Title: What is the Todd class of an orbifold?

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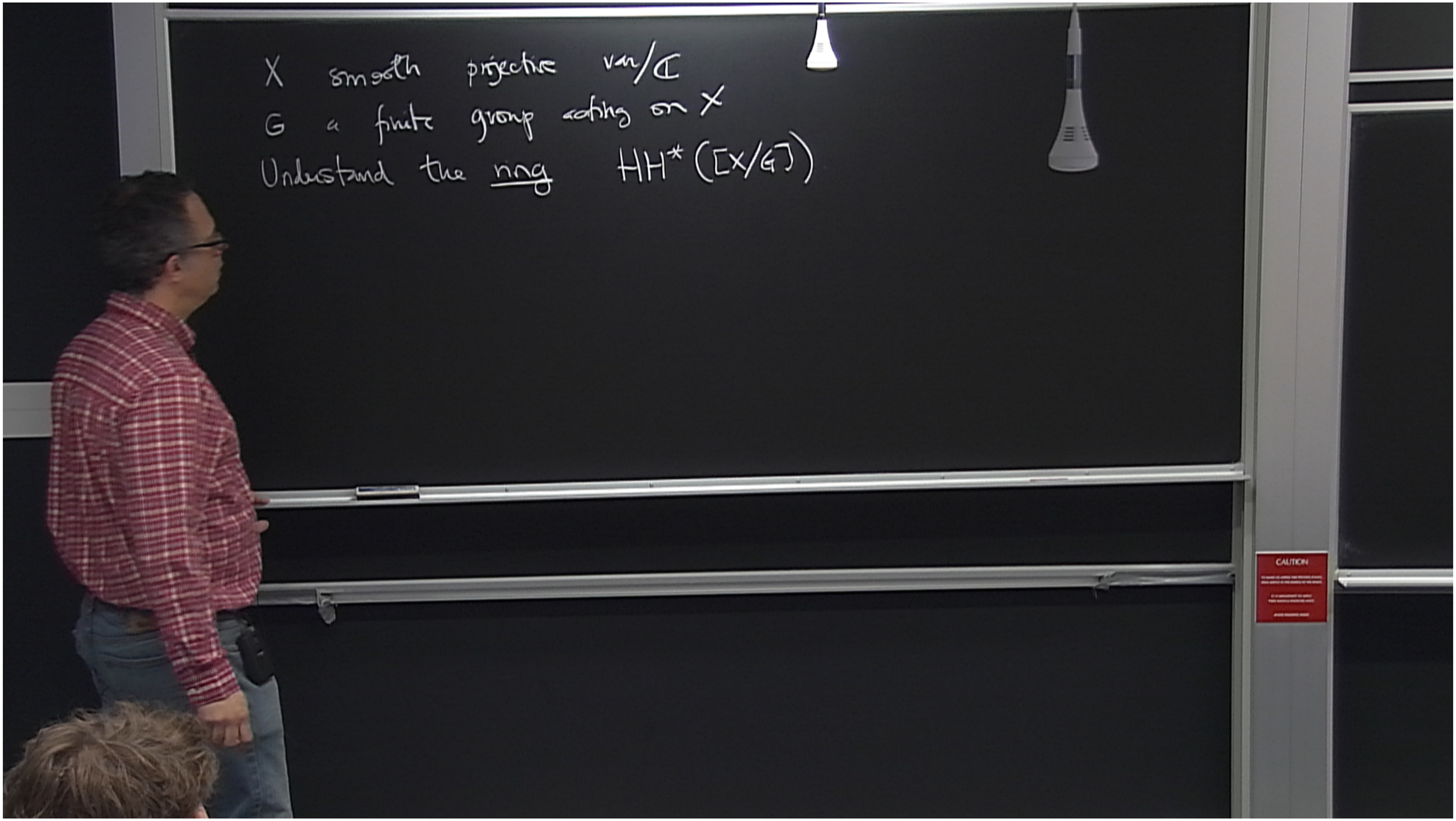
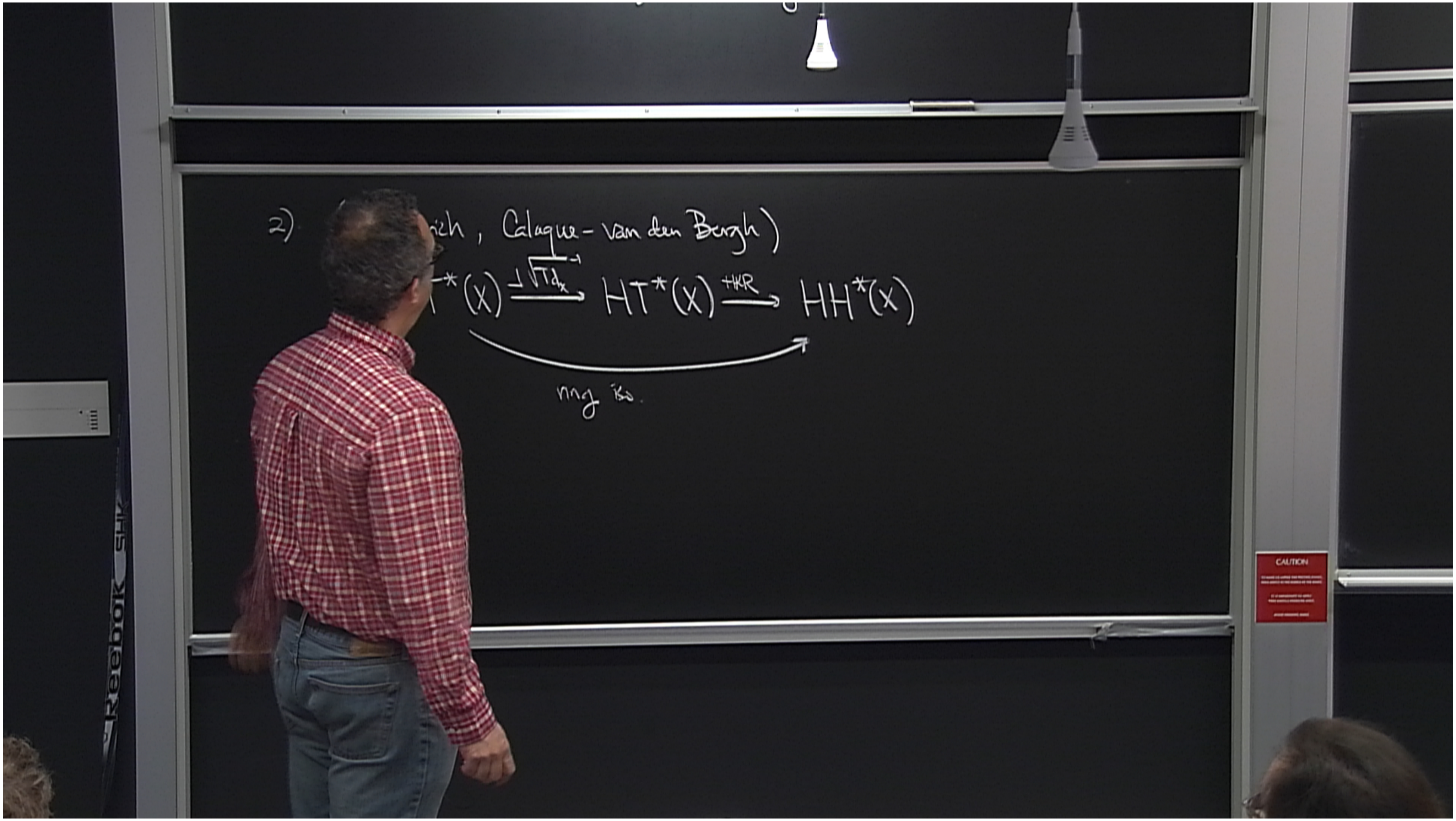
Abstract: The Todd class enters algebraic geometry in two places, in the Hirzebruch-Riemann-Roch formula and in the correction of the HKR isomorphism needed to make the Hochschild cohomology isomorphic to polyvector field cohomology (Kontsevichâ€™s claim, proved by Calaque and van den Bergh). In the case of orbifolds the Riemann-Roch formula is known, but not the analogue of Kontsevichâ€™s result. However, we can try to use the former as a guide towards a conjectural formulation for the latter.

