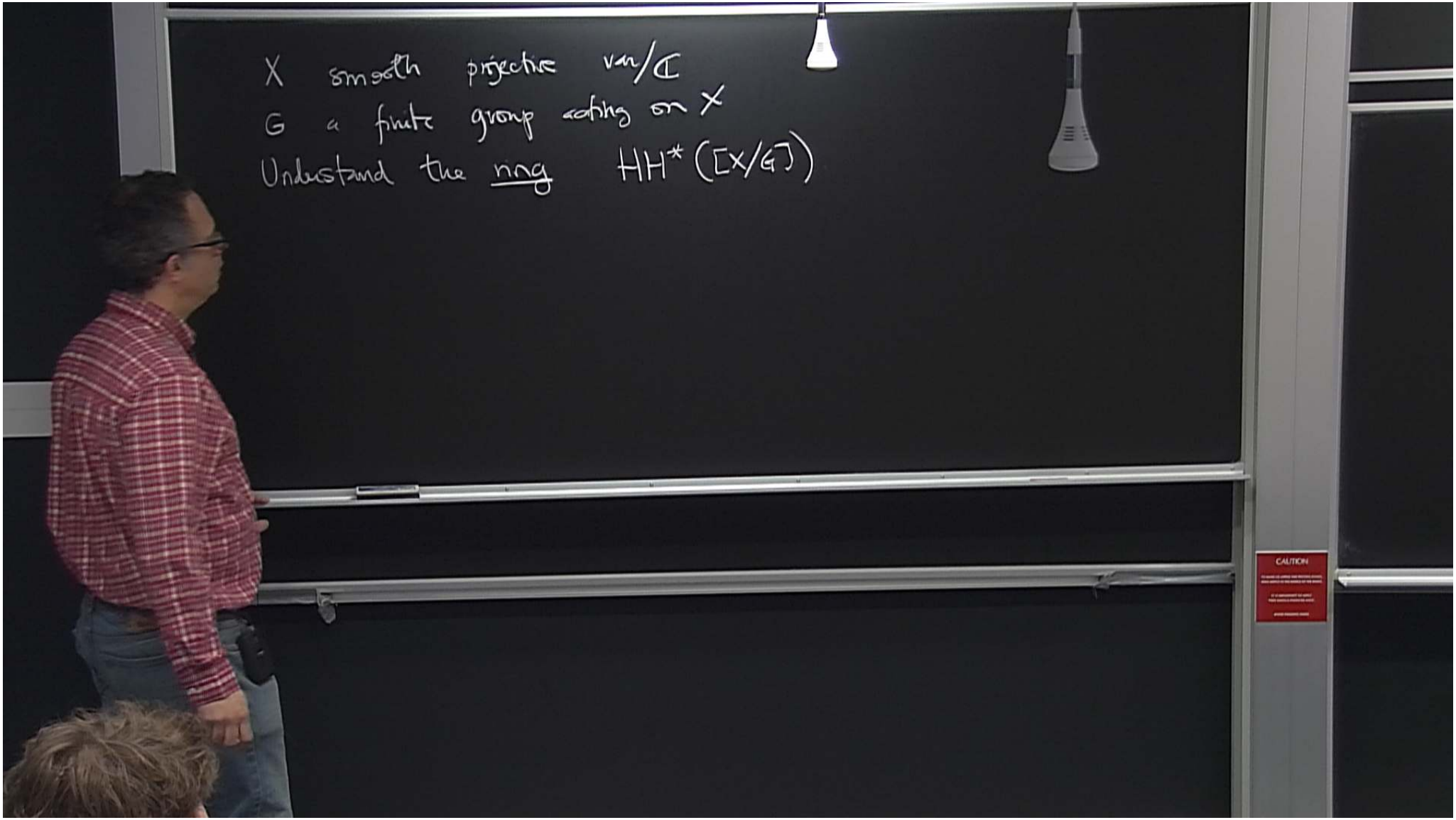
Title: What is the Todd class of an orbifold?

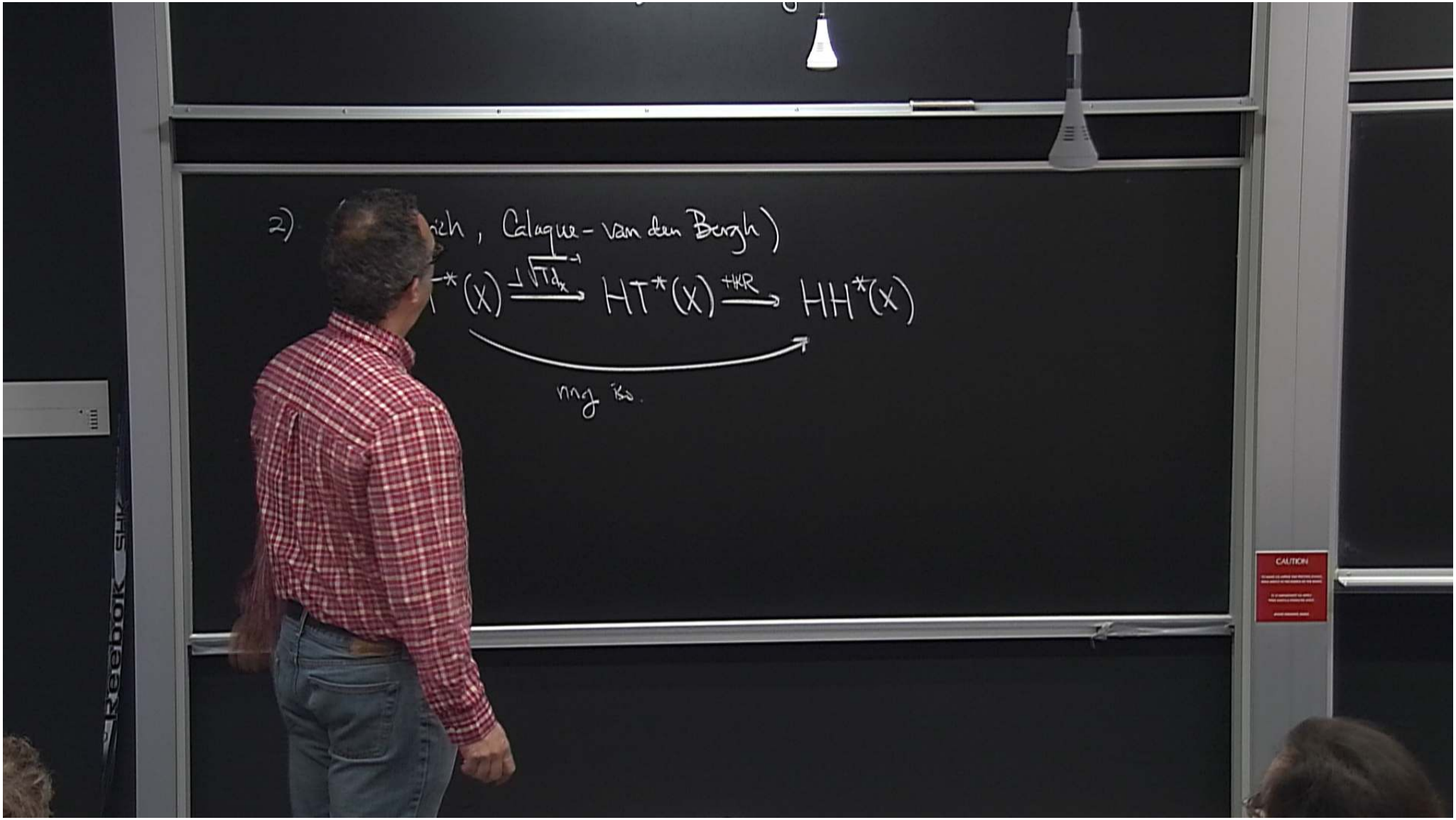
Date: Apr 19, 2016 09:30 AM

URL: http://pirsa.org/16040074

Abstract: The Todd class enters algebraic geometry in two places, in the Hirzebruch-Riemann-Roch formula and in the correction of the HKR isomorphism needed to make the Hochschild cohomology isomorphic to polyvector field cohomology (Kontsevichâ€™s claim, proved by Calaque and van den Bergh). In the case of orbifolds the Riemann-Roch formula is known, but not the analogue of Kontsevichâ€™s result. However, we can try to use the former as a guide towards a conjectural formulation for the latter.

The problem with this approach is that in the case of an orbifold it is not obvious what the Todd class actually is. This happens because the Riemann-Roch formula mixes the Todd class with the Chern character and it is difficult to separate one from the other. In my talk I shall discuss what the study of loop groups of orbifolds predicts the correct Todd class to be, and then I shall explain how the orbifold Riemann-Roch formula can be rewritten to make this prediction consistent.


