Title: Shifted structures and quantization

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Abstract: I will discuss the comparison of shifted Poisson and symplectic geometry and applications to the shifted quantization of moduli spaces.

# Shifted deformation quantization

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# Weak and strong quantization (i)

Fundamental statement:

**Theorem:** [Kontsevich,Yekutiely] Let $(X, \pi)$ be a smooth Poisson scheme/$\mathbb{C}$. Then

- There exists a formal deformation quantization of $(X, \pi)$ as a stack of algebroids. That is, there exists a stack of algebroids $\mathcal{X}/\mathbb{C}[[\hbar]]$ with $\mathcal{X}/\hbar = X$ and infinitesimal $\pi$.
- The formal deformation quantizations of $(X, \pi)$ are classified by Poisson deformations of $(X, \pi)$ over $\mathbb{C}[[\hbar]]$.

# Weak and strong quantization (ii)

Shifted Poisson structures arise when we study deformations of $X$ in which we allow only partial non-commutativity in the deformed structure.
To understand this it is useful to view Kontsevich's deformation quantization as a two step process:

**(weak quantization):** Deform the symmetric monoidal dg category $(L_{\mathrm{qcoh}}(X), \otimes)$ of sheaves on $X$ to a $\mathbb{C}[[\hbar]]$-linear dg category $\mathcal{L}$.

**(strong quantization):** Deform the structure sheaf $\mathcal{O}_X$ to a sheaf $\mathcal{A}_X$ of associative $\mathbb{C}[[\hbar]]$ algebras.

**Remark:** If the strong quantization exists, then we can take $\mathcal{L} = \mathcal{A}_X - \mathrm{mod}$.

# Shifted weak and strong quantization (i)

Fix $n \in \mathbb{Z}$.

**Conventions:**

- An $n$-shifted Poisson bracket on a cdga $A$ is a graded Lie bracket on $A$ of degree $(-n)$ which is a graded derivation of the product structure.
- $\mathbb{P}_n$ will denote the operad controlling $n-1$-shifted (unbounded) Poisson cdga.
- $\mathbb{E}_n$ will denote the topological operad of little $n$-dimensionl disks.

**Remark:** With these conventions $\mathbb{P}_n$ is the homology of $\mathbb{E}_n$ for $n \geqslant 2$, and in particular the homology of an $\mathbb{E}_n$ algebra is naturally a $(n-1)$-shifted Poisson cdga.

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# Shifted weak and strong quantization (ii)

**Recall:** Typically a family of $\mathbb{E}_n$ algebras (for $n \geqslant 2$) over the formal disk will specialize to a $\mathbb{P}_n$-algebra at the closed point.

**Caution:** One needs conditions on an $\mathbb{E}_n$-algebra over $\mathbb{C}((\hbar))$ to ensure a $\mathbb{P}_n$-algebra specialization at $\hbar = 0$. For instance we may require that the $\hbar = 0$ specialization is a $\mathbf{cdga}^{\leqslant 0}$. More generally: the deformation space is graded and the deformations that have $(n-1)$-shifted Poisson limits are the ones of degree 2.

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# Affine version of the quantization problem

(weak)$_n$ Show that every $\mathbb{P}_{n+1}$ algebra $A/\mathbb{C}$ lifts to a $BD_{n+1}$-algebra.

(strong)$_n$ Show that for every $\mathbb{P}_{n+1}$ algebra $A/\mathbb{C}$ the category $A-$mod deforms as a $\mathbb{E}_n$-monoidal category.

**Note:** For $n \geqslant 1$ this follows immediately from the formality of $\mathbb{E}_{n+1}$: choosing a formality isomorphism $\mathbb{E}_{n+1} \cong \mathbb{P}_{n+1}$ trivializes $BD_n$ over $\mathbb{A}^1$ so we can promote $\mathbb{P}_{n+1}$-algebras to $\mathbb{E}_{n+1}$.

