Title: Singular support of categories

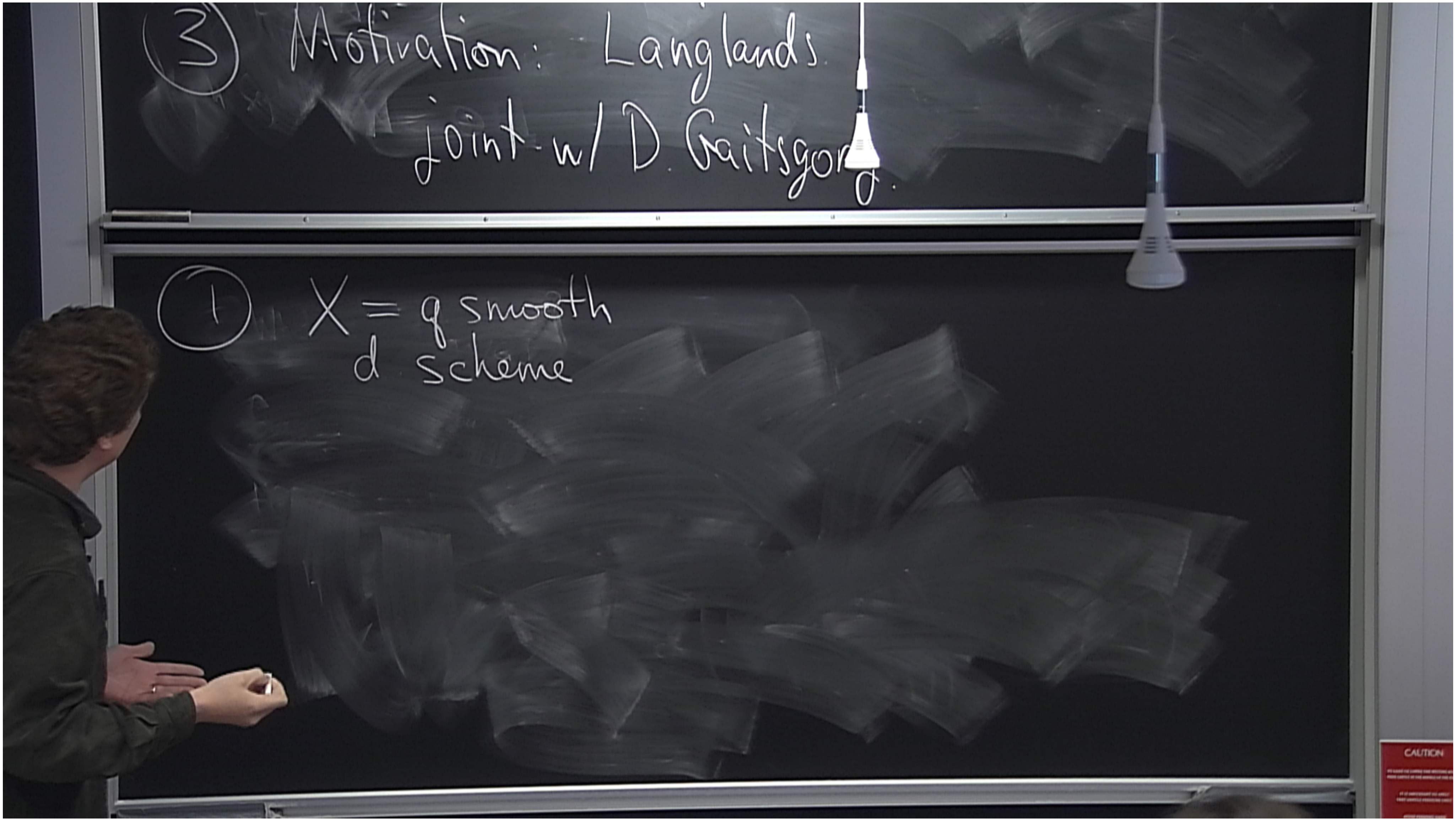
Date: Apr 19, 2016 11:00 AM

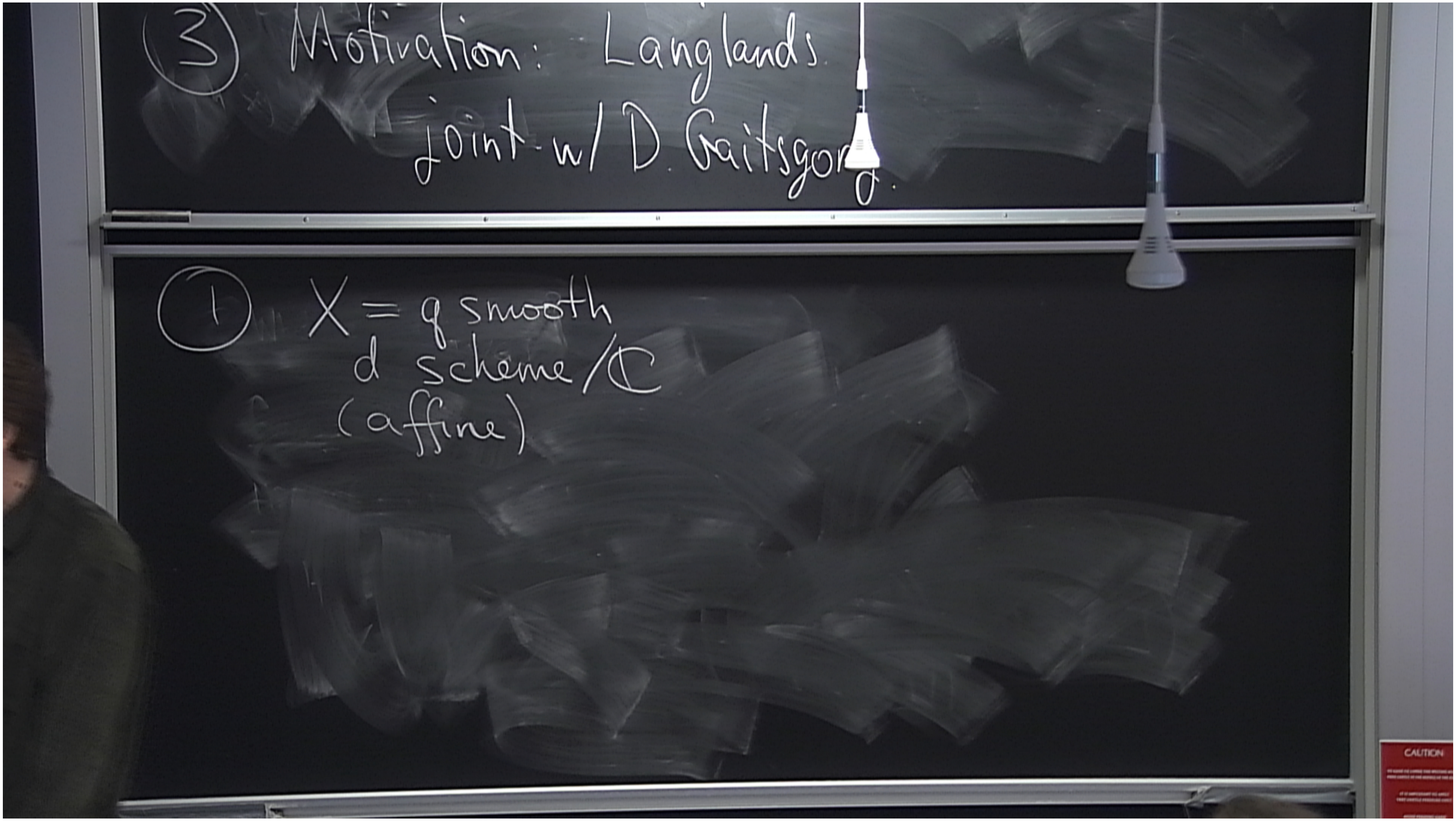
URL: http://pirsa.org/16040075

Abstract: In many situations, geometric objects on a space have some kind of singular support, which refines the usual support. For instance, for smooth X, the singular support of a D-module (or a perverse sheaf) on X is as a conical subset of the cotangent bundle; similarly, for quasi-smooth X, the singular support of a coherent sheaf on X is a conical subset of the cohomologically shifted cotangent bundle. I would like to describe a higher categorical version of this notion.

Let X be a smooth variety, and let Z be a closed conical isotropic subset of the cotangent bundle of X. I will define a 2-category associated with Z; its objects may be viewed as `categories over X with singular support in Z'. In particular, if Z is the zero section, we simply consider categories over Z in the usual sense.

This talk is based on a joint project with D.Gaitsgory. The project is motivated by the local geometric Langlands correspondence; I plan to sketch the relation with the Langlands correspondence in the talk.


