Title: An overview of derived analytic geometry

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Abstract: After the pioneering work of J. Lurie in [DAG-IX], the possibility of a derived version of analytic geometry drew the attention of several mathematicians. The goal of this talk is to provide an overview of the state of art of derived analytic geometry, addressing both the complex and the non-archimedean setting.

After providing a series of motivations for derived analytic geometry, I will survey the main results obtained in my PhD thesis: derived versions of GAGA theorems, the existence of the analytic cotangent complex and an analytic version of Lurie's representability theorem. If time will permit, I will conclude the talk by discussing the possible future directions.

Parts of the results I will talk about have been obtained in collaboration with T. Y. Yu.

An overview of derived analytic geometry

Deformation quantization of shifted Poisson structures

Perimeter Institute

Mauro Porta (IMJ - Florence)

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Plan of the talk

- Motivations for derived analytic geometry
- A glimpse of the foundational background
- Derived algebraic geometry vs derived analytic geometry
- Analytic deformation theory
- Future directions

This talk is based on my Ph.D. thesis, parts of which are already available:

- P. , *Derived $\mathbb{C}$-analytic geometry I: GAGA theorems*, arXiv 1506.09042;
- P. , *Derived $\mathbb{C}$-analytic geometry II: square-zero extensions*, arXiv 1507.06602;
- P. - Yu, *Higher analytic stacks and GAGA theorems*, arXiv 1412.5166;
- P. - Yu, *Derived non-archimedean analytic spaces*, arXiv 1601.00859.

Motivations for derived analytic geometry

Before motivations, some precisions. Analytic geometry means:

- $\mathbb{C}$-analytic geometry, or
- k-analytic geometry, where k is a non-archimedean field with a non-trivial valuation (Berkovich spaces).

Derived geometry: in the sense of Toën-Vezzosi [HAG II] and Lurie [DAG].

Two motivations for derived analytic geometry:

1. On the $\mathbb{C}$-analytic side, nonabelian Hodge theory (after Simpson, Pantev, Katzarkov, Toën, Vaquié, Vezzosi...).
2. On the k-analytic side, non-archimedean mirror symmetry (after Kontsevich-Soibelman).

Nonabelian Hodge theory. The question is to define a notion of mixed Hodge filtration on a suitable geometric stack F and show that for suitable X this structure is inherited by $\mathrm{Map}(X_{\mathrm{dR}}, F)$. The problem is with the definition of the weight filtration. On one side it is believed that this should be interpreted as a formal filtration, leading inevitably to the consideration of the derived mapping stack. On the other side, the integral structure requires a version of Simpson's Riemann-Hilbert correspondence, which lives naturally in the analytic realm.

Non-archimedean mirror symmetry. Kontsevich and Soibelman suggested that in the non-archimedean setting it might be easier to undertake the enumerative problems involved in the construction of a mirror variety. E.g., it is possible sometimes to reduce the definition of such invariants to the notion of stable map. As in the algebraic case, the stack $\overline{\mathcal{M}}_{0,n}(X, \beta)$ is not smooth unless strong hypotheses are put on X. To construct good virtual classes, derived geometry is required. Given that $\overline{\mathcal{M}}_{0,n}(X, \beta)$ is a proper Deligne-Mumford non-archimedean stack by T. Y. Yu arXiv 1401.6452, we need a version of Lurie's representability theorem in the analytic setting.

What doese a derived analytic space look like?

Let's start with a naïf guess:

1. After Lurie, we know how to define derived schemes as simplicially ringed spaces (or, better, ∞-topoi).
2. The category of classical $\mathbb{C}$-analytic spaces embeds fully faithfully in the category of locally ringed spaces.
3. The category of classical Berkovich spaces embeds fully faithfully in the category of locally ringed topoi [DNAnG, Lemma 4.5].

Definition (Wrong)

A derived analytic space is a pair $(\mathcal{X}, \mathcal{O}_\mathcal{X})$, where:

1. $\mathcal{X}$ is the topos of sheaves on either a $\mathbb{C}$-analytic space or on the quasi-étale site of a Berkovich space;
2. $\mathcal{O}_\mathcal{X}$ is a sheaf of simplicial algebras on $\mathcal{X}$.

These two objects should satisfy the following compatibilities. Let $\mathcal{O}_X$ denote the sheaf of analytic functions on the $\mathbb{C}$-analytic or Berkovich space underlying $\mathcal{X}$. Then we ask:

1. $\pi_0(\mathcal{O}_\mathcal{X}) \simeq \mathcal{O}_X$;
2. every $\pi_i(\mathcal{O}_\mathcal{X})$ is coherent as sheaf of $\mathcal{O}_X$-modules.