The problem with this approach is that in the case of an orbifold it is not obvious what the Todd class actually is. This happens because the Riemann-Roch formula mixes the Todd class with the Chern character and it is difficult to separate one from the other. In my talk I shall discuss what the study of loop groups of orbifolds predicts the correct Todd class to be, and then I shall explain how the orbifold Riemann-Roch formula can be rewritten to make this prediction consistent.


X smooth projective var/$\mathbb{C}$
G a finite group acting on X
Understand the ring $HH^*([X/G])$
CAUTION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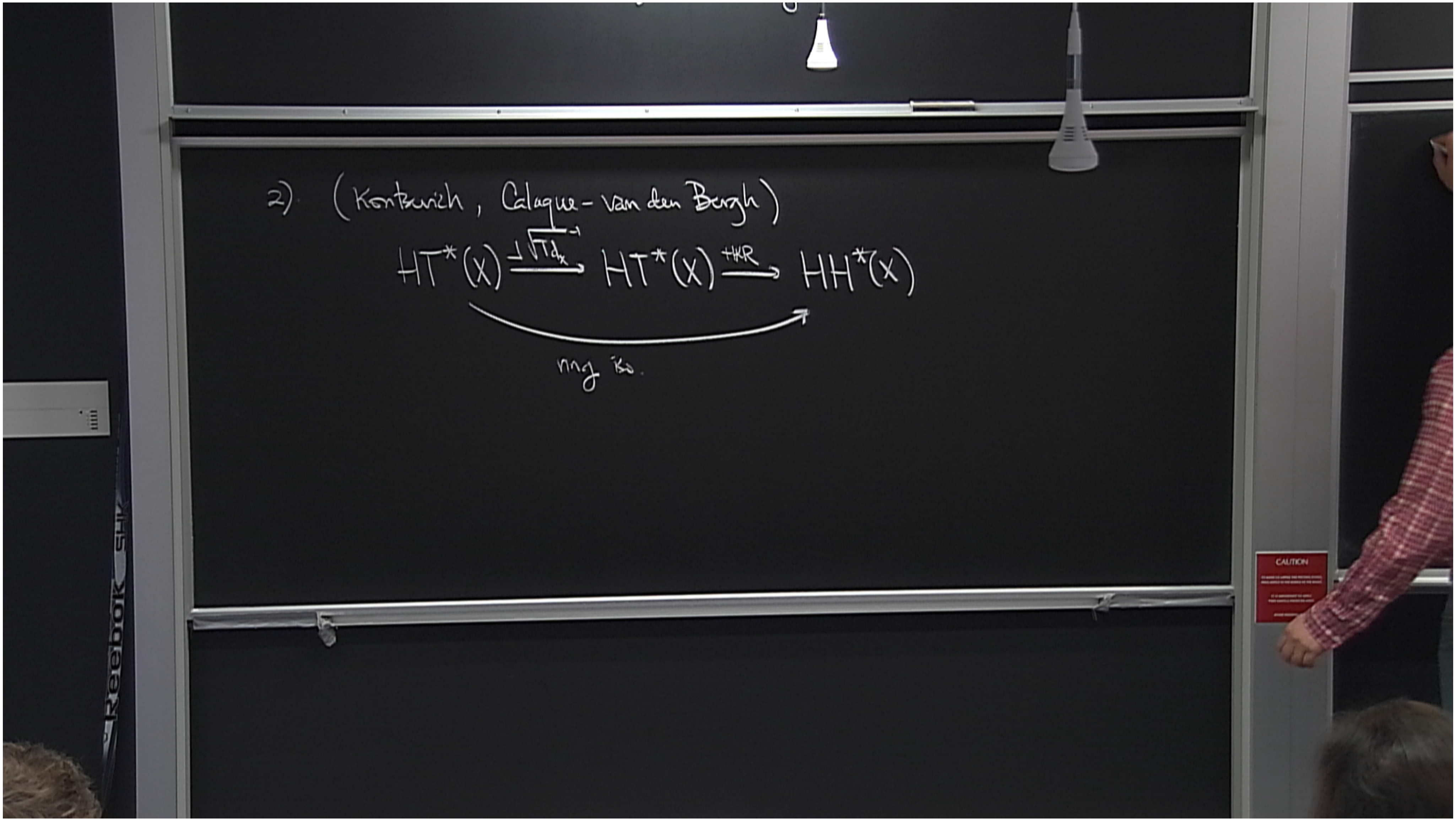
$HT^*(X) \xrightarrow{HKR} HH^*(X)$
ring iso.


2) (Kontsevich, Calaque - van den Bergh)
$HT^*(X) \xrightarrow{\lrcorner \sqrt{Td_X}} HT^*(X) \xrightarrow{HKR} HH^*(X)$
ring iso.

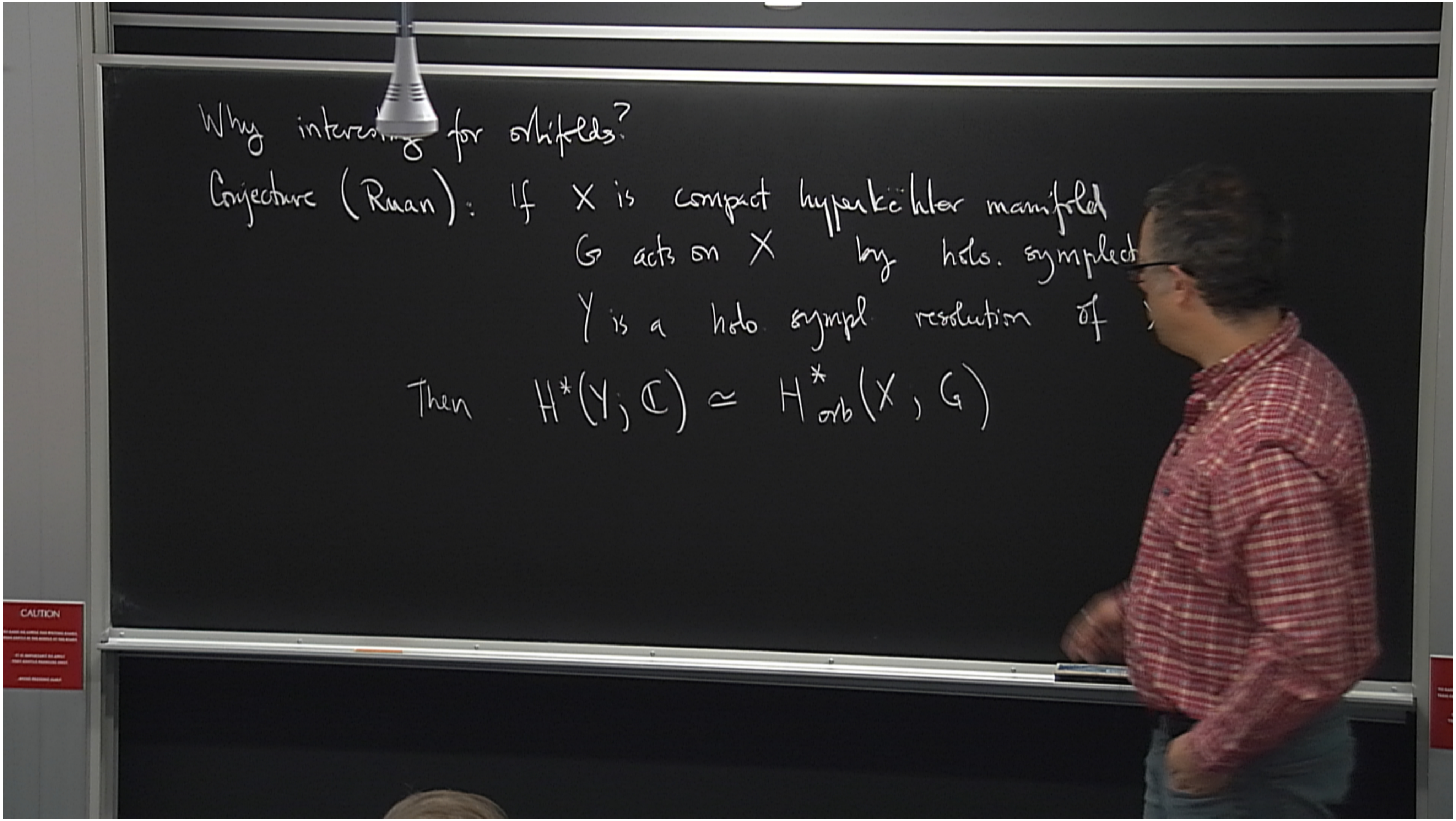
Why interesting for orbifolds?