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# General shifted quantization

**Problem:** Let $(X, \omega)$ be an $n$-shifted symplectic derived stack. Construct a canonical, formal 1-parameter quantization of $(X, \omega)$, where this means:

- $n > 0$: A deformation of $\mathcal{O}_X$ over $\mathbb{C}[[\hbar]]$ as a sheaf of $\mathbb{E}_{n+1}$ algebras.
- $n = 0$: A deformation of $L_{\mathrm{perf}}(X)$ as a dg category over $\mathbb{C}[[\hbar]]$.
- $n < 0$: (red shift trick) A deformation of $\mathcal{O}_X$ over $\mathbb{C}[[\hbar_{2n}]]$ as a sheaf of $\mathbb{E}_{1-n}$-algebras, where $|\hbar_{2n}| = 2n$.

**Note:** The notion of a formal deformation of a dg category$/\mathbb{C}$ is still under development. A good proxy for such a deformation is a $\mathbb{C}[u]$-linear structure with $|u| = 2$ ($\mathbb{C}[u]$ is the $\mathbb{E}_2$ Koszul dual of $\mathbb{C}[[\hbar]]$).

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# Red shift trick (i)

$\mathrm{compl}_{\mathbb{C}}^{\mathrm{gr}}$ - the category $\mathbb{Z}$-graded complexes of $\mathbb{C}$-vector spaces.

$\otimes$ - the usual symmetric monoidal structure on $\mathrm{compl}_{\mathbb{C}}^{\mathrm{gr}}$ (with **no** grading signs in the symmetry of in the external grading and with the usual grading signs in the symmetry for the homological grading).

Consider $\Phi : \mathrm{compl}_{\mathbb{C}}^{\mathrm{gr}} \to \mathrm{compl}_{\mathbb{C}}^{\mathrm{gr}}$ given by

$$(\Phi(E))(n) = E(n)[2n].$$

**Note:** This is a monoidal auto equivalence. In particular if $\mathbb{O}$ is any operad in $\mathrm{compl}_{\mathbb{C}}^{\mathrm{gr}}$, then $\Phi(\mathbb{O})$ is also a well defined operad in $\mathrm{compl}_{\mathbb{C}}^{\mathrm{gr}}$.

# Red shift trick (ii)

One checks that $\Phi(\mathbb{P}_n) = \mathbb{P}_{n+2}$.

**Consequences:**

- Formality of $\mathbb{P}_n$ implies formality of $\mathbb{P}_{n\pm 2}$.
- $\Phi$ gives an equivalence between the category of graded $\mathbb{P}_n$ algebras and the category of graded $\mathbb{P}_{n+2}$ algebras.

## Shifted polyvectors

$X$ - derived stack/$\mathbb{C}$, locally of finite presentation.
$n \in \mathbb{Z}$
Consider the global **$n+1$-shifted polyvector fields on $X$**:

$$\mathrm{Pol}(X, n+1) := \mathbb{R}\Gamma\left(X, \mathrm{Sym}\, \mathbb{T}_X[-1-n]\right)$$

When equipped with the Schouten-Nijenhuis bracket this is graded Poisson dg algebra which after a shift by $n+1$ becomes a graded dgLie agebra. Thus

$$\mathrm{Pol}(X, n+1)[n+1] \in \mathrm{dgLie}^{\mathrm{gr}}_{\mathbb{C}}.$$

# Shifted Poisson structures

**Definition:**

**(a)** An ***n*-shifted Poisson structure on $X$** is a morphism of graded dg Lie algebras $\pi : \mathbb{C}[-1](2) \to \mathrm{Pol}(X, n+1)[n+1]$.

**(b)** $\pi$ is **non-degenerate** if the associated element in cohomology $\pi_0$ induces a quasi-isomorphism $\pi_0^\flat : \mathbb{L}_X \widetilde{\to} \mathbb{T}_X[-n]$.

$\pi : \mathbb{C}(2) \to \mathrm{Pol}(X, n+1)[n+2]$ gives rise to an element $\pi_0 \in \mathbb{H}^{-n}(X, \Phi_n^{(2)}(\mathbb{T}_X))$, where

$$\Phi_n^{(2)}(\mathbb{T}_X) := \begin{cases} \mathrm{Sym}^2_{\mathcal{O}_X} \mathbb{T}_X, & \text{if } n \text{ is odd} \\ \wedge^2_{\mathcal{O}_X} \mathbb{T}_X, & \text{if } n \text{ is even.} \end{cases}$$

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# Equivalence theorems (i)

To quantize all the interesting shifted symplectic structures on moduli spaces we need two comparison results. The first allows us to pass from symplectic to Poisson structures:

> **Theorem:** [CPTVV] Let $X$ be a derived Artin stack locally of finite presentation. Then there exists a natural map of spaces
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> $$\sigma : \mathsf{Poiss}(X, n)^{\mathsf{nd}} \to \mathsf{Sympl}(X, n)$$
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> which is a weak homotopy equivalence.