There are many (equivalent) ways of explaining the problem with the above definition:

1. the associated deformation theory is not what we expect;

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There are many (equivalent) ways of explaining the problem with the above definition:

1. the associated deformation theory is not what we expect;
2. Lurie's representability theorem fails;

Indeed, let's denote $\mathcal{C}$ the ∞-category of derived analytic spaces in the above, wrong, sense. Consider the following construction:

- Let $(X, \mathcal{O}_X)$ be a classical $\mathbb{C}$-analytic space, and let F be a (discrete) coherent sheaf on X.
- Then the split square-zero extension $X[F[1]] := (X, \mathcal{O}_X \oplus F[1])$ is again in $\mathcal{C}$.
- Let $d \colon \mathcal{O}_X \to \mathcal{O}_X \oplus F[1]$ be any section of the projection $\mathcal{O}_X \oplus F[1] \to \mathcal{O}_X$.
- The pushout

$$\begin{array}{ccc} X[F[1]] & \longrightarrow & X \\ \downarrow & & \downarrow \\ X & \longrightarrow & X_d[F] \end{array}$$

(which is discrete) should belong to $\mathcal{C}$.

Main issue: the above square-zero extensions are completely classified by $\mathbb{L}_{\mathcal{O}_X}$. This is an object which is *too big*. We want to replace it with a smaller version:

Definition

Let $(X, \mathcal{O}_X)$ be an analytic space and let F be a (discrete) coherent sheaf on X. An analytic derivation on X with values in F is a derivation $\mathcal{O}_X \to F$ which furthermore satisfies

$$d(f(x)) = f'(x)dx$$

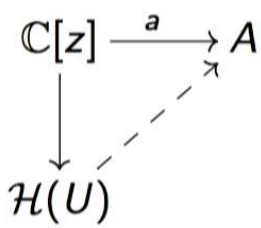
for every $x \in \mathcal{O}_X(U)$ and every analytic function $f\colon \mathbf{A}^1 \to \mathbf{A}^1$.

Analytic structure sheaves

Essentially two different approaches to formalize the idea:

- Interpret $\mathcal{O}_{\mathcal{X}}$ as a sheaf of simplicial (or dg) Ind-Banach algebras (Wallbridge, Ben Bassat-Kremnizer).
- Interpret $\mathcal{O}_{\mathcal{X}}$ as a sheaf of simplicial algebras equipped with an axiomatic holomorphic functional calculus (Lurie, P., P. - Yu).

Here it is a recap on functional calculus. Let A is a commutative Banach $\mathbb{C}$-algebra. Any element $a \in A$ determines a solid arrow



If $U \subset \mathbb{C}$ is an open subset and $\mathcal{H}(U)$ denotes the ring of holomorphic functions on U, we have a restriction $\mathbb{C}[z] \to \mathcal{H}(U)$.

Question: when does the dotted arrow exists?

Answer: Let $\sigma(a) \subset \mathbb{C}$ denote the spectrum of $a \in A$: it is the image of the Gelfand transform of a, seen as a function $\mathcal{M}(A) \to \mathbb{C}$. Then the lifting problem has solution if and only if $\sigma(a) \subset U$.

Axiomatic holomorphic functional calculus: (Rough) An algebra A together with subsets $A(U) \subset A^n$ for every $U \subset \mathbb{C}^n$ and (composable) holomorphic operations $f_A \colon A(U) \to A$ for every $f \colon U \to \mathbb{C}$.

Definition

Let $\mathcal{T}_{\mathrm{an}}(\mathbb{C})$ be the category of open subsets of $\mathbb{C}^n$, with holomorphic functions between them.

Reformulation: an axiomatic holomorphic functional calculus on a commutative $\mathbb{C}$-algebra A is a functor

$$\mathcal{O} \colon \mathcal{T}_{\mathrm{an}}(\mathbb{C}) \to \mathrm{Set}$$

such that $\mathcal{O}(\mathbb{C}) = A$.

The k-analytic analogue of $\mathcal{T}_{\mathrm{an}}(\mathbb{C})$ introduced in [DNAnG] is the following:

Definition

$\mathcal{T}_{\mathrm{an}}(k)$ is the full subcategory of An_k spanned by separated, quasi-smooth strictly k-analytic spaces which are paracompact.