Conjecture (Ruan): If X is compact hyperkähler manifold

G acts on X by holo. symplectic auto.

Y is a holo sympl resolution of X/G

Then $H^*(Y;\mathbb{C}) \simeq H^*_{orb}(X, G)$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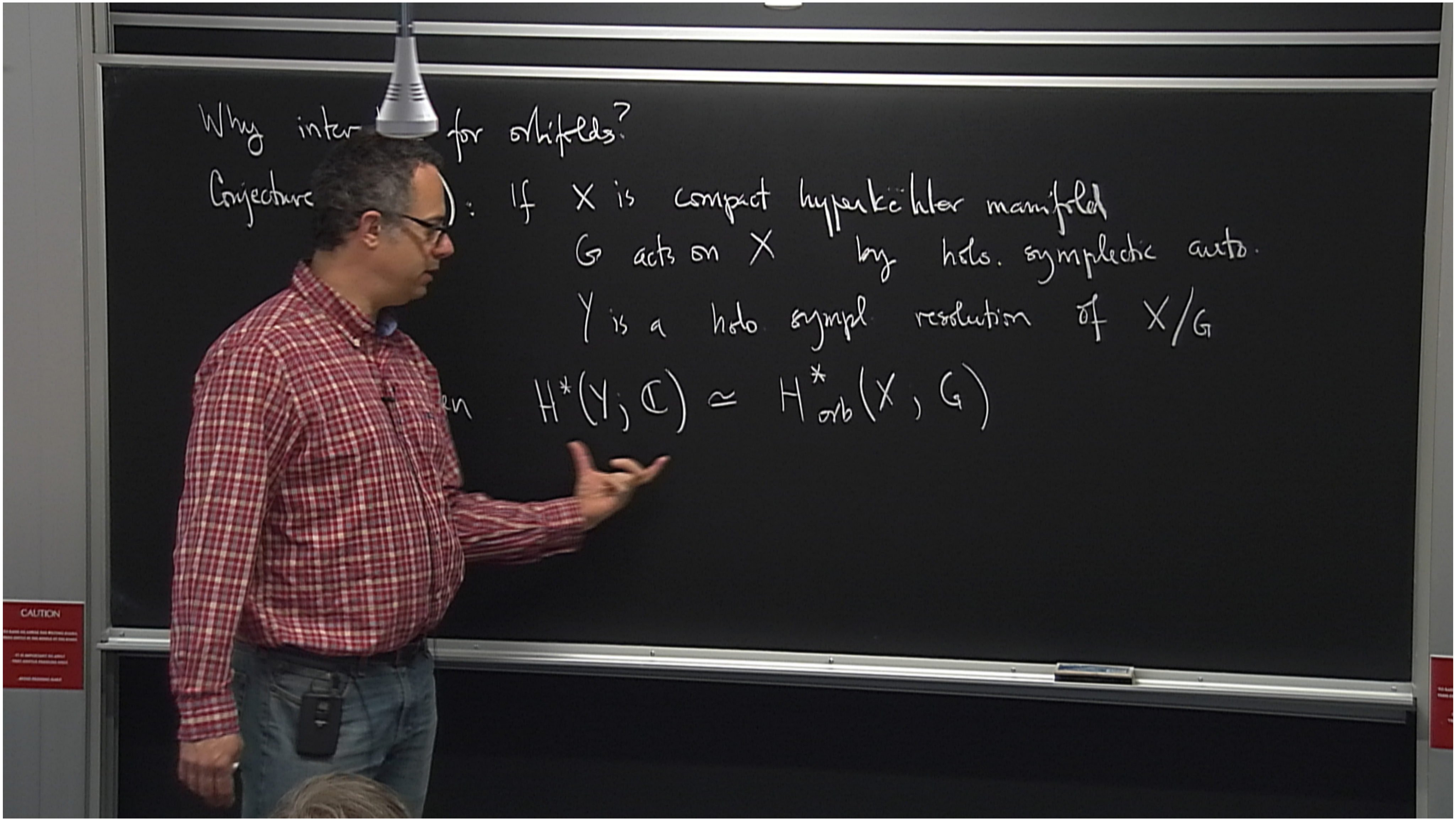

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Example: (Fantechi - Göttsche):