**Remark:** A version of this theorem was recently proven by J. Pridham by a different method.

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# Equivalence theorems (ii)

**Theorem:** [Melani,CPTVV] Let $X$ be a derived Artin stack. Then there exists a natural map of spaces **Details**

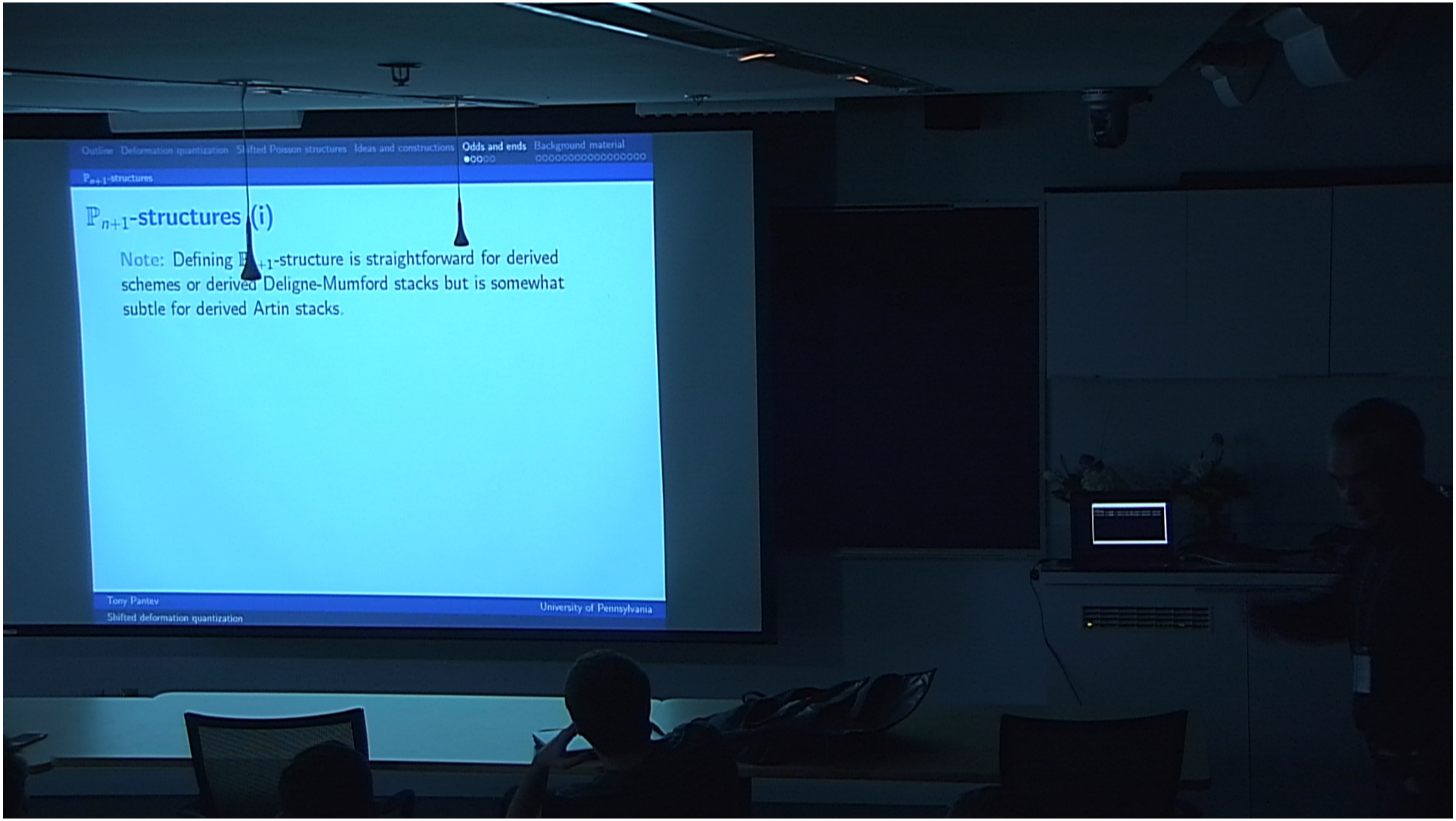
$$\mu : \mathrm{Poiss}(X, n) \to \mathbb{P}_{n+1}(X).$$