X smooth projective var/ℂ
G a finite group acting on X
Understand the ring HH*([X/G])
CAUTION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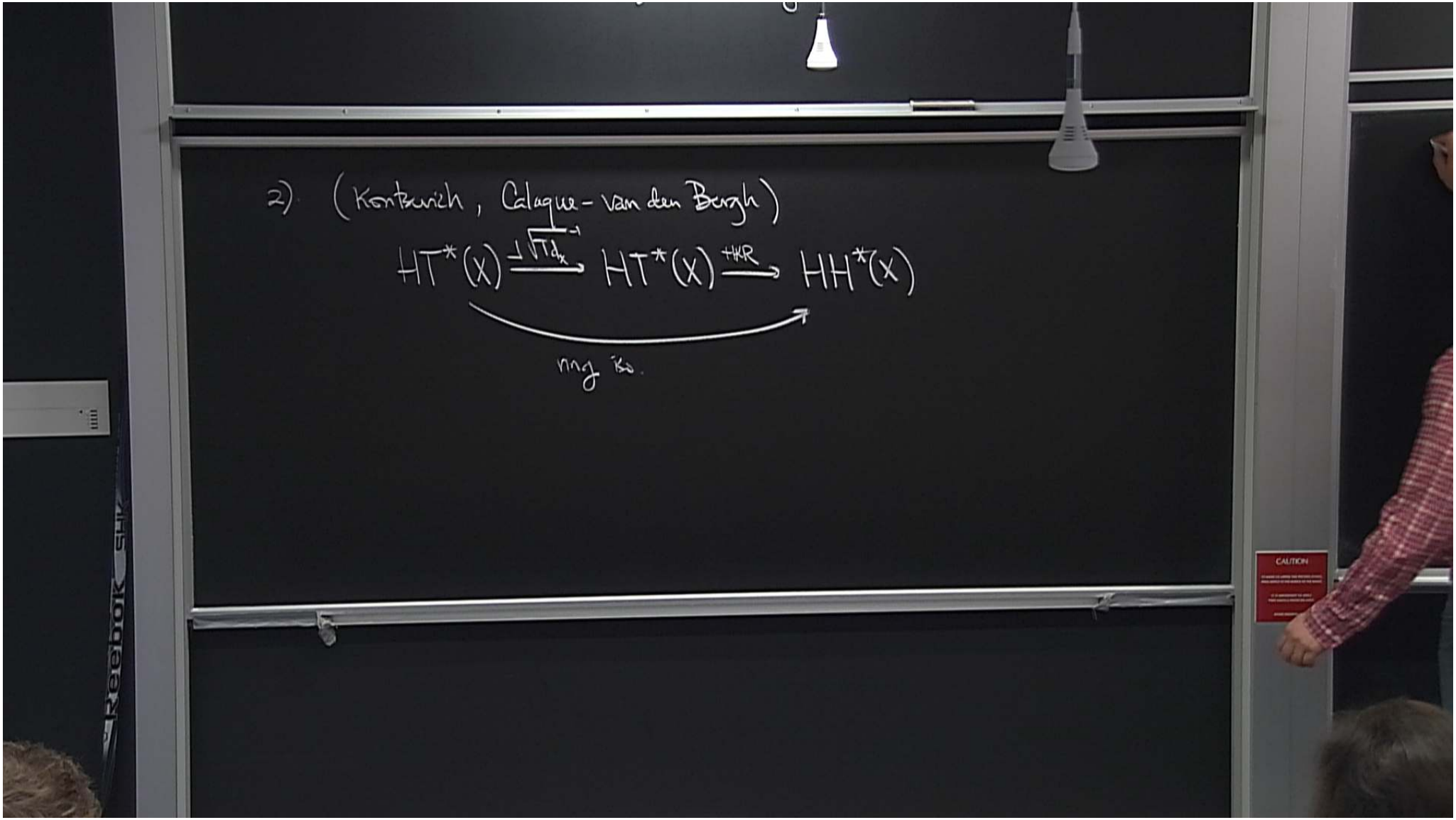
$T^*(X) \xrightarrow{\perp \sqrt{T}\, d_x} HT^*(X) \xrightarrow{+HKR} HH^*(X)$
mg iso.
CAUTION


2) (Kontsevich, Calaque-van den Bergh)
$HT^*(X) \xrightarrow{\perp \sqrt{Td_X}} HT^*(X) \xrightarrow{HKR} HH^*(X)$
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Reebok
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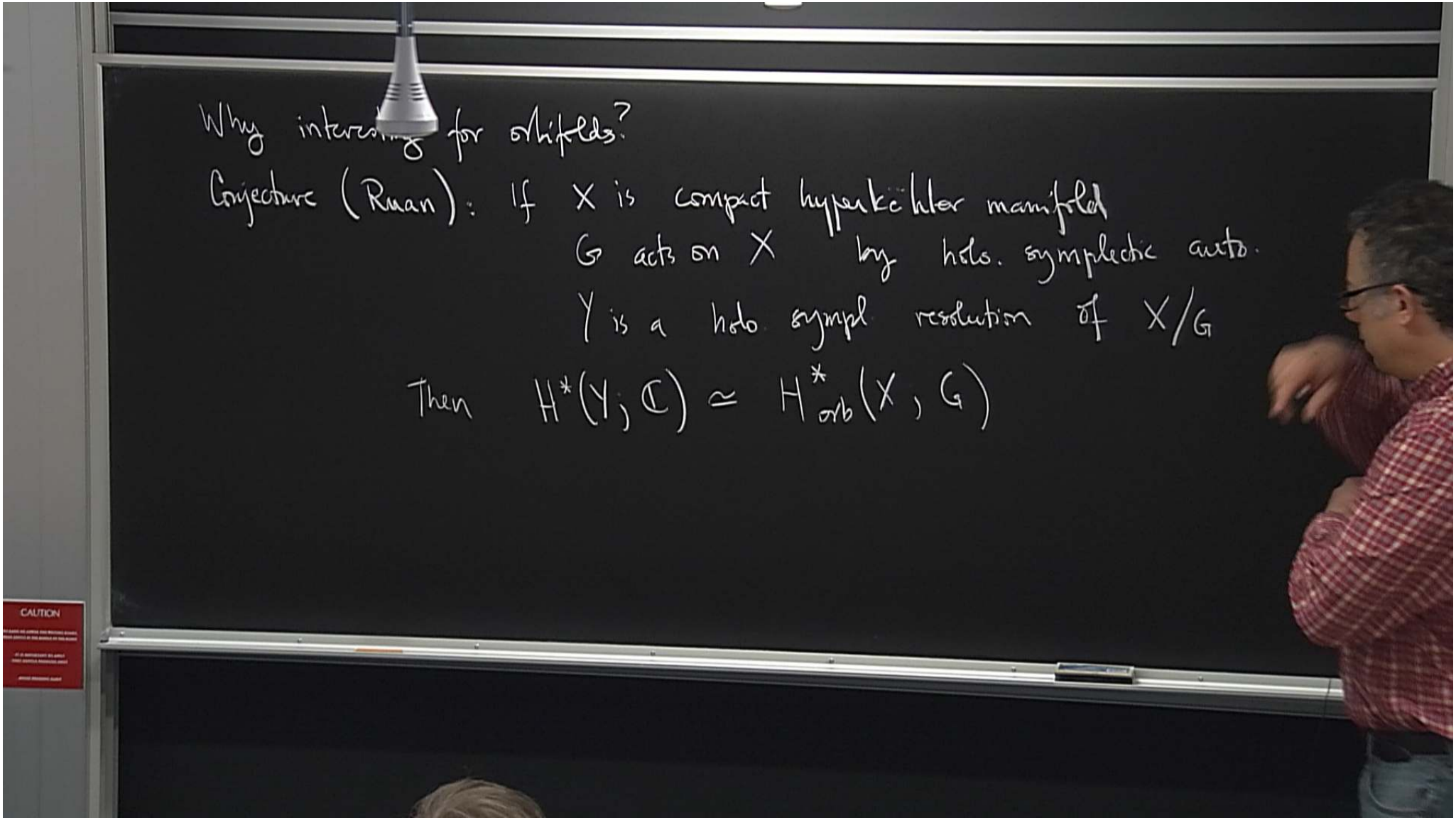
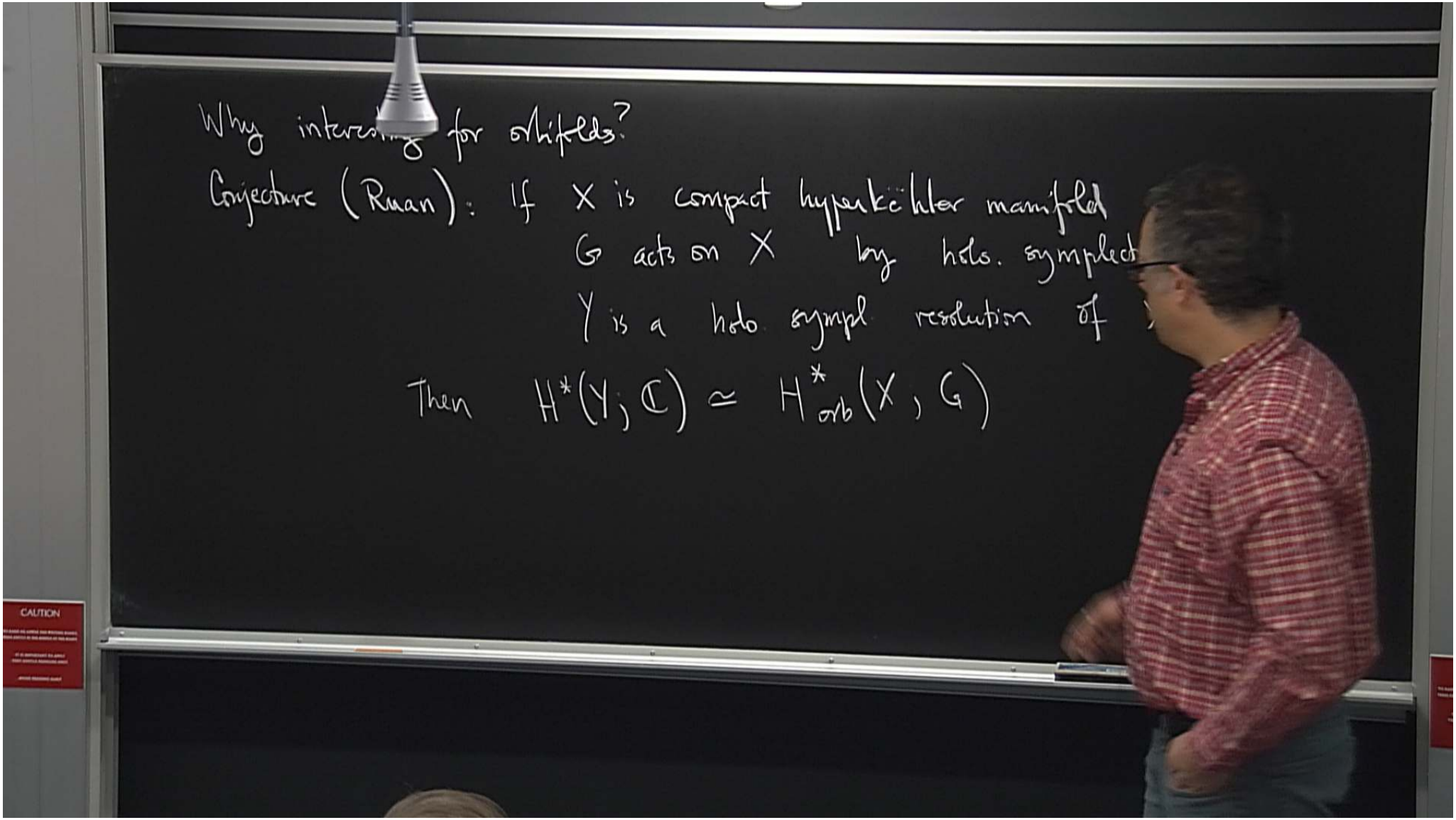

