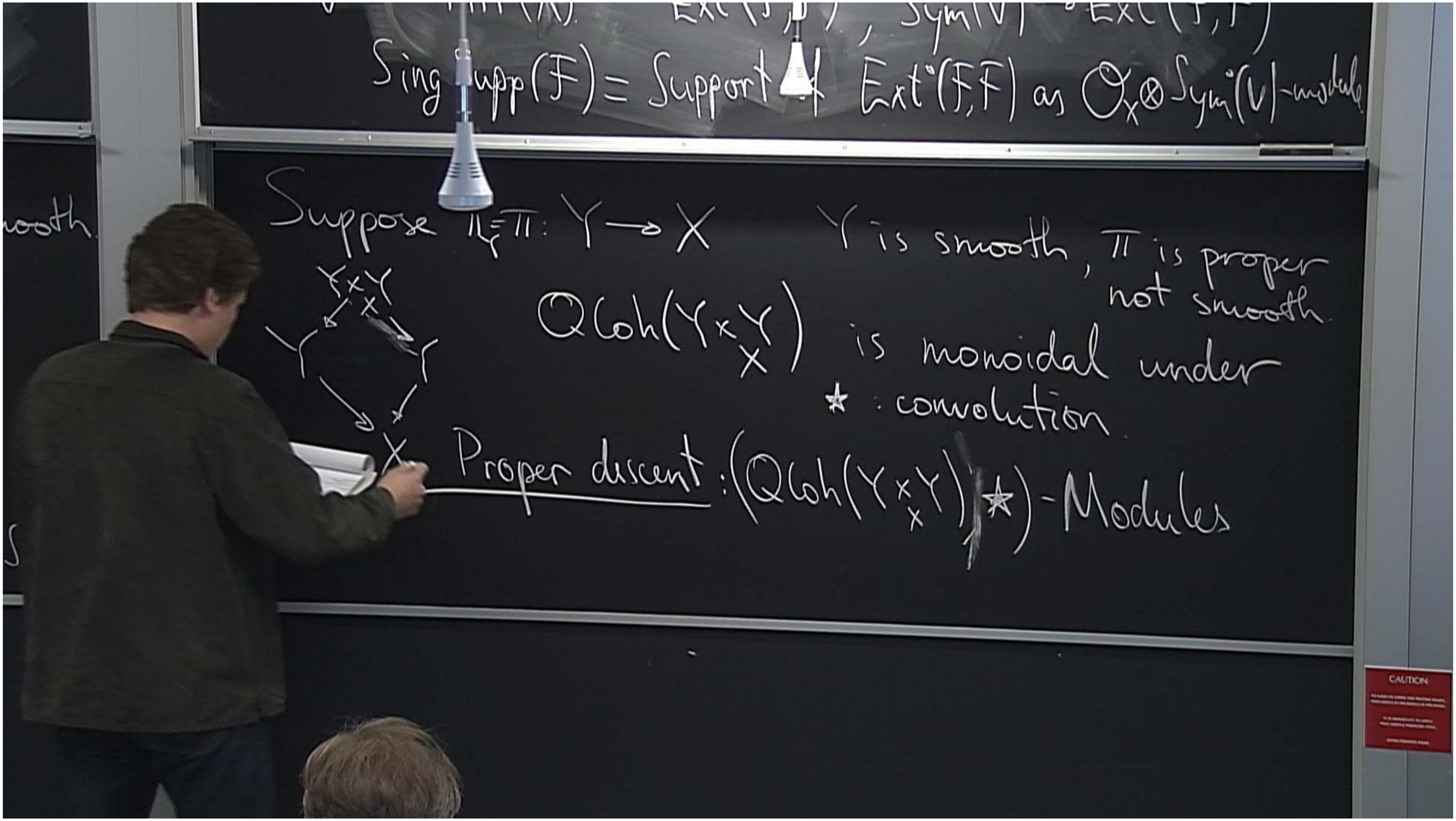


$V \to HH^2(X) \to Ext^2(\mathcal{F},\mathcal{F})$; $Sym(V) \to Ext^\bullet(\mathcal{F},\mathcal{F})$
$Sing\,Supp(\mathcal{F}) =$ Support of $Ext^\bullet(\mathcal{F},\mathcal{F})$ as $\mathcal{O}_X \otimes Sym(V)$-module
Sing Supp behaves as expected: $Z_1 \subset Z_2$:
$IndCoh_{Z_1}(X) \hookrightarrow IndCoh_{Z_2}(X)$
right adjoint (colocalization).