$X = S^n$ $\quad$ S = K3 surface

$G = \Sigma_n$ acting by permutations

X/G = sym. prod. of S n times

$S^{[n]}$ = Hilbert scheme of n pts on X

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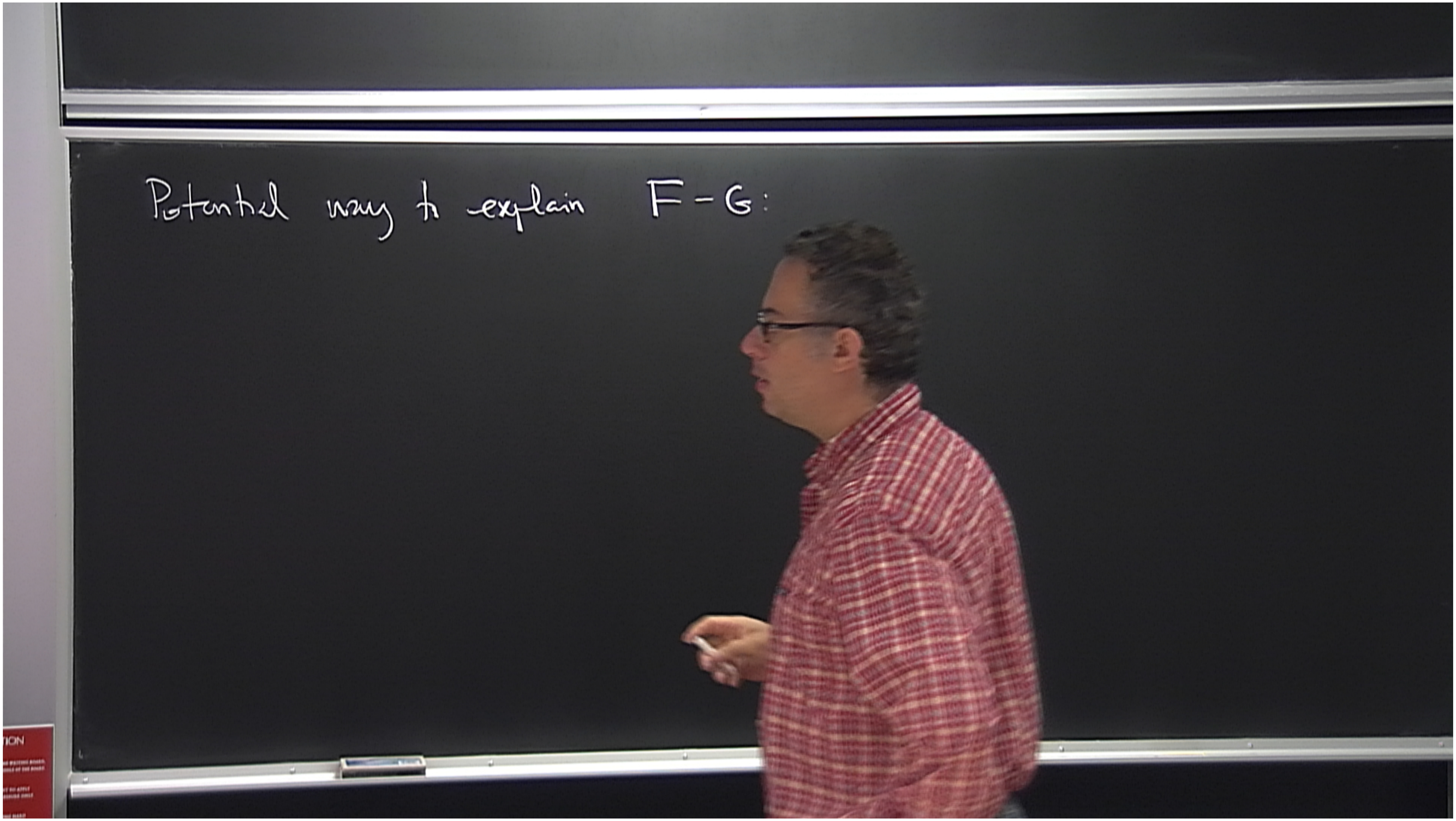
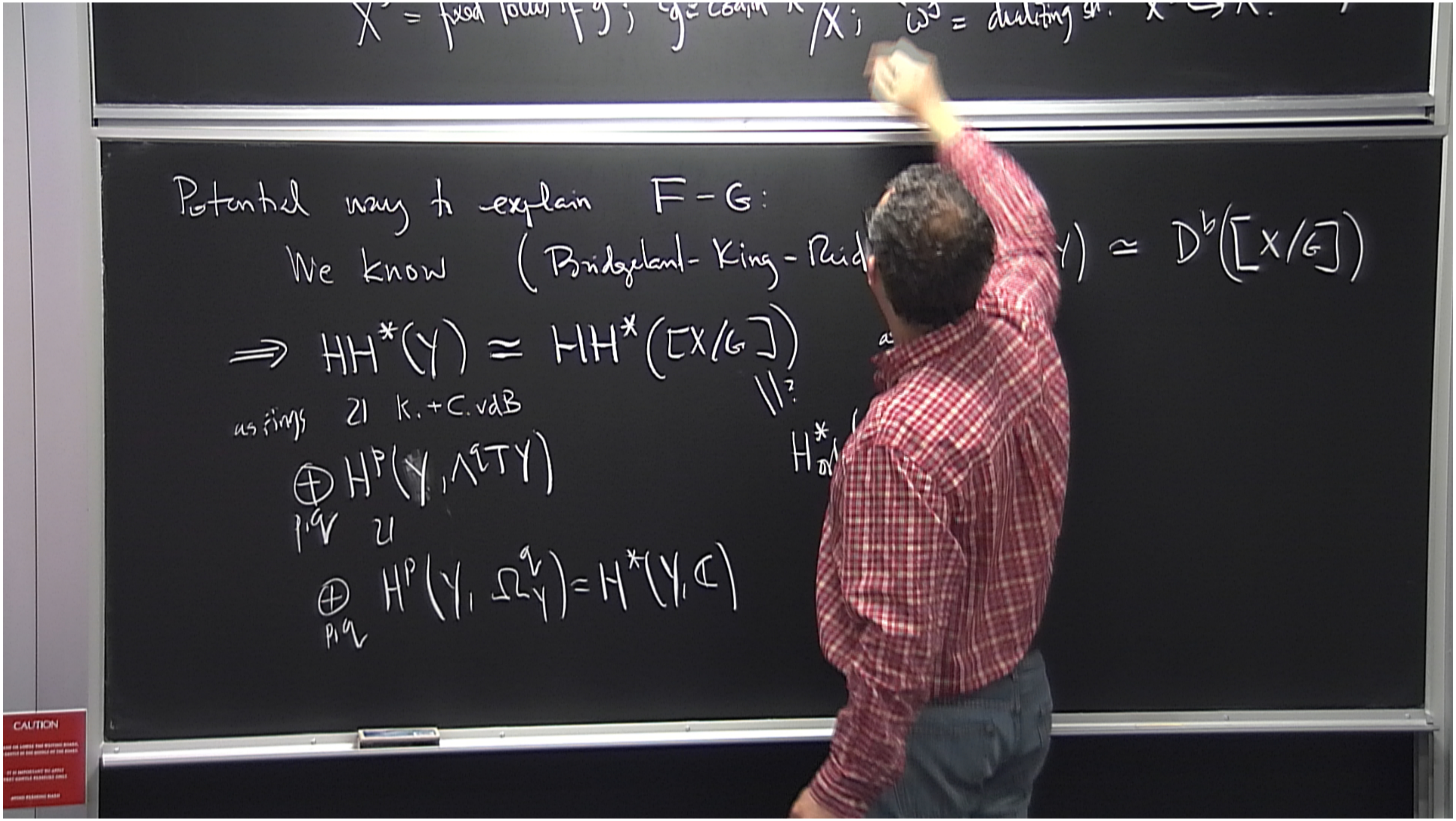
LHS – Lehn - Sorger

RHS – Fantechi - Göttsche.

iso by a $\mathbb{C}$ - iso.


Potential way to explain F-G:


Potential way to explain F-G:
We know (Bridgeland-King-Reid
) ≃ D^b([X/G])
⟹ HH*(Y) ≃ HH*([X/G])
as rings ≅ K. + C. vdB
⊕ H^p(Y, Λ^q TY)
p,q ≅
⊕ H^p(Y, Ω^q_Y) = H*(Y, ℂ)
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CAUTION

X^g = fixed locus of g; c_g = codim X^g/X; ω^g = dualizing sh. $X^g \hookrightarrow X$.

Potential way to explain F-G:

We know (Bridgeland-King-Reid) $D^b(Y) \simeq D^b([X/G])$

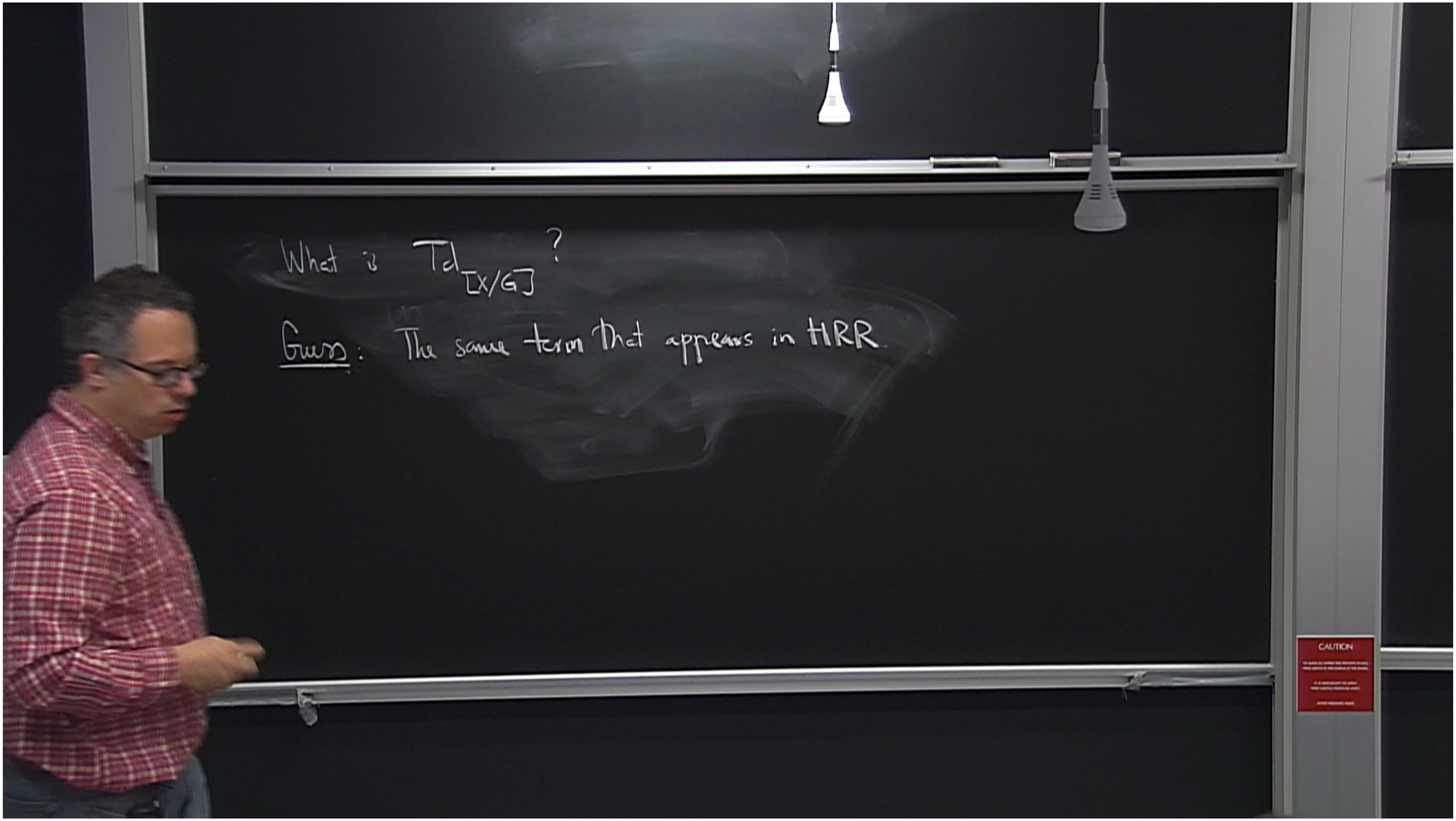
$$\Longrightarrow HH^*(Y) \simeq HH^*([X/G]) \quad \text{as rings}$$