If $X$ is locally of finite presentation and 1-affine, then $\mu$ is a weak homotopy equivalence.

**Gaitsgory:** $X$ is 1-affine if the global section functor

$$\Gamma(X, -) : \mathcal{Q}coh_X\text{-}\mathbf{mod} \to \mathrm{Qcoh}(X)\text{-}\mathbf{mod}$$

is an equivalence.


Outline Deformation quantization Shifted Poisson structures Ideas and constructions Odds and ends Background material
$\mathbb{P}_{n+1}$-structures
$\mathbb{P}_{n+1}$-structures (i)
Note: Defining $\mathbb{P}_{n+1}$-structure is straightforward for derived schemes or derived Deligne-Mumford stacks but is somewhat subtle for derived Artin stacks.
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Shifted deformation quantization

# $\mathbb{P}_{n+1}$-structures (i)

Given $A \in \mathrm{cdga}_{\mathbb{C}}^{\leqslant 0}$, the space of $\mathbb{P}_{n+1}$-structures on $A$ is the mapping space

$$\mathbb{P}_{n+1}(A) = \mathrm{Map}_{\mathrm{dgOp}_{\mathbb{C}}}\left(\mathbb{P}_{n+1}, \mathrm{End}_{\mathbb{E}_1}(A)\right).$$

The comparison between $n$-shifted Poisson structures and $\mathbb{P}_{n+1}$-structures in the affine case is provided by Melani's theorem:

**Theorem:** [Melani] For any $A \in \mathrm{cdga}_{\mathbb{C}}^{\leqslant 0}$, there is a natural map of spaces

$$\mu : \mathrm{Poiss}(A, n) \to \mathbb{P}_{n+1}(A),$$

which is a weak equivalence if $\mathbb{L}_A$ is perfect.

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# Equivalence theorems (ii)

**Theorem:** [Melani,CPTVV] Let $X$ be a derived Artin stack. Then there exists a natural map of spaces **Details**

$$\mu : \mathrm{Poiss}(X, n) \to \mathbb{P}_{n+1}(X).$$

If $X$ is locally of finite presentation and 1-affine, then $\mu$ is a weak homotopy equivalence.

**Remark** • The two comparison results together convert any $n$-shifted symplectic structure $\omega$ into a $\mathbb{P}_{n+1}$ structure $\mu \circ \sigma^{-1}(\omega)$ on $X$.

• This $\mathbb{P}_{n+1}$-structure combined with the formality of $\mathbb{E}_{n+1}$ then give rise to a shifted quantization of $X$.

## From Poisson to symplectic

**Goal:** Explain the geometry leading to the equivalence

$$\mathsf{Poiss}(X,n)^{\mathsf{nd}} \cong \mathsf{Sympl}(X,n).$$

**Note:**

- To simplify the exposition will assume that $X$ is a derived scheme which is locally of finite presentation.
- Such a derived scheme $X$ can be represented by a pair $(t_0X, \mathcal{O}_X)$

# $\mathbb{P}_{n+1}$-structures

**Definition:** A $\mathbb{P}_{n+1}$-structure on a derived scheme $X$ is a pair $(\mathcal{O}'_X, \alpha)$, where

- $\mathcal{O}'_X$ is a sheaf of strict $\mathbb{P}_{n+1}$-algebras on $t_0X$;
- $\alpha : \mathcal{O}'_X \to \mathcal{O}_X$ is a quasi-isomorphism of sheaves of $\mathbf{cdga}^{\leqslant 0}$.