2 Categories over spaces
3 Motivation: Langlands
joint w/ D. Gaitsgory
2 X is smooth (affine)
categorified
more
less
X q-smooth
Sheaves of $\mathcal{O}_X$-linear cat /X
is
QCoh(X)-Modules
?
QCoh(X)
IndCoh(X), Sing Supp


Sing Supp(F) = Support of Ext°(F,F) as O_X ⊗ Sym°(V)-module
Suppose π: Y → X
Y is smooth, π is prop
not smoo
Y ×_X Y
QCoh(Y ×_X Y) is monoidal under
★ : convol
CAUTION


Sing Supp(F) = Support of Ext°(F,F) as O_X ⊗ Sym°(V)-modules
Suppose π: Y → X
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QCoh(Y ×_X Y) is monoidal under
⋆ : convolution
Proper descent: (QCoh(Y ×_X Y), ⋆)-Modules

$\mathrm{Sing\,Supp}(\mathcal{F}) = \text{Support of } \mathrm{Ext}^\bullet(\mathcal{F},\mathcal{F}) \text{ as } \mathcal{O}_X \otimes \mathrm{Sym}^\bullet(V)\text{-modules}$

Suppose $\pi: Y \to X$ $\quad$ Y is smooth, π is proper, not smooth

$$\begin{tikzcd} & Y \times_X Y \arrow[dl] \arrow[dr] & \\ Y \arrow[dr] & & Y \arrow[dl] \\ & X & \end{tikzcd}$$

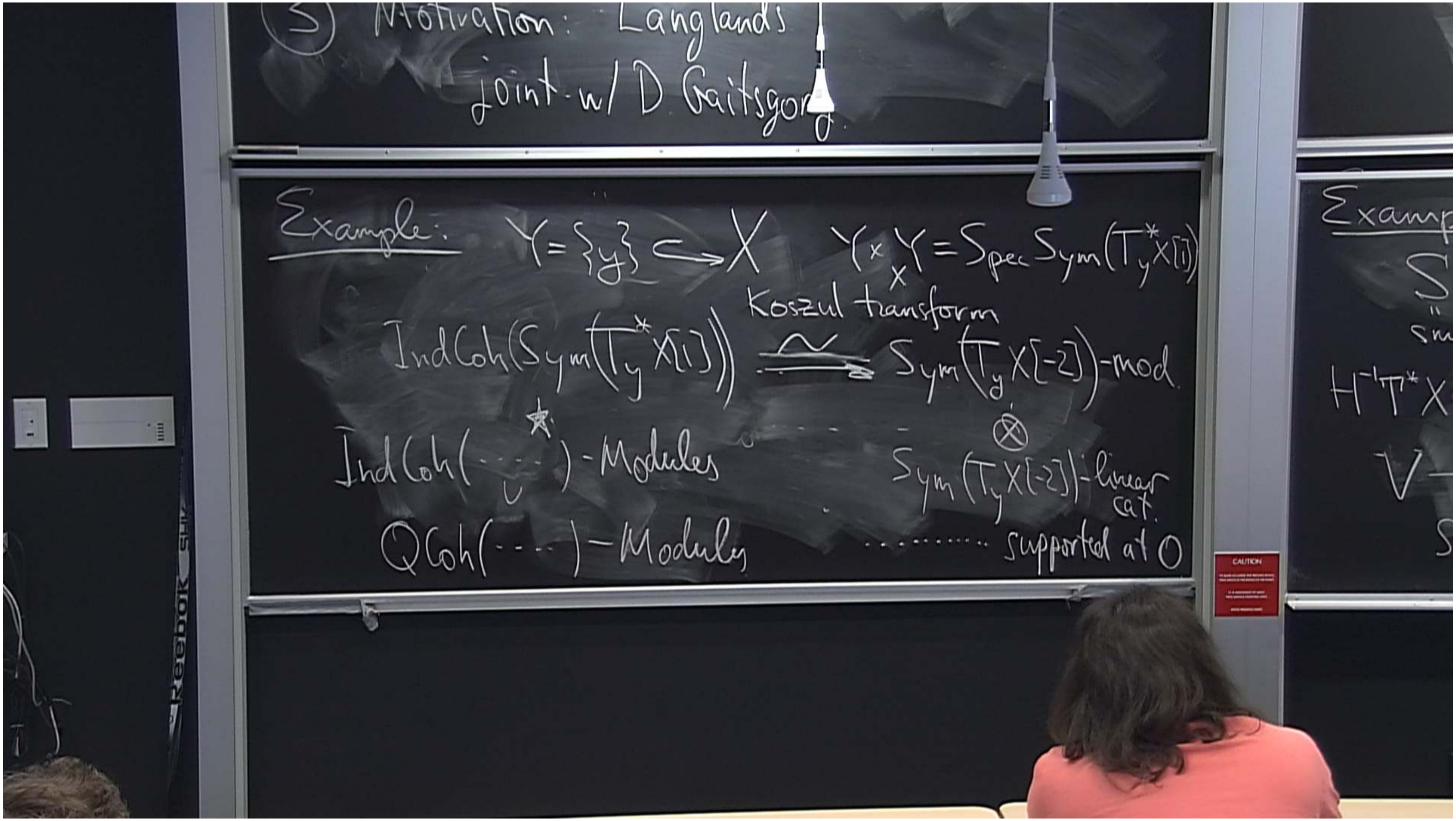
$\mathrm{QCoh}(Y \times_X Y)$ is monoidal under $\star$: convolution