Why interesting for orbifolds?
Conjecture (Ruan): If X is compact hyperkähler manifold
G acts on X by hol. symplectic auto.
Y is a hol. sympl. resolution of X/G
Then $H^*(Y;\mathbb{C}) \simeq H^*_{orb}(X,G)$
CAUTION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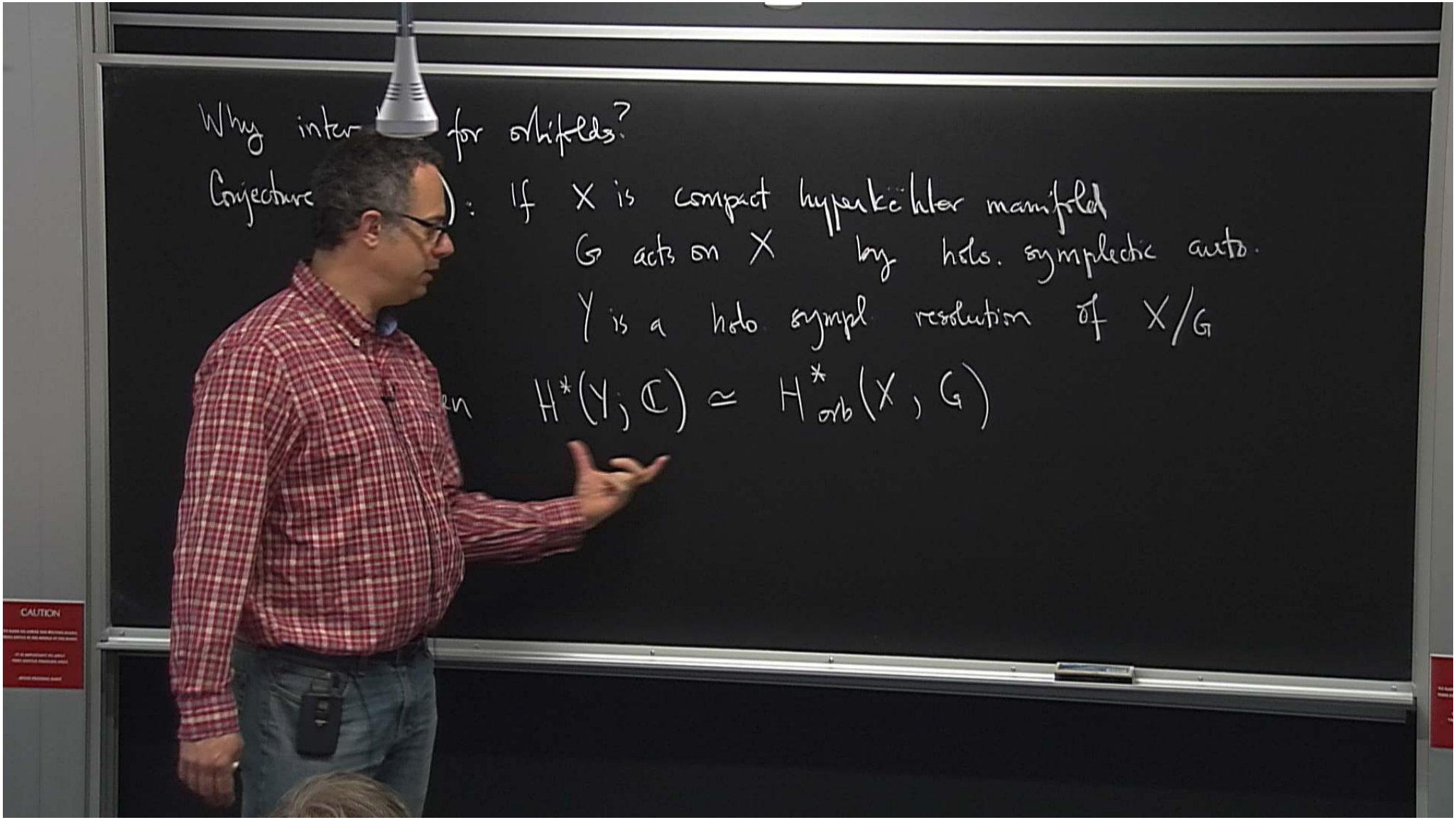
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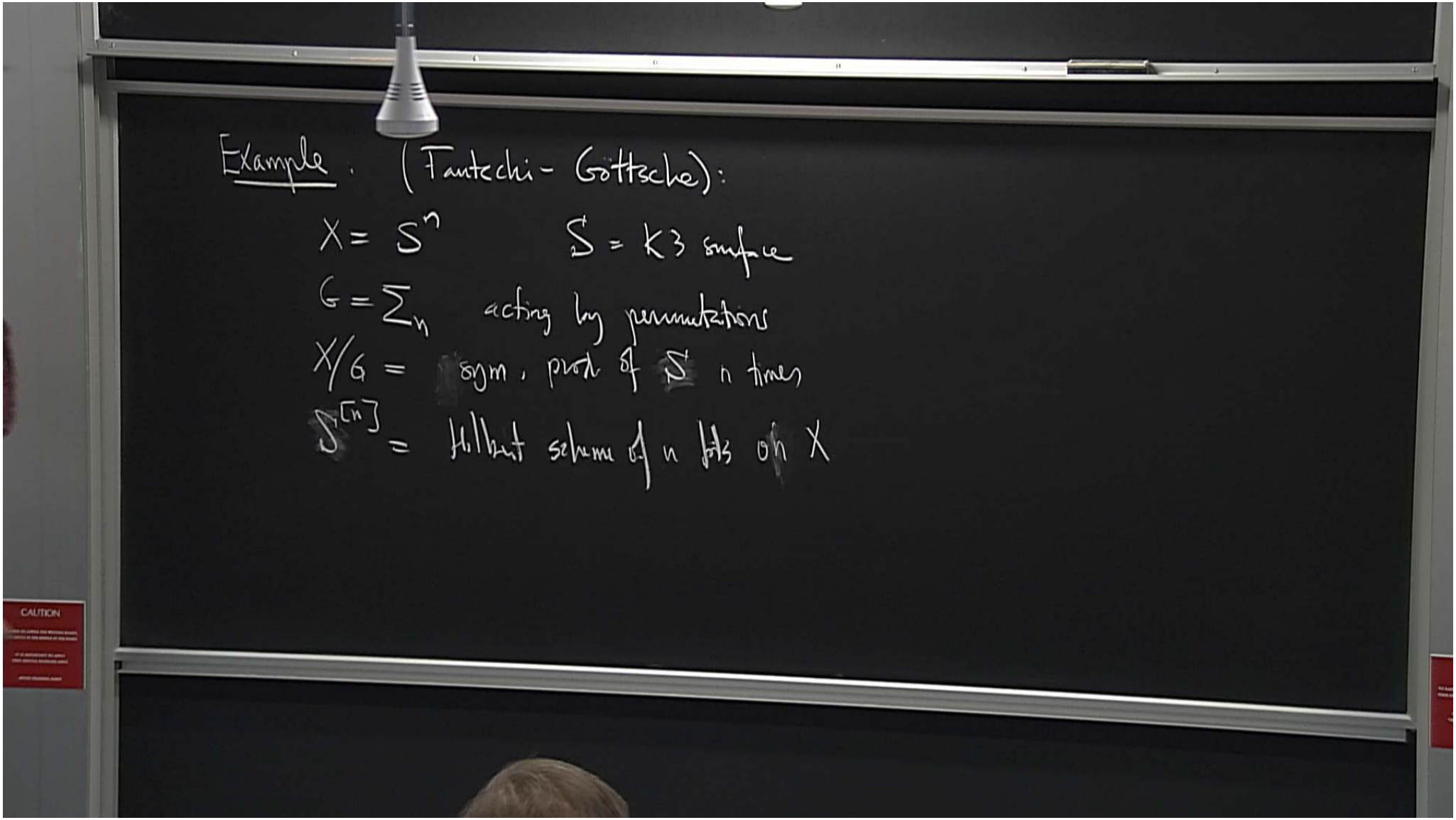
Example. (Fantechi-Göttsche):