There are many possible variations, all giving rise to the same theory of derived non-archimedean analytic spaces.

Definition

Let $\mathcal{T}_{\mathrm{an}}$ be either $\mathcal{T}_{\mathrm{an}}(\mathbb{C})$ or $\mathcal{T}_{\mathrm{an}}(k)$. Let $\mathcal{X}$ be an ∞-topos. A sheaf of analytic rings on $\mathcal{X}$ is a functor $\mathcal{O}\colon \mathcal{T}_{\mathrm{an}} \to \mathcal{X}$ which preserves:

1. products;
2. pullbacks along open immersions (in the $\mathbb{C}$-analytic case) or quasi-étale maps (in the k-analytic case).

The pair $(\mathcal{X}, \mathcal{O})$ is said to be an analytically ringed ∞-topos.

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To any analytically ringed topos $(\mathcal{X}, \mathcal{O})$, we can always associate a simplicially ringed topos $(\mathcal{X}, \mathcal{O}^{\mathrm{alg}})$.

Derived analytic spaces

Definition (Rough)

A *derived analytic space* is a an analytically ringed space $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$

1. $(\mathcal{X}, \pi_0(\mathcal{O}_{\mathcal{X}}^{\mathrm{alg}}))$ is a classical $\mathbb{C}$-analytic space (or a Berkovich space);
2. The sheaves $\pi_i(\mathcal{O}_{\mathcal{X}}^{\mathrm{alg}})$ are coherent as $\pi_0(\mathcal{O}_{\mathcal{X}}^{\mathrm{alg}})$-modules.

Theorem (Lurie, P. - Yu)

1. *There exists an ∞-category $\mathrm{dAn}_{\mathbb{C}}$ (resp. dAn_k) of derived $\mathbb{C}$-analytic (resp. k-analytic) spaces.*
2. *$\mathrm{dAn}_{\mathbb{C}}$ and dAn_k admit fiber products.*
3. *There are fully faithful embeddings $\mathrm{An}_{\mathbb{C}} \to \mathrm{dAn}_{\mathbb{C}}$ and $\mathrm{An}_k \to \mathrm{dAn}_k$.*
4. *If $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is a derived analytic space, then so is $(\mathcal{X}, \tau_{\leq n}\mathcal{O}_{\mathcal{X}})$ for every $n \geq 0$.*

Analytification

Definition (Grothendieck, SGA 1, Exposé XII)

Let $X \in \mathrm{Sch}_{\mathbb{C}}^{\mathrm{f.t.}}$. The analytification X^{an} is the unique $\mathbb{C}$-analytic space such that for every other $Y \in \mathrm{An}_{\mathbb{C}}$ we have

$$\mathrm{Hom}_{\mathrm{An}_{\mathbb{C}}}(Y, X^{\mathrm{an}}) \simeq \mathrm{Hom}_{\mathrm{LRS}}(Y, X)$$

Theorem (Lurie)

1. *Let $X \in \mathrm{dSch}_{\mathbb{C}}^{\mathrm{aft}}$. There exists an analytically ringed space X^{an} such that*
$$\mathrm{Map}_{\mathrm{Top}(\mathcal{T}_{\mathrm{an}})}(Y, X^{\mathrm{an}}) \simeq \mathrm{Map}_{\mathrm{Top}(\mathcal{T}_{\text{ét}})}(Y^{\mathrm{alg}}, X)$$
2. *X^{an} is a derived $\mathbb{C}$-analytic space.*

Theorem (P., arXiv 1506.09042)

1. *If* $X \in \mathrm{Sch}_{\mathbb{C}}^{\mathrm{aft}}$ *then Lurie's analytification coincides with Grothendieck's.*
2. *Let* $X \in \mathrm{dSch}_{\mathbb{C}}^{\mathrm{aft}}$. *The canonical map* $(X^{\mathrm{an}})^{\mathrm{alg}} \to X$ *is flat (in the derived sense).*
3. *Let*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle g}\downarrow & & \downarrow \\ Z & \longrightarrow & W \end{array}$$

be a pushout square in $\mathrm{dSch}_{\mathbb{C}}^{\mathrm{aft}}$. *Suppose that* f *and* g *are square-zero extensions. Then* $(-)^{\mathrm{an}}$ *commutes with this pushout.*

Corollary (P., arXiv 1506.09042)

We can compute the analytification of a derived scheme starting by analytifying the underlying classical scheme and proceeding by induction on the Postnikov tower.

