Title: Formal derived stack and Formal localization

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Abstract: A crucial ingredient in the theory of shifted Poisson structures on general derived Artin stacks is the method of formal localization. Formal localization is interesting in its own right as a new, very power ful tool that will prove useful in order to globalize tricky constructions and results, whose extension from the local case presents obstructions that only vanish formally locally.

graded mixed objects in M: ε - M

$E = (E(p))$ graded object.

$\varepsilon : E(p) \longrightarrow E(p+1)[1]$ such that $\varepsilon^2 = 0$

$M = C(k)$: graded mixed complex.

$\forall X \in M \quad \forall K \in C(k) \quad X^{K}$

$X[1] = X^{k[-1]}$

- $\mathcal{E}\text{-}cdga^{gr}_{n} \longrightarrow cdga_{\pi}$.

  $(A(p)) \longmapsto A(0)$

The ∞-functor $\mathcal{E}\text{-}cdga^{gr}_{\mathfrak{X}} \longrightarrow cdga_{\mathfrak{X}}$.
commutes with limits and is accessible.

$\Longrightarrow \exists$ exists a left adjoint. : $DR^{int} : cdga_{\mathfrak{X}} \longrightarrow \mathcal{E}\text{-}cdga^{gr}_{\mathfrak{X}}$

the cotangent complex of A internal in M.

ε structure on $DR^{int}(A)$

induces a map $DR^{int}(A)(0) \longrightarrow DR^{int}(A)(1)[1]$

It induces a derivation by composition by $A \longrightarrow DR^{int}(A)(0)$

$$\Longrightarrow \quad \mathbb{L}^{int}_A[-1] \longrightarrow DR^{int}(A)$$

$$\phi_A : \mathrm{Sym}_A\left(\mathbb{L}^{int}_A[-1]\right) \longrightarrow DR^{int}(A).$$

$$\mathbb{L}^{int}_{A/B}$$

$$T^{(p)}(A,-n) \subset \underline{\mathrm{Hom}}_{\Pi}\left(A^{\otimes p}, A[np]\right) \in \Pi.$$

Σ_p acts on $T^p(A,-n)$

$$\mathrm{Pol}^{int}(A,n) = \bigoplus_{p\geq 0} \left(T^p(A,-n)\right)^{\Sigma_p} \in \Pi.$$

$$\mathrm{Pol}^{int}(A/B,n)$$

$|\cdot| =$ realization: $\Pi \longrightarrow C(k) = dg_k$

absolute de Rham object $\quad DR(A) := |DR^{int}(A)|$

$DR(A/B) := |DR^{int}(A/B)|$

absolute cot. complex $\quad \mathbb{L}_A = |\mathbb{L}^{int}_A| \qquad \mathbb{L}_{A/B} = |\mathbb{L}^{int}_{A/B}|$

graded mixed objects in Π: $\quad \varepsilon - \Pi^{gr}$

$E = (E(p))$ graded object.

$H^0(A)$ is a k alg of finite type

$\forall n \quad H^{-n}(A)$ is finitely generated over $H^0(A)$

$$d\,\mathrm{Aff}_k = \left(\mathrm{cdga}_k^{<0,\,aft}\right)$$

$B_{\leq k}$ Po

2) F infinitesimally cohesive:
$$\begin{tikzcd} B \arrow[r] \arrow[d] & B_1 \arrow[d] \\ B_2 \arrow[r] & B_0 \end{tikzcd}$$
cart. square in $\mathrm{cdga}^{\leq 0}_k$ aft

with $\pi_0(B_i) \longrightarrow \pi_0(B_0)$ is surjective with nilpotent kernel
$i = 1, 2$

$$\begin{tikzcd} F(B) \arrow[r] & F(B_1) \arrow[d] \\ {} \arrow[r] & F(B_0) \end{tikzcd}$$
cart. square

$$\begin{tikzcd} F(B) \arrow[r] \arrow[d] & F(B_1) \arrow[d] \\ F(B_2) \arrow[r] & F(B_0) \end{tikzcd}$$

cart. square

Remarks:

1) Algebraic stack is a formal derived stack.