$X = S^n$ S = K3 surface
$G = \Sigma_n$ acting by permutations
X/G = sym. prod of S n times
$S^{[n]}$ = Hilbert scheme of n pts on X
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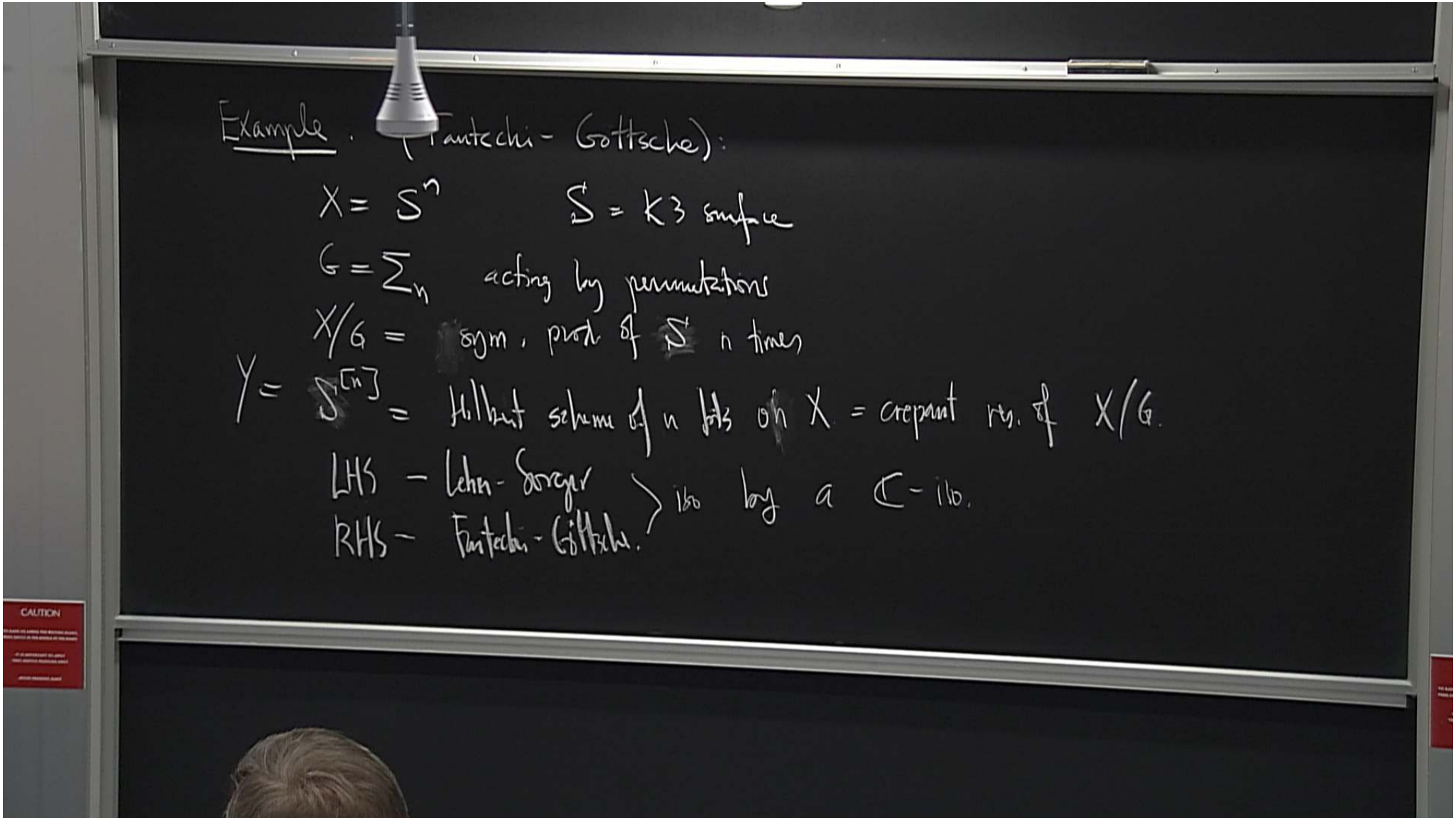
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LHS - Lehn - Sorger
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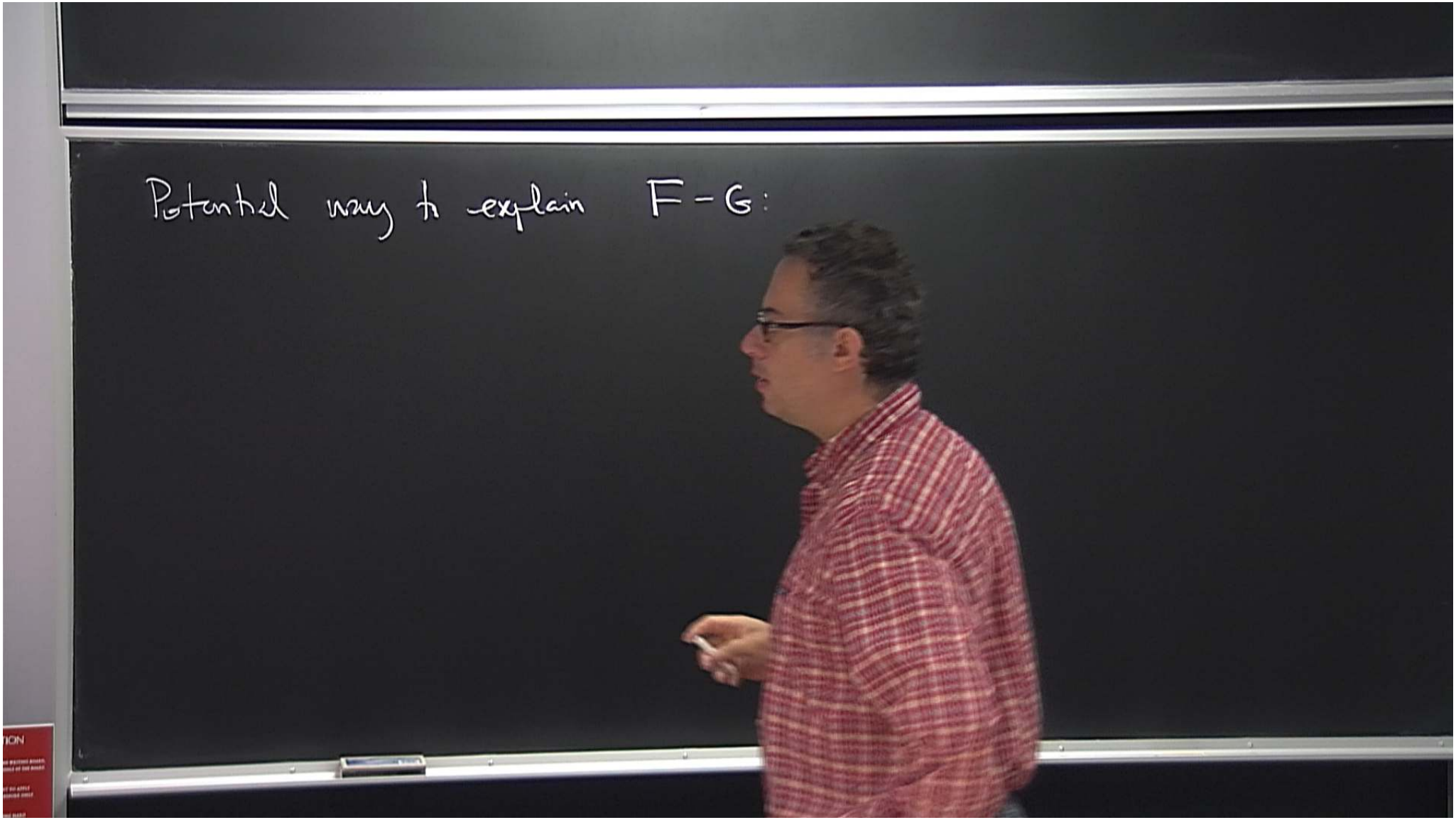
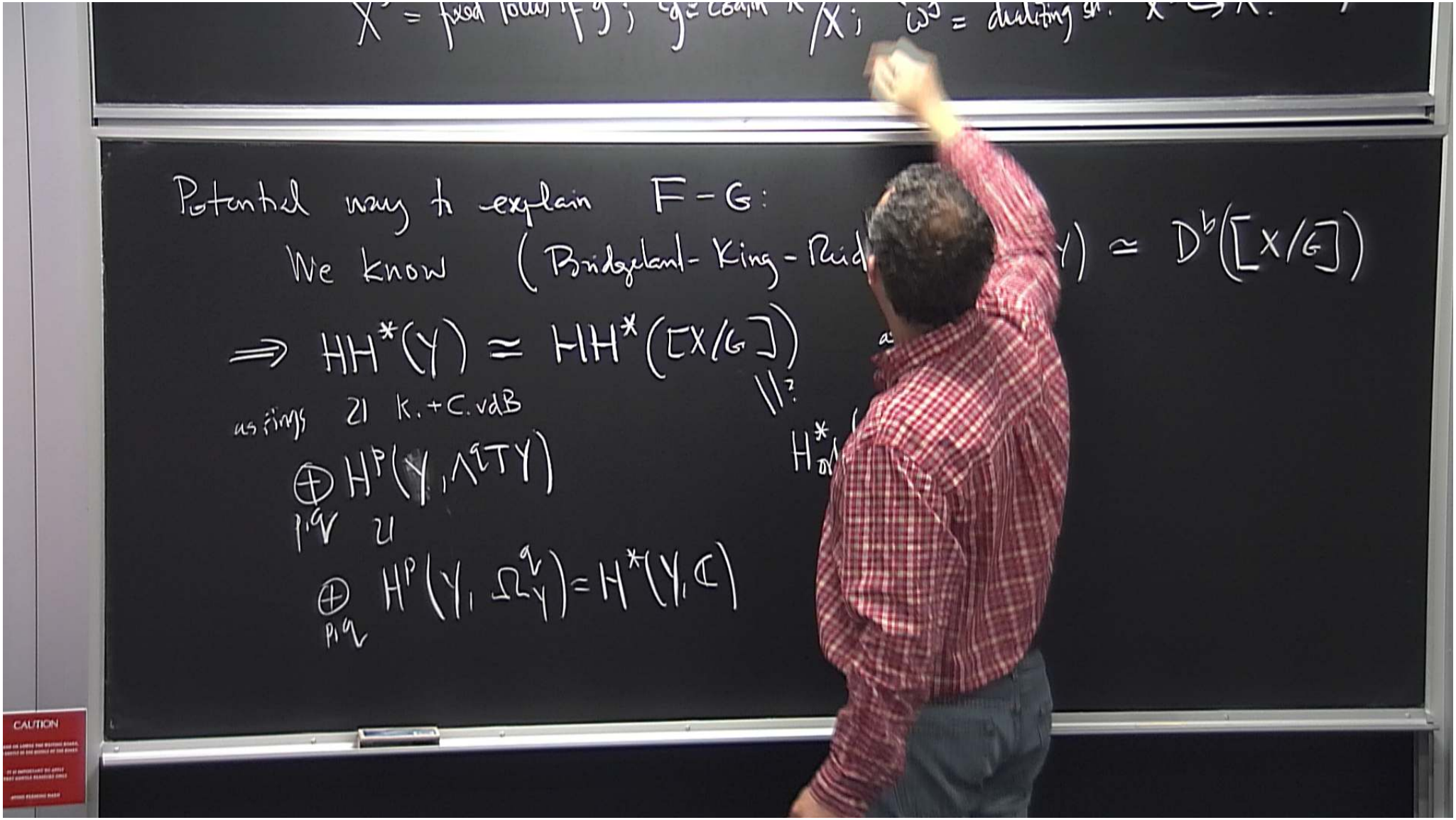