Derived GAGA theorems

Definition

Let $X = (\mathcal{X}, \mathcal{O}_\mathcal{X}) \in \mathrm{dAn}_\mathbb{C}$. We let $\mathcal{O}_\mathcal{X}\text{-Mod} := \mathcal{O}_\mathcal{X}^{\mathrm{alg}}\text{-Mod}$. We let $\mathrm{Coh}(X)$ be the full subcategory of $\mathcal{O}_\mathcal{X}\text{-Mod}$ spanned by those $\mathcal{F}$ such that $\mathrm{H}^i(\mathcal{F})$ is coherent over $(\mathcal{X}, \pi_0(\mathcal{O}_\mathcal{X}^{\mathrm{alg}}))$.

Definition

Let $f \colon X \to Y$ be a morphism in $\mathrm{dSch}_\mathbb{C}$ or in $\mathrm{dAn}_\mathbb{C}$. We will say that f *is proper* if $\mathrm{t}_0(f)$ is proper in the usual sense.

Remark

It is possible to generalize the above definition to Deligne-Mumford and even Artin stacks. One of the goal of P. - Yu arXiv 1412.5166 is to establish the proper direct image / Grauert theorem in this setting. The derived analogue (for Coh^+) is easily deduced by dévissage to the heart.

Theorem (GAGA I, P., arXiv 1506.09042)

Let $f\colon X \to Y$ *be a proper morphism in* $\mathrm{dSch}_{\mathbb{C}}^{\mathrm{f.t.}}$. *The diagram of stable* ∞*-categories*

$$\begin{array}{ccc} \mathrm{Coh}^+(X) & \xrightarrow{(-)^{\mathrm{an}}_X} & \mathrm{Coh}^+(X^{\mathrm{an}}) \\ \downarrow{\scriptstyle \mathrm{R}f_*} & & \downarrow{\scriptstyle \mathrm{R}f^{\mathrm{an}}_*} \\ \mathrm{Coh}^+(Y) & \xrightarrow{(-)^{\mathrm{an}}_Y} & \mathrm{Coh}^+(Y^{\mathrm{an}}) \end{array}$$

commutes.

Theorem (GAGA II, P., arXiv 1506.09042)

Let X *be a proper derived scheme. The analytification functor*

$$(-)^{\mathrm{an}}_X\colon \mathrm{Coh}(X) \to \mathrm{Coh}(X^{\mathrm{an}})$$

is an equivalence of stable ∞*-categories.*

Analytic deformation theory

Question: how to define analytic (derived) derivations?

Theorem (P. (arXiv 1507.06602), P. - Y. (to appear))

Let $(\mathcal{X}, \mathcal{O}_\mathcal{X}) \in \mathrm{dAn}_k$. *After passing to stabilizations, the forgetful functor* $\overline{\Phi} : \mathrm{AnRing}(\mathcal{X})_{/\mathcal{O}_\mathcal{X}} \to \mathrm{CAlg}(\mathcal{X})_{/\mathcal{O}_\mathcal{X}^{\mathrm{alg}}}$ *induces an equivalence*

$$\mathrm{Sp}(\mathrm{AnRing}(\mathcal{X})_{/\mathcal{O}_\mathcal{X}}) \simeq \mathrm{Sp}(\mathrm{CAlg}(\mathcal{X})_{/\mathcal{O}_\mathcal{X}^{\mathrm{alg}}}) \simeq \mathcal{O}_\mathcal{X}^{\mathrm{alg}}\text{-Mod}$$

Definition

Let $X = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$, let $M \in \mathrm{Coh}(M)$. Let $d \colon \mathcal{O}_{\mathcal{X}} \to \Omega^\infty(M)$ be an analytic derivation. The associated analytic square-zero extension $e : \mathcal{O} \to \mathcal{O}_{\mathcal{X}}$ of $\mathcal{O}_{\mathcal{X}}$ is determined as the pullback

$$\begin{array}{ccc} \mathcal{O} & \longrightarrow & \mathcal{O}_{\mathcal{X}} \\ {\scriptstyle e}\downarrow & & \downarrow{\scriptstyle d} \\ \mathcal{O}_{\mathcal{X}} & \xrightarrow{0} & \Omega^\infty(M) \end{array}$$

Analytic cotangent complex

Theorem (P., P. - Yu, to appear)

Let $f : X \to Y$ *be a morphism in* dAn_k*. There exists an analytic cotangent complex* $\mathbb{L}^{\mathrm{an}}_{X/Y} \in \mathrm{Coh}^-(X)$.

