Title: Categorification of shifted symplectic geometry using perverse sheaves

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Abstract: Let (X,w) be a -1-shifted symplectic derived scheme or stack over C in the sense of Pantev-Toen-Vaquie-Vezzosi with an "orientation" (square root of det L_X). We explain how to construct a perverse sheaf P on the classical truncation X=t_0(X), over a base ring A. The hypercohomology H*(P) is regarded as a categorification of X.

Now suppose i : L --> X is a Lagrangian in (X,w) in the sense of PTVV, with a "relative orientation". We outline a programme (work in progress) to construct a natural morphism

\mu : A_L[vdim L] --> i^!(P)

of constructible complexes on L=t_0(L). If i is proper this is equivalent to a hypercohomology in H^{-vdim L}(P). These natural morphisms / hypercohomology classes \mu satisfy various identities under products, composition of Lagrangian correspondences, etc.

This programme will have interesting applications. In particular:

(a) Take (X,w) to be the derived moduli stack of coherent sheaves on a Calabi-Yau 3-fold Y, so that the orientation is essentially "orientation data" in the sense of Kontsevich-Soibelman 2008. Then we regard H*(P) as being the Cohomological Hall Algebra of Y (cf Kontsevich and Soibelman 2010 for quivers). Consider

i : Exact --> (X,w) x (X,-w) x (X,w)

the moduli stack of exact sequences of coherent sheaves on Y, with projections to first, second and third factors. This is a Lagrangian in -1-shifted symplectic. Suppose we have a relative orientation. Then the hypercohomology element \mu associated to Exact should give the COHA multiplication on H*(P), and identities on \mu should imply associativity of multiplication.

(b) Let (S,w) be a classical symplectic C-scheme, or complex symplectic manifold, of dimension 2n, and L --> S, M --> S be algebraic / complex Lagrangians (or derived Lagrangians in the PTVV sense), proper over S. Suppose we are given "orientations" on L,M, i.e. square roots of the canonical bundles K_L,K_M. Then the derived intersection X = L x_S M is -1-shifted symplectic and oriented, so we get a perverse sheaf P on X. We regard the shifted hypercohomology H^{*-n}(P) as being a version of the "Lagrangian Floer cohomology" HF*(L,M), and the morphisms L --> M in a "Fukaya category" of (S,w).

If L,M,N are oriented Lagrangians in (S,w), then the triple intersection L x_S M x_S N is Lagrangian in the triple product (L x_S M) x (M x_S N) x (N x_S L). The associated hypercohomology element should correspond to the product HF*(L,M) x HF*(M,N) --> HF*(L,N) which is composition of morphisms in the "Fukaya category". Using these techniques we intend to define "Fukaya categories" of algebraic symplectic / complex symplectic manifolds, with many nice properties.

Different parts of this programme are joint work with subsets of Lino Amorim, Oren Ben-Bassat, Chris Brav, Vittoria Bussi, Delphine Dupont, Pavel Safronov, and Balazs Szendroi.

Plan of talk:

1. Shifted symplectic geometry

2. A Darboux theorem for shifted symplectic schemes

3. Categorification using perverse sheaves: objects

4. Categorification using perverse sheaves: morphisms

PTVV's shifted symplectic geometry

Pantev, Toën, Vaquié and Vezzosi (arXiv:1111.3209) defined a version of symplectic geometry in the derived world.

Let $\mathbf{X}$ be a derived $\mathbb{K}$-scheme or $\mathbb{K}$-stack. The cotangent complex $\mathbb{L}_{\mathbf{X}}$ has exterior powers $\Lambda^p\mathbb{L}_{\mathbf{X}}$. The *de Rham differential* $\mathrm{d}_{dR} : \Lambda^p\mathbb{L}_{\mathbf{X}} \to \Lambda^{p+1}\mathbb{L}_{\mathbf{X}}$ is a morphism of complexes. Each $\Lambda^p\mathbb{L}_{\mathbf{X}}$ is a complex, so has an internal differential $\mathrm{d} : (\Lambda^p\mathbb{L}_{\mathbf{X}})^k \to (\Lambda^p\mathbb{L}_{\mathbf{X}})^{k+1}$. We have $\mathrm{d}^2 = \mathrm{d}_{dR}^2 = \mathrm{d}\circ\mathrm{d}_{dR} + \mathrm{d}_{dR}\circ\mathrm{d} = 0$.

A *p-form of degree* k on $\mathbf{X}$ for $k \in \mathbb{Z}$ is an element $[\omega^0]$ of $H^k(\Lambda^p\mathbb{L}_{\mathbf{X}}, \mathrm{d})$. A *closed p-form of degree* k on $\mathbf{X}$ is an element

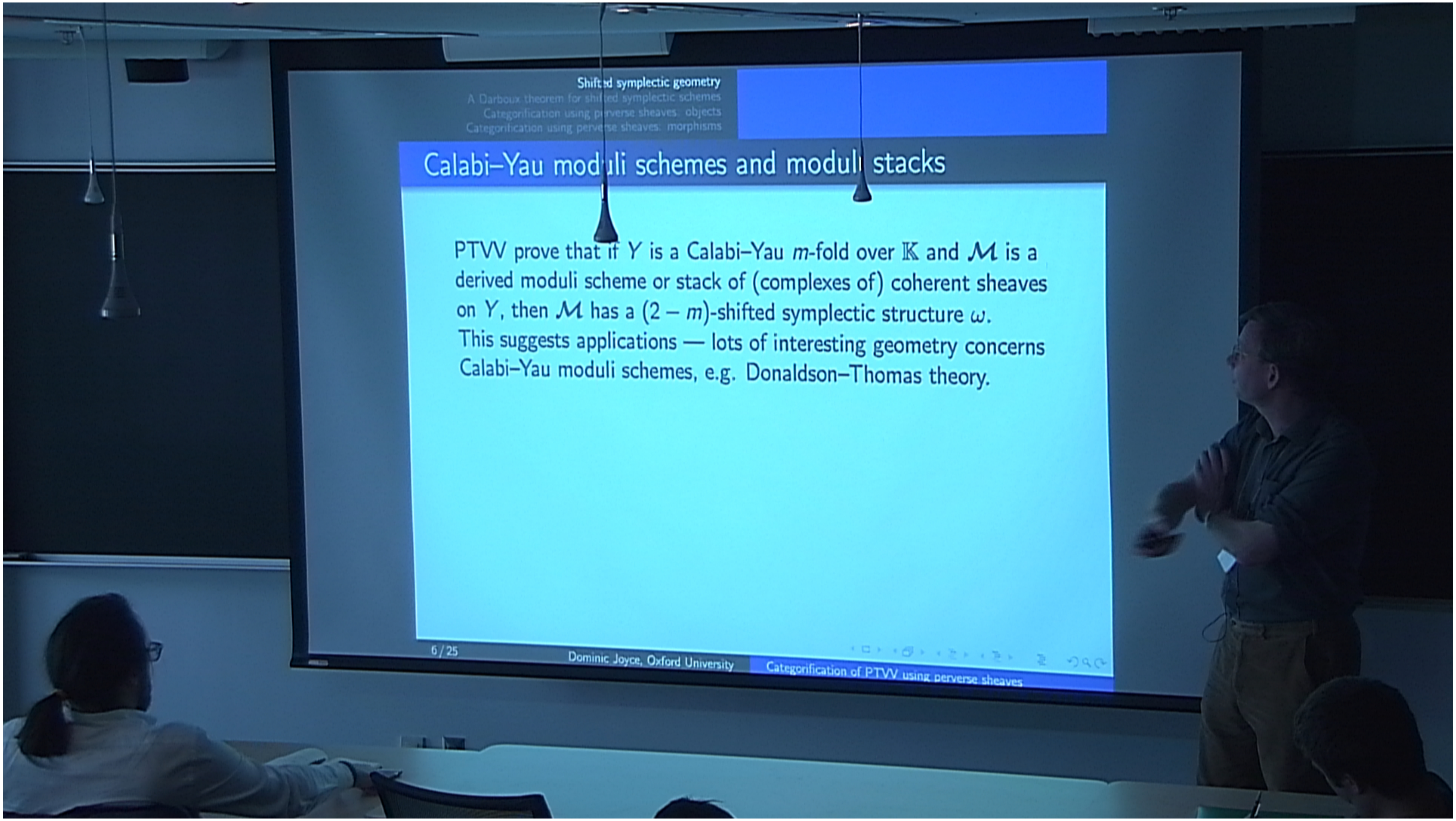
$$[(\omega^0, \omega^1, \ldots)] \in H^k\big(\textstyle\bigoplus_{i=0}^{\infty} \Lambda^{p+i}\mathbb{L}_{\mathbf{X}}[i], \mathrm{d} + \mathrm{d}_{dR}\big).$$

There is a projection $\pi : [(\omega^0, \omega^1, \ldots)] \mapsto [\omega^0]$ from closed p-forms $[(\omega^0, \omega^1, \ldots)]$ of degree k to p-forms $[\omega^0]$ of degree k.

Nondegenerate 2-forms and symplectic structures

Let $[\omega^0]$ be a 2-form of degree k on $\mathbf{X}$. Then $[\omega^0]$ induces a morphism $\omega^0 : \mathbb{T}_{\mathbf{X}} \to \mathbb{L}_{\mathbf{X}}[k]$, where $\mathbb{T}_{\mathbf{X}} = \mathbb{L}_{\mathbf{X}}^\vee$ is the tangent complex of $\mathbf{X}$. We call $[\omega^0]$ *nondegenerate* if $\omega^0 : \mathbb{T}_{\mathbf{X}} \to \mathbb{L}_{\mathbf{X}}[k]$ is a quasi-isomorphism.