Potential way to explain F-G:


Potential way to explain F-G:
We know (Bridgeland-King-Reid
$\simeq D^b([X/G])$
$\Rightarrow HH^*(Y) \simeq HH^*([X/G])$
as rings 2) K.+C. vdB
$\bigoplus_{p,q} H^p(Y, \Lambda^q TY)$
2)
$\bigoplus_{p,q} H^p(Y, \Omega_Y^q) = H^*(Y, \mathbb{C})$
CAUTION

$X' = \text{fcn following } f$; $g = \text{coarse } X'/X$; $\omega^{\vee} = \text{dualizing sh. } X' \to X$.

Potential way to explain F-G:

We know (Bridgeland-King-Reid) $D^b(Y) \simeq D^b([X/G])$

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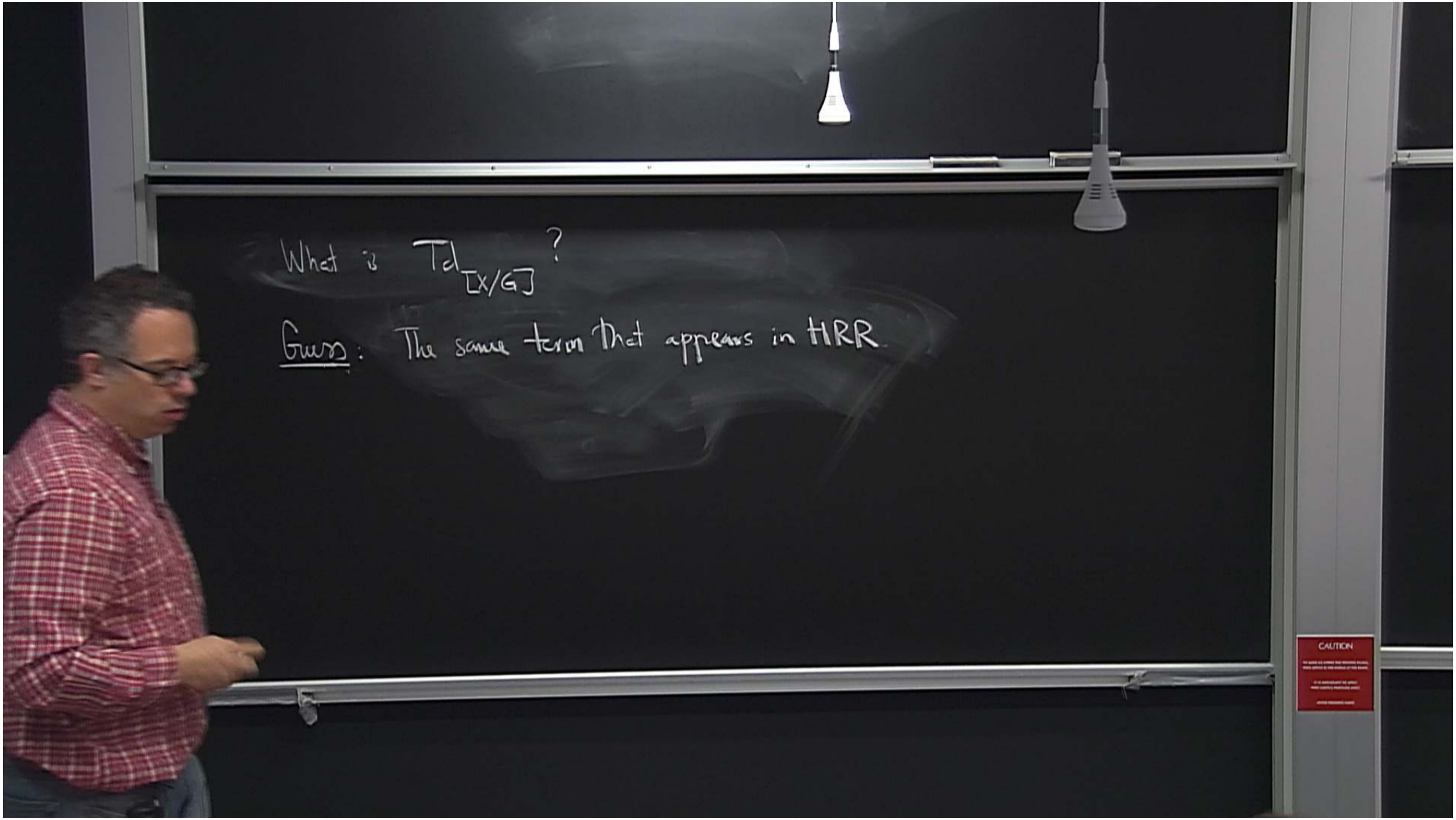
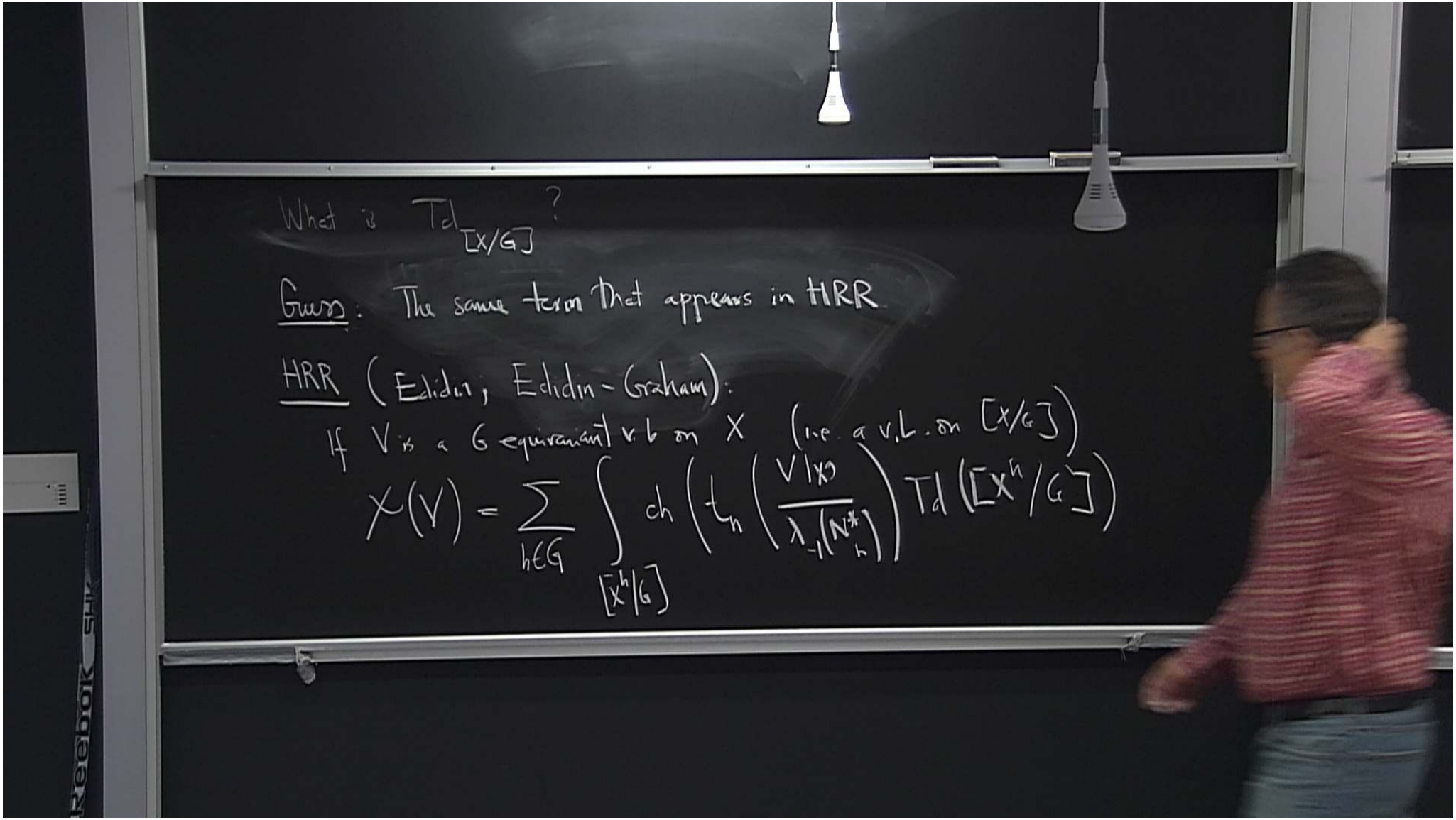
$\| ?$

$H^*_{orb}(X, G)$.

CAUTION


What is $Td_{[X/G]}$?
Guess: The same term that appears in HRR.
CAUTION


What is $\mathrm{Td}_{[X/G]}$?
Guess: The same term that appears in HRR
HRR (Edidin, Edidin-Graham):
If V is a G equivariant v.b on X (i.e. a v.b. on $[X/G]$)
$$\chi(V) = \sum_{h \in G} \int_{[X^h/G]} \mathrm{ch}\left(\frac{V|_{X^h}}{\lambda_{-1}(N^*_h)}\right) \mathrm{Td}\left([X^h/G]\right)$$

$$\chi(V) = \sum_{h \in G} \int_{[X^h/G]} ch\left(t_h\left(\frac{V|_{X^h}}{\lambda_{-1}(N_h^*)} \right) \right) Td\left([X^h/G]\right)$$

Tempting to guess $Td_{[X/G]}$ is $Td_{[X^h/G]}$ along X^h.