③ Motivation: Langlands
joint w/ D. Gaitsgory.
① X = q smooth
d scheme
CAUTION



③ Motivation: Langlands
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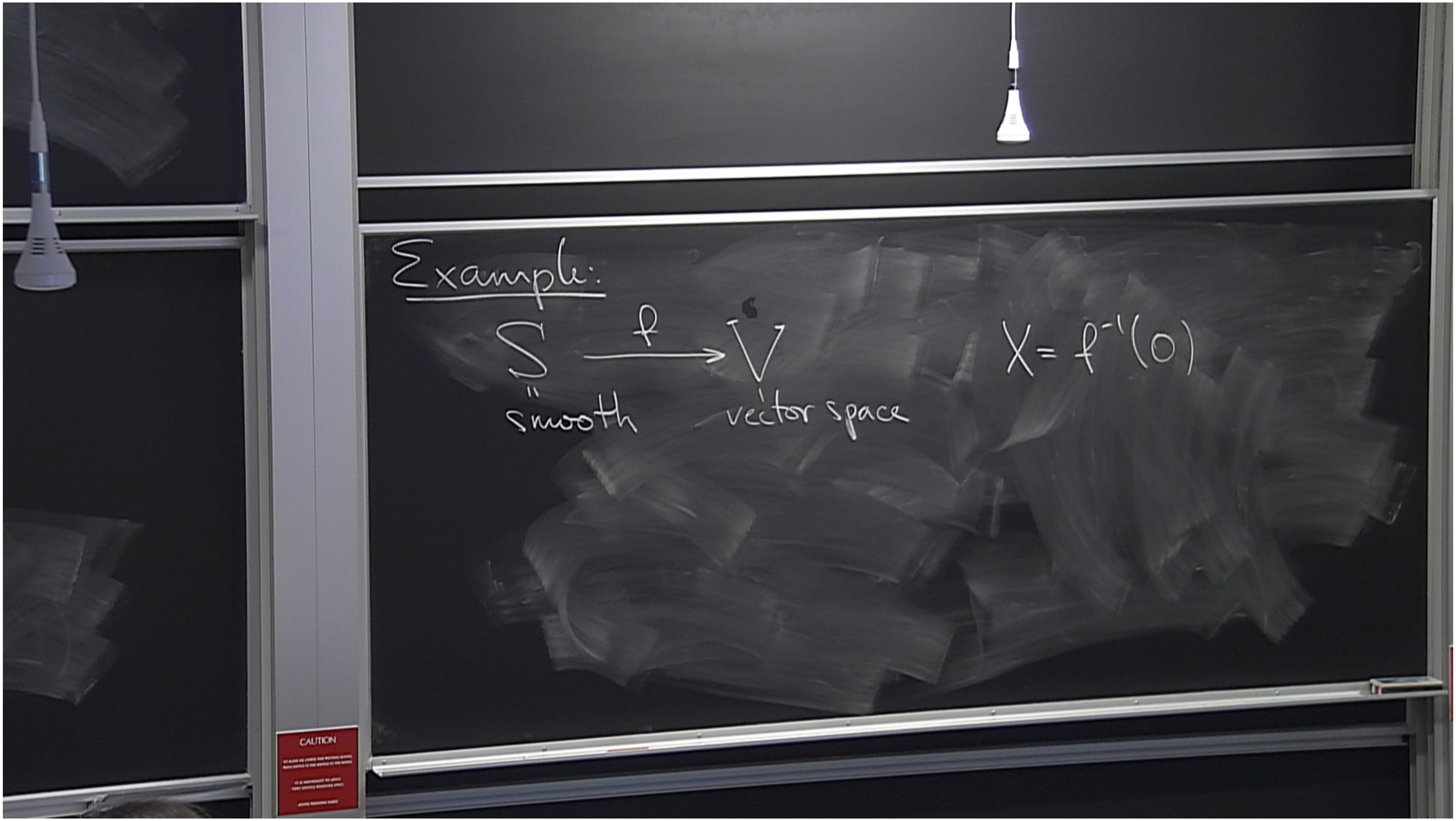
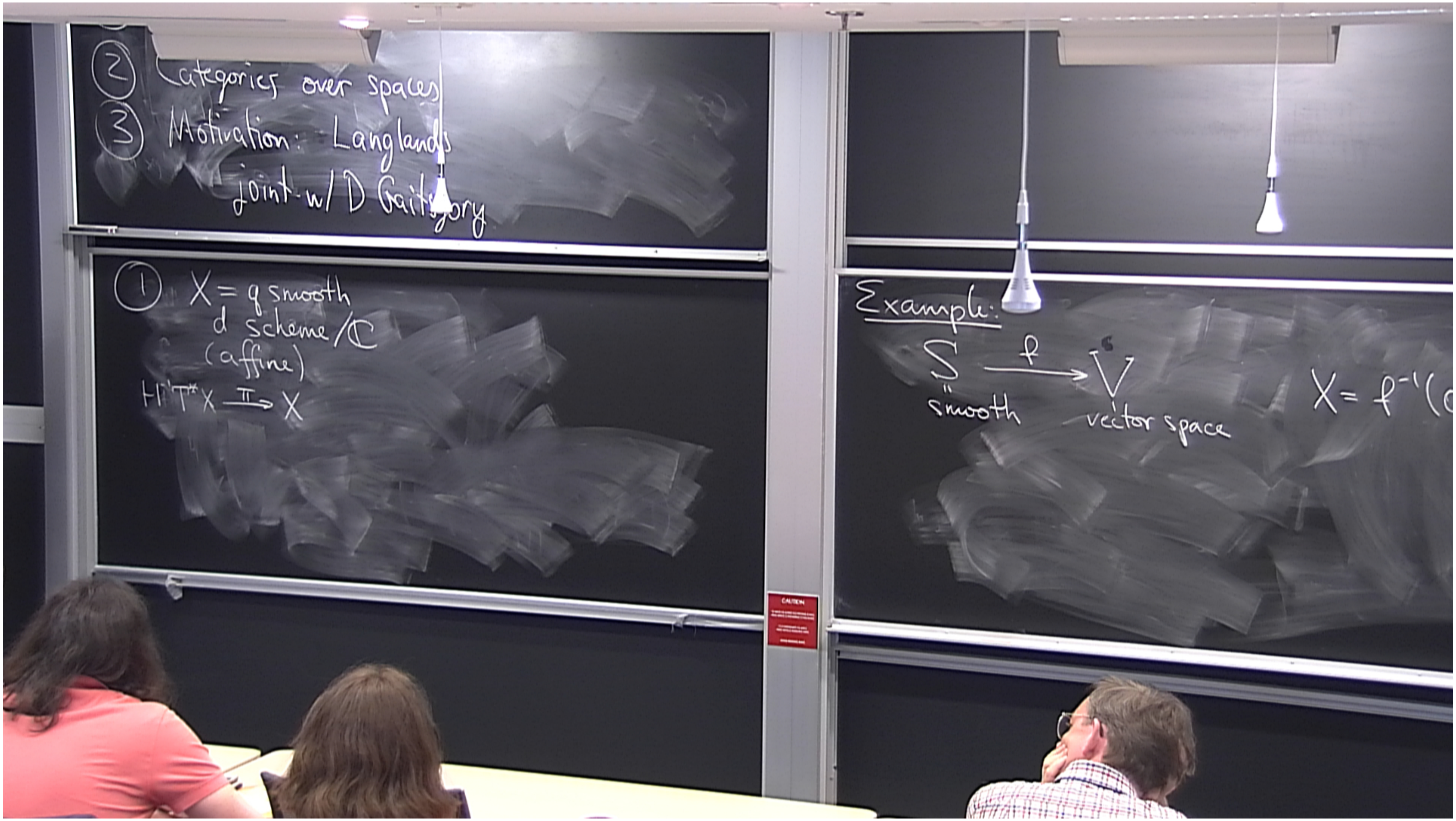
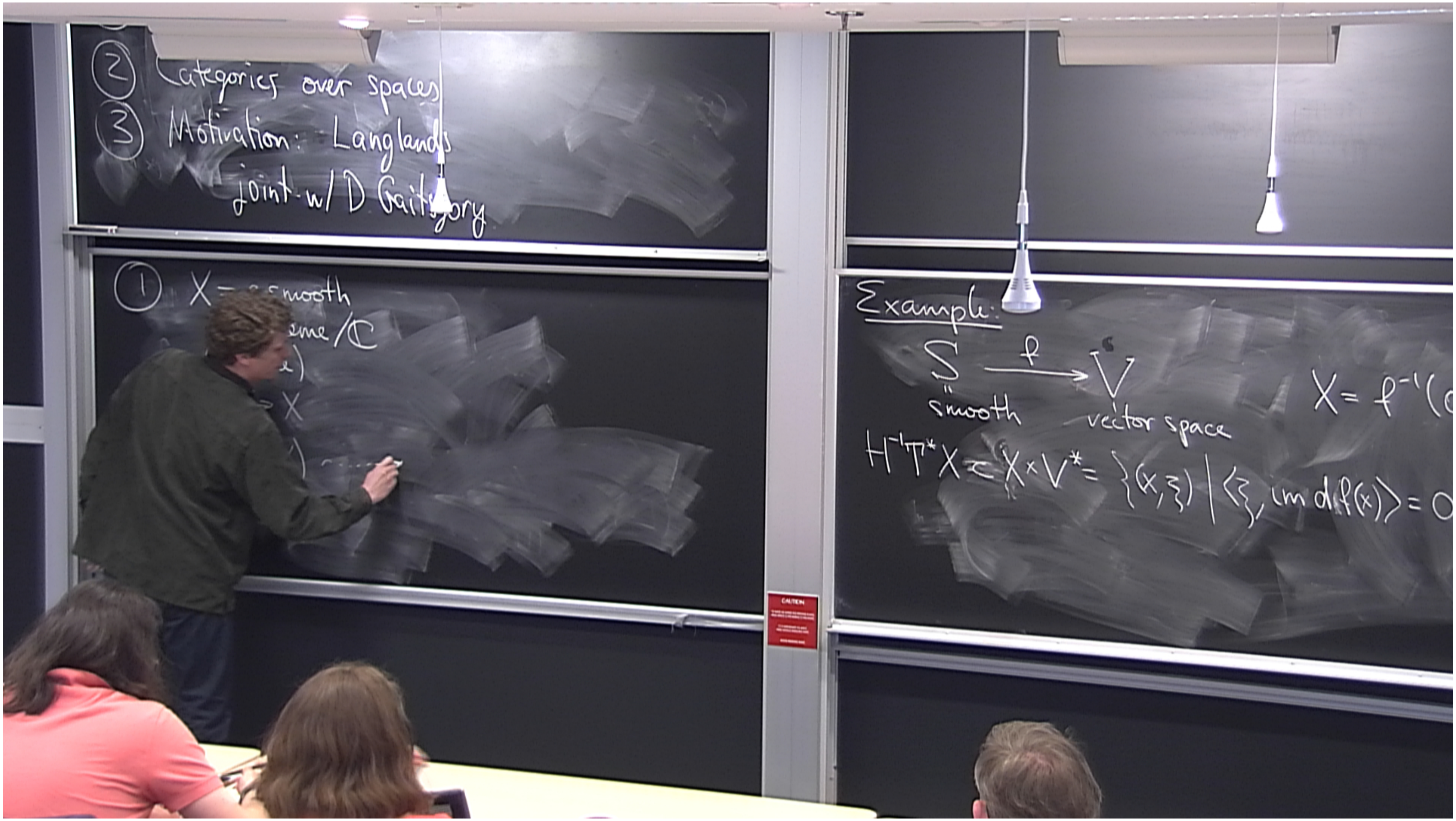
d scheme/$\mathbb{C}$
(affine)


Example:
$S \xrightarrow{f} V$
smooth
vector space
$X = f^{-1}(0)$
CAUTION


2 Categories over spaces
3 Motivation: Langlands
joint w/ D Gaitsgory
1 X = q smooth
d scheme/ℂ
(affine)
T*X → X
π
Example:
S → V
f
smooth
vector space