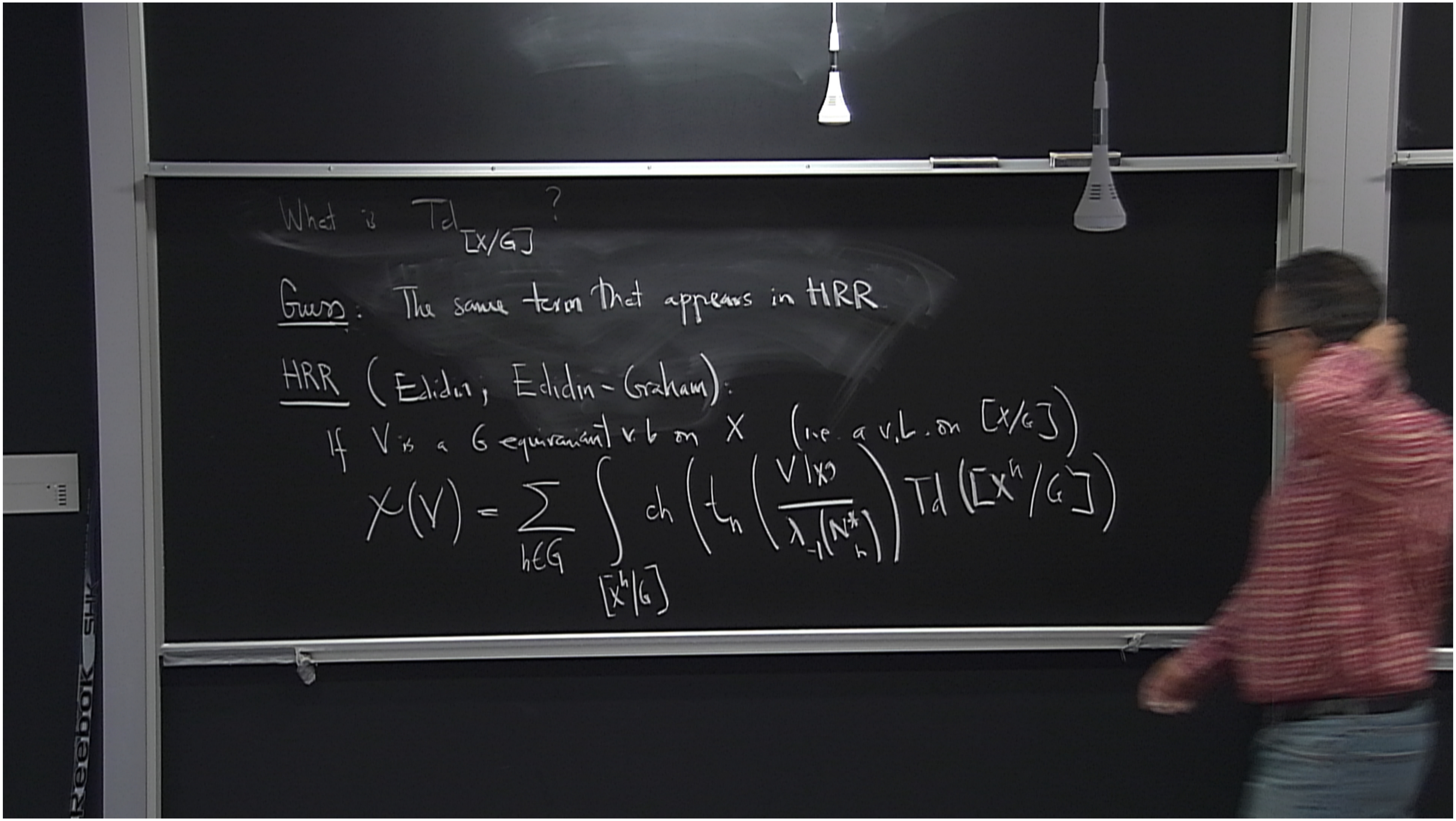
as rings $\cong$ K. + C. vdB $\qquad \overset{?}{\|}$

$$\bigoplus_{p,q} H^p(Y, \Lambda^q TY) \qquad H^*_{orb}(X, G).$$

$$\cong$$

$$\bigoplus_{p,q} H^p(Y, \Omega^q_Y) = H^*(Y, \mathbb{C})$$


What is $Td_{[X/G]}$?
Guess: The same term that appears in HRR.


What is $Td_{[X/G]}$?
Guess: The same term that appears in HRR
HRR (Edidin, Edidin-Graham):
If V is a G equivariant v.b on X (i.e. a v.b. on $[X/G]$)
$\chi(V) = \sum_{h\in G} \int_{[X^h/G]} ch\left(t_h\left(\frac{V|_{X^h}}{\lambda_{-1}(N^*_h)}\right) Td([X^h/G])\right)$

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Tempting to guess $Td_{[X/G]}$ is $Td_{[X^h/G]}$ along X^h.

WRONG. N_h = normal bundle of X^h/X.

$$\lambda_{-1}(N_h^*) = \sum (-1)^i \Lambda^i N_h^*.$$

$$t_h(W) = \sum_i \xi^i W_i$$ for a v.b. W on X^h, h acts trivially

Atiyah-Segal. $W = \oplus W_i$ is the dec. of W according to e.v. of action of h.