WRONG. N_h = normal bundle of X^h/X.

$$\lambda_t(N_h^*) = \sum (-t)^i \Lambda^i N_h^*.$$

$$t_h(W) = \sum_i \xi^i W_i \quad \text{for a v.b. } W \text{ on } X^h$$

h acts trivially

Atiyah-Segal. $W = \oplus W_i$ is the dec. of W according to e.v. of action of h.

$$\chi(\mathcal{V}) = \sum_{h \in G} \int_{[X^h/G]} ch\left(t_h\left(\frac{\mathcal{V}|_{X^h}}{\lambda_{-1}(N_h^*)}\right)\right) Td\left([X^h/G]\right)$$

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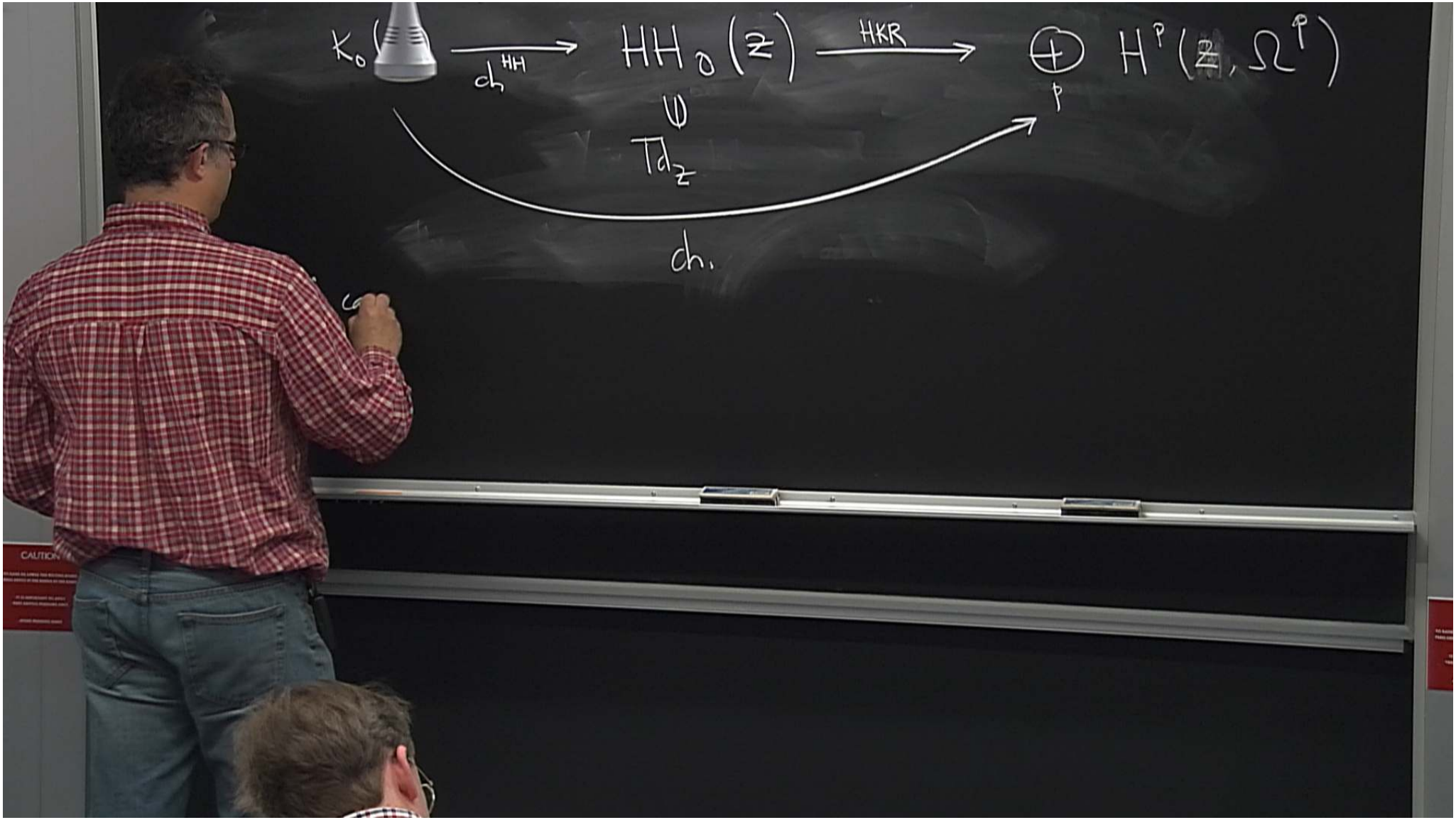
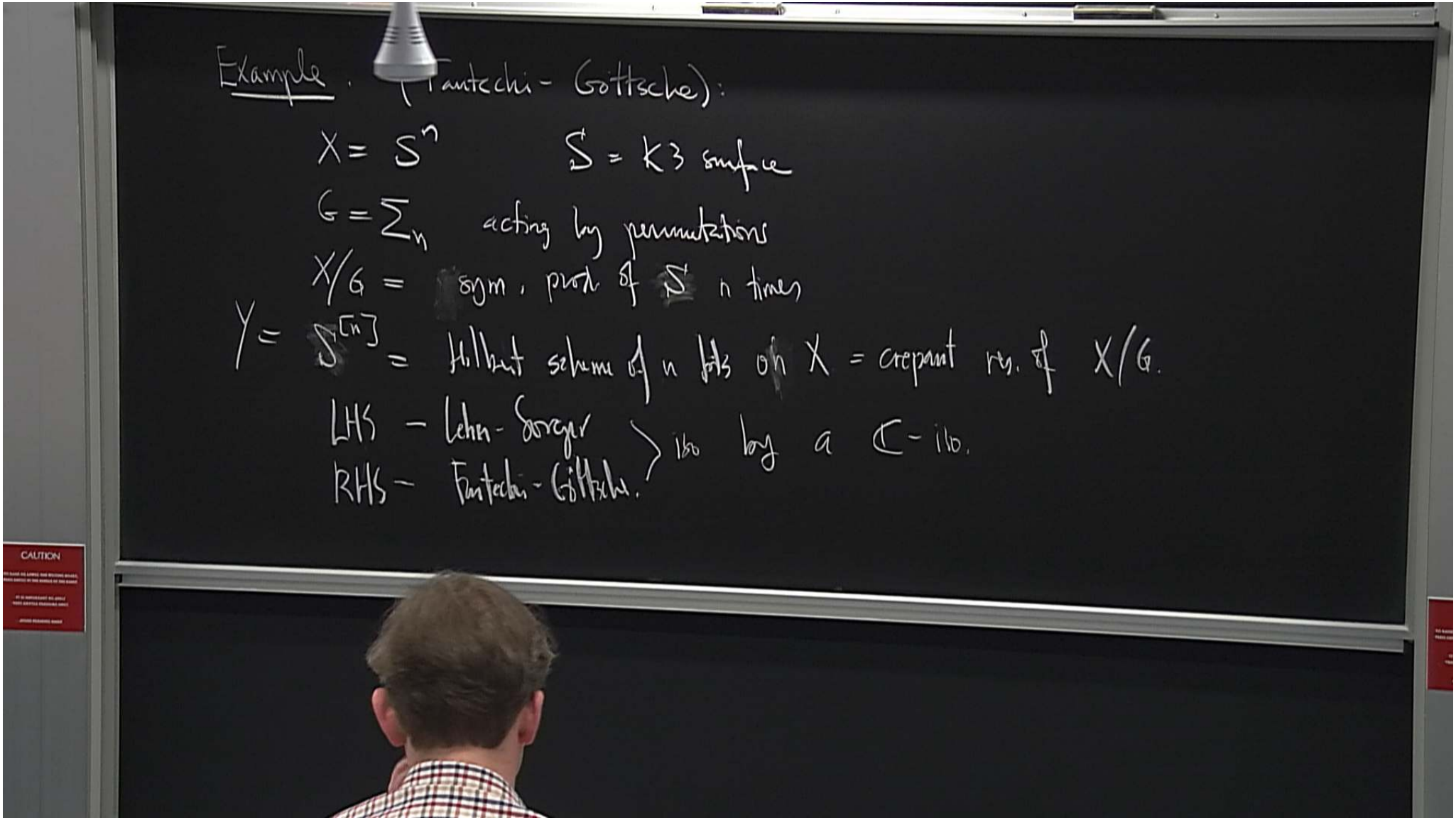

K_0
ch^{HH}
$HH_0(\mathbb{Z})$
HKR
$\bigoplus_p H^p(\mathbb{Z}, \Omega^p)$
Ψ
$Td_{\mathbb{Z}}$
$ch.$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} iso by a $\mathbb{C}$-iso.
CAUTION

Three questions:

1). Analogue of HKR ?

2). What is analogue of the "obvious" product on HT^* ?

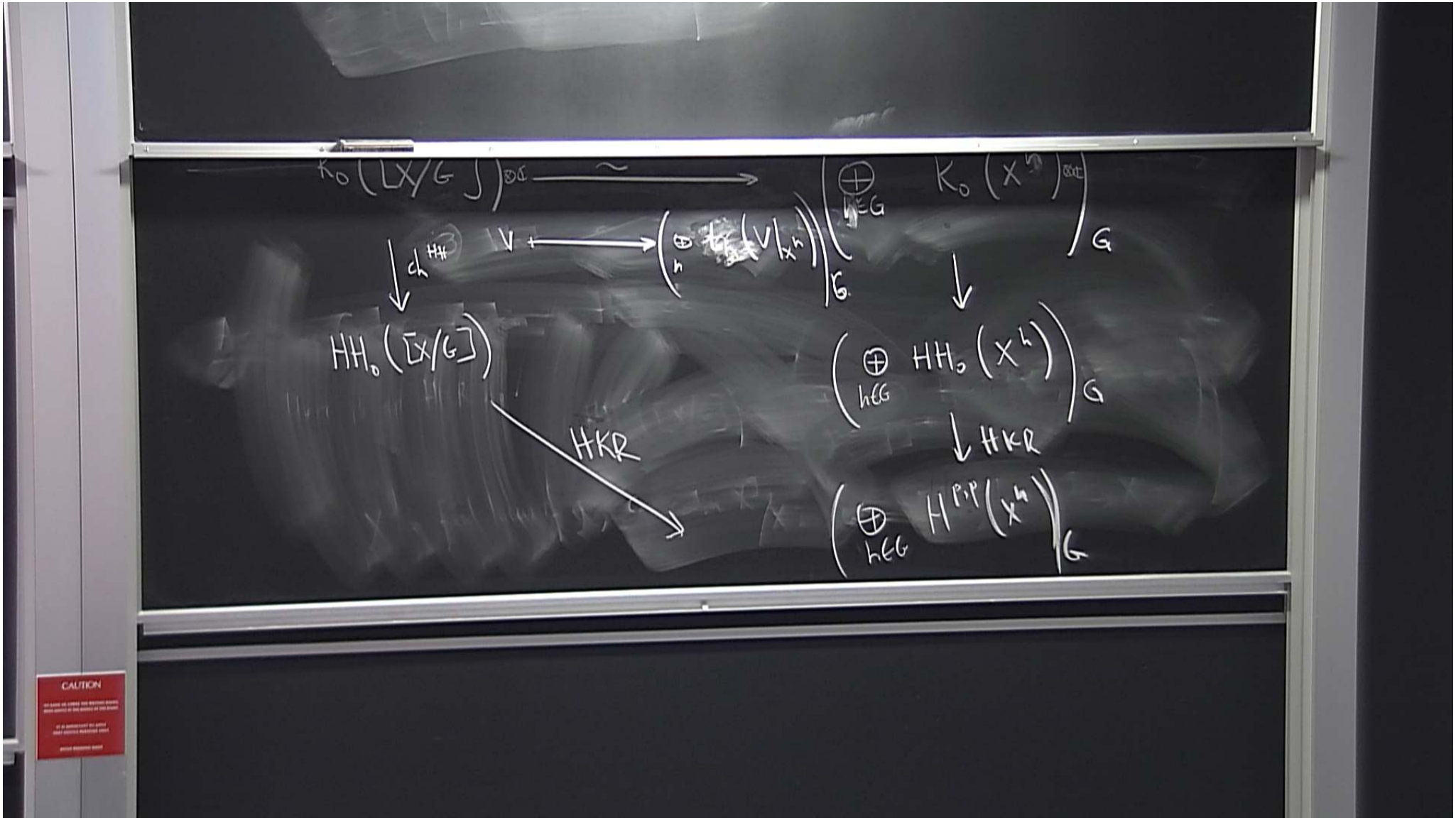
3). What is the $\hat{A}$ - correction?