**Goal:** Define a map of spaces

$$\begin{array}{ccc} \mathbb{P}_{n+1}(X)^{\mathrm{nd}} & \longrightarrow & \mathrm{Sympl}(X, n) \\ & & \cap \\ & & A^{2,cl}(X, n) \end{array}$$

# Construction at the level of points (i)

Let $(\mathcal{O}'_X, \alpha) \in \mathbb{P}_{n+1}(X)$, and let $\mathbb{L}'_X$ and $\mathbb{T}'_X$ be the tangent and cotangent complexes of $\mathcal{O}'_X$. Comments

We have a sheaf of graded $\text{dgLie}_{\mathbb{C}}$ algebras

$$\text{Pol}'(X, n)[n+1] = \left(\left(\text{Sym}\, \mathbb{T}'_X[-n-1]\right)[n+1], [\bullet, \bullet], d\right)$$

on the Zariski site of $X$. By Melani's theorem the strict $\mathbb{P}_{n+1}$-structure on $X$ is encoded in a $\text{dgLie}^{\text{gr}}_{\mathbb{C}}$-map

$$\pi : \mathbb{C}[-1](2) \to \text{Pol}'(X, n)[n+1].$$

**Note:** $\pi$ is an actual map of $\text{dgLie}^{\text{gr}}_{\mathbb{C}}$, not just a map in the homotopy category.

# Technical subtlety

Note: $\mathcal{O}_X$ is locally f.p. but $\mathcal{O}'_X$ will not be locally f.p. as a cdga. In fact, since $\mathcal{O}'_X$ is chosen to be cofibrant as a $\mathbb{P}_{n+1}$-algebra, it will not be locally f.p. as a cdga but will only be **weakly** locally f.p..

This only guarantees that $\mathbb{T}'_X$ and $\mathbb{L}'_X$ are weakly perfect, i.e. are complexes of $\mathcal{O}_X$-modules of possibly infinite rank (the cohomology sheaves are of course of finite rank). In particular we can not view a shifted Poisson structure as a map to $(\mathrm{Sym}\,\mathbb{T}'_X[-1-n])[n+1]$ but rather as a map to shifted polylinear map $\mathbb{L}'_X \times \cdots \times \mathbb{L}'_X \to \mathcal{O}'_X$.

Note: We deal with this carefully in the paper. To simplify the exposition, I will pretend here that this issue does not arise.

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## Construction at the level of points (iv)

By the Dold-Kan correspondence we have

$$A^{2,cl}_{\mathcal{O}'}(X,n) = \left|\left(\mathrm{Sym}^{\geqslant 2}_{\mathcal{O}'}\mathbb{L}'_X[-1]\right)[n+2]\right|,$$

which in turn can be identified with $A^{2,cl}_{\mathcal{O}}(X,n)$ via $\alpha$.

**Conclusion:** $\alpha((\dagger)^{-1}\circ\pi[1])$ is a closed non-degenerate $n$-shifted 2-form on $X$. This gives a map

$$\sigma : \mathbb{P}_{n+1}(X)^{\mathrm{nd}} \to \mathrm{Sympl}(X,n)$$

at the level of points.

**Next:**
- Extend $\sigma$ to a map of spaces (=ssets).
- Show that $\sigma$ is functorial for étale maps in $X'$
- Prove that $\sigma$ is an equivalence.

# Map of spaces (i)

**Digression:** Given a simplicial set $M$, can talk of locally constant sheaves (of anything) on $M$:

- Represent $M$ as a nerve of a 1-category $C$;
- Suppose $A$ is a category with weak equivalences (e.g. a model category). Define a **locally constant sheaf on $M$ with values objects in $A$** as a functor $F : C \to A$ such that $F(\mathrm{Mor}(C)) \subset \mathrm{Weakeq}(A)$.

Note: • We can use either $C$ or $C^{\mathrm{op}}$ since $\mathrm{Nerve}(C) \cong \mathrm{Nerve}(C^{\mathrm{op}})$.

• For any simplicial set $M$ we can talk about locally constant sheaves on $M$ of $\mathbf{cdga}^{\leqslant 0}$, $\mathbf{cdga}_{\mathrm{gr}}^{\leqslant 0}$, $\varepsilon - \mathbf{cdga}_{\mathrm{gr}}^{\leqslant 0}$, $\mathbb{P}_{n+1}$-algebras, etc.