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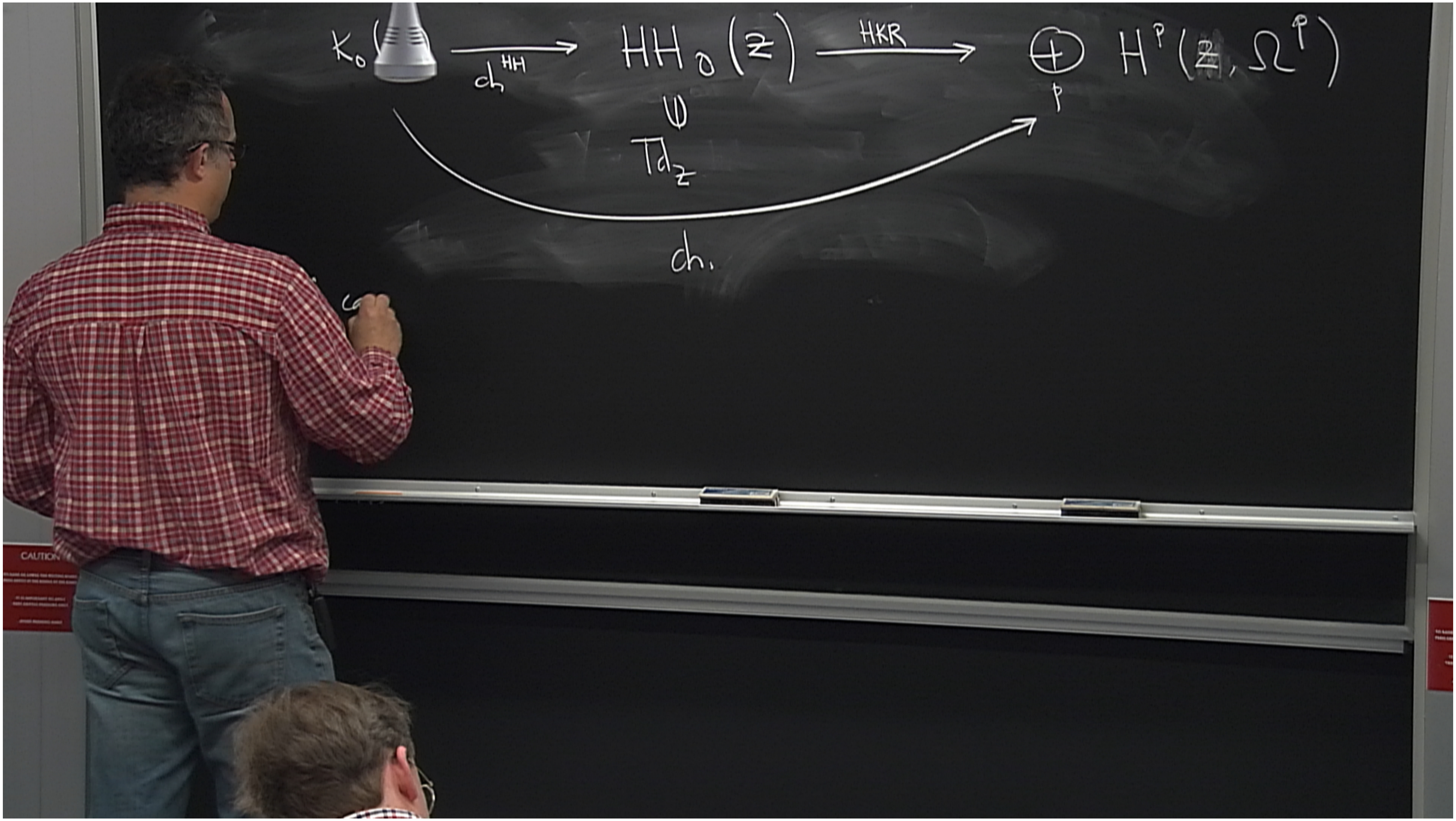
$K_0($
$\xrightarrow{ch^{HH}}$
$HH_0(Z) \xrightarrow{HKR} \bigoplus_p H^p(Z, \Omega^p)$
$\Downarrow$
Td_Z
ch.

Example: (Fantechi - Göttsche):

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$$\left.\begin{array}{l}\text{LHS} - \text{Lehn-Sorger} \\ \text{RHS} - \text{Fantechi-Göttsche.}\end{array}\right\} \text{iso by a } \mathbb{C}\text{-iso.}$$

Three questions: 1). Analogue of HKR ?

2). What is analogue of the "obvious" product on HT^* ?

3). What is the Td - correction?

Ans 1: Ganter, Arinkin, –, Hablicsek

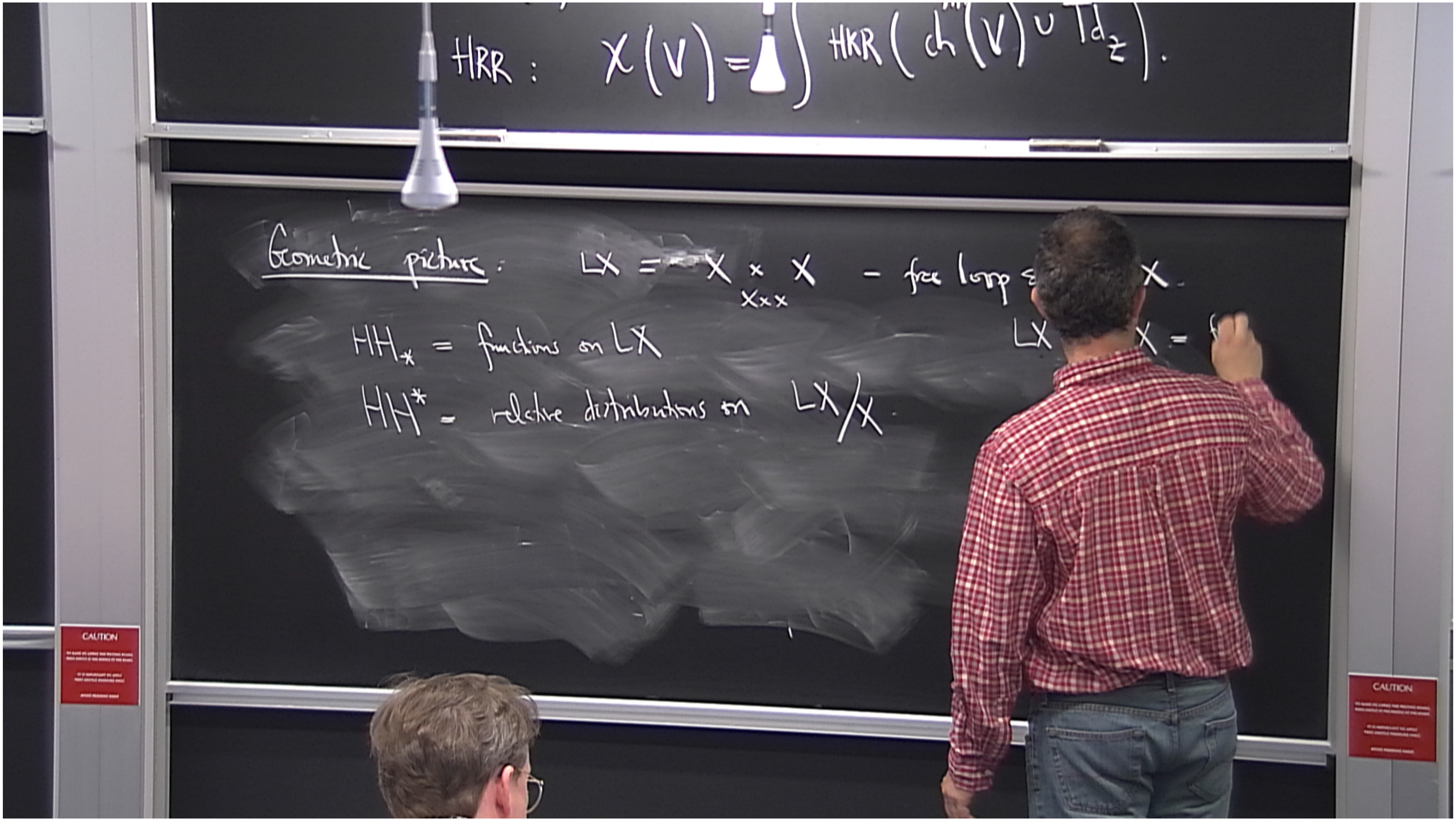
There is an HKR iso.