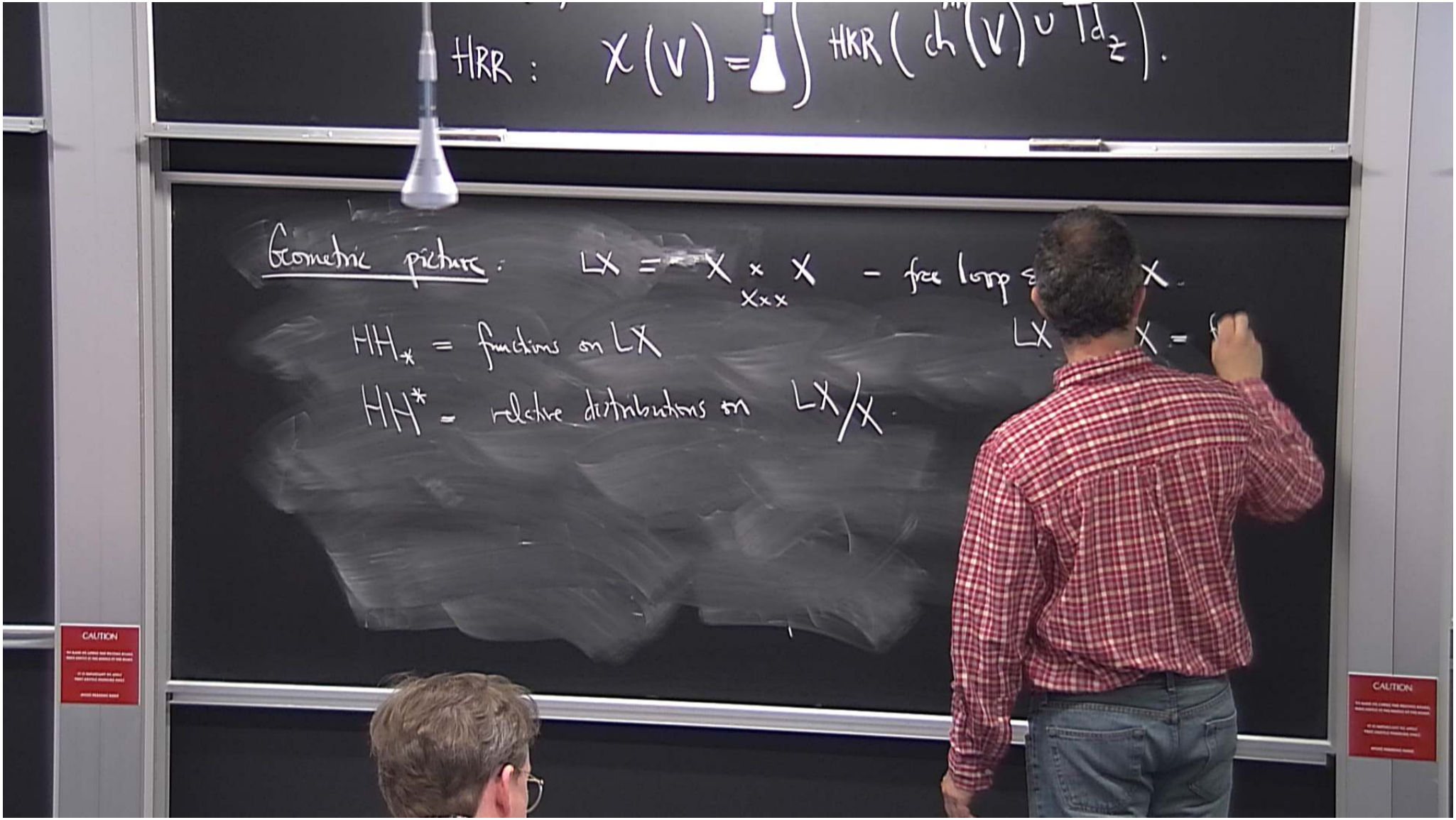
Ans 1: Ginzburg, Arinkin, –, Halpern-Leistner

There is an HKR iso.

$$HH^*([X/G]) \simeq \left(\bigoplus_{g \in G} \bigoplus_{p+q=*} H^{p-c_g}\left(X^g, \hat{\Lambda} TX^g \otimes \omega^g\right) \right)^G$$

X^g = fixed locus of g; c_g = codim X^g/X; ω^g = dualizing sh. $X^g \hookrightarrow X$.


HH₀([X/G])
HKR
HKR


HRR : $\chi(V) = \int$ HKR (ch (V) ∪ Td
Geometric picture: $LX = X \underset{X \times X}{\times} X$ - free loop
$HH_* =$ functions on LX
$HH^* =$ relative distributions on LX/X
CAUTION
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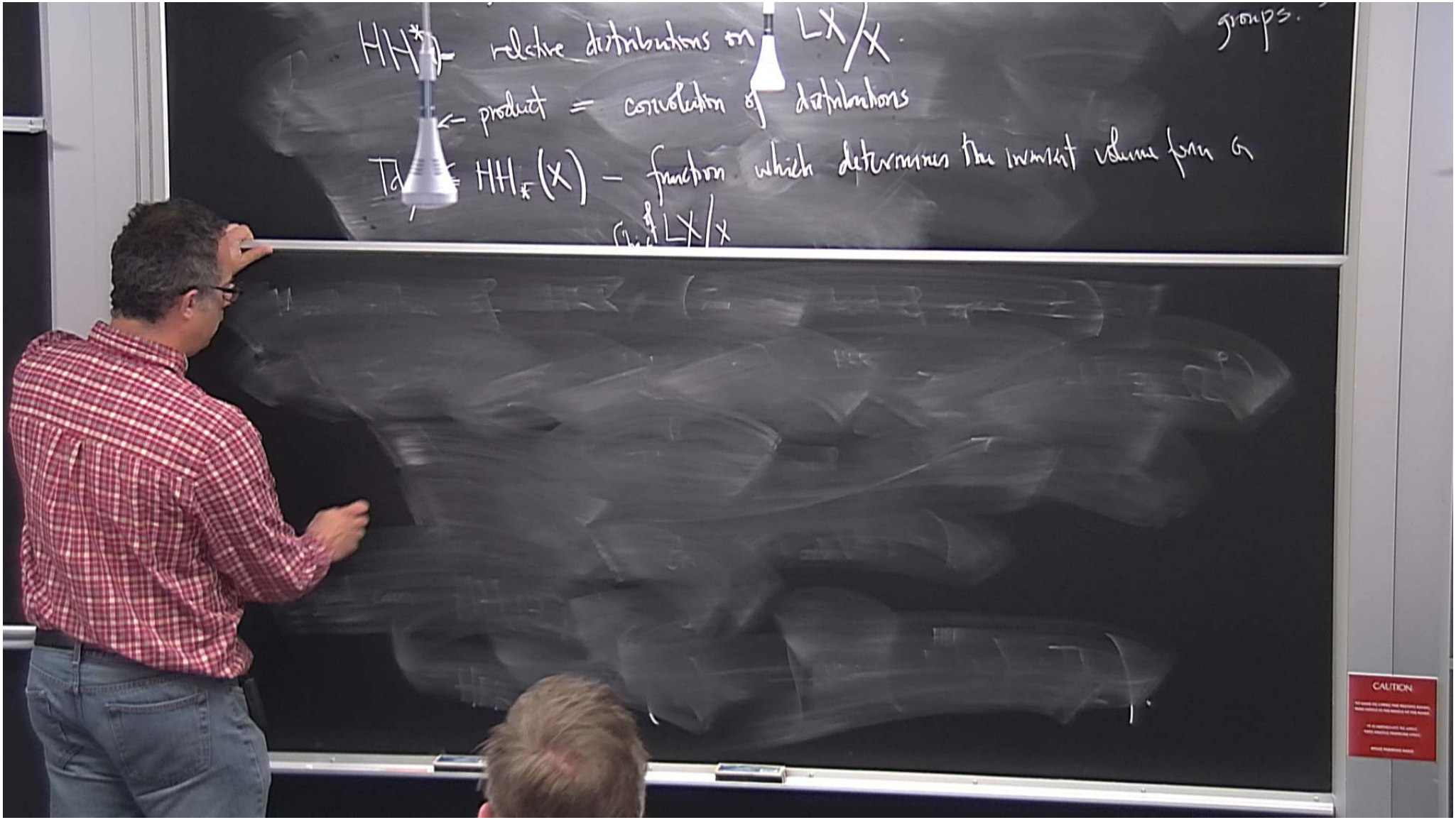
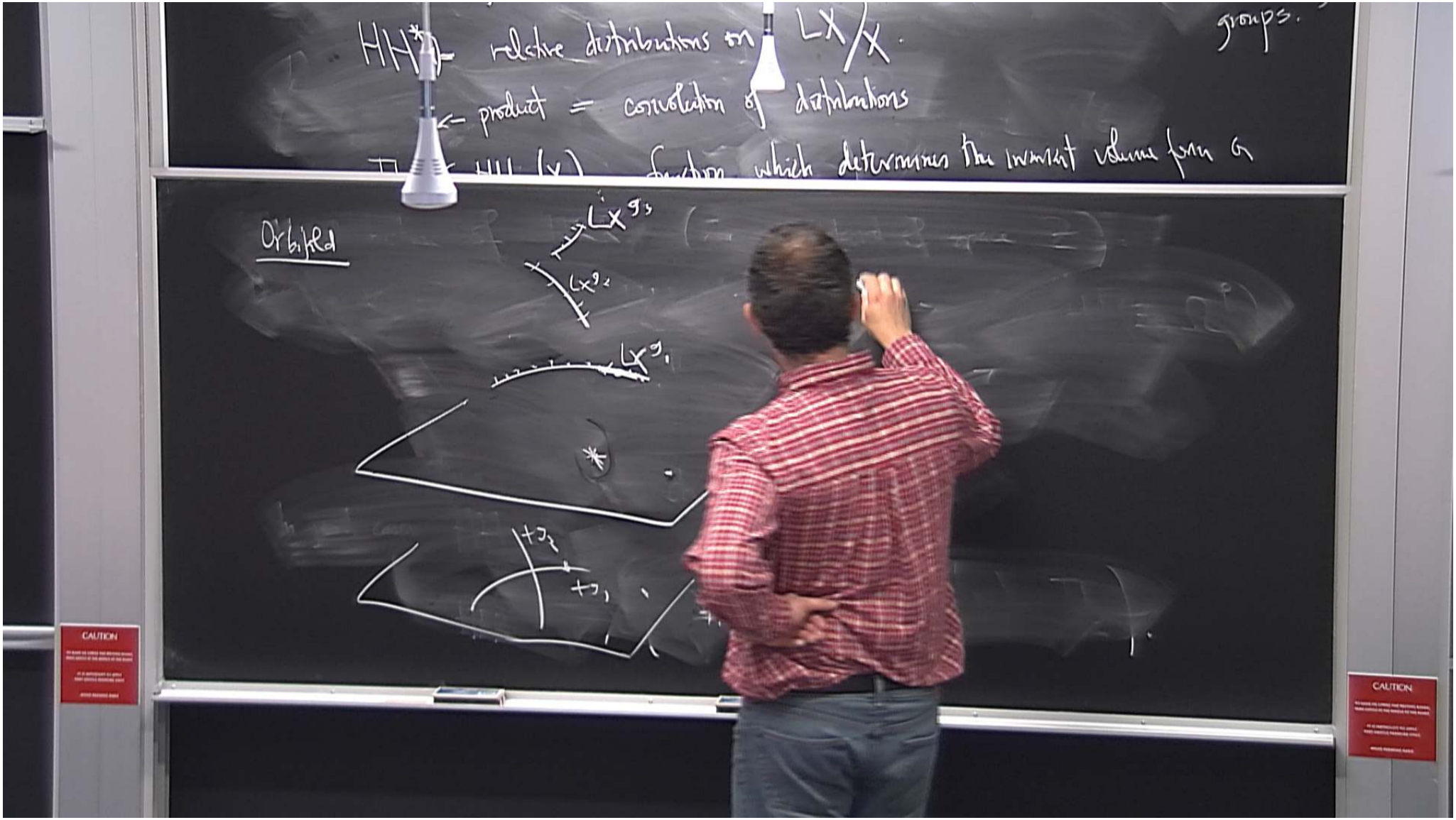