# Map of spaces (ii)

**Claim:** [CPTVV] There is a universal sheaf $\mathcal{A} \to \mathbb{P}_{n+1}(X)$ of $n$-shifted Poisson $\mathbf{cdga}_{\mathbb{C}}^{\leqslant 0}$.

**Explanation:**

- $\mathbb{P}_{n+1}(X) = \text{Nerve}\,(\text{category of pairs } (\mathcal{O}'_X, \alpha))$.
- The locally constant sheaf $\mathcal{A}$ is given by the functor

$$\mathcal{A} : \quad \begin{pmatrix} \text{category of pairs} \\ (\mathcal{O}'_X, \alpha) \end{pmatrix} \longrightarrow \begin{pmatrix} \text{category of sheaves of } n\text{-shifted Poisson} \\ \mathbf{cdga}^{\leqslant 0} \text{ on } X \end{pmatrix}, \quad (\mathcal{O}'_X, \alpha) \mapsto \mathcal{O}'_X.$$

## Map of spaces (iv)

The pointwise construction applied to a diagram of non-degenerate $n$-shifted Poisson $\mathbf{cdga}^{\leqslant 0}$ with étale maps yields a diagram of $n$-shifted symplectic forms. This gives the desired map of spaces

$$\sigma : \mathbb{P}^{\mathrm{nd}}_{n+1}(X) \to \mathrm{Sympl}(X, n)$$

The comparison theorem now follows from the following

**Theorem:** [CPTVV] $\sigma$ induces an equivalence

$$\mathbb{P}^{\mathrm{nd}}_{n+1}(-) \to \mathrm{Sympl}(-, n)$$

of stacks on $(\mathrm{derSp})_{\text{ét}}$.

# The equivalence theorem (i)

$\mathbf{n = 0}$: a $\mathbb{P}_1$ is an ordinary Poisson structure, and $X$ is underived and smooth. In this case the map is simply the usual inversion of Poisson structures.

$\mathbf{n < 0}$: To show that the map of stacks $\mathbb{P}^{\mathrm{nd}}_{n+1}(-) \to \mathrm{Sympl}(-, n)$ is an equivalence, we must show that it is fully faithful and essentially surjective.

# The equivalence theorem (ii)

**Essential surjectivity:** Can be checked locally since these are stacks.

Use the Darboux lemma of Brav-Bussi-Joyce: up to a quasi-isomorphism a pair $(A \in \mathbf{cdga}^{\leqslant 0}, \omega \in \mathrm{Sympl}(A, n))$ is equivalent to a pair $(\widetilde{A}, \widetilde{\omega})$, where $\widetilde{\omega}$ is strictly closed and strictly non-degenerate. In particular $\widetilde{\omega}^{-1}$ is a strict $n$-shifted Poisson structure on $A$.

# The case of loops (i)

Fix $(\mathcal{O}'_X, \alpha)$ with $\pi$ - a strict $n+1$ structure on $\mathcal{O}'_X$. Consider the completed (product) total complexes of forms and polyvector fields on $\mathcal{O}'_X$. Contraction with $\pi$ gives a natural map

$$\begin{array}{c} \left(\mathrm{Sym}^{\sqcap}(\mathbb{L}'_X[-1])[n+1], d + d_{DR}\right) \\ \Big\downarrow{\scriptstyle \pi^\flat} \\ \left(\mathrm{Sym}^{\sqcap}(\mathbb{T}'_X[-1-n])[n+1], d + [\pi, \bullet]\right) \end{array}$$

which is a filtered quasi-isomorphism of complexes which respects the stupid filtrations.

# The case of loops (ii)

In particular we have a quasi-iso

$$\mathrm{Sym}^{\sqcap,\geqslant 2}(\mathbb{L}'_X[-1])[n+1] \xrightarrow{\pi^\flat} \mathrm{Sym}^{\sqcap,\geqslant 2}(\mathbb{T}'_X[-1-n])[n+1]$$