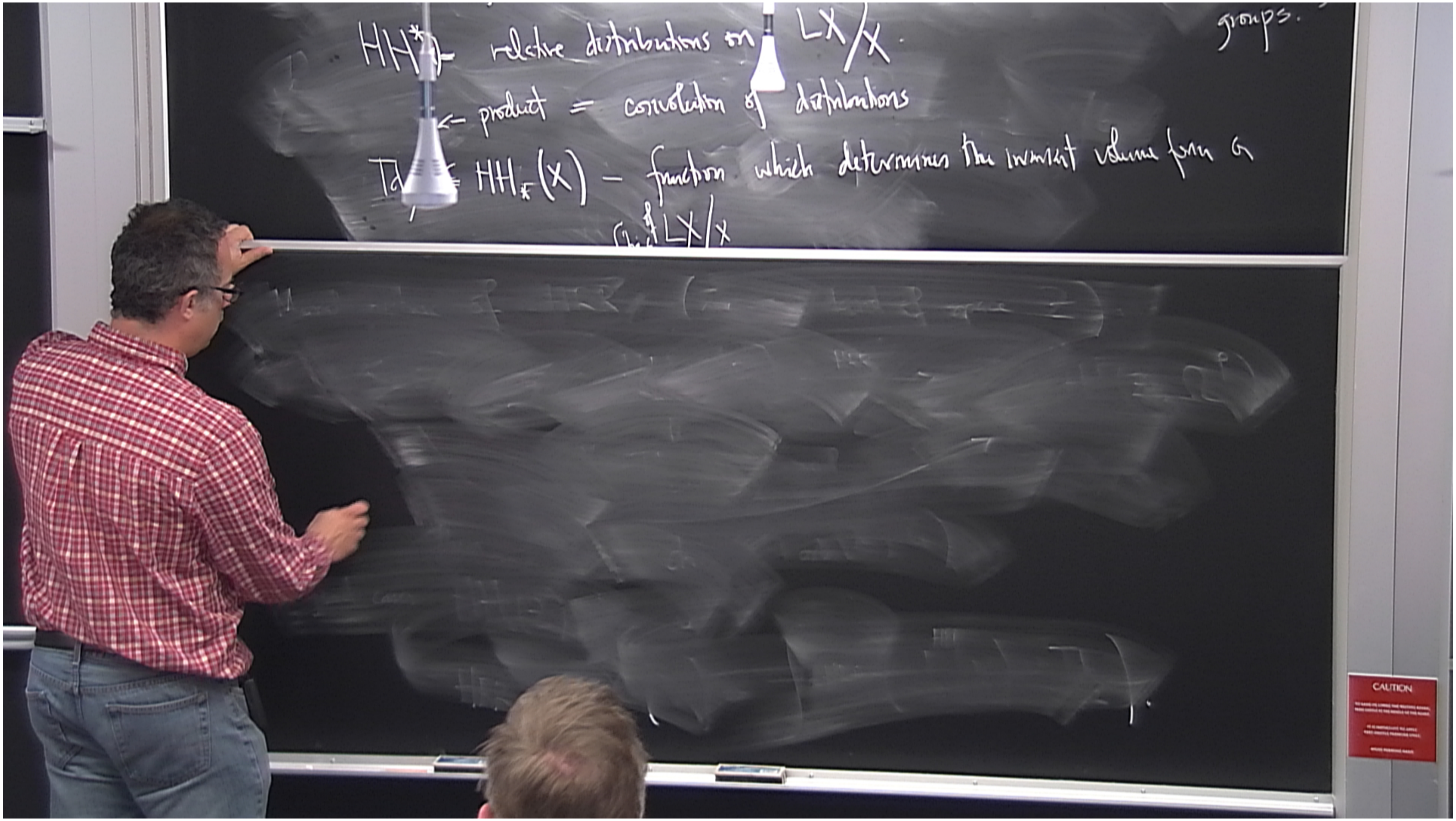
$$HH^*([X/G]) \simeq \left(\bigoplus_{g\in G} \bigoplus_{p+q=*} H^{p-c_g}\left(X^g, \Lambda^q TX^g \otimes \omega^g\right) \right)^G$$

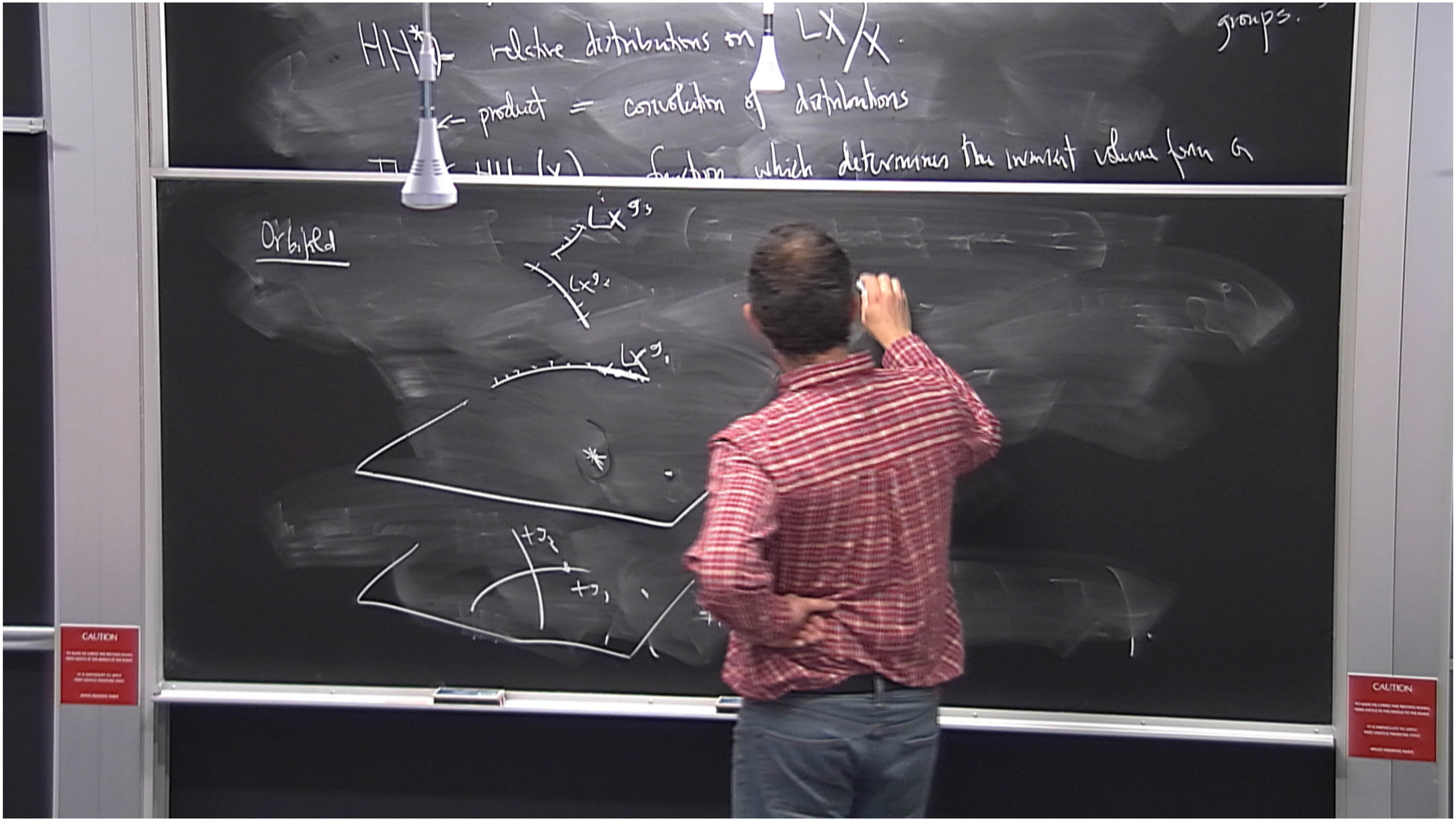
X^g = fixed locus of g; c_g = codim X^g/X; ω^g = dualizing sh. $X^g \hookrightarrow X$.

$$
\begin{tikzcd}
K_0([X/G])\otimes\mathbb{C} \arrow[rr, "\sim"] \arrow[d, "ch^{HH}"] & & \left(\bigoplus_{h\in G} K_0(X^h)\otimes\mathbb{C}\right)_G \arrow[d] \\
HH_0([X/G]) \arrow[drr, "HKR"] & & \left(\bigoplus_{h\in G} HH_0(X^h)\right)_G \arrow[d, "HKR"] \\
& & \left(\bigoplus_{h\in G} H^{p,p}(X^h)\right)_G
\end{tikzcd}
$$

$$V \longmapsto \left(\bigoplus_h \mathrm{tr}_h(V|_{X^h})\right)_G$$


HRR : X(V) = ∫ HKR(ch(V) ∪ Td_Z).
Geometric picture: LX = X ×_{X×X} X – free loop space X.
HH_* = functions on LX
HH^* = relative distributions on LX/X.