HH^* relative distributions on LX/X
groups.
← product = convolution of distributions
$HH_*(X)$ – function which determines the invariant volume form on
LX/X
CAUTION


HH* relative distributions on LX/X.
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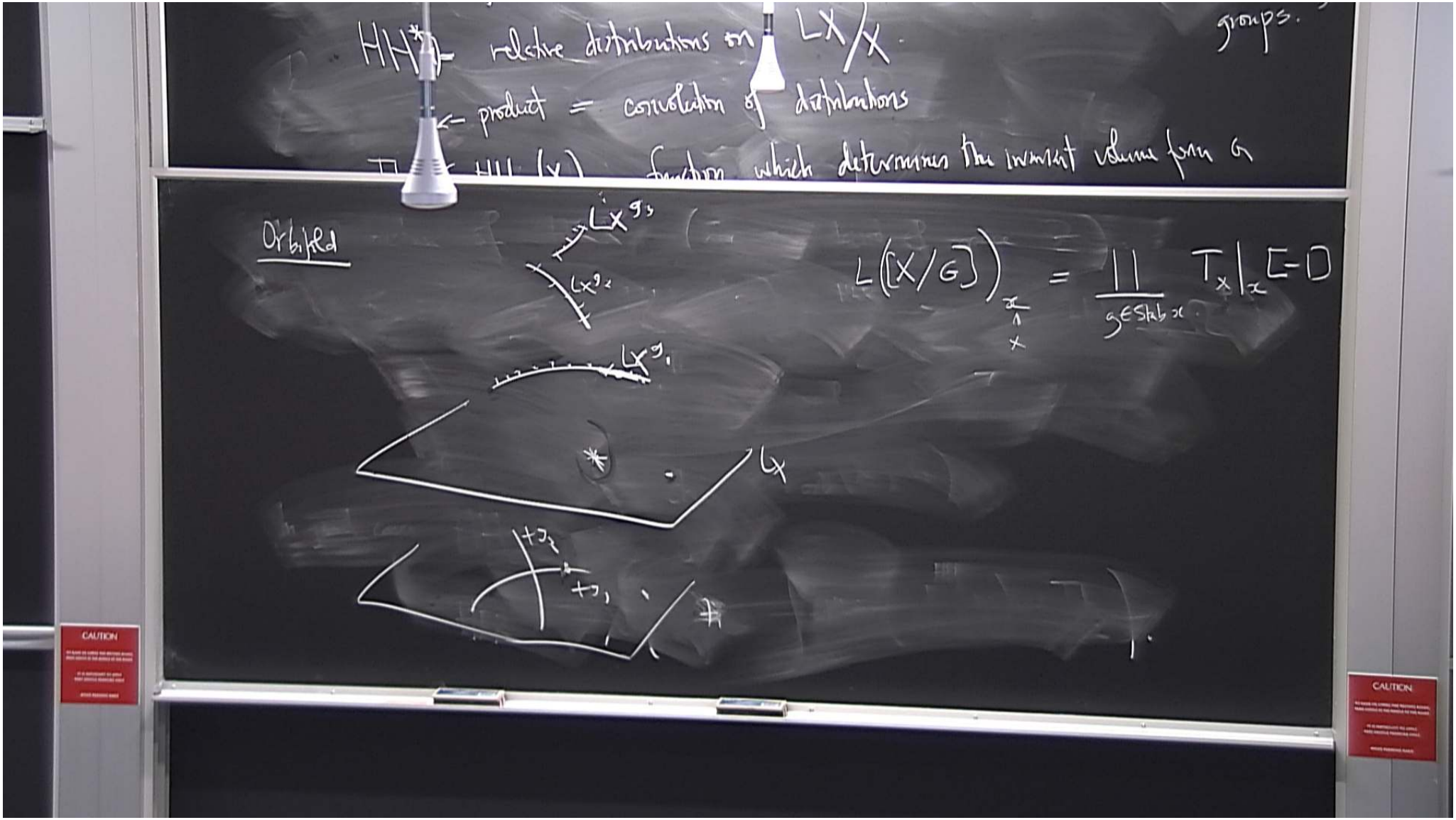
← product = convolution of distributions
which determines the invariant volume form on G
Orbifold
LX g3
LX g2
LX g1
+γ3
+γ2
+γ1
CAUTION
CAUTION


HH^* relative distributions on LX/X
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Π $HH(X)$ function which determines the invariant volume form on G
Orbifold
LX^{g_3}
LX^{g_2}
LX^{g_1}
LX
$L([X/G]) = \coprod_{g \in Stab_x} T_x|_x^{G-D}$
α
X

HH^* relative distributions on LX/X

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groupoids.

Π ← $HH(X)$ function which determines the invariant volume form on

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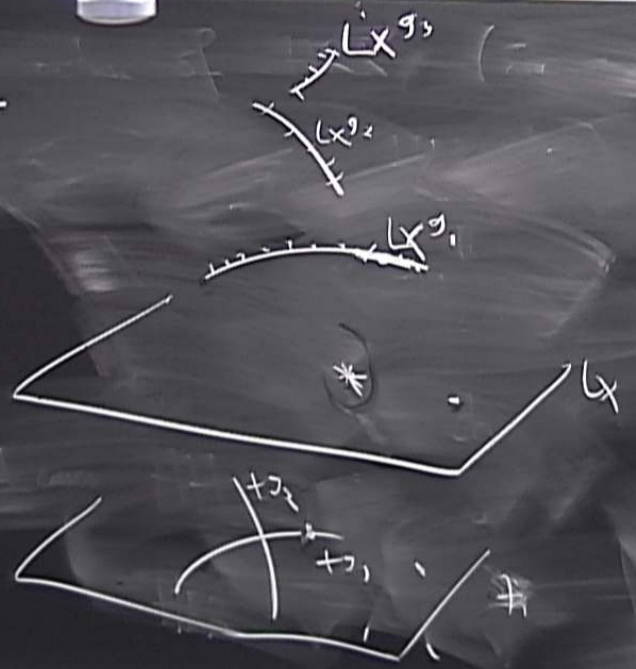

$$L([X/G])_x = \frac{1}{|G_x|} T_x|_x \subset 0$$

$x \uparrow x$

$g \in Stab_x$

$\Rightarrow$ invariant volume from (Td) should mix from T_x on all components, not from $T_{x?}$!!