Proper descent: $(\mathrm{QCoh}(Y \times_X Y), \star)$-Modules

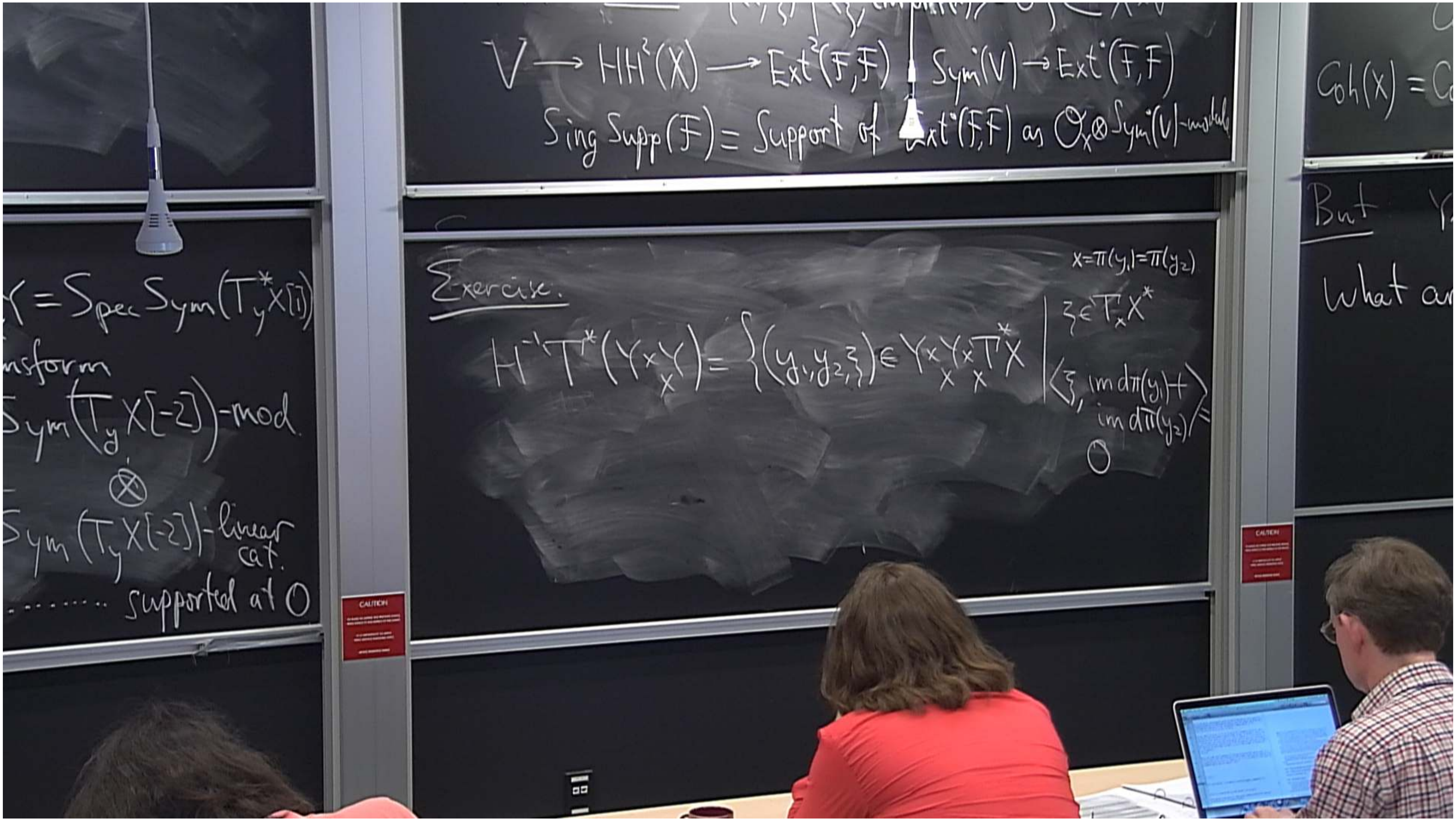
$\{\mathrm{QCoh}(X)\text{-Modules "supported over } \pi(Y)\}$