HH* - relative distributions on LX/X.
groups.
<- product = convolution of distributions
HH*(X) - function which determines the invariant volume form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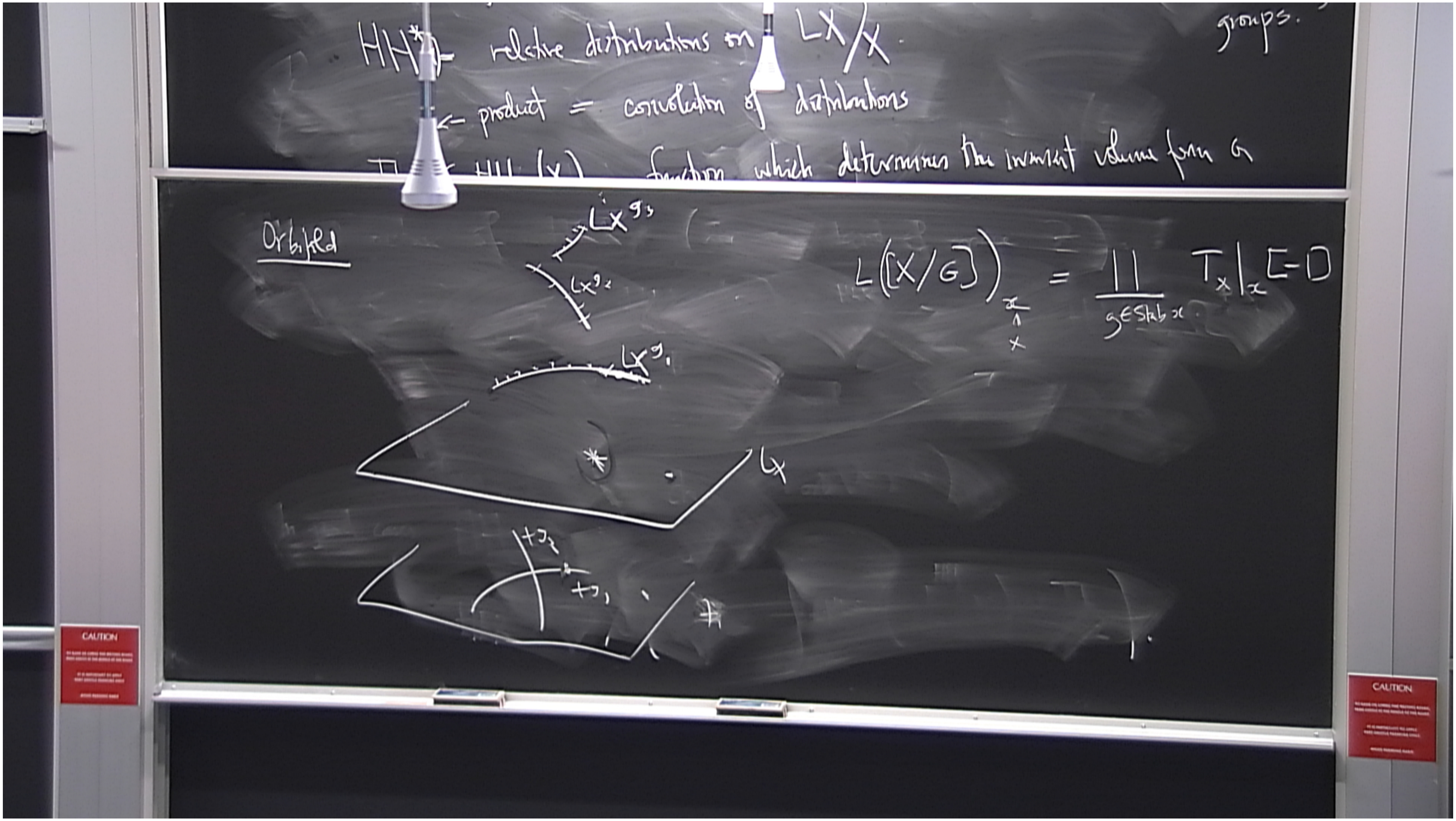
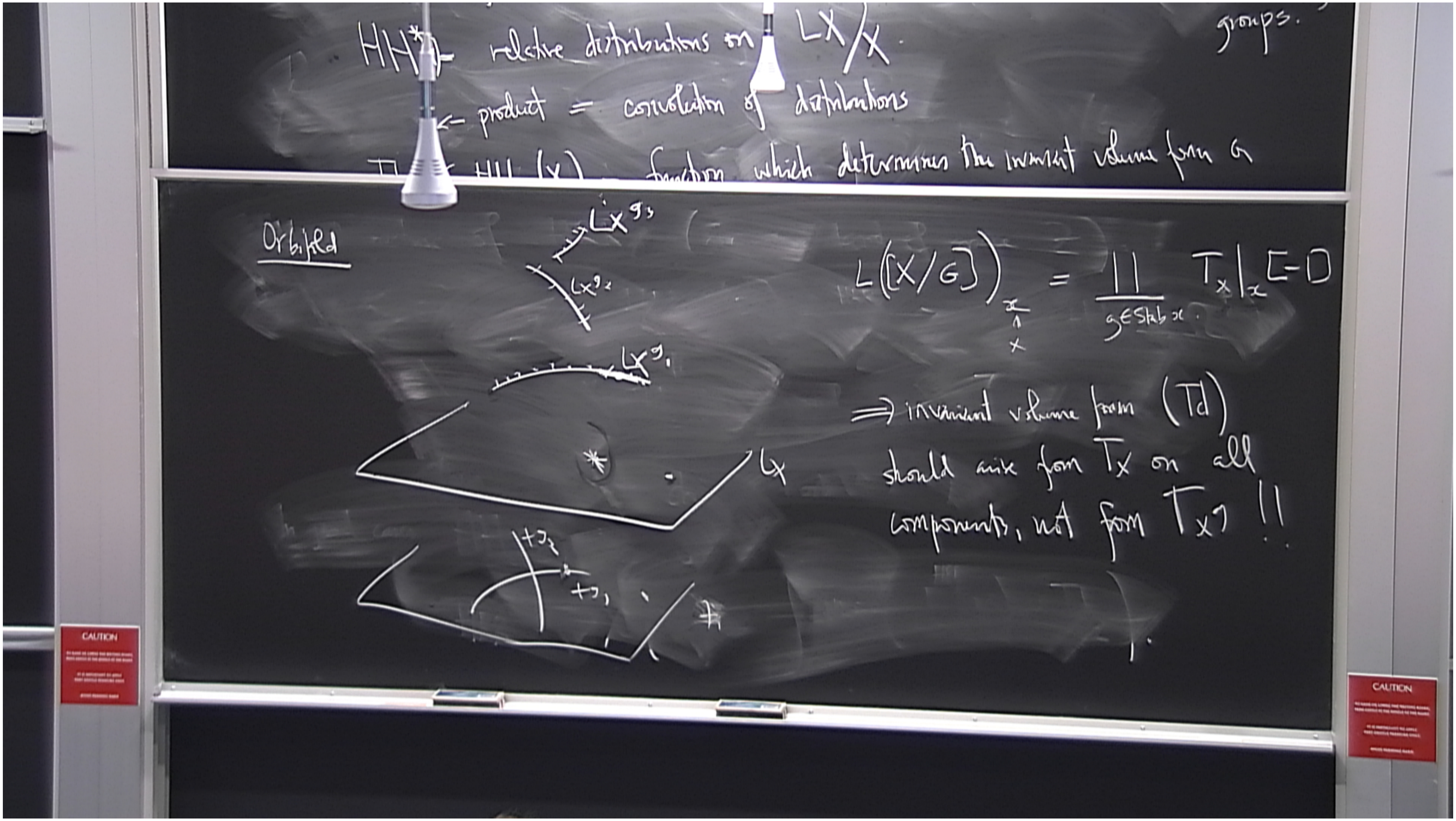
Orbifold
LX^g3
LX^g2
LX^g1



groups.
HH^* relative distributions on LX/X.
← product = convolution of distributions
$HH_*(X)$ function which determines the invariant volume form G
Orbifold
$L([X/G])_x = \coprod_{g \in \mathrm{Stab}\, x} T_x|_x[-1]$
Lx^{g_3}
Lx^{g_2}
Lx^{g_1}
Lx
$+g_2$
$+g_1$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L([X/G]) = ∐ T_x|_x [-1]
g ∈ Stab x
⟹ invariant volume form (Td)
should arise from T_X on all
components, not from T_{X^γ} !!