If $\mathbf{X}$ is a derived scheme then the complex $\mathbb{L}_{\mathbf{X}}$ lives in degrees $(-\infty, 0]$ and $\mathbb{T}_{\mathbf{X}}$ in degrees $[0, \infty)$. So $\omega^0 : \mathbb{T}_{\mathbf{X}} \to \mathbb{L}_{\mathbf{X}}[k]$ can be a quasi-isomorphism only if $k \leqslant 0$, and then $\mathbb{L}_{\mathbf{X}}$ lives in degrees $[k, 0]$ and $\mathbb{T}_{\mathbf{X}}$ in degrees $[0, -k]$. If $k = 0$ then $\mathbf{X}$ is a smooth classical $\mathbb{K}$-scheme, and if $k = -1$ then $\mathbf{X}$ is quasi-smooth.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Calabi–Yau moduli schemes and moduli stacks
PTVV prove that if Y is a Calabi–Yau m-fold over $\mathbb{K}$ and $\mathcal{M}$ is a derived moduli scheme or stack of (complexes of) coherent sheaves on Y, then $\mathcal{M}$ has a $(2-m)$-shifted symplectic structure ω.
This suggests applications — lots of interesting geometry concerns Calabi–Yau moduli schemes, e.g. Donaldson–Thomas theory.
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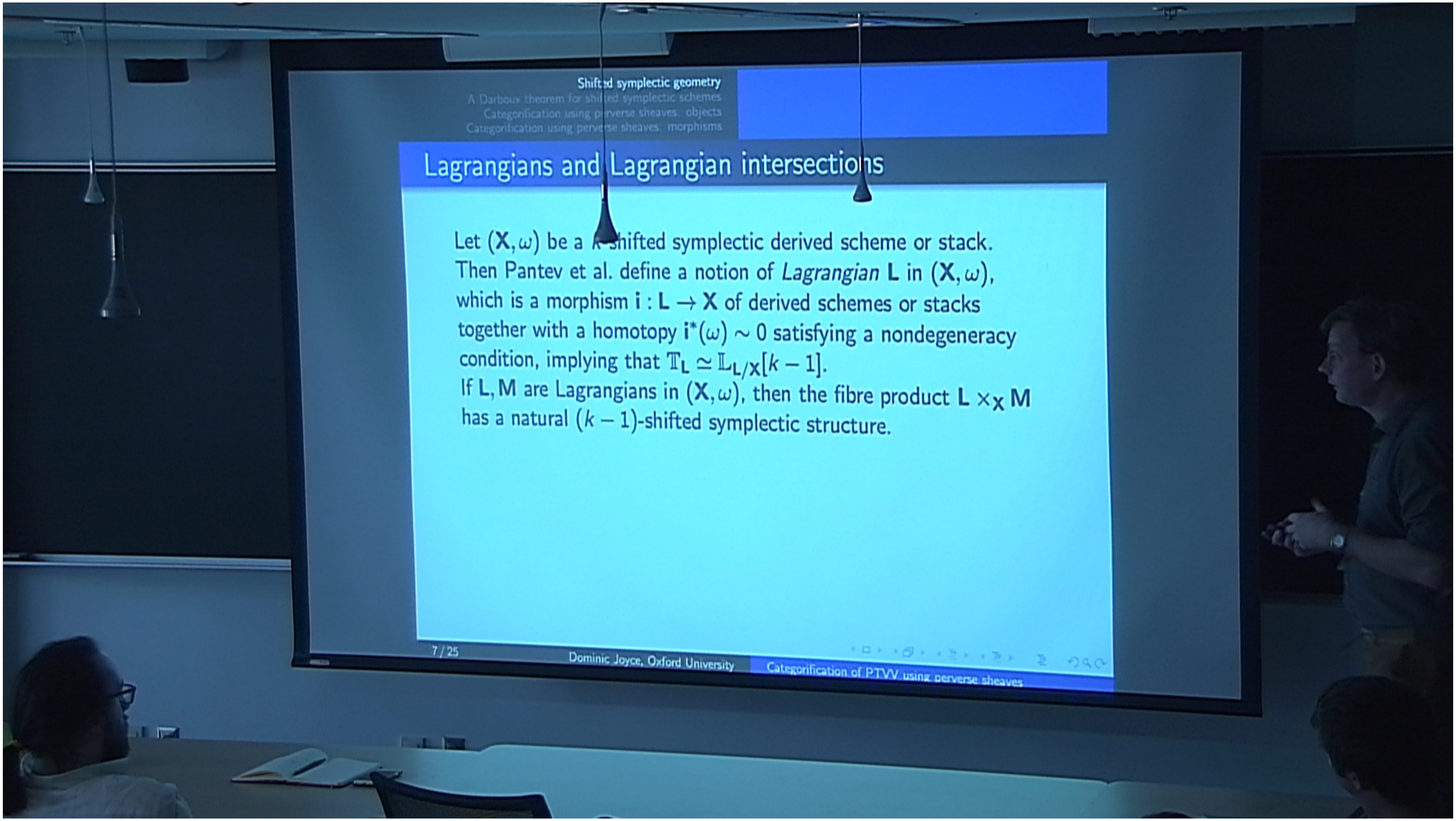
We can understand the associated nondegenerate 2-form $[\omega^0]$ in terms of *Serre duality*. At a point $[E] \in \boldsymbol{\mathcal{M}}$, we have $h^i(\mathbb{T}_{\boldsymbol{\mathcal{M}}})|_{[E]} \cong \mathrm{Ext}^{i-1}(E,E)$ and $h^i(\mathbb{L}_{\boldsymbol{\mathcal{M}}})|_{[E]} \cong \mathrm{Ext}^{1-i}(E,E)^*$. The Calabi–Yau condition gives $\mathrm{Ext}^i(E,E) \cong \mathrm{Ext}^{m-i}(E,E)^*$, which corresponds to $h^{i+1}(\mathbb{T}_{\boldsymbol{\mathcal{M}}})|_{[E]} \cong h^{i+1}(\mathbb{L}_{\boldsymbol{\mathcal{M}}}[2-m])|_{[E]}$. This is the cohomology at $[E]$ of the quasi-isomorphism $\omega^0 : \mathbb{T}_{\boldsymbol{\mathcal{M}}} \to \mathbb{L}_{\boldsymbol{\mathcal{M}}}[2-m]$.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Lagrangians and Lagrangian intersections
Let $(\mathbf{X},\omega)$ be a k-shifted symplectic derived scheme or stack. Then Pantev et al. define a notion of *Lagrangian* $\mathbf{L}$ in $(\mathbf{X},\omega)$, which is a morphism $\mathbf{i}:\mathbf{L}\rightarrow\mathbf{X}$ of derived schemes or stacks together with a homotopy $\mathbf{i}^*(\omega)\sim 0$ satisfying a nondegeneracy condition, implying that $\mathbb{T}_{\mathbf{L}}\simeq\mathbb{L}_{\mathbf{L}/\mathbf{X}}[k-1]$.
If $\mathbf{L},\mathbf{M}$ are Lagrangians in $(\mathbf{X},\omega)$, then the fibre product $\mathbf{L}\times_{\mathbf{X}}\mathbf{M}$ has a natural $(k-1)$-shifted symplectic structure.
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Lagrangians and Lagrangian intersections

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If (S,ω) is a classical smooth symplectic scheme, then it is a 0-shifted symplectic derived scheme in the sense of PTVV, and if $L,M\subset S$ are classical smooth Lagrangian subschemes, then they are Lagrangians in the sense of PTVV. Therefore the (derived) Lagrangian intersection $L\cap M=L\times_S M$ is a -1-shifted symplectic derived scheme.

2. A Darboux theorem for shifted symplectic schemes

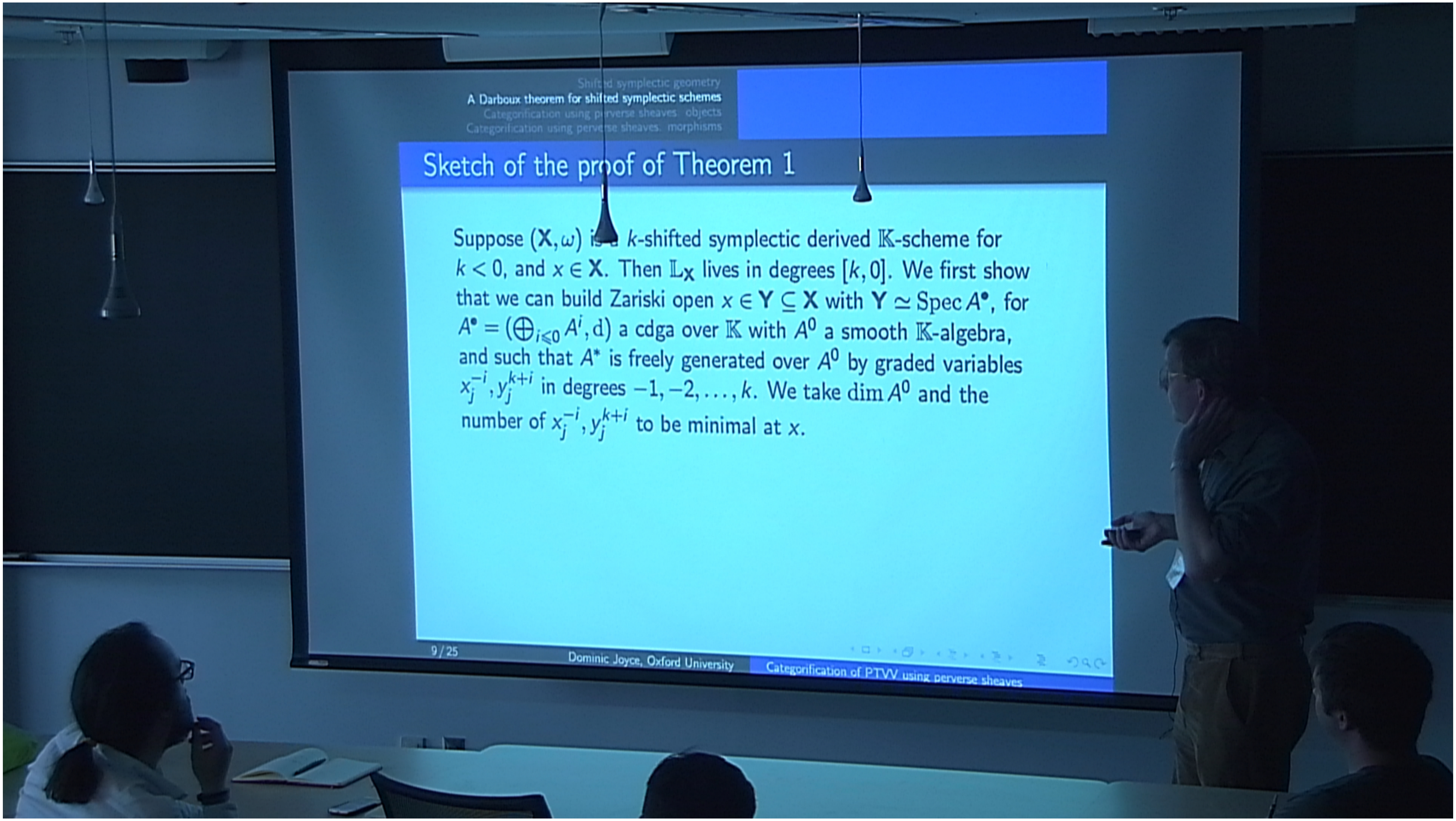
Theorem 1 (Brav, Bussi and Joyce arXiv:1305.6302)