Categories over spaces
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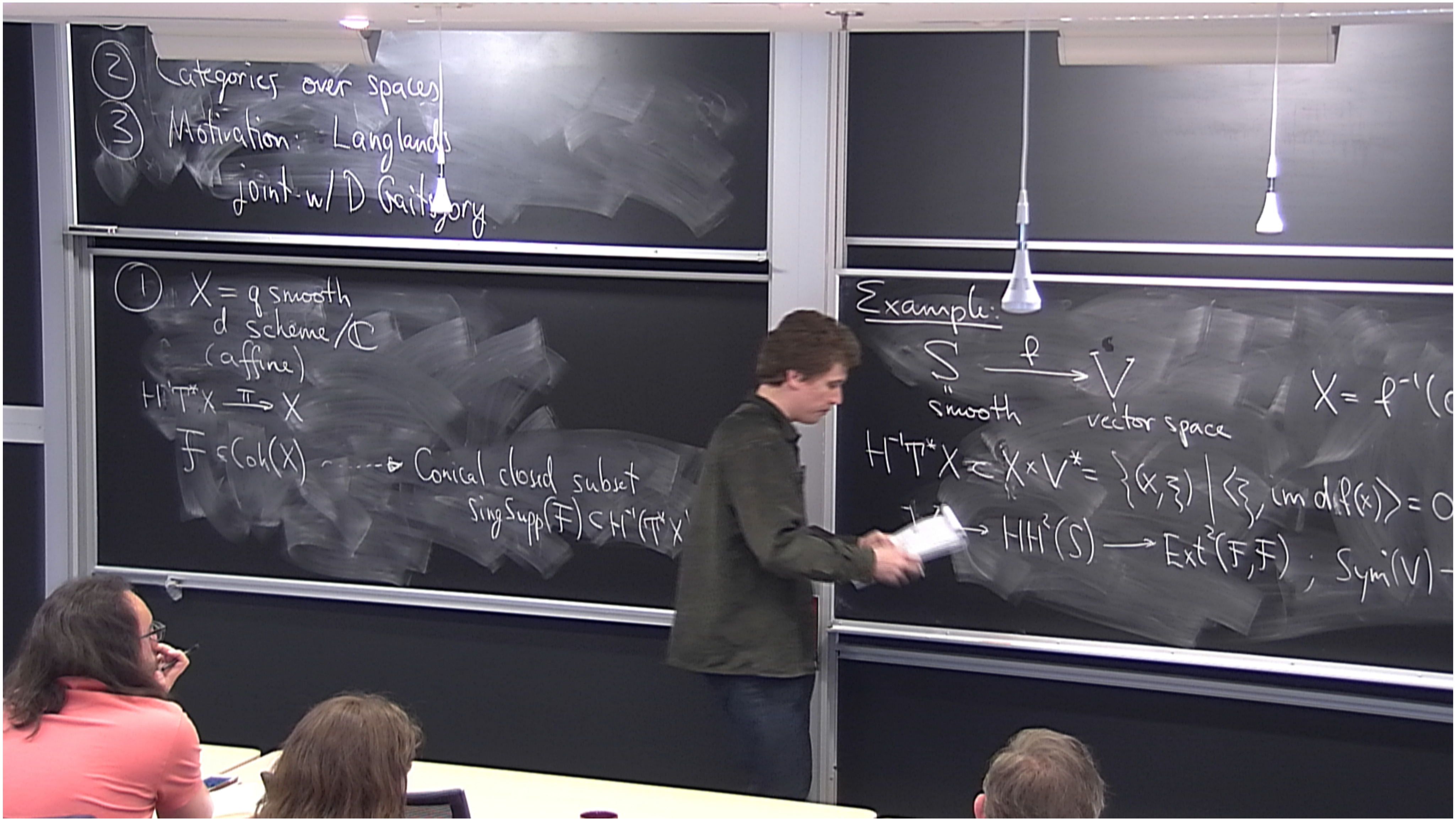
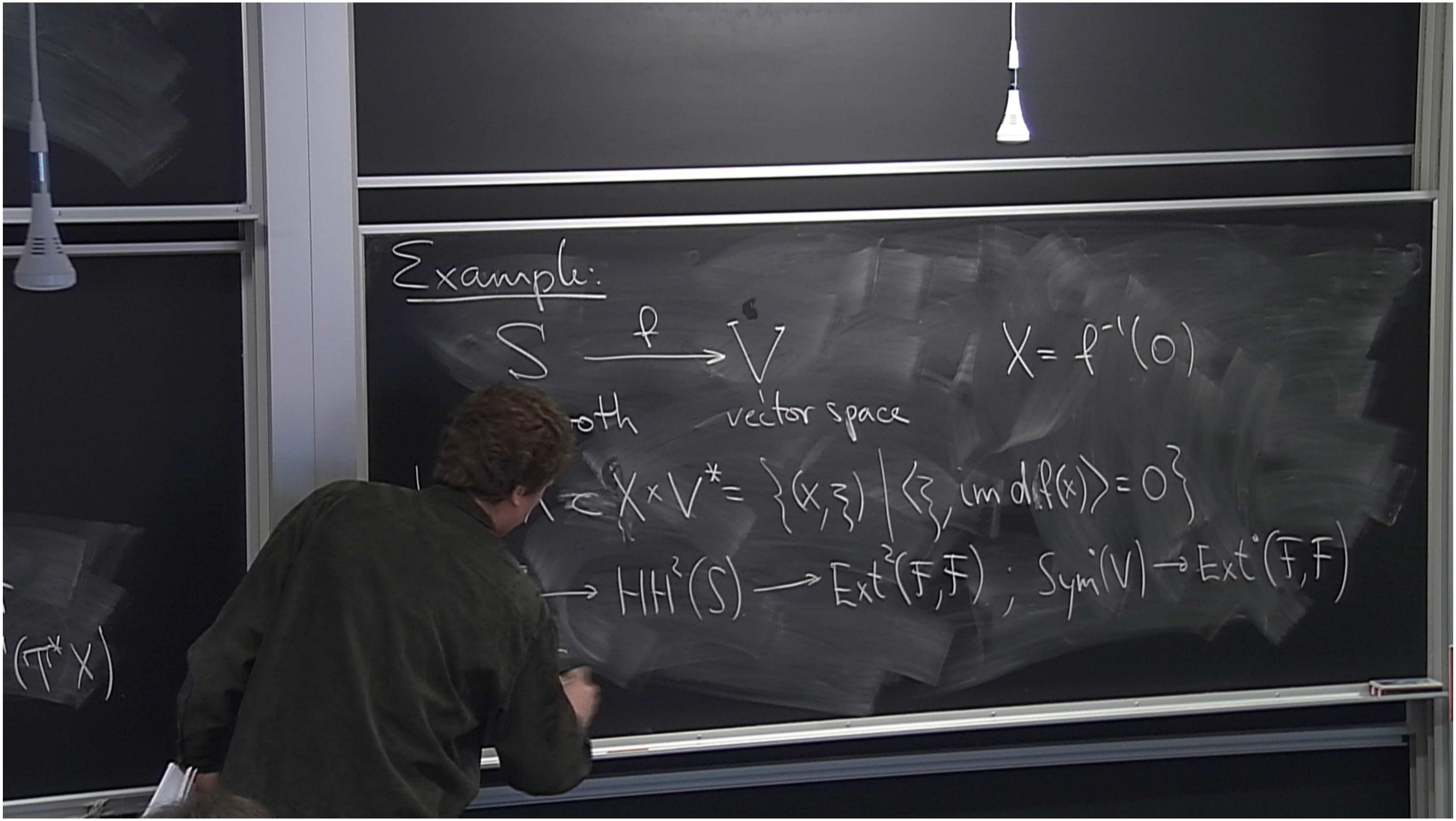
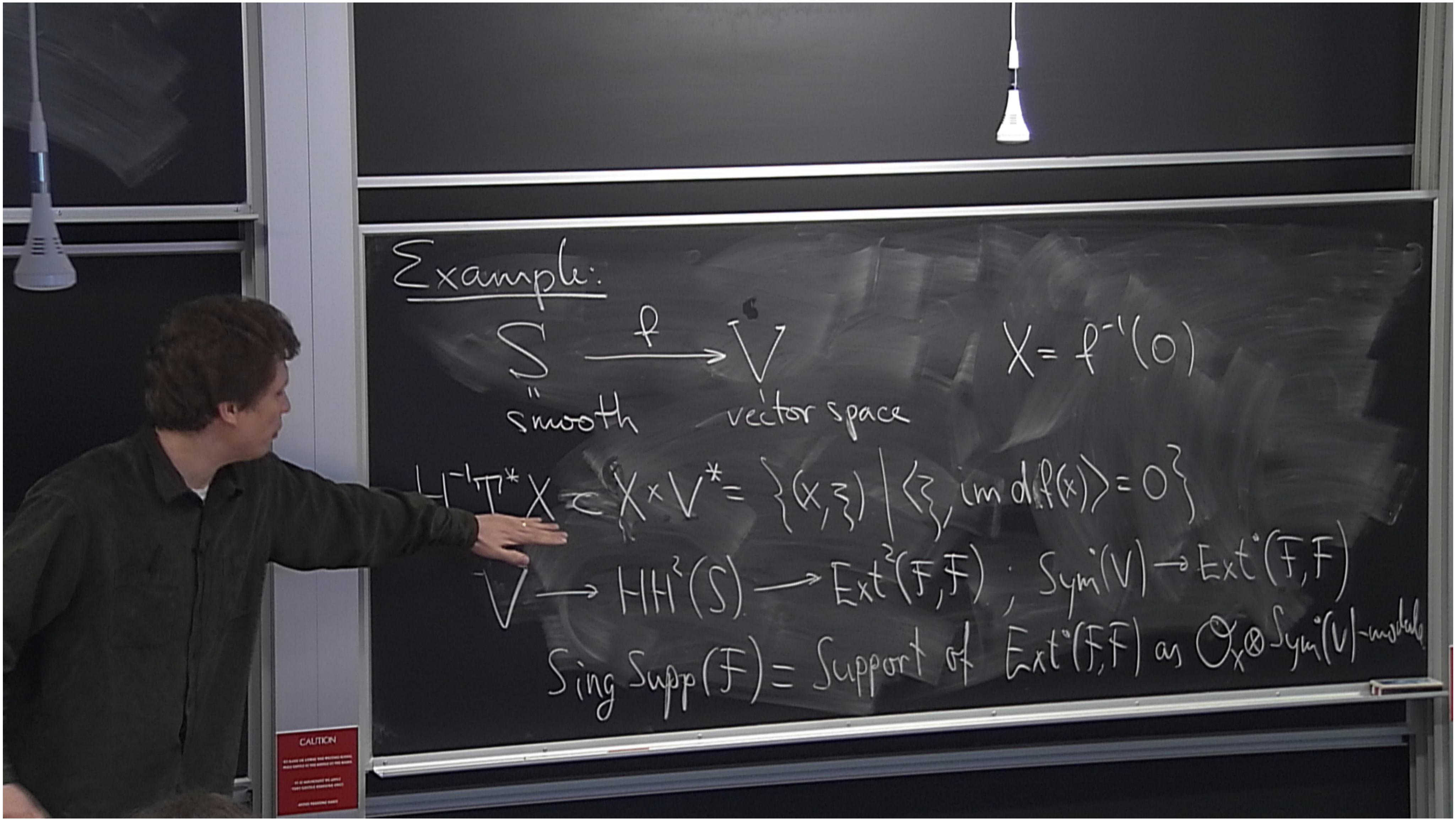
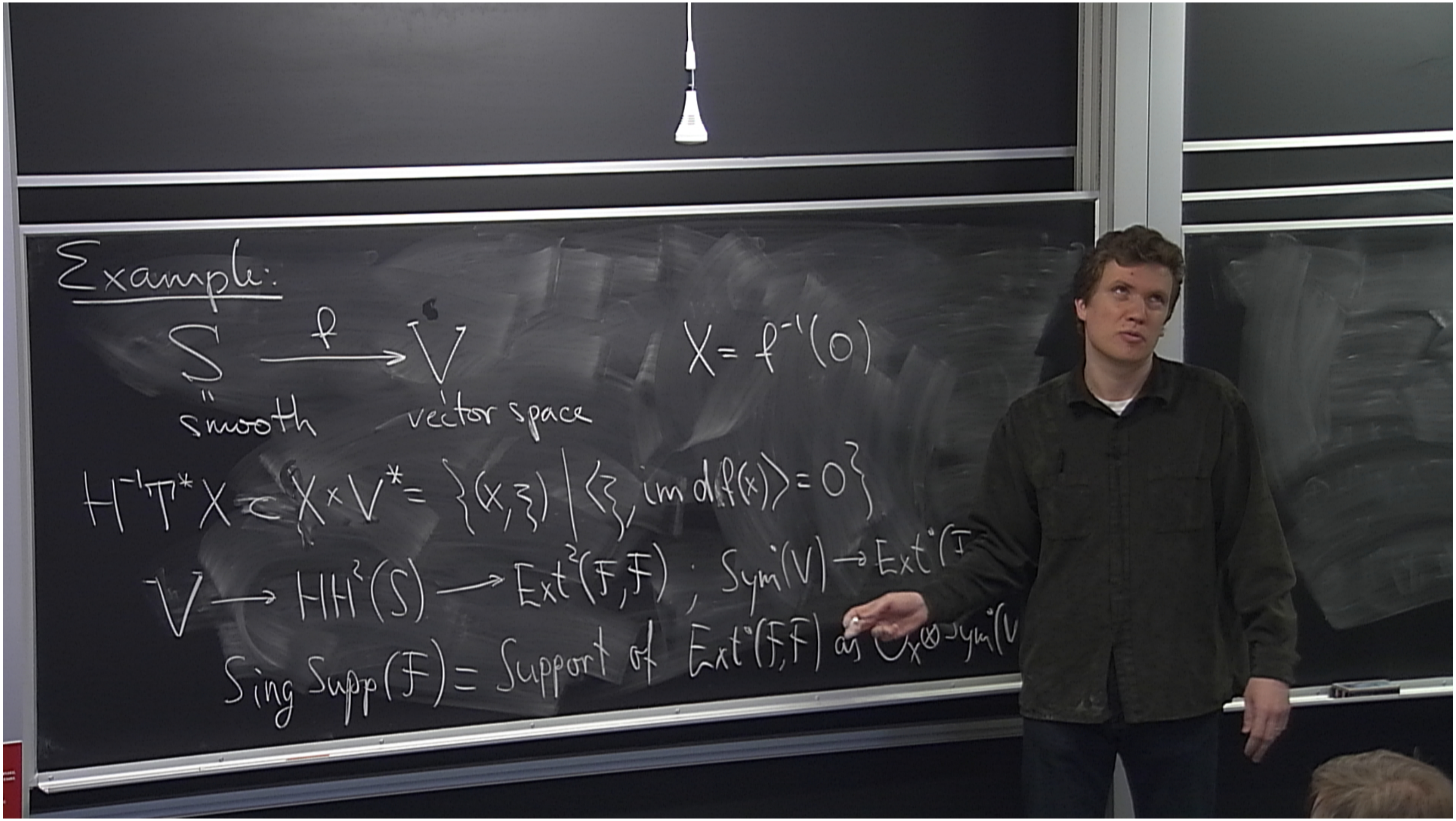