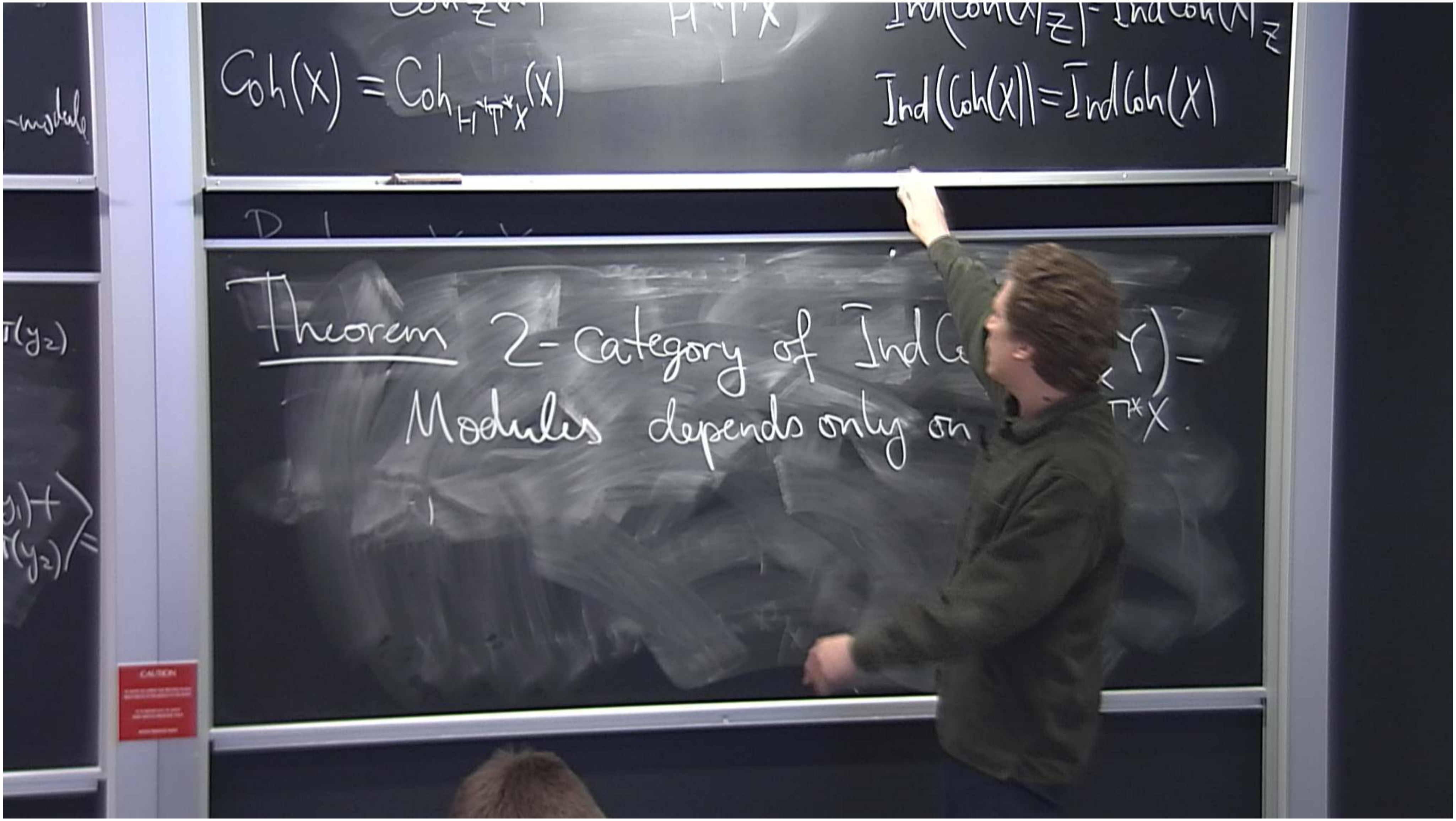
But: $Y \times_X Y$ is q. smooth, but not smooth.