Theorem (P., P. - Yu to appear)

1. *Let* $f : X \to Y$ *be a morphism in* $\mathrm{dSch}^{\mathrm{aft}}_{\mathbb{C}}$*. The canonical morphism*

$$\mathbb{L}^{\mathrm{an}}_{X^{\mathrm{an}}/Y^{\mathrm{an}}} \to (\mathbb{L}_{X/Y})^{\mathrm{an}}$$

   is an equivalence.

2. *If* $f : X \to Y$ *is a closed immersion on* dAn_k *then* $\mathbb{L}^{\mathrm{an}}_{X/Y} \simeq \mathbb{L}_{X^{\mathrm{alg}}/Y^{\mathrm{alg}}}$.

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2. *If $f : X \to Y$ is a closed immersion on dAn_k then $\mathbb{L}^{\mathrm{an}}_{X/Y} \simeq \mathbb{L}_{X^{\mathrm{alg}}/Y^{\mathrm{alg}}}$.*
3. *A morphism $f\colon X \to Y$ in dAn_k is smooth if and only if $\mathrm{t}_0(f)$ is smooth and $\mathbb{L}^{\mathrm{an}}_{X/Y}$ is perfect in cohomological amplitude 0.*

Lurie's representability theorem

Question: how to define derived analytic Artin stacks?

Answer: let $\mathrm{dStn}_{\mathbb{C}}$ be the full subcategory of $\mathrm{dAn}_{\mathbb{C}}$ spanned by those $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ such that $(\mathcal{X}, \pi_0(\mathcal{O}_{\mathcal{X}}^{\mathrm{alg}}))$ is a Stein space.
Let $\mathrm{dAn}_{\mathbb{C}}^{\mathrm{loc}}$ be the full subcategory of $\mathrm{dAn}_{\mathbb{C}}$ spanned by those $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ where $\mathcal{X}$ is an *n*-localic ∞-topos for some *n*.

Theorem (P. arXiv 1506.09042)

The full subcategory $\mathrm{dAn}_{\mathbb{C}}^{\mathrm{loc}}$ *of* $\mathrm{dAn}_{\mathbb{C}}$ *is equivalent to the category of Deligne-Mumford stacks with respect to* $(\mathrm{dStn}_{\mathbb{C}}, \tau_{\mathrm{an}}, \mathbb{P}_{\text{ét}})$ *(see [DNAnG] for the non-archimedean counterpart).*

Definition

A derived analytic Artin stack is a geometric stack with respect to the context $(\mathrm{dStn}_{\mathbb{C}}, \tau_{\mathrm{an}}, \mathbb{P}_{\mathrm{sm}})$.

Lemma

$\mathrm{dAn}_{\mathbb{C}}$ *is closed under pushouts along closed immersions.*

Theorem (P. - Yu, to appear)

Let $F\colon \mathrm{dStn}_{\mathbb{C}}^{\mathrm{op}} \to \mathcal{S}$ *be a sheaf for* τ_{an}. *The following conditions are equivalent:*

1. *F is a derived analytic Artin stack;*
2. *F satisfies the following conditions:*
   - (i) $\mathrm{t}_0(F)$ *is an analytic Artin stack;*
   - (ii) *F is convergent;*
   - (iii) *F is infinitesimally cartesian;*
   - (iv) *F admits a global analytic cotangent complex.*

Near future directions

- (joint with T. Y. Yu) Construction of some enumerative invariants in non-archimedean geometry.
- Analytification of the mapping stack and derived Riemann Hilbert correspondence.
- (joint with F. Petit) Analytic (multiplicative, S^1-equivariant) HKR theorem.

Additional results

Theorem

1. *Let $X \in \mathrm{dAn}_k$ be proper. Then X is algebraizable if and only if $\mathrm{t}_0(X)$ is algebraizable.*
2. *Let $X \in \mathrm{dAn}_{\mathbb{C}}$ and let $x \in X$ be a point. Then $\mathbb{T}_{X,x}[-1]$ admits a canonical structure of dgLie algebra.*
3. *Let*

$$\begin{array}{ccc} X' & \xrightarrow{q'} & X \\ \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\ Y' & \xrightarrow{q} & Y \end{array}$$

be a cartesian diagram of derived $\mathbb{C}$-analytic spaces, where p is proper. For every $F \in \mathrm{Coh}(X)$, the natural transformation $q^* p_*(F) \to p'_* q'^*(F)$ is an equivalence.

Thanks for the attention!