2 Categories over spaces
3 Motivation: Langlands
joint w/ D Gaitsgory
1 X = q smooth
d scheme/ℂ
(affine)
H⁻¹T*X —π→ X
F ∈ Coh(X) ----→ Conical closed subset
SingSupp(F) ⊂ H⁻¹(T*X)
Example:
S —f→ V
smooth
vector space
X = f⁻¹(
H⁻¹T*X ⊂ X × V* = {(x, ξ) | ⟨ξ, im df(x)⟩ = 0
HH²(S) ⟶ Ext²(F, F) ; Sym(V)


Example:
S f V
X = f⁻¹(0)
oth
vector space
X × V* = {(x, ξ) | ⟨ξ, im df(x)⟩ = 0}
HH²(S) → Ext²(F,F) ; Sym(V) → Ext(F,F)
(T*X)


Example:
$S \xrightarrow{f} V$
smooth
vector space
$X = f^{-1}(0)$
$\mu^{-1}T^*X \subset X \times V^* = \{(x,\xi) \mid \langle \xi, \mathrm{im}\, df(x) \rangle = 0\}$
$V \longrightarrow HH^2(S) \longrightarrow \mathrm{Ext}^2(\mathcal{F},\mathcal{F})$ ; $\mathrm{Sym}^\bullet(V) \to \mathrm{Ext}^\bullet(\mathcal{F},\mathcal{F})$
$\mathrm{Sing\,Supp}(\mathcal{F})$ = Support of $\mathrm{Ext}^\bullet(\mathcal{F},\mathcal{F})$ as $\mathcal{O}_X \otimes \mathrm{Sym}^\bullet(V)$-module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$V \longrightarrow HH^2(S) \longrightarrow Ext^2(F,F)$ ; $Sym^\bullet(V) \to Ext^\bullet(F$
$Sing\ Supp(F) =$ Support of $Ext^\bullet(F,F)$ as $O_X \otimes Sym^\bullet(V$

## Example:

$$\underset{\text{smooth}}{S} \xrightarrow{\;f\;} \underset{\text{vector space}}{V} \qquad X = f^{-1}(0)$$

$$H^{-1}T^*X = \{(x,\xi) \mid \langle \xi, \operatorname{im} df(x) \rangle = 0\} \subset X \times V^*$$

$$V \to HH^2(X) \to \operatorname{Ext}^2(\mathcal{F},\mathcal{F}) \;;\; \operatorname{Sym}^\bullet(V) \to \operatorname{Ext}^\bullet(\mathcal{F},\mathcal{F})$$

$$\operatorname{Sing\,Supp}(\mathcal{F}) = \text{Support of } \operatorname{Ext}^\bullet(\mathcal{F},\mathcal{F}) \text{ as } \mathcal{O}_X \otimes \operatorname{Sym}^\bullet(V)\text{-module}$$

Get a full subcategory $\mathrm{Coh}_Z(X) \subset \mathrm{Coh}(X)$

for any closed conical $Z \subset H^{-1}(T^*X)$ $\quad \mathrm{Coh}_Z(X) = \{\mathcal{F} \in \mathrm{Coh}(X) \mid \mathrm{SingSupp}\,\mathcal{F} \subset Z\}$

$\mathrm{Coh}_0(X)$

$X \times V^*$

$F, F)$

$\mathrm{Sym}(V)$-module

Get a full subcategory $\mathrm{Coh}_Z(X) \subset \mathrm{Coh}(X)$

$$\mathrm{Coh}_Z(X) = \{ \mathcal{F} \in \mathrm{Coh}(X) \mid \mathrm{SingSupp}\, \mathcal{F} \subset Z \}$$

for any closed conical $Z \subset H^{-1}(T^*X)$

ind completion

$\mathrm{Perf}(X) = \mathrm{Coh}_0(X)$

$\mathrm{Coh}_Z(X)$

$\mathrm{Coh}(X) = \mathrm{Coh}_{H^{-1}T^*X}(X)$

$\mathrm{Ind}(\mathrm{Perf}(X)) = \mathrm{QCoh}(X)$

$\mathrm{Ind}(\mathrm{Coh}(X)_Z) = \mathrm{IndCoh}(X)_Z$

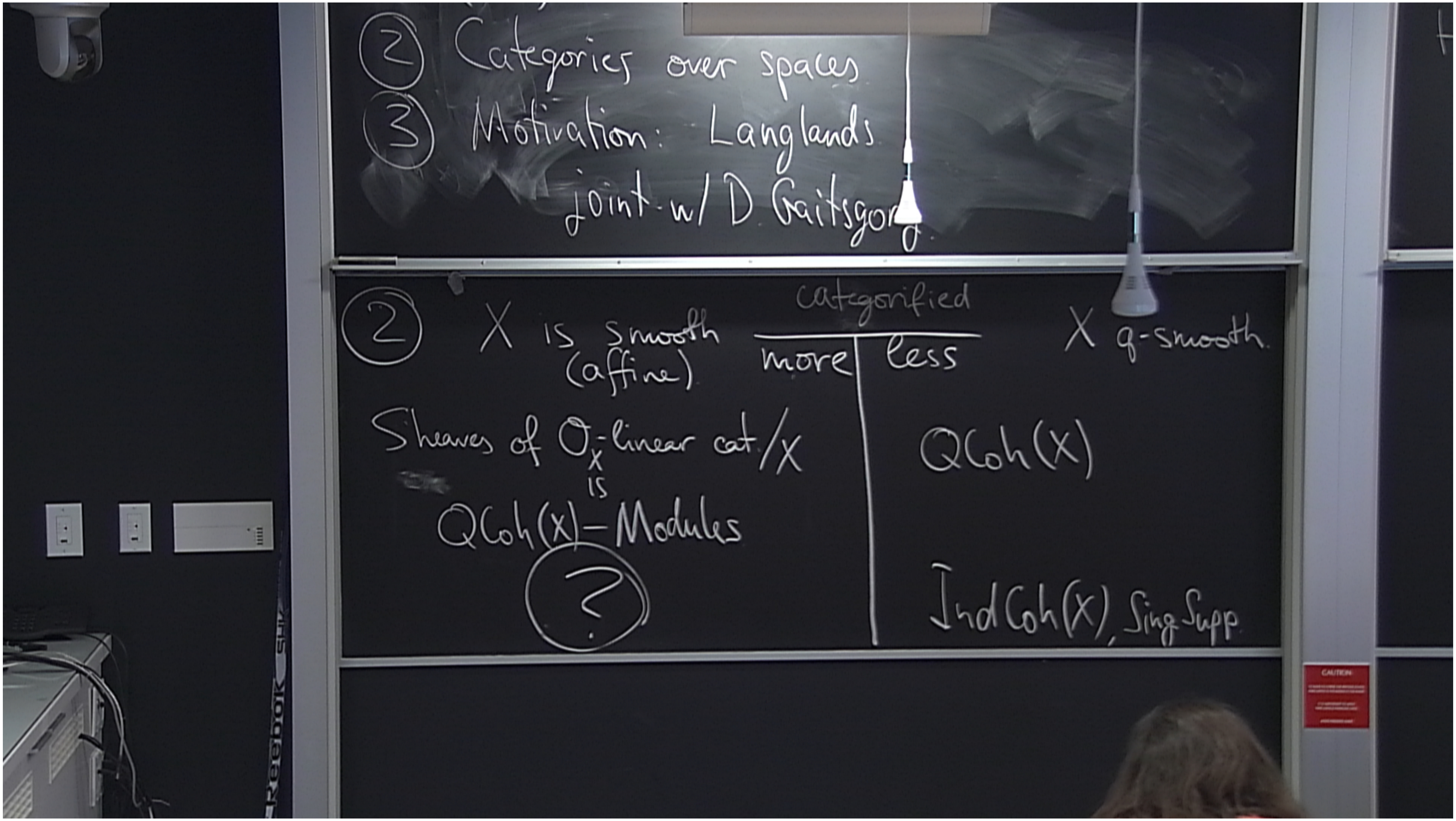
$\mathrm{Ind}(\mathrm{Coh}(X)) = \mathrm{IndCoh}(X)$

$$V \longrightarrow HH^2(X) \longrightarrow \mathrm{Ext}^2(\mathcal{F},\mathcal{F}) \ ; \ \mathrm{Sym}^\bullet(V) \to \mathrm{Ext}^\bullet(\mathcal{F},\mathcal{F})$$