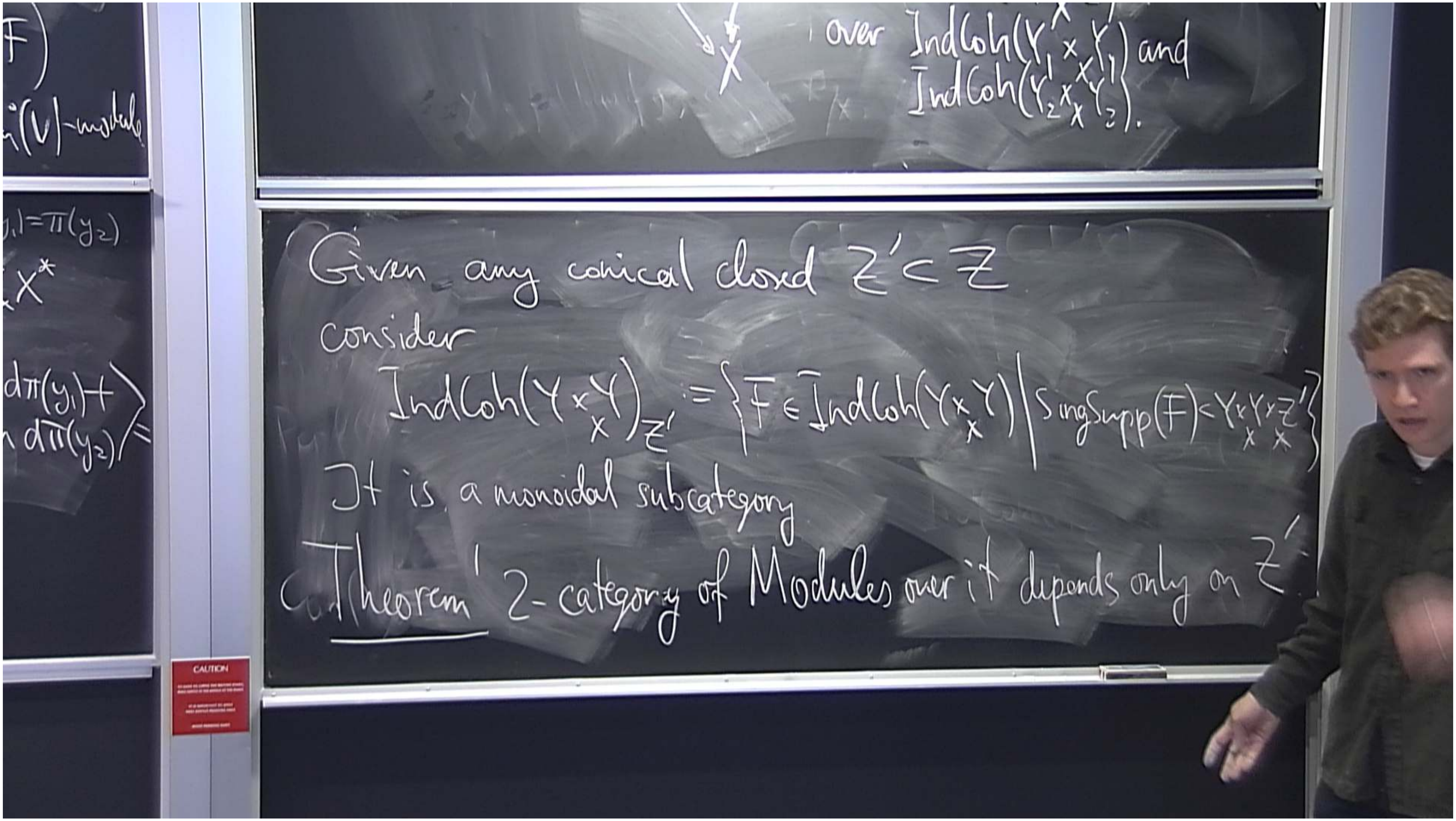
What are $(\mathrm{IndCoh}(Y \times_X Y), \star)$-Modules?