2) $FdSt_k \subset dSt_k$. stable by all limits.

A $\mathrm{cdga}_k^{\leq 0}$

$$A^{red} = H^0(A) / \mathrm{Nilp}(H^0(A)) \in \mathrm{ca}_k$$

$$(\ )_{red} \ \mathrm{cdga}_k^{\leq 0} \longrightarrow \mathrm{ca}_k^{red.}$$

$$A \longmapsto A^{red.}$$

left adjoint to $i : \mathrm{ca}_k^{red} \longrightarrow \mathrm{cdga}_k^{\leq 0}$

$i^*, i_*, i^!$ on $\mathrm{St}_{red,k}$ and on $d\mathrm{St}_k$

$i^*, i_*, i_!$ on $St_{red,k}$ and on dSt_k

Definition: 1) The functor $(-)_{DR} := i_* i^* : dSt_k \longrightarrow dSt_k$
is called the de Rham stack functor. $F \to F_{DR}$.

2) The functor $(\ )_{red} := i_! i^* : dSt_k \longrightarrow dSt_k$
is called the reduced stack functor. $F_{red} \to F$

3) $f : F \to G$. The formal completion of G along f

$$
\begin{tikzcd}
F \arrow[dr] & & \\
& \hat{G}_f \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & F_{DR} \arrow[d] \\
& G \arrow[r] & G_{DR}
\end{tikzcd}
$$

$$F_{DR}(A) = F(A^{red})$$

$$(\operatorname{Spec} A)_{red} = \operatorname{Spec}(A^{red})$$

$$(F_{DR}) \xrightarrow{\sim} (F_{DR})_{DR}$$

$$(F_{red})_{red} \xrightarrow{\sim} (F_{red})$$

$$F_{red} \xrightarrow{\sim} (F_{DR})_{red}$$

$$(F_{red})_{DR} \xrightarrow{\sim} F_{DR}.$$

$$F_{red} \longrightarrow (\hat{G}_f)_{red}$$

Def. 1) $F \in dFSt_k$ is affine if:

- $F_{red} = \operatorname{Spec} A_F$ is affine. $F_{red} \longrightarrow F$.
- F has a cotangent complex which is coherent and bounded above.

2) $F \longrightarrow X = \operatorname{Spec} A$.

- F good formal derived stack over X if F is affine formal stack with $\operatorname{Spec} A_F \xrightarrow{\sim} X_{red}$
- F is perfect ———————— if moreover $\mathbb{L}_{F/}$

$F: cdga_k^{\leq 0} \longrightarrow SSets$

$(\quad) \longrightarrow T$

as functor.

Remarks: 1) Algebraic stack is a fo... ...ack.

2) $FdSt_k \subset dSt_k$

$$F_A := F \times_{F_{DR}} \mathrm{Spec}\, A$$

<u>Proposition</u>: $F_A \simeq \widehat{\left(\mathrm{Spec}\, A \times F\right)}_{(i,u)}$ $\quad (i,u) : \mathrm{Spec}\, A^{red} \longrightarrow \mathrm{Spec}\, A \times F$

<u>Corollary</u>: If F is an alg stack each fiber F_A is a good formal derived stack over A.

Moreover if F is locally of f type then it is perfect.

$$\Phi_F : L_{qc}(F) \longrightarrow \mathbb{D}(F)\text{-}\mathrm{Mod}^{gr}_{\varepsilon\text{-}dg}$$

which induces an equivalence:

$$L_{perf}(F) \longrightarrow \left(\mathbb{D}(F)\text{-}\mathrm{Mod}^{gr}_{\varepsilon\text{-}dg}\right)^{perf}$$

<u>Def</u>: E' is perfect if $E = \mathbb{D}(F) \otimes_{A_F} E_0$

$E_0 \in L_{perf}(A_F)$

Theorem: $F \to \mathrm{Spec}\, A$ perfect form. der. stack.