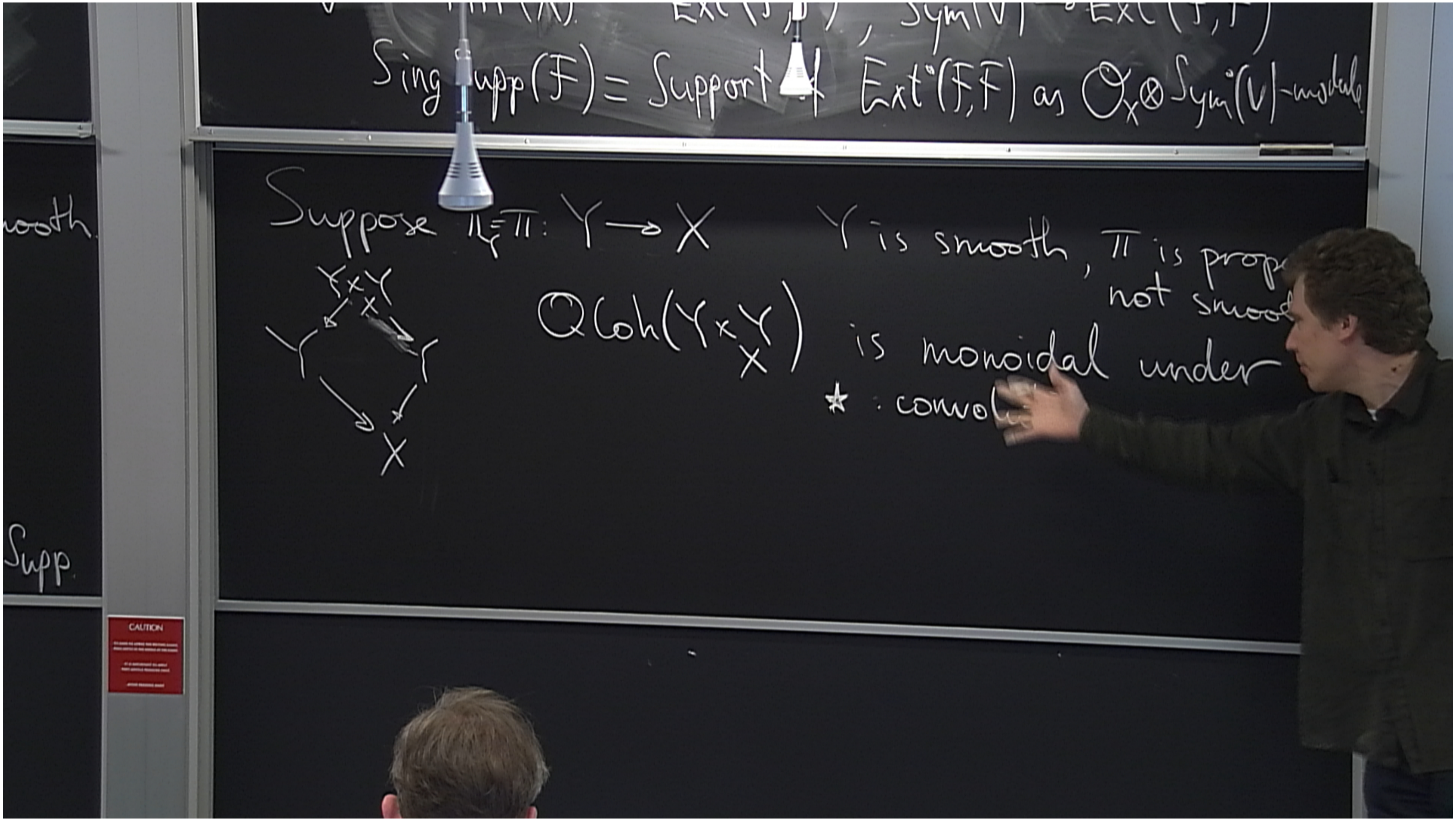
$$\mathrm{Sing\,Supp}(\mathcal{F}) = \text{Support of } \mathrm{Ext}^\bullet(\mathcal{F},\mathcal{F}) \text{ as } \mathcal{O}_X \otimes \mathrm{Sym}^\bullet(V)\text{-module}$$

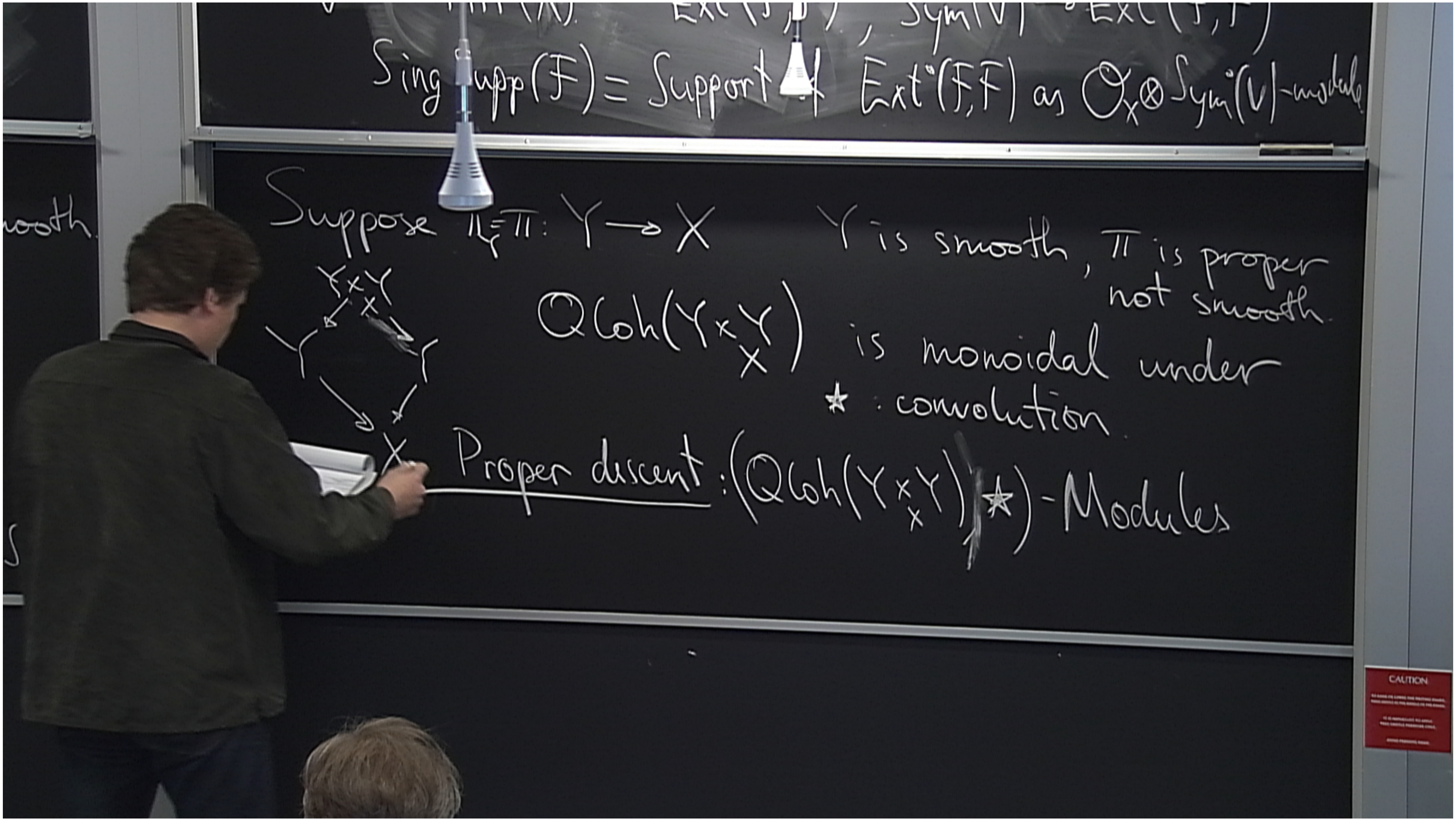
Sing Supp behaves as expected: $Z_1 \subset Z_2$:

$$\mathrm{IndCoh}_{Z_1}(X) \underset{\text{right adjoint (colocalization)}}{\overset{\hookrightarrow}{\longleftarrow}} \mathrm{IndCoh}_{Z_2}(X)$$



2 Categories over spaces
3 Motivation: Langlands
joint w/ D. Gaitsgory
2 X is smooth (affine)
categorified
more
less
X q-smooth
Sheaves of O_X-linear cat /X
is
QCoh(X)–Modules
?
QCoh(X)
IndCoh(X), Sing Supp


Ext*(F,F) as O_X ⊗ Sym*(V)-module
Suppose π: Y → X
Y is smooth, π is
not
Y ×_X Y
Y
Y
X
QCoh(Y ×_X Y)
is monoidal under
⋆ : convo
nooth.
Supp.
CAUTION


Sing Supp(F) = Support of Ext°(F,F) as O_X ⊗ Sym°(V)-modules
Suppose π: Y → X
Y is smooth, π is proper
not smooth
Y×Y
X
QCoh(Y×Y)
X
is monoidal under
⋆ : convolution
Proper descent : (QCoh(Y×Y), ⋆)-Modules
X

$\mathrm{Ext}^\bullet(\mathcal{F},\mathcal{F})$, $\mathrm{Sym}(V) \to \mathrm{Ext}^\bullet(\mathcal{F},\mathcal{F})$

$\mathrm{Sing\,Supp}(\mathcal{F}) = \mathrm{Support}$ of $\mathrm{Ext}^\bullet(\mathcal{F},\mathcal{F})$ as $\mathcal{O}_X \otimes \mathrm{Sym}^\bullet(V)$-modules

Suppose $\pi : Y \to X$, $Y$ is smooth, $\pi$ is proper, not smooth.