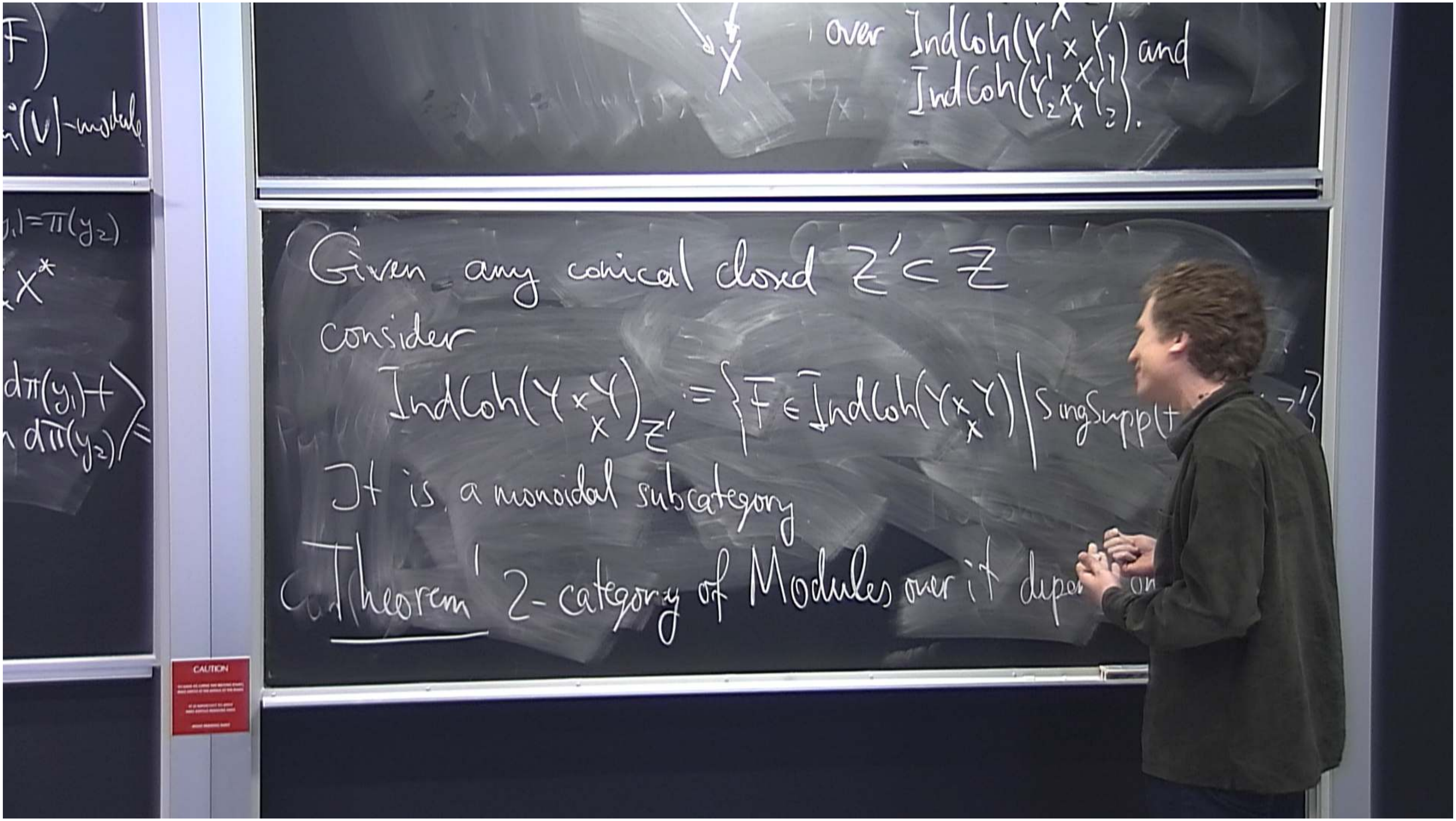


3) Motivation: Langlands
joint w/ D. Gaitsgory
Example: $Y=\{y\} \hookrightarrow X$ $Y \times_X Y = \mathrm{Spec}\,\mathrm{Sym}(T_y^*X[1])$
Koszul transform
$\mathrm{IndCoh}(\mathrm{Sym}(T_y^*X[1])) \simeq \mathrm{Sym}(T_yX[-2])$-mod
$\mathrm{IndCoh}(\cdots)$-Modules $\mathrm{Sym}(T_yX[-2])$-linear cat.
$\mathrm{QCoh}(\cdots)$-Modules   supported at 0
Example
$H^{-1}T^*X$


Y = Spec Sym(T*_y X[1])
nsform
Sym(T_y X[-2])-mod
⊗
Sym(T_y X[-2])-linear cat.
supported at 0
V → HH²(X) → Ext²(F,F)
Sym(V) → Ext(F,F)
Sing Supp(F) = Support of Ext(F,F) as O_X ⊗ Sym(V)-module
Exercise.
H⁻¹T*(Y ×_X Y) = {(y₁, y₂, ξ) ∈ Y ×_X Y ×_X T*X
x = π(y₁) = π(y₂)
ξ ∈ T*_x X
⟨ξ, im dπ(y₁) + im dπ(y₂)⟩ = 0
Coh(X) =
But
What a


Coh(X) = Coh_{H^{-1}T^*X}(X)
Ind(Coh(X)) = IndCoh(X)
Theorem 2-Category of IndCoh
Modules depends only on


over $\mathrm{IndCoh}(Y_1' \times_X Y_1)$ and
$\mathrm{IndCoh}(Y_2' \times_X Y_2)$.
Given any conical closed $Z' \subset Z$
consider
$\mathrm{IndCoh}(Y \times_X Y)_{Z'} = \{ F \in \mathrm{IndCoh}(Y \times_X Y) \mid \mathrm{SingSupp}(F) \subset Y \times_X Y \times_X Z' \}$
It is a monoidal subcategory
Theorem' 2-category of Modules over it depends only on Z'