But

$$\begin{pmatrix}\text{stack of loops in}\\ \mathbb{P}_{n+1}(X)\ \ \text{based}\\ \text{at } \pi\end{pmatrix} \overset{\text{Dold-Kan}}{\longleftrightarrow} \left(\mathrm{Sym}^{\sqcap,\geqslant 2}(\mathbb{T}'_X[-1-n])[n+1], d + [\pi, \bullet]\right)$$

and

$$\begin{pmatrix}\text{stack of loops}\\ \text{in Sympl}(X, n)\\ \text{based at } \sigma(\pi)\end{pmatrix} \overset{\text{Dold-Kan}}{\longleftrightarrow} \left(\mathrm{Sym}^{\sqcap,\geqslant 2}(\mathbb{L}'_X[-1])[n+1], d + d_{DR}\right).$$

## Formal geometry (i)

**Question:** Why working formally at a point is relevant to our original question?

Let $X$ be a derived scheme¿ the natural map $X \to X_{DR}$ realizes $X$ as a family of formal derived schemes over $X_{DR}$. **Details**

The previous argument actually help us prove the following formal equivalence theorem:

**Theorem:** [CPTVV] Let $X$ be a derived DM stack locally of finite presentation. Then there exists a natural equivalence of stacks:

$$\mathsf{Poiss}(X/X_{DR}, n)^{\mathrm{nd}} \to \mathsf{Sympl}(X/X_{DR}, n).$$

# Formal geometry (ii)

**Key remark:** The moduli stacks of Poisson and symplectic structures on $X/\mathbb{C}$ are isomorphic to the moduli stacks of Poisson and symplectic structures on $X/X_{DR}$.

# Formal geometry (iii)

Need to show that for any $A \in \mathbf{cdga}^{\leqslant 0}$ and any formal stack $Z \to S = \mathbb{R}\mathbf{Spec}(A)$ such that $Z_{\mathrm{red}} = S_{\mathrm{red}}$ we have an isomorphism

$$\mathrm{Poiss}(Z/S, n)^{\mathrm{nd}} \xrightarrow{\sim} \mathrm{Sympl}(Z/S, n)$$

of stacks over $S$.

**Note:** This is easier since $Z \to S$ is given by **algebra data**.

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# Formal geometry (iv)

**Remark:**

- Expect that $Z \to \mathbb{D}_Z$ gives an isomorhism of the stack parametrizing $Z \to S$ with $Z \in \mathrm{dforSt}_{\mathbb{C}}$, $Z_{\mathrm{red}} = S_{\mathrm{red}}$ with the stack of $A$-linear mixed graded cdga.
- We prove that there is a symmetric monoidal equivalence $L_{\mathrm{perf}}(Z)$ and $\varepsilon - \mathrm{perf}(\mathbb{D})$.
- We prove that shifted forms, closed forms, and Poisson structures on $Z/S$ are the same as shifted forms, closed forms, and Poisson structures on $\mathbb{D}_Z/S$.

# Formal geometry (iv)

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- Expect that $Z \to \mathbb{D}_Z$ gives an isomorhism of the stack parametrizing $Z \to S$ with $Z \in \mathrm{dforSt}_{\mathbb{C}}$, $Z_{\mathrm{red}} = S_{\mathrm{red}}$ with the stack of $A$-linear mixed graded cdga.
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# Formal geometry (vi)

The exact triangle for the relative cotangent complexes of the maps $S \to Z \to S$ gives an identification

$$\mathbb{L}_{S/Z} \cong \mathbb{L}_{Z/S}[1]_{|S}.$$

Thus

$$\mathbb{D}_Z = \mathrm{Sym}^{\sqcap}(\mathbb{L}_{Z/S})_{|S}.$$

Since $S$ is reduced we have that $Z \to S$ is of finite type and so $\mathbb{L}_{Z/S}$ is perfect.
Dualization converts the mixed structure on $\mathbb{D}_Z$ into a dgLie structure on $\mathbb{T}_{Z/S}$, which is easily seen to be the usual bracket.