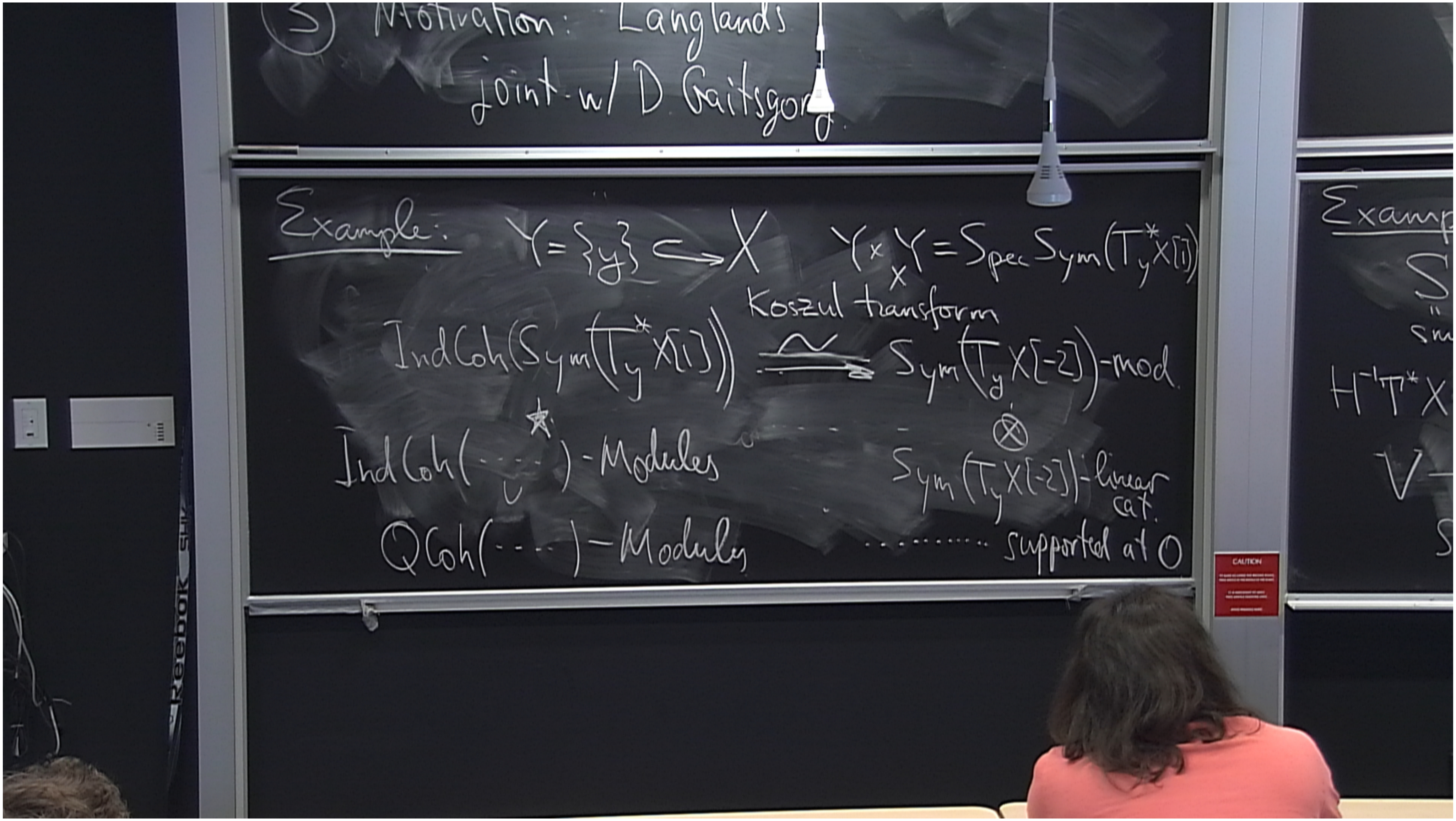
$$\begin{tikzcd} & Y \times_X Y \arrow[dl] \arrow[dr] & \\ Y \arrow[dr] & & Y \arrow[dl] \\ & X & \end{tikzcd}$$

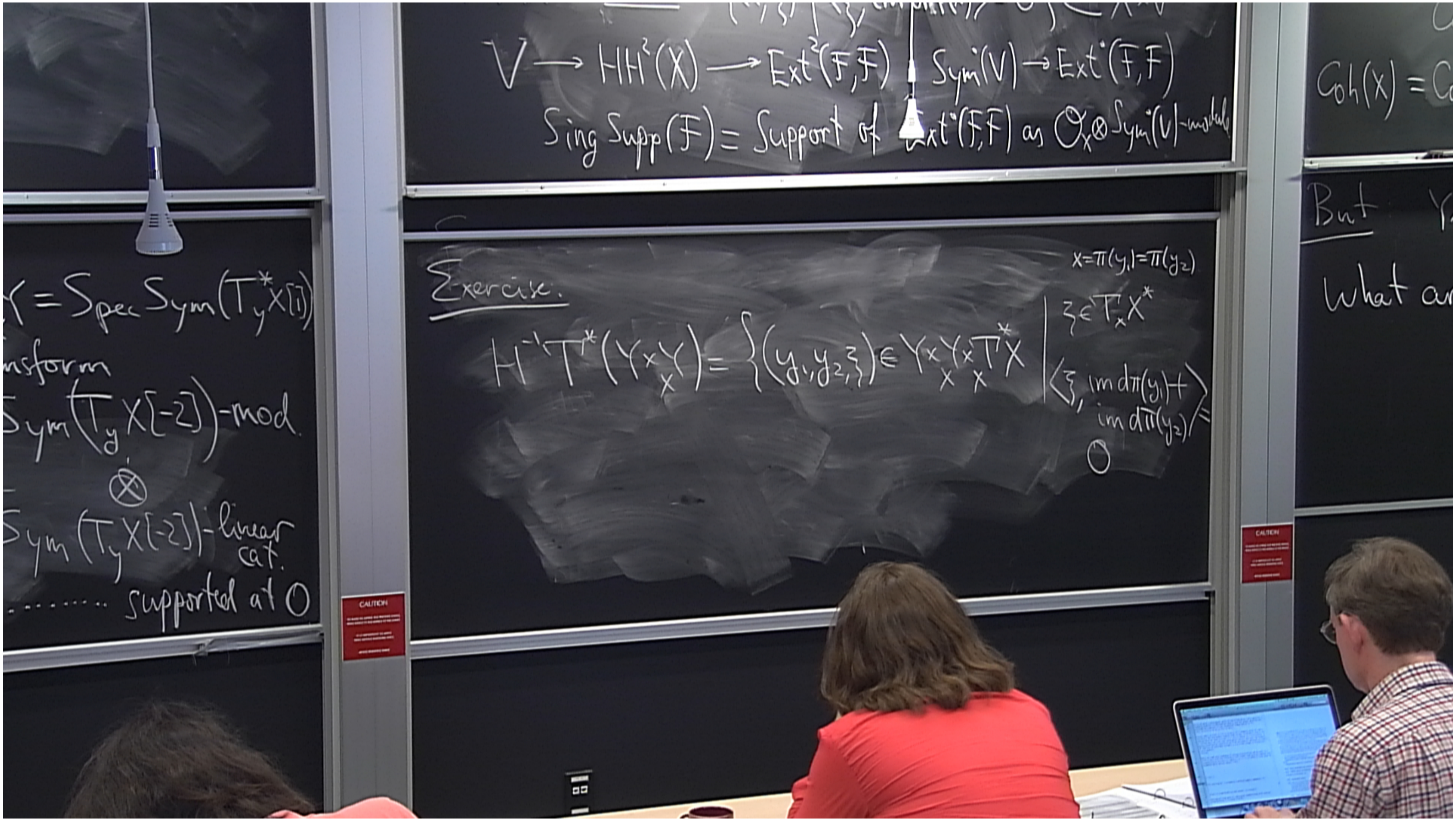
$\mathrm{QCoh}(Y \times_X Y)$ is monoidal under $\star$ : convolution

<u>Proper descent</u>: $(\mathrm{QCoh}(Y \times_X Y), \star)$-Modules $\simeq$ $\{\mathrm{QCoh}(X)$-Modules "supported over $\pi(Y)$"$\}$