Let $(\mathbf{X},\omega)$ be a k-shifted symplectic derived $\mathbb{K}$-scheme for $k<0$. If $k \not\equiv 2 \mod 4$, then each $x\in\mathbf{X}$ admits a Zariski open neighbourhood $\mathbf{Y}\subseteq\mathbf{X}$ with $\mathbf{Y}\simeq \operatorname{Spec} A^\bullet$ for $A^\bullet=(A^*,\mathrm{d})$ an explicit cdga generated by graded variables x_j^{-i}, y_j^{k+i} for $0\leqslant i\leqslant -k/2$, and $\omega|_{\mathbf{Y}}=[(\omega^0,0,0,\ldots)]$ where x_j^l, y_j^l have degree l, and

$$\omega^0=\sum_{i=0}^{[-k/2]}\sum_{j=1}^{m_i}\mathrm{d}_{dR}y_j^{k+i}\mathrm{d}_{dR}x_j^{-i}.$$

Also the differential d in $A^\bullet$ is given by Poisson bracket with a Hamiltonian H in A of degree $k+1$. If $k\equiv 2 \mod 4$, we have two statements, one étale local with ω^0 standard, and one Zariski local with the components of ω^0 in the degree $k/2$ variables depending on some invertible functions.

Ben-Bassat–Brav–Bussi–Joyce extend this to derived Artin $\mathbb{K}$-stacks.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Suppose $(\mathbf{X},\omega)$ is a k-shifted symplectic derived $\mathbb{K}$-scheme for $k<0$, and $x\in\mathbf{X}$. Then $\mathbb{L}_{\mathbf{X}}$ lives in degrees $[k,0]$. We first show that we can build Zariski open $x\in\mathbf{Y}\subseteq\mathbf{X}$ with $\mathbf{Y}\simeq\operatorname{Spec}A^\bullet$, for $A^\bullet=(\bigoplus_{i\leqslant 0}A^i,\mathrm{d})$ a cdga over $\mathbb{K}$ with A^0 a smooth $\mathbb{K}$-algebra, and such that A^* is freely generated over A^0 by graded variables x_j^{-i},y_j^{k+i} in degrees $-1,-2,\ldots,k$. We take $\dim A^0$ and the number of x_j^{-i},y_j^{k+i} to be minimal at x.
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Using theorems about periodic cyclic cohomology, we show that on $Y\simeq\operatorname{Spec}A^\bullet$ we can write $\omega|_Y=[(\omega^0,0,0,\ldots)]$, for ω^0 a 2-form of degree k with $\mathrm{d}\omega^0=\mathrm{d}_{dR}\omega^0=0$. Minimality at x implies ω^0 is strictly nondegenerate near x, so we can change variables to write $\omega^0=\sum_{i,j}\mathrm{d}_{dR}y_j^{k+i}\mathrm{d}_{dR}x_j^{-i}$. Finally, we show d in $A^\bullet$ is a symplectic vector field, which integrates to a Hamiltonian H.

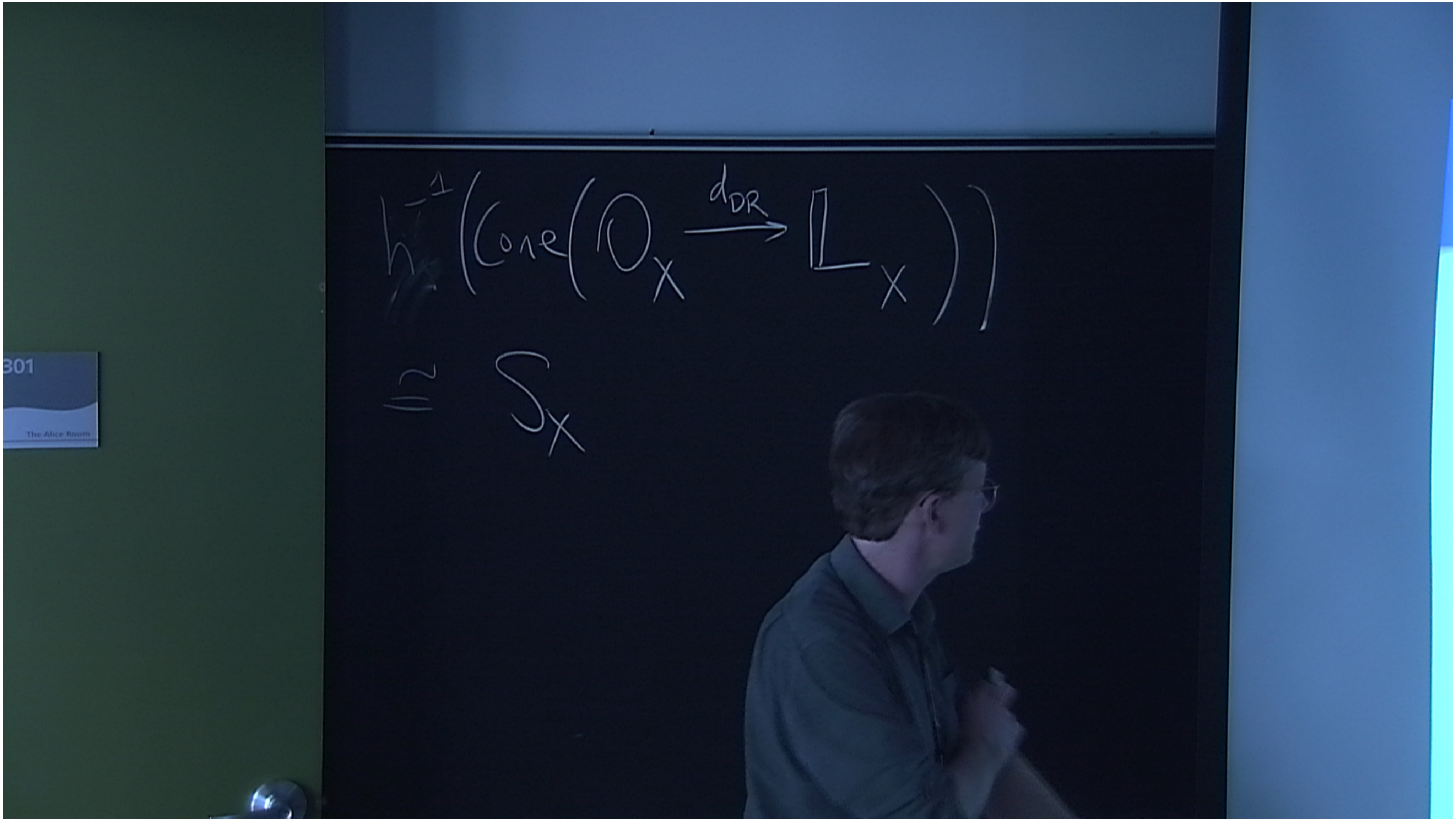
The case of -1-shifted symplectic derived schemes

When $k = -1$ the Hamiltonian H in Theorem 1 has degree 0. Then Theorem 1 reduces to:

Corollary

Suppose $(\mathbf{X}, \omega)$ *is a* -1*-shifted symplectic derived* $\mathbb{K}$*-scheme. Then* $(\mathbf{X}, \omega)$ *is Zariski locally equivalent to a derived critical locus* $\mathbf{Crit}(H : U \to \mathbb{A}^1)$*, for* U *a smooth classical* $\mathbb{K}$*-scheme and* $H : U \to \mathbb{A}^1$ *a regular function. Hence, the underlying classical* $\mathbb{K}$*-scheme* $X = t_0(\mathbf{X})$ *is Zariski locally isomorphic to a classical critical locus* $\mathrm{Crit}(H : U \to \mathbb{A}^1)$.

This implies that classical Calabi–Yau 3-fold moduli schemes are, Zariski locally, critical loci of regular functions on smooth schemes.


$h^{-1}(\mathrm{Cone}(\mathcal{O}_X \xrightarrow{d_{DR}} \mathbb{L}_X))$
$\cong S_X$
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Definition (Joyce arXiv:1304.4508)

An (*algebraic*) *d-critical locus* (X,s) is a classical $\mathbb{K}$-scheme X and a global section $s\in H^0(\mathcal{S}_X^0)$ such that X may be covered by Zariski open $R\subseteq X$ with an isomorphism $i:R\to \mathrm{Crit}(f:U\to\mathbb{A}^1)$ identifying $s|_R$ with $f+I_{R,U}^2$, for f a regular function on a smooth $\mathbb{K}$-scheme U.