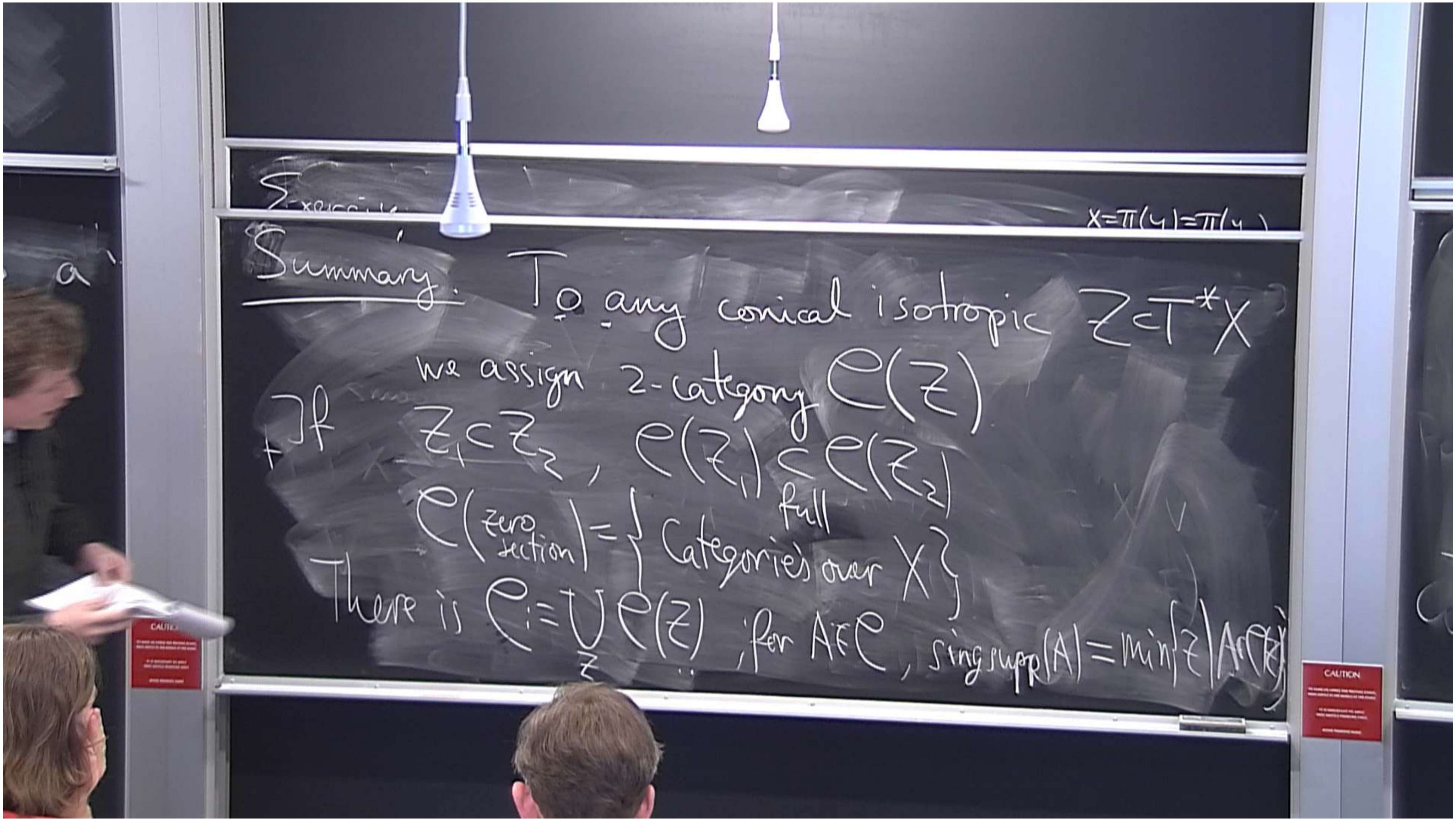

over IndCoh(Y'1 ×X Y1) and
IndCoh(Y'2 ×X Y2).
Given any conical closed Z' ⊂ Z
consider
IndCoh(Y ×X Y)Z' = {F ∈ IndCoh(Y ×X Y) | SingSupp(F
It is a monoidal subcategory
Theorem' 2-category of Modules over it depen