$$DR(\mathbb{D}(F)/\mathbb{D}(A)) \xrightarrow{\sim} DR(F/A)$$

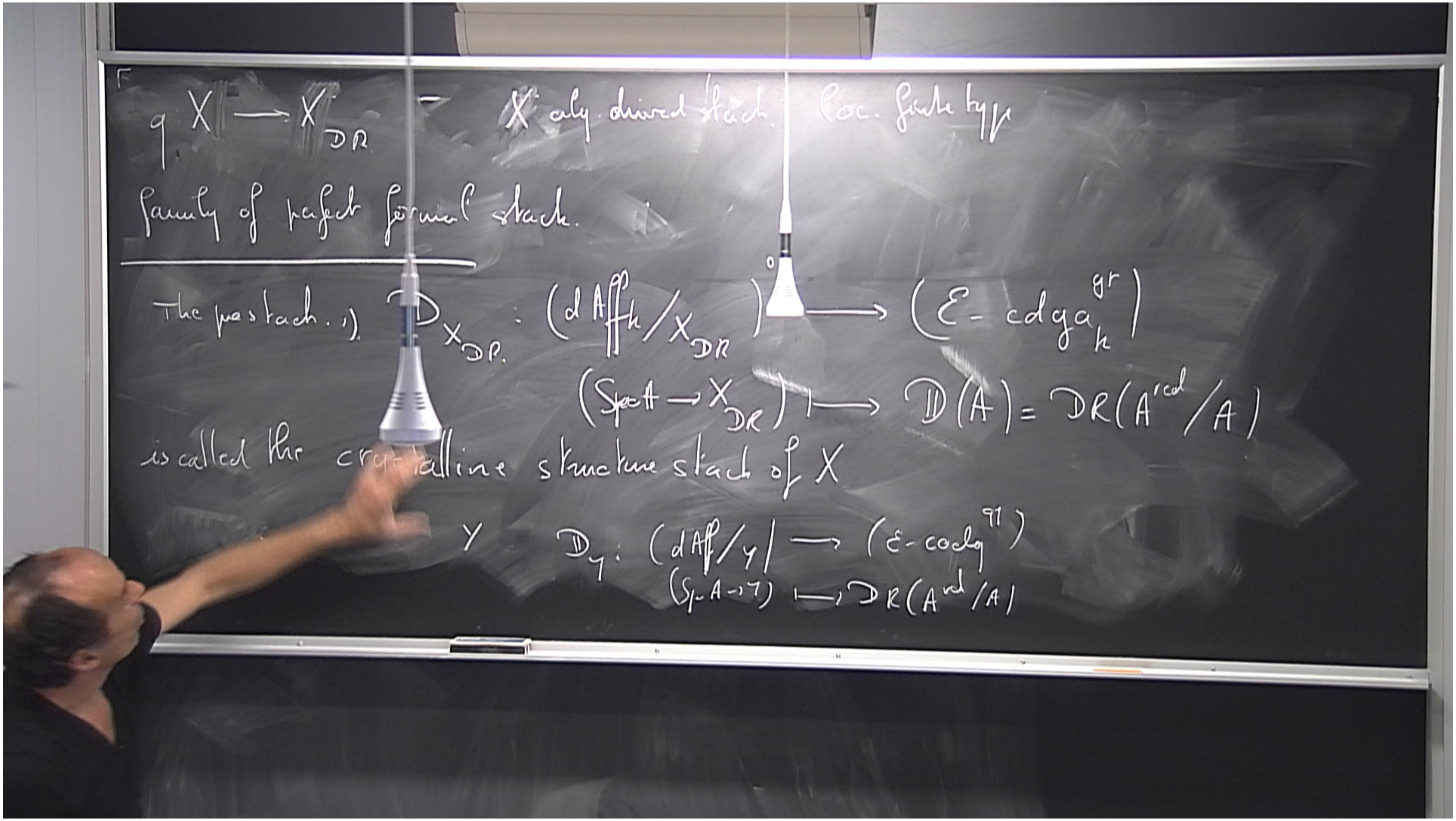
$$DR(F/A) = \varinjlim_{\mathrm{Spec}\, B \to A} DR(B/A)$$

$F: cdga_k^{\leq 0} \to SSet$

$(\quad) \to T$

∞-functor.