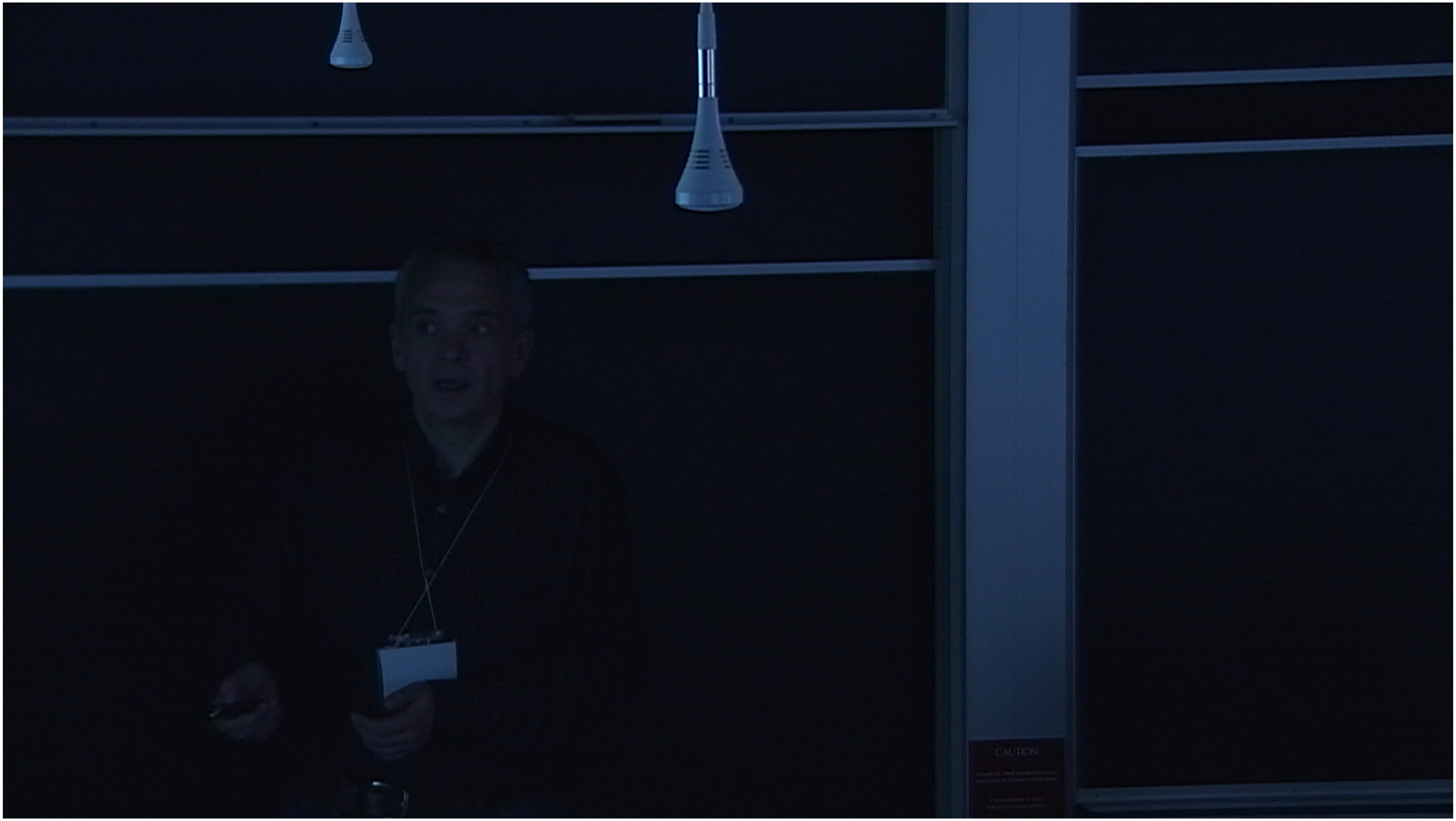
# Formal geometry (viii)

**Step 2:** If $S$ is not reduced, then $Z \to S$ is locally of almost finite presentation and so $\mathbb{L}_{Z/S}$ will not be perfect. In this case we can not dualize and use the dgLie argument.

Instead: extend the formal equivalence statement to derived or nilpotent thickenings by using deformation theory.

**Strategy:**

- Use Postnikov induction to decmpose $S_{\mathrm{red}} \to S$ into a sequence of square zero extensions;
- Analyze the map of stacks $\mathrm{Poiss}(-, n)^{\mathrm{nd}} \to \mathrm{Sympl}(-, n)$. By **Step 1** this map is an iso over the reduction, so we only need to show that the map is an iso on tangent complexes and that both sides have obstruction theories (explicit calculation).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