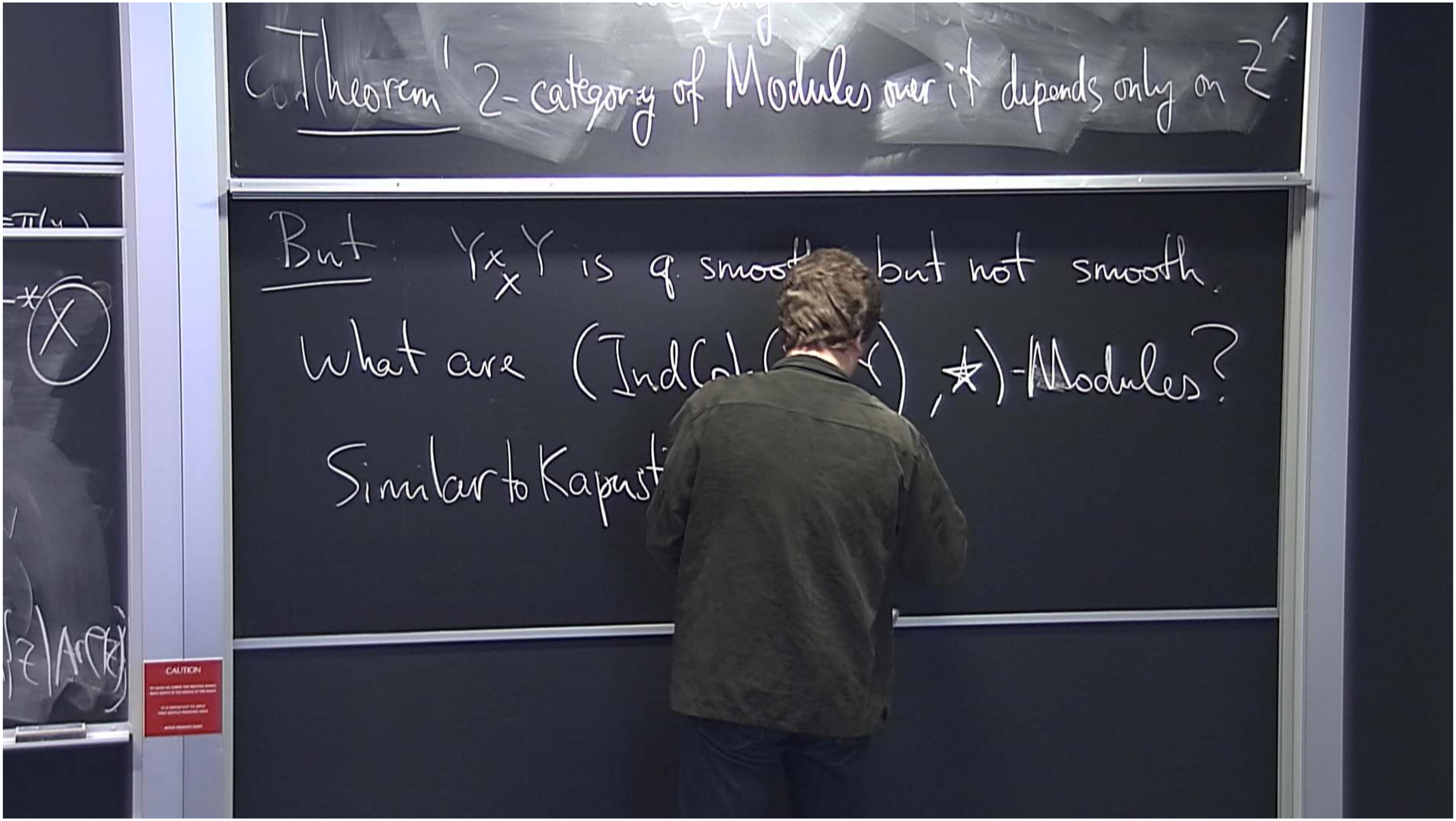
Summary: To any conical isotropic $Z \subset T^*X$ we assign 2-category $\mathcal{C}(Z)$.

If $Z_1 \subset Z_2$, $\mathcal{C}(Z_1) \underset{\text{full}}{\subset} \mathcal{C}(Z_2)$

$\mathcal{C}(\text{zero section}) = \{\text{Categories over } X\}$

There is $\mathcal{C} := \bigcup_Z \mathcal{C}(Z)$


Summary: To any conical isotropic $Z \subset T^*X$
we assign 2-category $\mathcal{C}(Z)$
If $Z_1 \subset Z_2$, $\mathcal{C}(Z_1) \subset \mathcal{C}(Z_2)$ full
$\mathcal{C}(\text{zero section}) = \{\text{Categories over } X\}$
There is $\mathcal{C} := \bigcup_Z \mathcal{C}(Z)$; for $A \in \mathcal{C}$, sing supp$(A) = \min\{Z \mid A \in \mathcal{C}(Z)\}$


Theorem' 2-category of Modules over it depends only on Z'.
But: Y×X Y is q. smoo but not smooth.
What are (Ind(Co Y), ★)-Modules?
Similar to Kapust
CAUTION