3. Categorification using perverse sheaves: objects

Theorem 3 (Brav, Bussi, Dupont, Joyce, Szendrői arXiv:1211.3259)

Let $(\mathbf{X},\omega)$ be a -1-shifted symplectic derived $\mathbb{K}$-scheme. Then the 'canonical bundle' $\det(\mathbb{L}_{\mathbf{X}})$ is a line bundle over the classical scheme $X = t_0(\mathbf{X})$. Suppose we are given an **orientation** *of $(\mathbf{X},\omega)$, i.e. a square root line bundle $\det(\mathbb{L}_{\mathbf{X}})^{1/2}$. Then we can construct a canonical perverse sheaf $P^\bullet_{\mathbf{X},\omega}$ on X, such that if $(\mathbf{X},\omega)$ is Zariski locally modelled on $\mathbf{Crit}(f : U \to \mathbb{A}^1)$, then $P^\bullet_{\mathbf{X},\omega}$ is locally modelled on the perverse sheaf of vanishing cycles $\mathcal{PV}^\bullet_{U,f}$ of (U,f). Similarly, we can construct a natural $\mathscr{D}$-module $D^\bullet_{\mathbf{X},\omega}$ on X, and when $\mathbb{K}=\mathbb{C}$ a natural mixed Hodge module $M^\bullet_{\mathbf{X},\omega}$ on X.*

In fact we actually construct the perverse sheaf on the oriented d-critical locus (X,s) associated to $(\mathbf{X},\omega)$ in Theorem 2. We also define perverse sheaves on oriented complex analytic d-critical loci.

Categorifying Calabi–Yau 3-fold moduli spaces

Corollary

Let Y be a Calabi–Yau 3-fold over $\mathbb{K}$ and $\mathcal{M}$ a classical moduli $\mathbb{K}$-scheme of coherent sheaves, or complexes of coherent sheaves, on Y, with (symmetric) obstruction theory $\phi : \mathcal{E}^\bullet \to \mathbb{L}_{\mathcal{M}}$. Suppose we are given a square root $\det(\mathcal{E}^\bullet)^{1/2}$ for $\det(\mathcal{E}^\bullet)$ (i.e. **orientation data**, *K–S). Then we have a natural perverse sheaf $P^\bullet_{\mathcal{M},s}$ on $\mathcal{M}$.*

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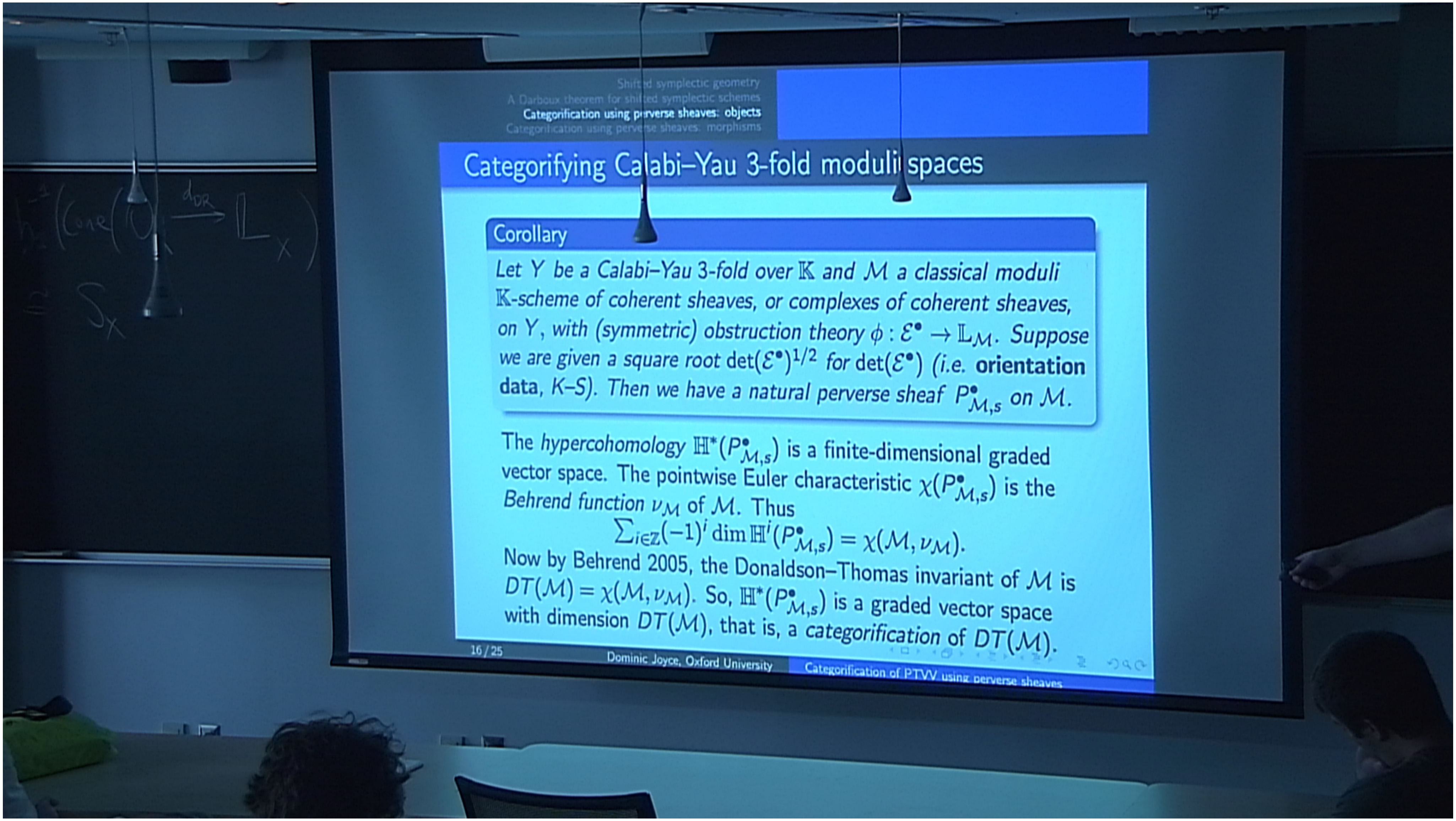
The *hypercohomology* $\mathbb{H}^*(P^\bullet_{\mathcal{M},s})$ is a finite-dimensional graded vector space. The pointwise Euler characteristic $\chi(P^\bullet_{\mathcal{M},s})$ is the *Behrend function* $\nu_{\mathcal{M}}$ of $\mathcal{M}$. Thus

$$\sum_{i\in\mathbb{Z}}(-1)^i \dim \mathbb{H}^i(P^\bullet_{\mathcal{M},s}) = \chi(\mathcal{M}, \nu_{\mathcal{M}}).$$

Now by Behrend 2005, the Donaldson–Thomas invariant of $\mathcal{M}$ is $DT(\mathcal{M}) = \chi(\mathcal{M}, \nu_{\mathcal{M}})$. So, $\mathbb{H}^*(P^\bullet_{\mathcal{M},s})$ is a graded vector space with dimension $DT(\mathcal{M})$, that is, a *categorification* of $DT(\mathcal{M})$.