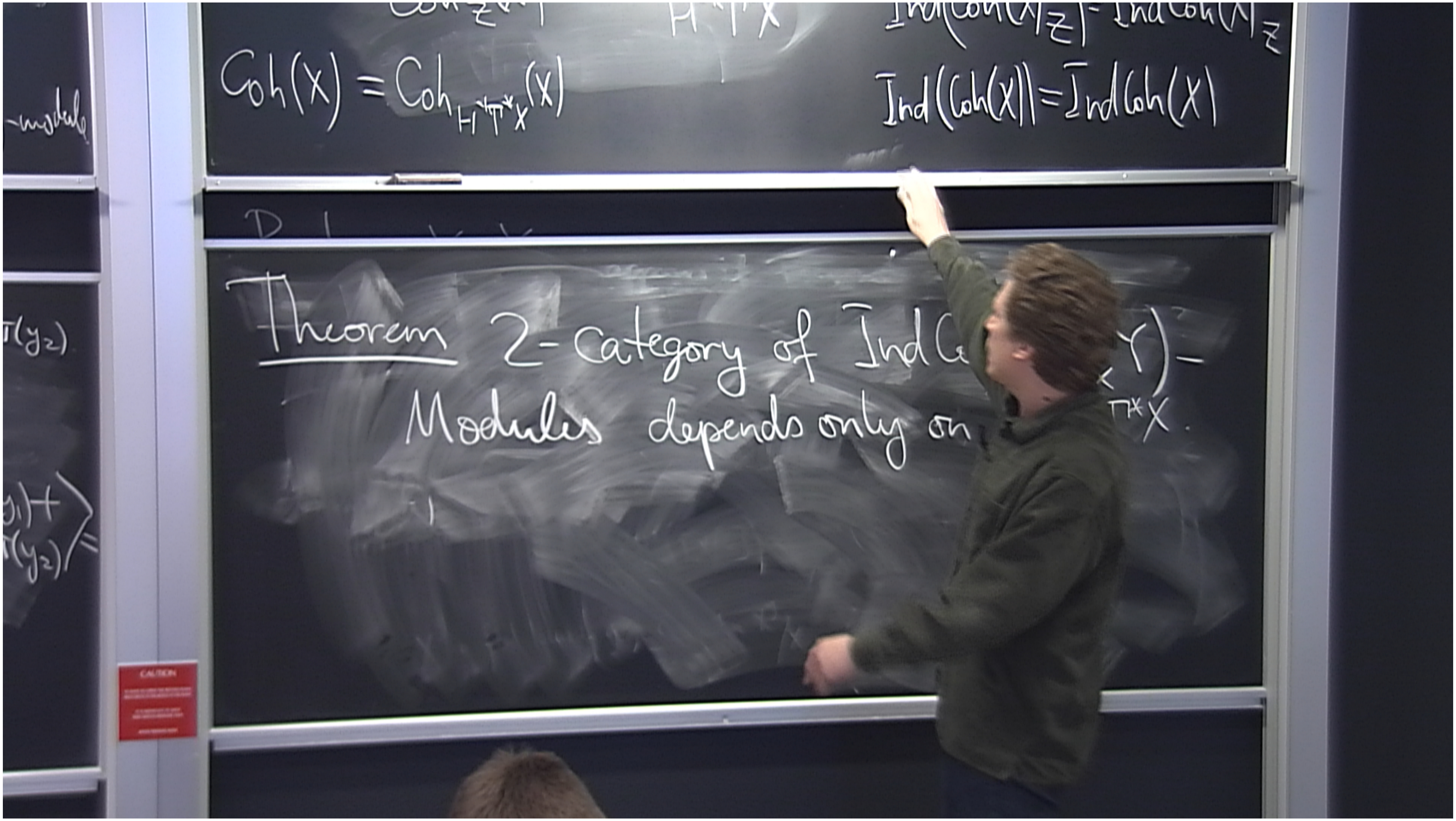
<u>But</u>: $Y \times_X Y$ is q. smooth, but not smooth.

What are $(\mathrm{IndCoh}(Y \times_X Y), \star)$-Modules?

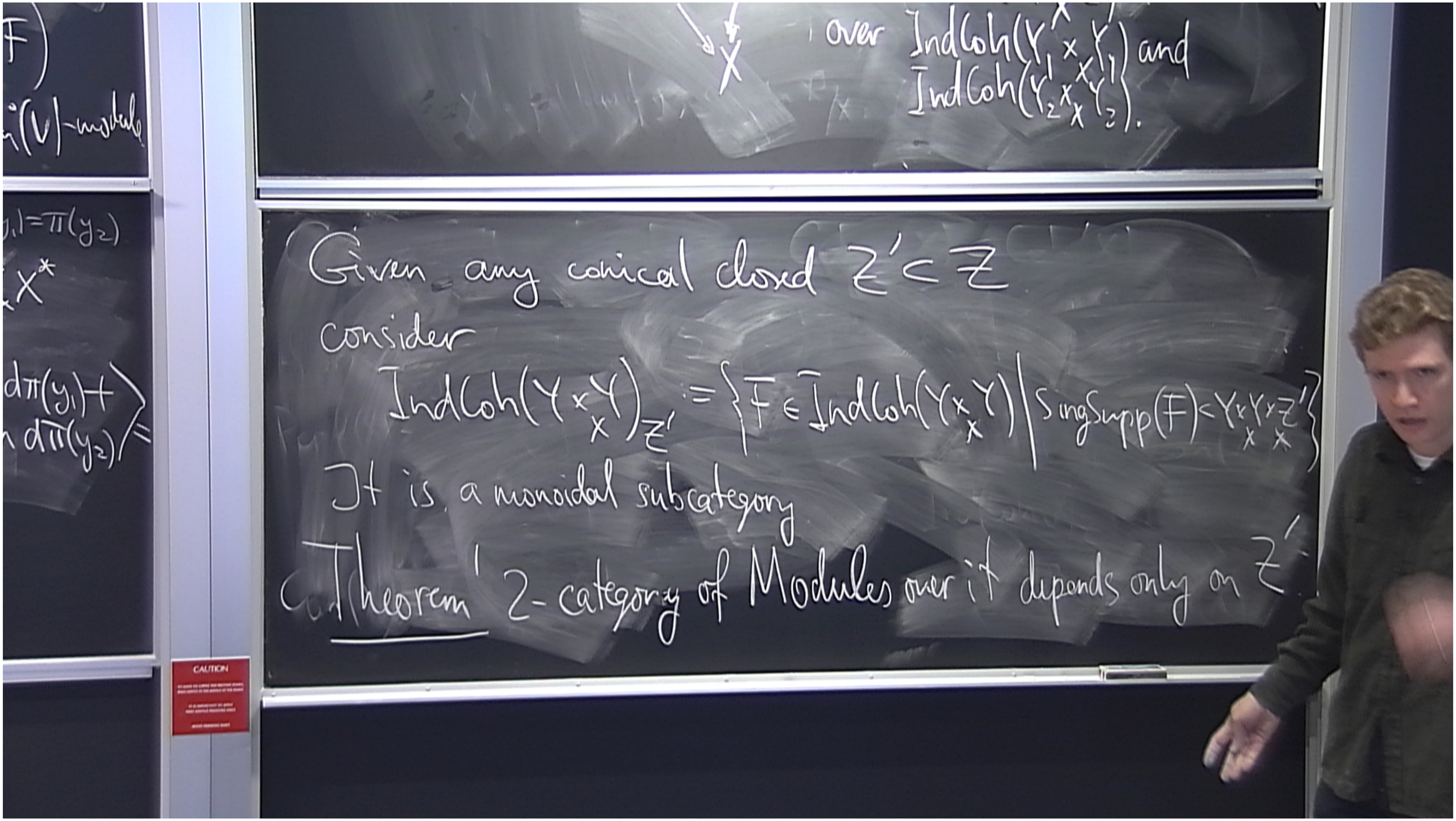

(3) Motivation: Langlands
joint w/ D. Gaitsgory
Example: $Y = \{y\} \hookrightarrow X$ $Y \times_X Y = \mathrm{Spec}\, \mathrm{Sym}(T_y^* X[1])$
Koszul transform
$\mathrm{IndCoh}(\mathrm{Sym}(T_y^* X[1])) \simeq \mathrm{Sym}(T_y X[-2])$-mod
$\mathrm{IndCoh}(\dots)$-Modules ........ $\mathrm{Sym}(T_y X[-2])$-linear cat.
$\mathrm{QCoh}(\dots)$-Modules ........ supported at 0

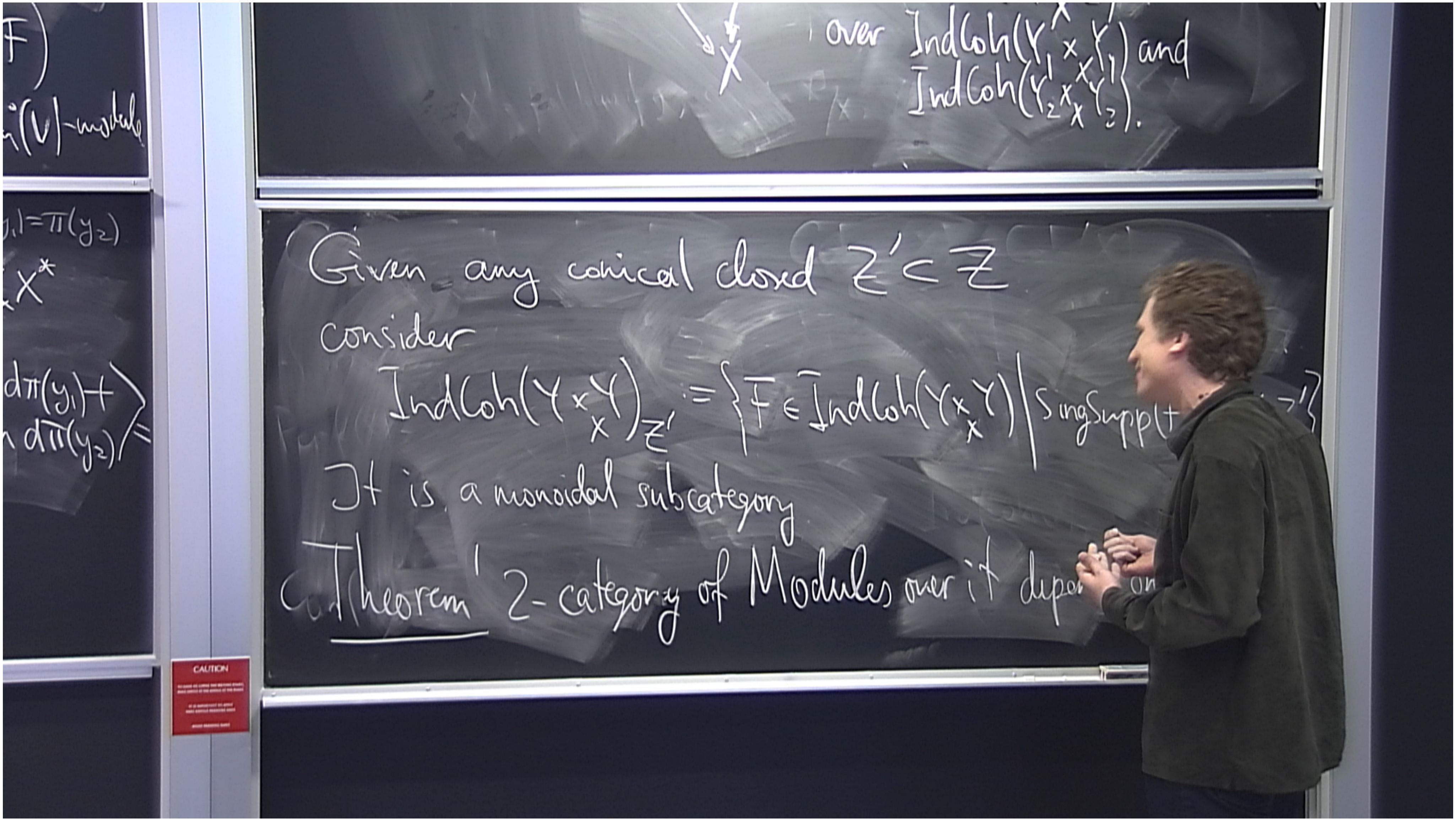


$V \to HH^2(X) \to Ext^2(F,F)$
$Sym^\bullet(V) \to Ext^\bullet(F,F)$
Sing Supp$(F)$ = Support of $Ext^\bullet(F,F)$ as $\mathcal{O}_X \otimes Sym^\bullet(V)$-module
$Coh(X) =$
$Y = Spec\, Sym(T_y^* X[1])$
nsform
$Sym(T_y X[-2])$-mod
$\otimes$
$Sym(T_y X[-2])$-linear cat.
supported at $0$
Exercise.
$x = \pi(y_1) = \pi(y_2)$
$H^{-1} T^*(Y \times_X Y) = \{(y_1, y_2, \xi) \in Y \times_X Y \times_X T^* X \mid \xi \in T_x^* X, \langle \xi, \operatorname{im} d\pi(y_1) + \operatorname{im} d\pi(y_2) \rangle = 0\}$
But
What ar
CAUTION