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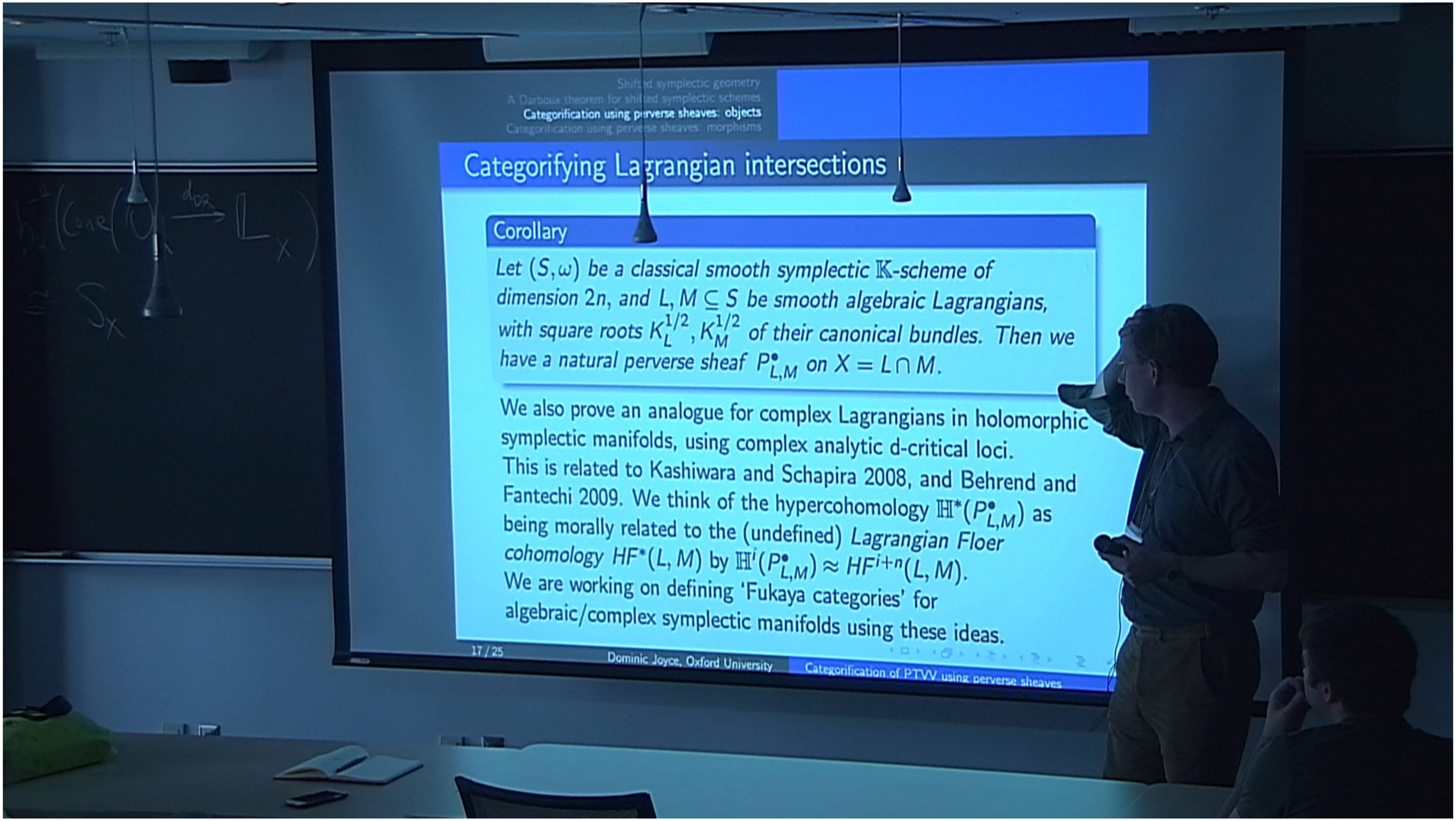
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We also prove an analogue for complex Lagrangians in holomorphic symplectic manifolds, using complex analytic d-critical loci.
This is related to Kashiwara and Schapira 2008, and Behrend and Fantechi 2009. We think of the hypercohomology $\mathbb{H}^*(P^\bullet_{L,M})$ as being morally related to the (undefined) Lagrangian Floer cohomology $HF^*(L,M)$ by $\mathbb{H}^i(P^\bullet_{L,M}) \approx HF^{i+n}(L,M)$.
We are working on defining 'Fukaya categories' for algebraic/complex symplectic manifolds using these ideas.
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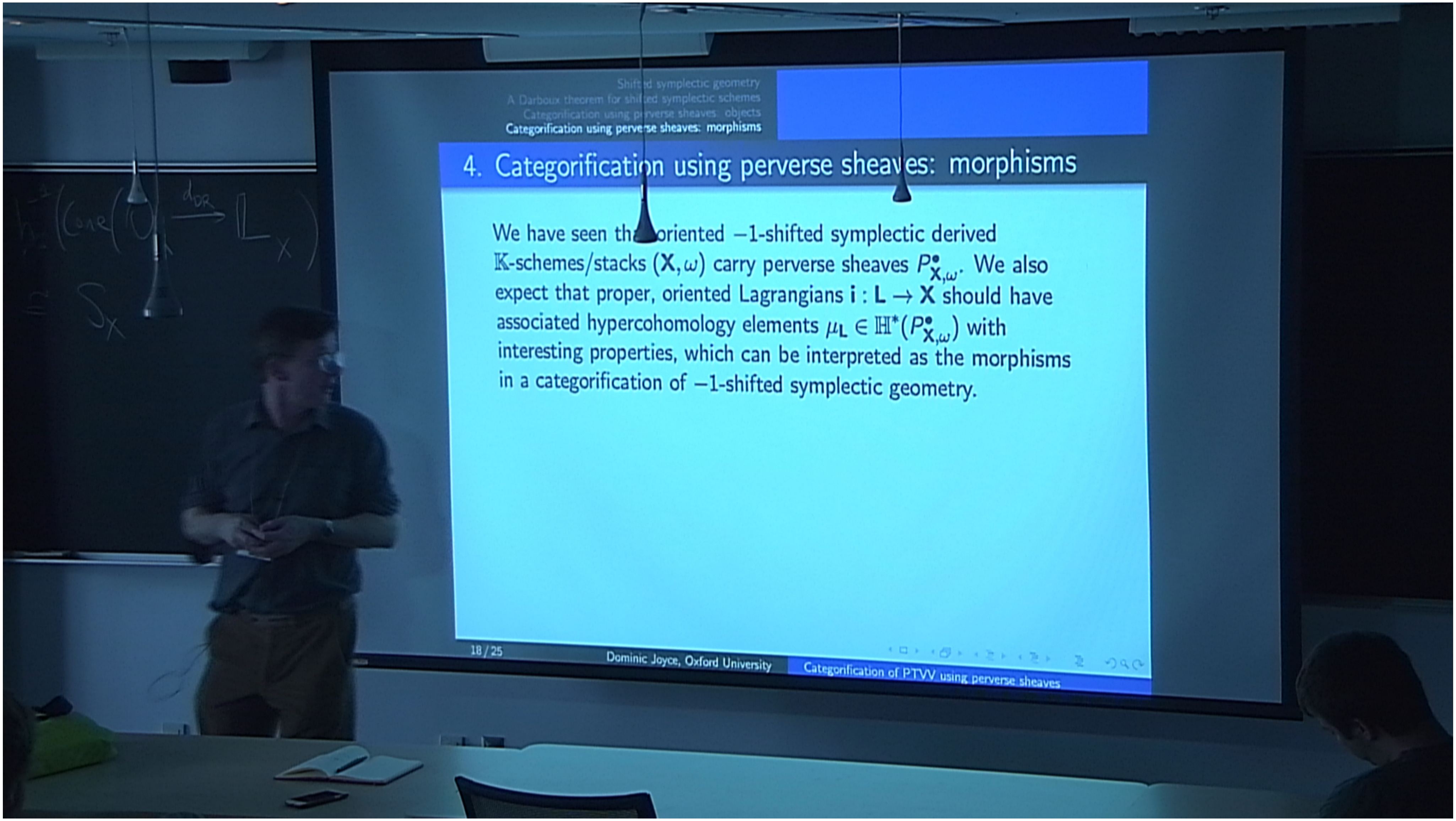
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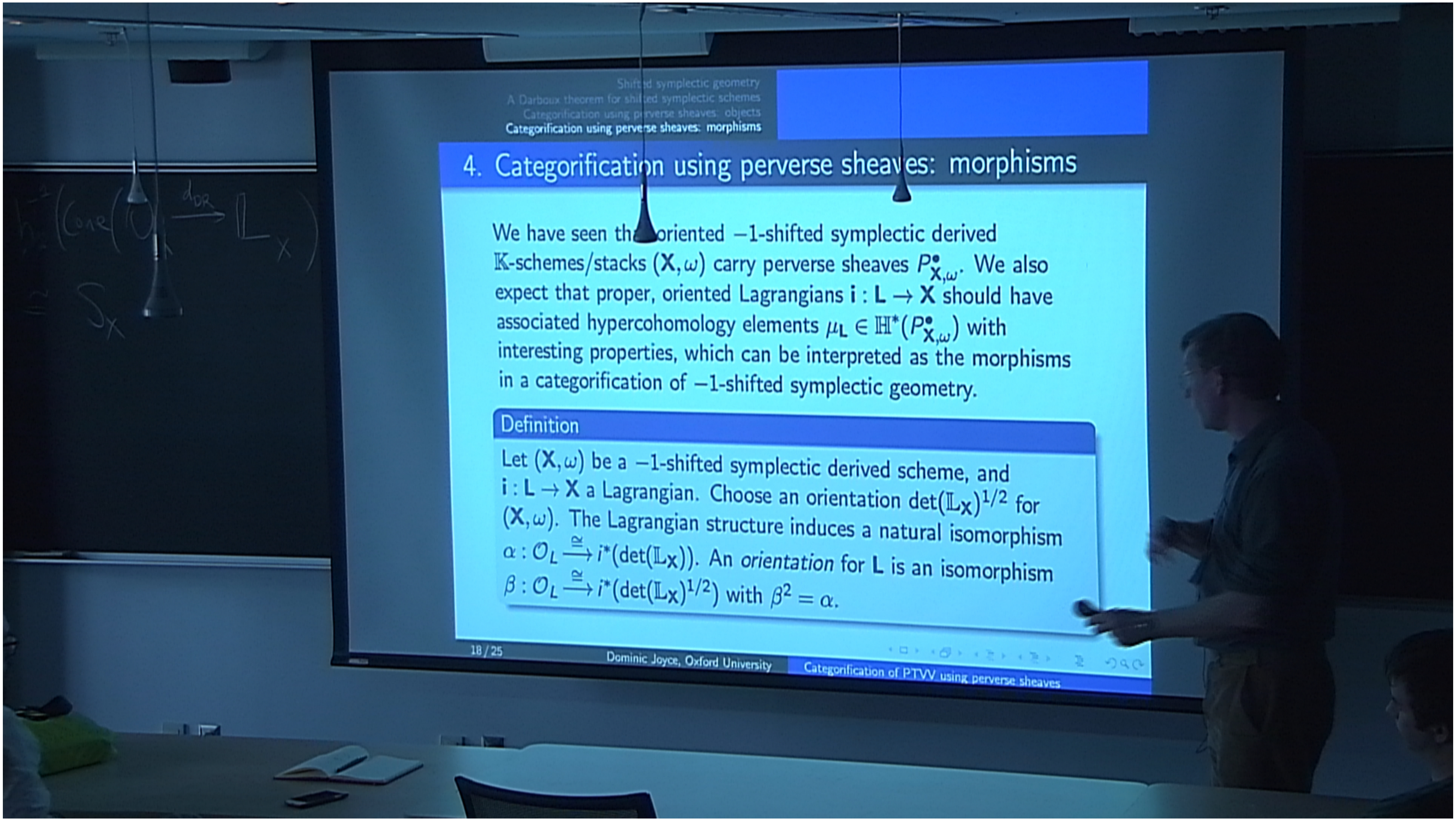
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Definition
Let $(\mathbf{X},\omega)$ be a -1-shifted symplectic derived scheme, and $\mathbf{i}:\mathbf{L}\to\mathbf{X}$ a Lagrangian. Choose an orientation $\det(\mathbb{L}_{\mathbf{X}})^{1/2}$ for $(\mathbf{X},\omega)$. The Lagrangian structure induces a natural isomorphism $\alpha:\mathcal{O}_L\xrightarrow{\cong}i^*(\det(\mathbb{L}_{\mathbf{X}}))$. An *orientation* for $\mathbf{L}$ is an isomorphism $\beta:\mathcal{O}_L\xrightarrow{\cong}i^*(\det(\mathbb{L}_{\mathbf{X}})^{1/2})$ with $\beta^2=\alpha$.
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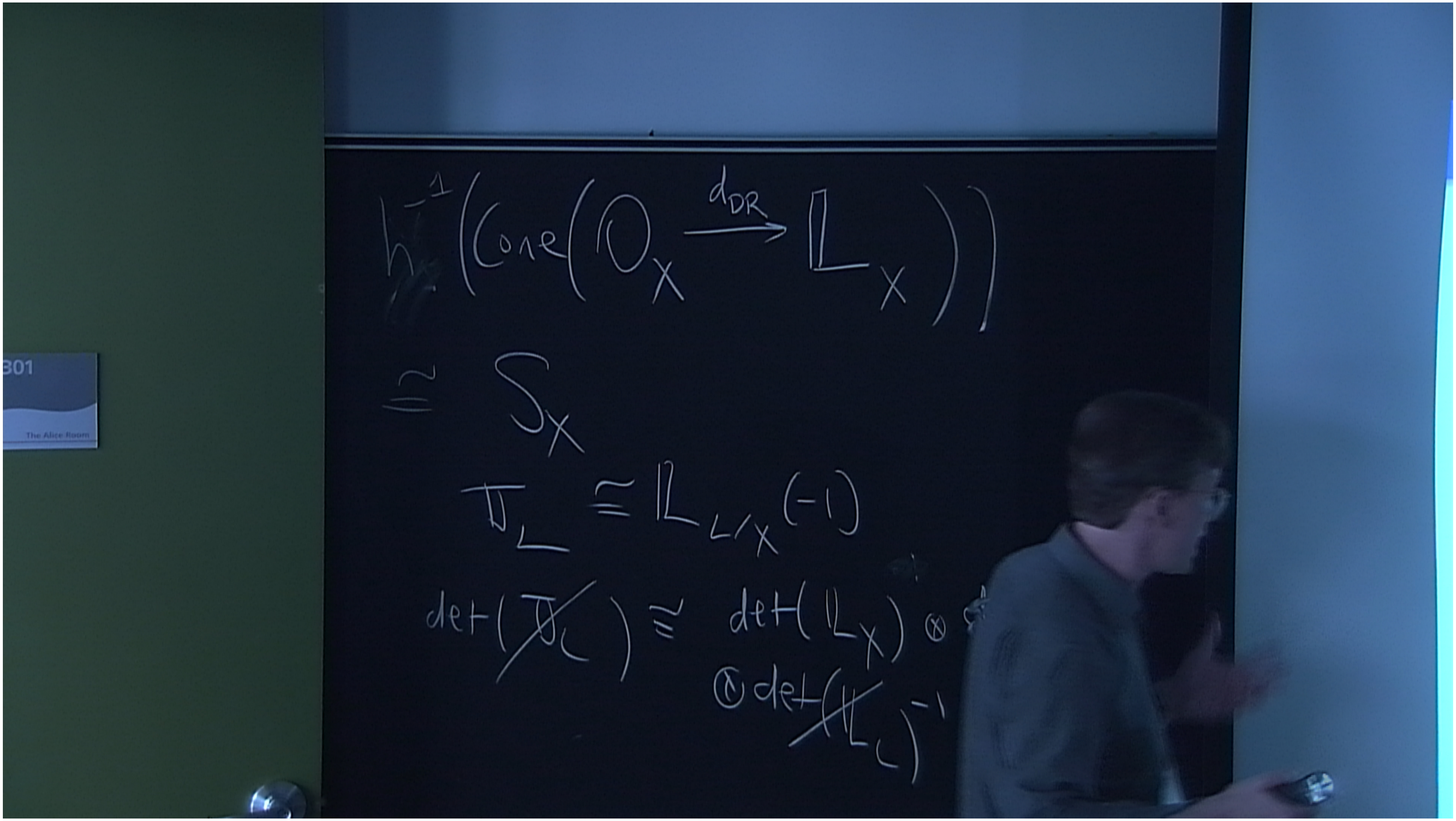
h^{-1}(Cone(O_X —d_DR→ L_X))
≅ S_X
T_L ≅ L_{L/X}(-1)
det(T_L) ≅ det(L_X) ⊗
⊗ det(L_L)^{-1}
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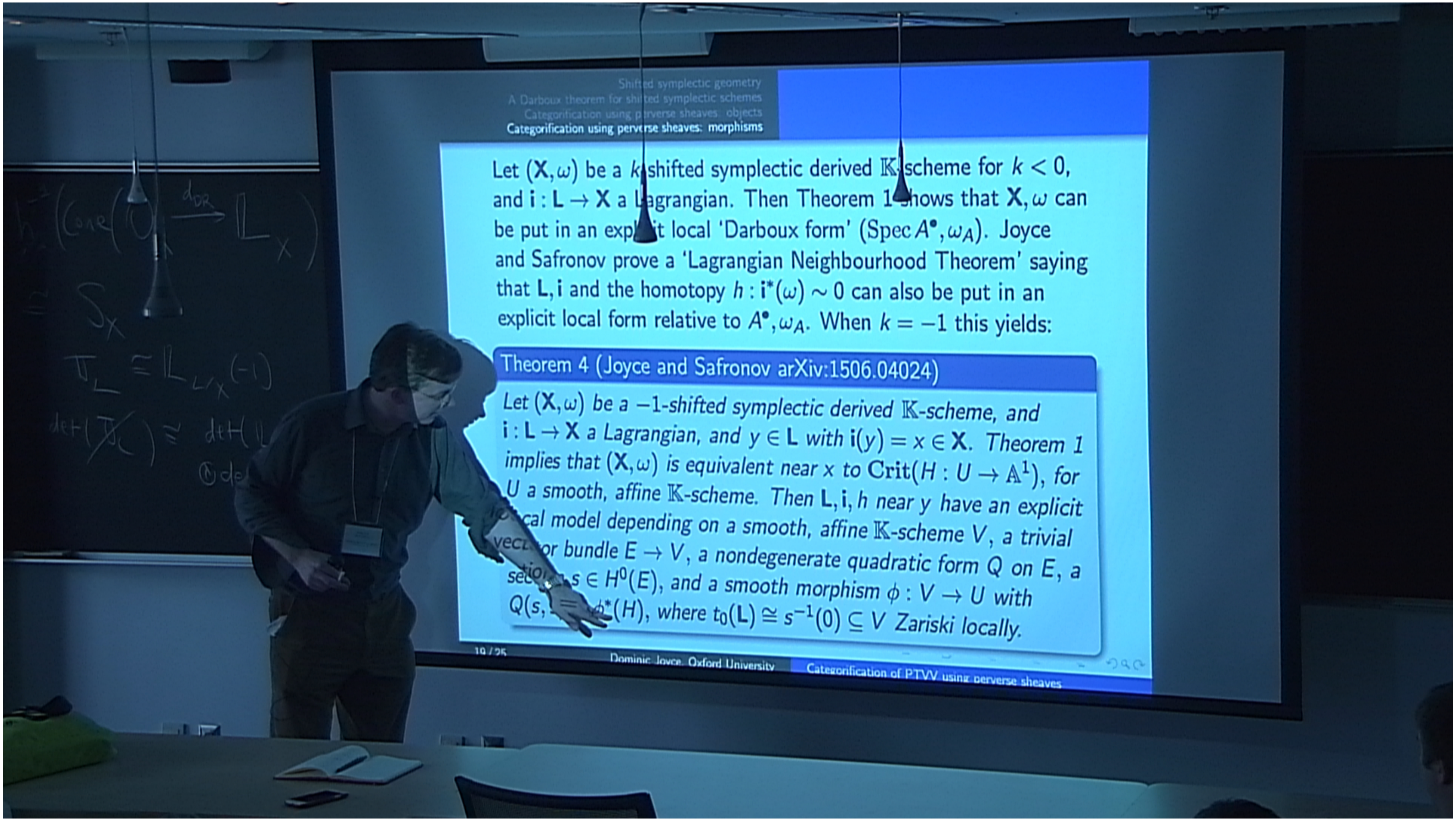
Let $(\mathbf{X},\omega)$ be a k-shifted symplectic derived $\mathbb{K}$-scheme for $k<0$, and $\mathbf{i}:\mathbf{L}\to\mathbf{X}$ a Lagrangian. Then Theorem 1 shows that $\mathbf{X},\omega$ can be put in an explicit local 'Darboux form' $(\mathrm{Spec}\, A^\bullet,\omega_A)$. Joyce and Safronov prove a 'Lagrangian Neighbourhood Theorem' saying that $\mathbf{L},\mathbf{i}$ and the homotopy $h:\mathbf{i}^*(\omega)\sim 0$ can also be put in an explicit local form relative to $A^\bullet,\omega_A$. When $k=-1$ this yields:

Theorem 4 (Joyce and Safronov arXiv:1506.04024)

Let $(\mathbf{X},\omega)$ be a -1-shifted symplectic derived $\mathbb{K}$-scheme, and $\mathbf{i}:\mathbf{L}\to\mathbf{X}$ a Lagrangian, and $y\in\mathbf{L}$ with $\mathbf{i}(y)=x\in\mathbf{X}$. Theorem 1 implies that $(\mathbf{X},\omega)$ is equivalent near x to $\mathbf{Crit}(H:U\to\mathbb{A}^1)$, for U a smooth, affine $\mathbb{K}$-scheme. Then $\mathbf{L},\mathbf{i},h$ near y have an explicit local model depending on a smooth, affine $\mathbb{K}$-scheme V, a trivial vector bundle $E\to V$, a nondegenerate quadratic form Q on E, a section $s\in H^0(E)$, and a smooth morphism $\phi:V\to U$ with $Q(s,s)=\phi^*(H)$, where $t_0(\mathbf{L})\cong s^{-1}(0)\subseteq V$ Zariski locally.


Shifted symplectic geometry
A Darboux theorem for shifted symplectic schemes
Categorification using perverse sheaves: objects
Categorification using perverse sheaves: morphisms
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Theorem 4 (Joyce and Safronov arXiv:1506.04024)
Let (X, ω) be a −1-shifted symplectic derived K-scheme, and i : L → X a Lagrangian, and y ∈ L with i(y) = x ∈ X. Theorem 1 implies that (X, ω) is equivalent near x to Crit(H : U → A¹), for U a smooth, affine K-scheme. Then L, i, h near y have an explicit local model depending on a smooth, affine K-scheme V, a trivial vector bundle E → V, a nondegenerate quadratic form Q on E, a section s ∈ H⁰(E), and a smooth morphism φ : V → U with Q(s, s) = φ*(H), where t₀(L) ≅ s⁻¹(0) ⊆ V Zariski locally.
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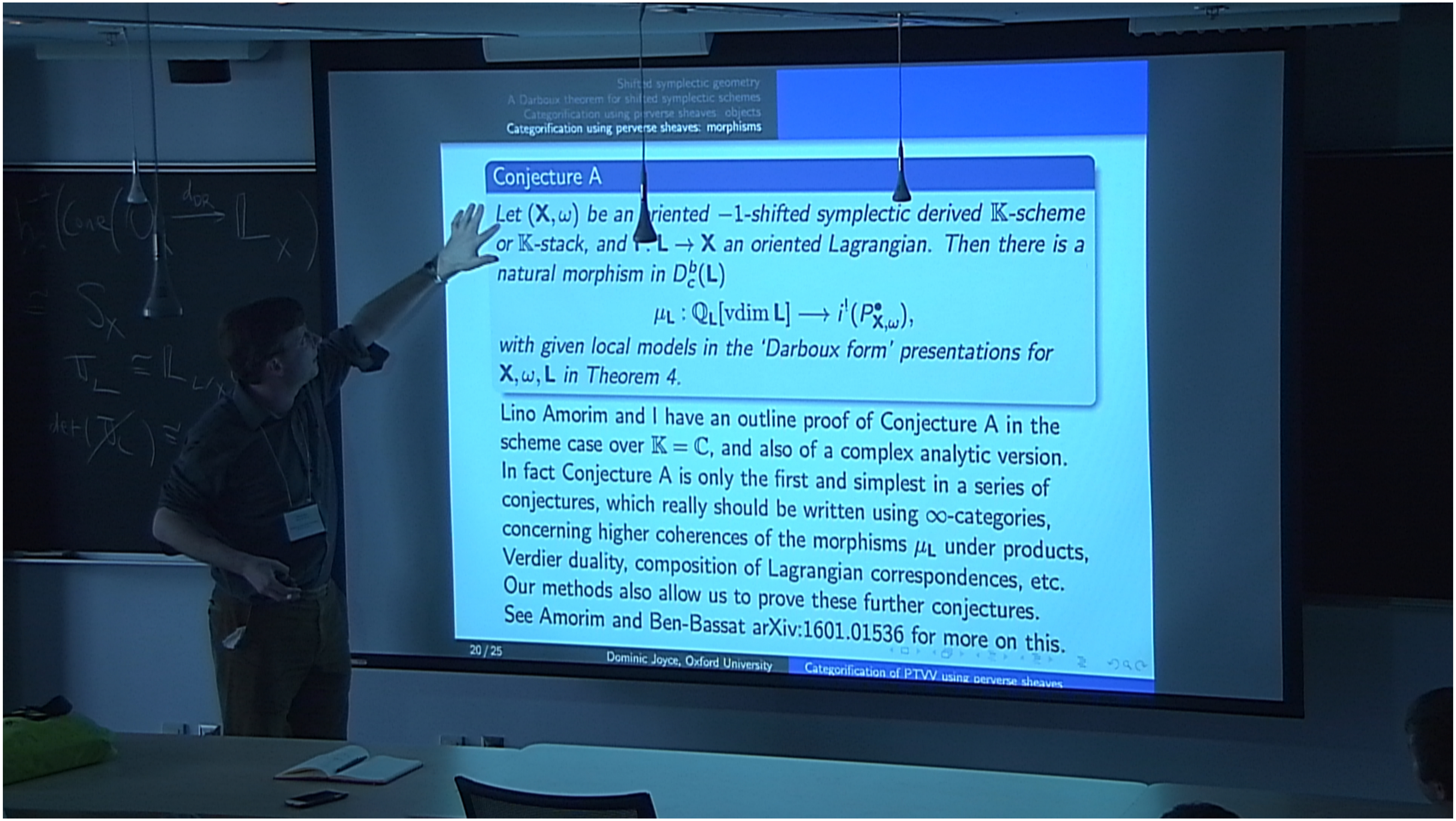

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Conjecture A
Let $(\mathbf{X},\omega)$ be an oriented -1-shifted symplectic derived $\mathbb{K}$-scheme or $\mathbb{K}$-stack, and $i:\mathbf{L}\to\mathbf{X}$ an oriented Lagrangian. Then there is a natural morphism in $D^b_c(\mathbf{L})$
$\mu_{\mathbf{L}}:\mathbb{Q}_{\mathbf{L}}[\mathrm{vdim}\,\mathbf{L}]\longrightarrow i^!(P^\bullet_{\mathbf{X},\omega}),$
with given local models in the 'Darboux form' presentations for $\mathbf{X},\omega,\mathbf{L}$ in Theorem 4.
Lino Amorim and I have an outline proof of Conjecture A in the scheme case over $\mathbb{K}=\mathbb{C}$, and also of a complex analytic version.
In fact Conjecture A is only the first and simplest in a series of conjectures, which really should be written using ∞-categories, concerning higher coherences of the morphisms $\mu_{\mathbf{L}}$ under products, Verdier duality, composition of Lagrangian correspondences, etc.
Our methods also allow us to prove these further conjectures.
See Amorim and Ben-Bassat arXiv:1601.01536 for more on this.
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Conjecture A

Let $(\mathbf{X},\omega)$ *be an oriented* -1*-shifted symplectic derived* $\mathbb{K}$*-scheme or* $\mathbb{K}$*-stack, and* $\mathbf{i}:\mathbf{L}\to\mathbf{X}$ *an oriented Lagrangian. Then there is a natural morphism in* $D^b_c(\mathbf{L})$

$$\mu_{\mathbf{L}}:\mathbb{Q}_{\mathbf{L}}[\mathrm{vdim}\,\mathbf{L}]\longrightarrow i^!(P^\bullet_{\mathbf{X},\omega}),$$

with given local models in the 'Darboux form' presentations for $\mathbf{X},\omega,\mathbf{L}$ *in Theorem 4.*

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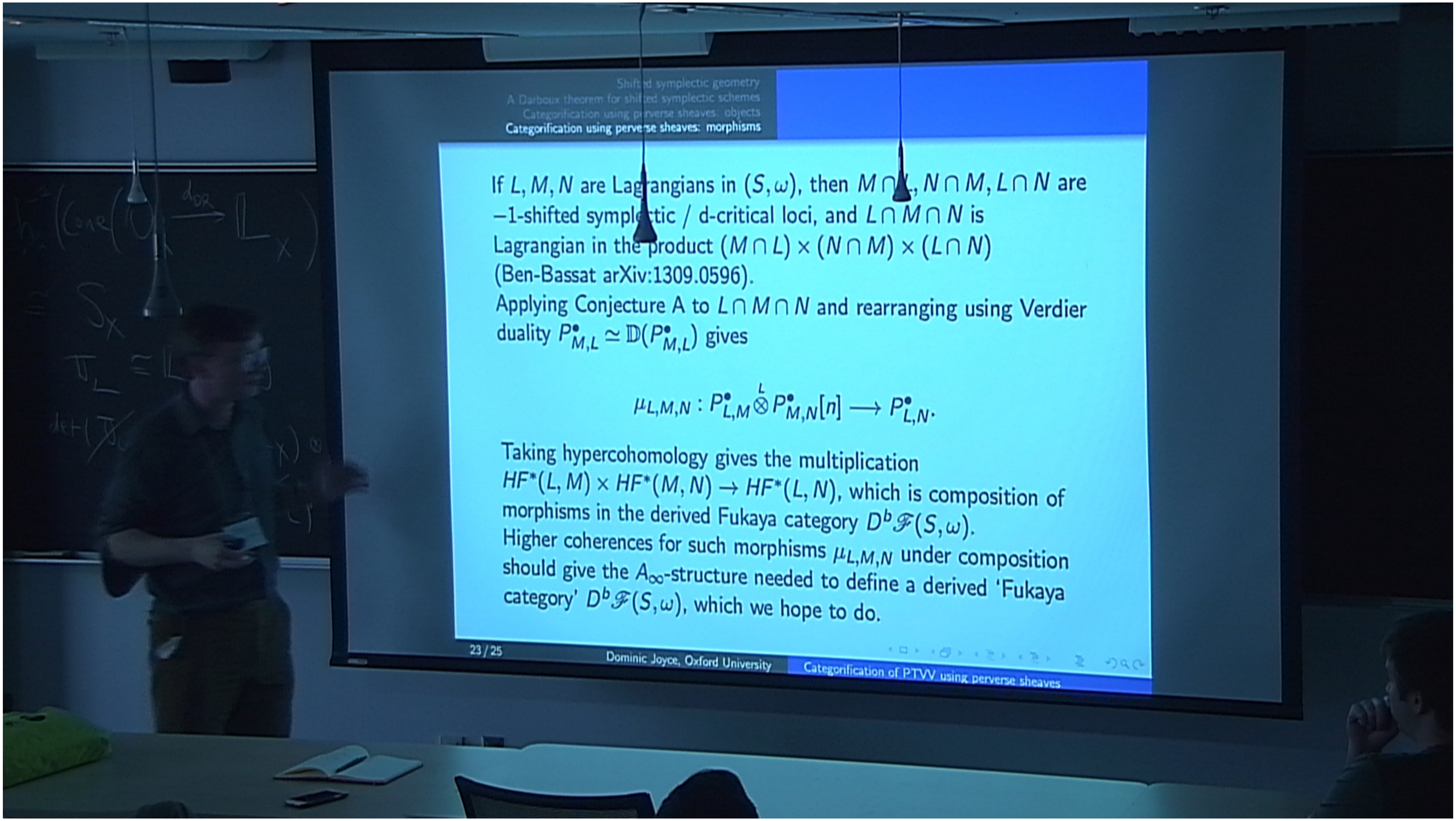
Consequences of Conjecture A: perverse COHAs for CY3's

Let Y be a Calabi–Yau 3-fold, and $\boldsymbol{\mathcal{M}}$ the moduli stack of coherent sheaves on Y, so $\boldsymbol{\mathcal{M}}$ is -1-shifted symplectic.
Let $\boldsymbol{\mathcal{E}}\mathbf{xact}$ be the derived stack of short exact sequences $0 \to F_1 \to F_2 \to F_3 \to 0$ in $\mathrm{coh}(Y)$, with projections $\boldsymbol{\pi}_1, \boldsymbol{\pi}_2, \boldsymbol{\pi}_3 : \boldsymbol{\mathcal{E}}\mathbf{xact} \to \boldsymbol{\mathcal{M}}$. Ben-Bassat (work in progress) shows $\boldsymbol{\pi}_1 \times \boldsymbol{\pi}_2 \times \boldsymbol{\pi}_3 : \boldsymbol{\mathcal{E}}\mathbf{xact} \to (\boldsymbol{\mathcal{M}}, \omega) \times (\boldsymbol{\mathcal{M}}, -\omega) \times (\boldsymbol{\mathcal{M}}, \omega)$ is Lagrangian.
Suppose we have 'orientation data' for Y, i.e. an orientation for $(\boldsymbol{\mathcal{M}}, \omega)$, with a compatibility condition on exact sequences, which is equivalent to an orientation on $\boldsymbol{\mathcal{E}}\mathbf{xact}$.
Then as in Theorem 3 we have a perverse sheaf $P^\bullet_{\boldsymbol{\mathcal{M}},s}$, with hypercohomology $\mathbb{H}^*(P^\bullet_{\boldsymbol{\mathcal{M}},s})$. Applying Conjecture A to $\boldsymbol{\mathcal{E}}\mathbf{xact}$ and using Verdier duality should (?) give an associative multiplication on $\mathbb{H}^*(P^\bullet_{\boldsymbol{\mathcal{M}},s})$, making it into a *Cohomological Hall Algebra*, as in Kontsevich–Soibelman arXiv:1006.2706, COHAs for CY3 quivers.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If L, M, N are Lagrangians in (S, ω), then $M \cap L, N \cap M, L \cap N$ are -1-shifted symplectic / d-critical loci, and $L \cap M \cap N$ is Lagrangian in the product $(M \cap L) \times (N \cap M) \times (L \cap N)$ (Ben-Bassat arXiv:1309.0596).
Applying Conjecture A to $L \cap M \cap N$ and rearranging using Verdier duality $P^\bullet_{M,L} \simeq \mathbb{D}(P^\bullet_{M,L})$ gives
$\mu_{L,M,N} : P^\bullet_{L,M} \overset{L}{\otimes} P^\bullet_{M,N}[n] \longrightarrow P^\bullet_{L,N}.$
Taking hypercohomology gives the multiplication $HF^*(L, M) \times HF^*(M, N) \to HF^*(L, N)$, which is composition of morphisms in the derived Fukaya category $D^b\mathscr{F}(S, \omega)$.
Higher coherences for such morphisms $\mu_{L,M,N}$ under composition should give the A_∞-structure needed to define a derived 'Fukaya category' $D^b\mathscr{F}(S, \omega)$, which we hope to do.
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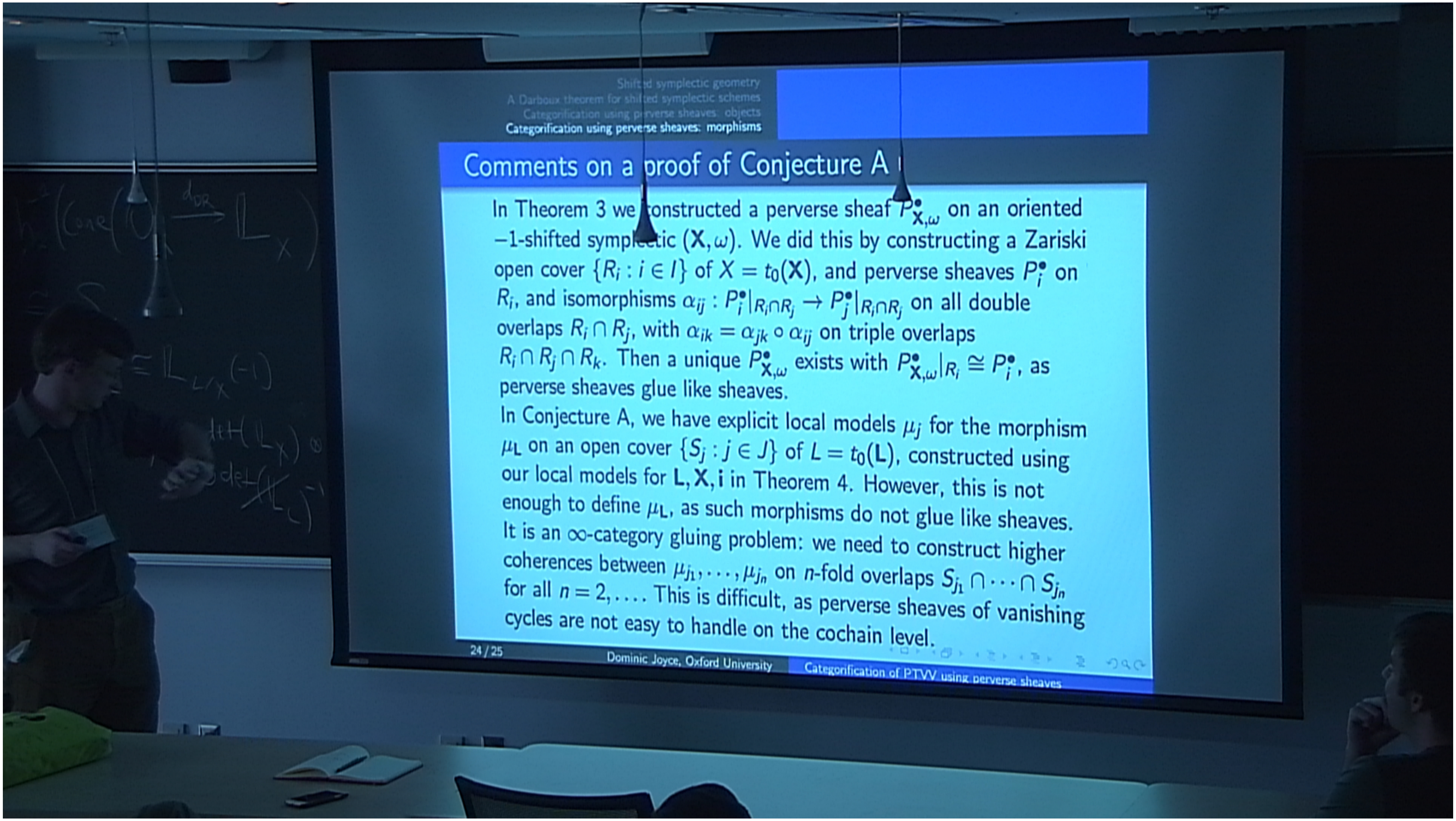
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Comments on a proof of Conjecture A
In Theorem 3 we constructed a perverse sheaf $P^\bullet_{\mathbf{X},\omega}$ on an oriented -1-shifted symplectic $(\mathbf{X},\omega)$. We did this by constructing a Zariski open cover $\{R_i : i \in I\}$ of $X = t_0(\mathbf{X})$, and perverse sheaves $P^\bullet_i$ on R_i, and isomorphisms $\alpha_{ij} : P^\bullet_i|_{R_i \cap R_j} \to P^\bullet_j|_{R_i \cap R_j}$ on all double overlaps $R_i \cap R_j$, with $\alpha_{ik} = \alpha_{jk} \circ \alpha_{ij}$ on triple overlaps $R_i \cap R_j \cap R_k$. Then a unique $P^\bullet_{\mathbf{X},\omega}$ exists with $P^\bullet_{\mathbf{X},\omega}|_{R_i} \cong P^\bullet_i$, as perverse sheaves glue like sheaves.
In Conjecture A, we have explicit local models μ_j for the morphism $\mu_{\mathbf{L}}$ on an open cover $\{S_j : j \in J\}$ of $L = t_0(\mathbf{L})$, constructed using our local models for $\mathbf{L}, \mathbf{X}, \mathbf{i}$ in Theorem 4. However, this is not enough to define $\mu_{\mathbf{L}}$, as such morphisms do not glue like sheaves. It is an ∞-category gluing problem: we need to construct higher coherences between $\mu_{j_1}, \ldots, \mu_{j_n}$ on n-fold overlaps $S_{j_1} \cap \cdots \cap S_{j_n}$ for all $n = 2, \ldots$. This is difficult, as perverse sheaves of vanishing cycles are not easy to handle on the cochain level.
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