Coh(X) = Coh
Ind(Coh(X)) = IndCoh(X)
Theorem 2-Category of
Modules depends only on


over IndCoh($Y_1' \times_X Y_1$) and
IndCoh($Y_2' \times_X Y_2$).
Given any conical closed $Z' \subset Z$
consider
$\mathrm{IndCoh}(Y \times_X Y)_{Z'} = \{ F \in \mathrm{IndCoh}(Y \times_X Y) \mid \mathrm{SingSupp}(F) \subset Y \times_X Y \times_X Z' \}$
It is a monoidal subcategory
Theorem: 2-category of Modules over it depends only on $Z'$.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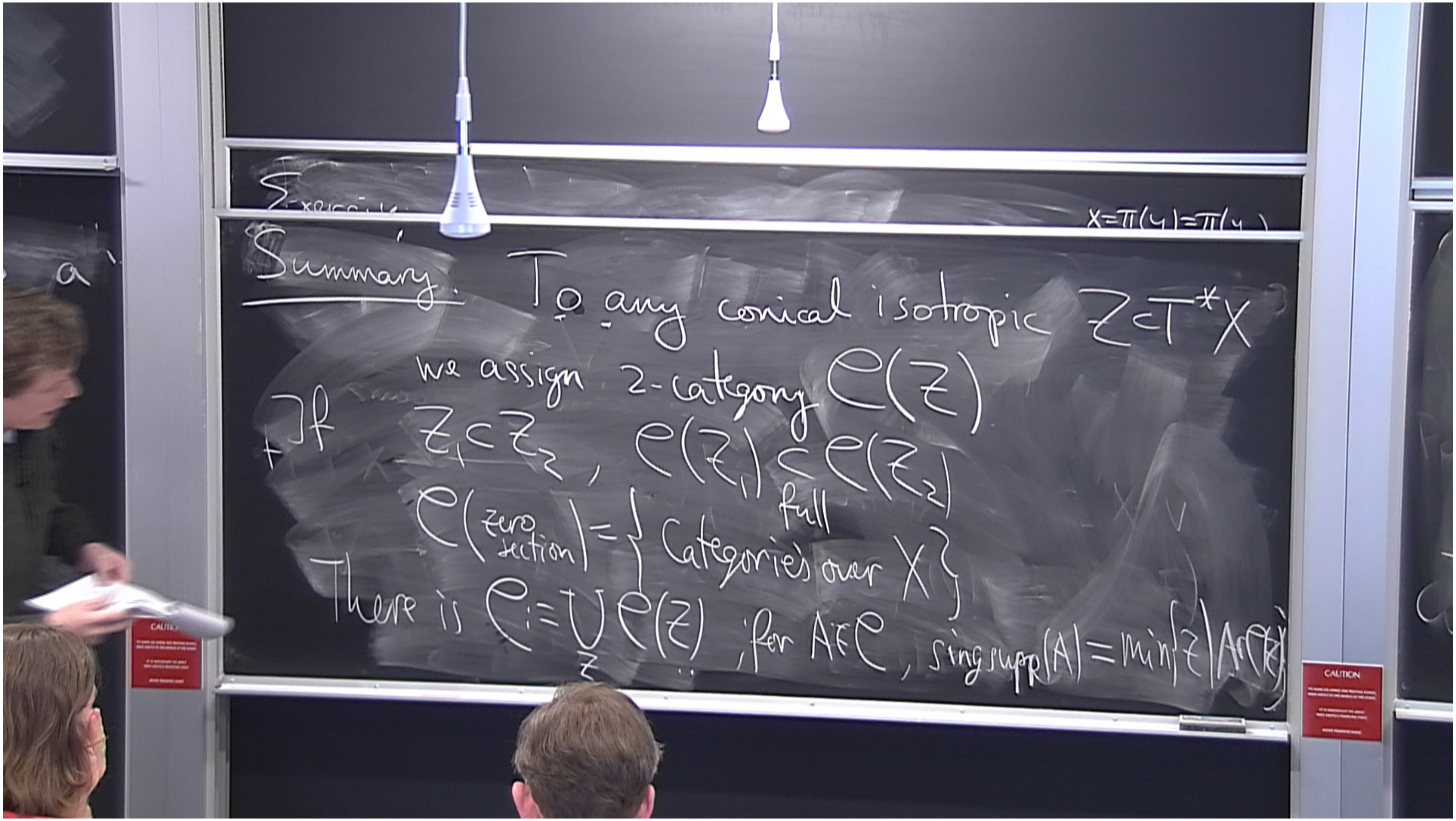
Given any conical closed $Z' \subset Z$
consider
$\mathrm{IndCoh}(Y \times_X Y)_{Z'} = \{ F \in \mathrm{IndCoh}(Y \times_X Y) \mid \mathrm{SingSupp}($
It is a monoidal subcategory
Theorem 2-category of Modules over it
$\pi(y_1) = \pi(y_2)$
$X^*$
$d\pi(y_1) +$
$d\pi(y_2)$

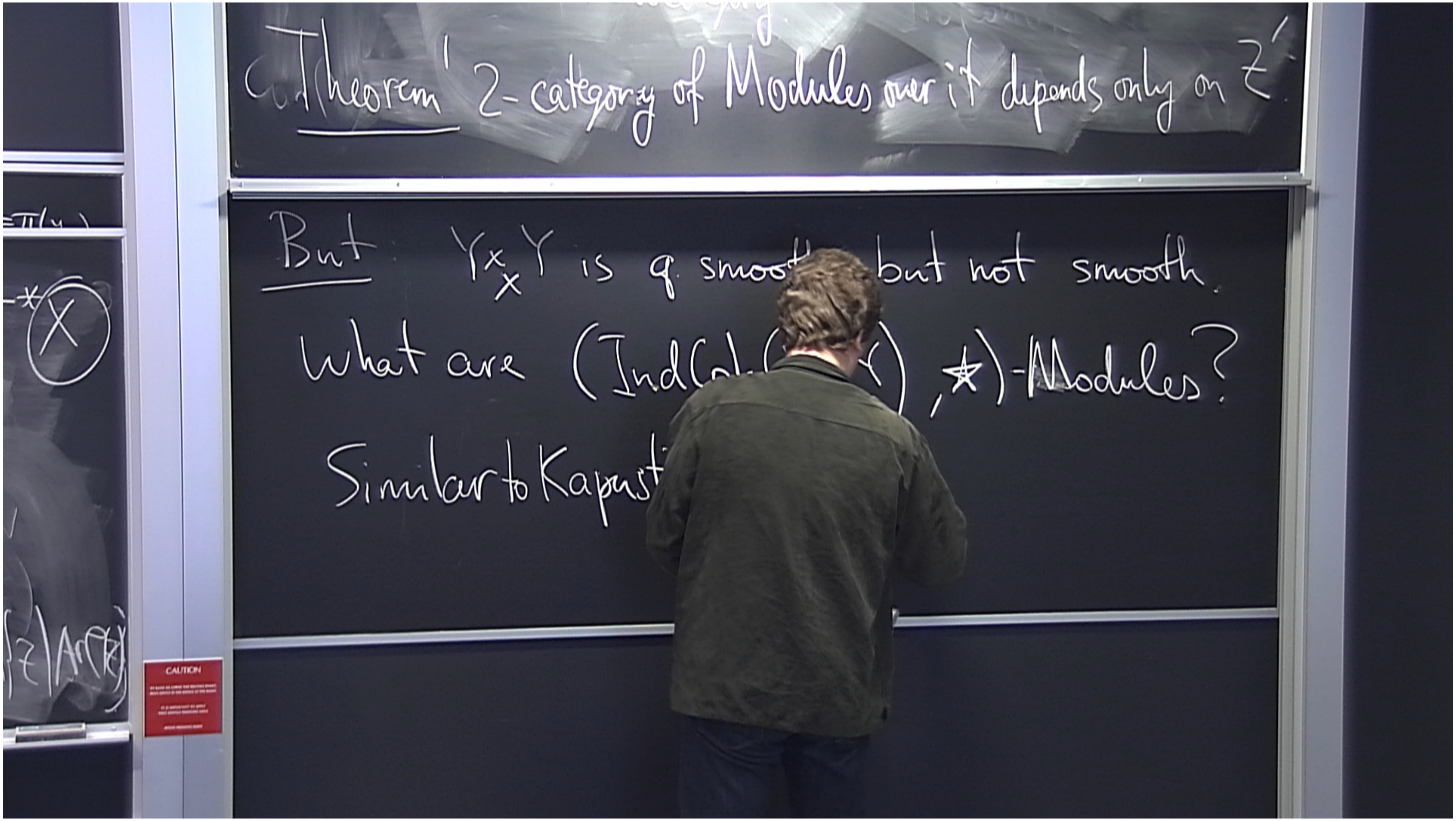
**Summary:** To any conical isotropic $Z \subset T^*X$ we assign 2-category $\mathcal{C}(Z)$.

If $Z_1 \subset Z_2$, $\mathcal{C}(Z_1) \underset{\text{full}}{\subset} \mathcal{C}(Z_2)$

$\mathcal{C}\left(\begin{smallmatrix}\text{zero}\\ \text{section}\end{smallmatrix}\right) = \{\text{Categories over } X\}$

There is $\mathcal{C} := \bigcup_{Z} \mathcal{C}(Z)$


$X = \pi(y) = \pi(y)$
Summary: To any conical isotropic $Z \subset T^*X$
we assign 2-category $\mathcal{C}(Z)$
If $Z_1 \subset Z_2$, $\mathcal{C}(Z_1) \underset{\text{full}}{\subset} \mathcal{C}(Z_2)$
$\mathcal{C}(\text{zero section}) = \{\text{Categories over } X\}$
There is $\mathcal{C} := \bigcup_Z \mathcal{C}(Z)$; for $A \in \mathcal{C}$, sing supp$(A) = \min\{Z \mid A \in \mathcal{C}(Z)\}$


Theorem' 2-category of Modules over it depends only on Z'.
But: $Y \times_X Y$ is q. smoo
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What are (IndCoh(
), ⋆)-Modules?
Similar to Kapust