with. $\mathrm{Spec}\, A_F \xrightarrow{\sim} X_{red}$

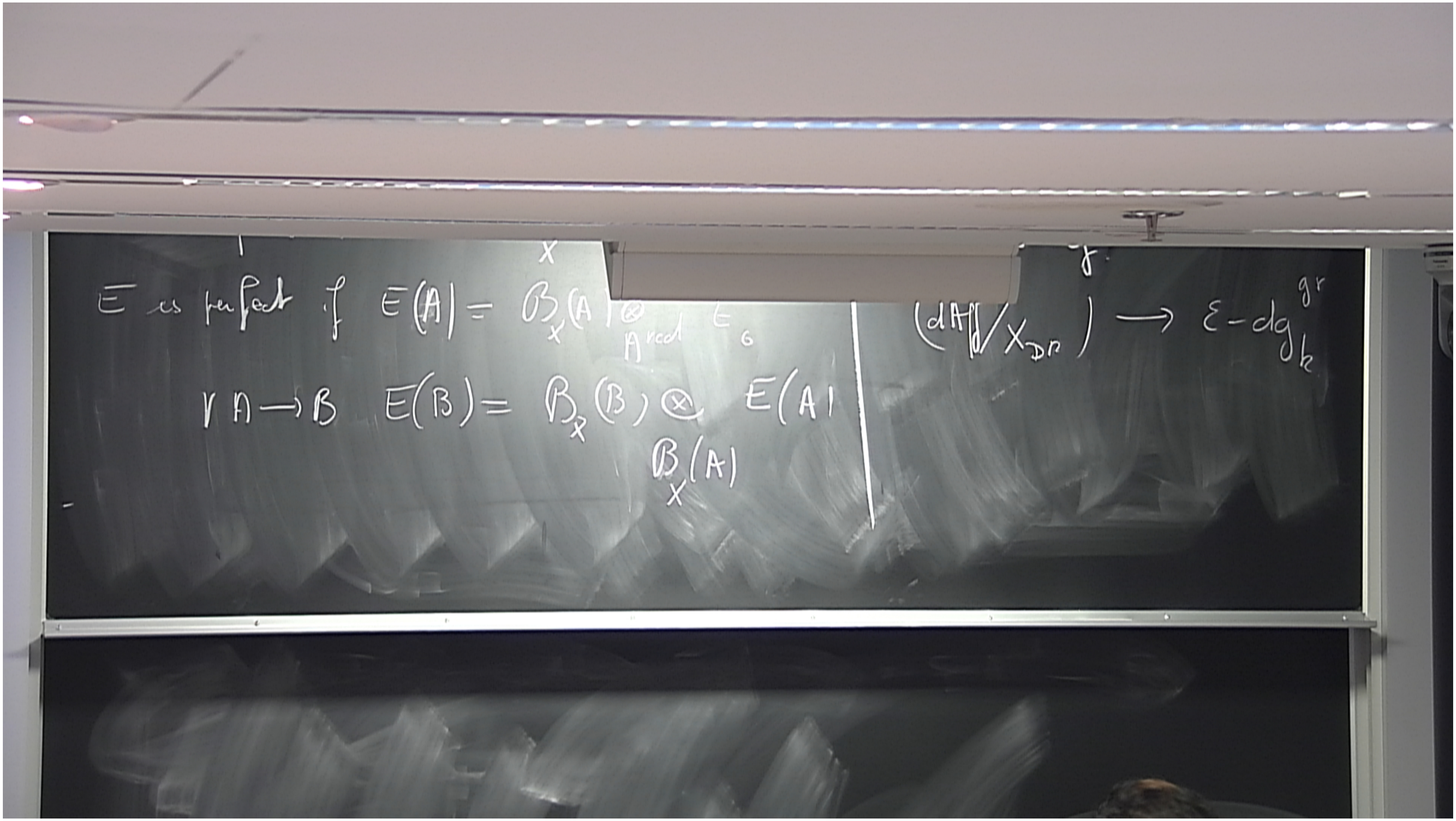

$q: X \to X_{DR}$ — X alg. derived stack, loc. finite type
family of perfect formal stack.
The prestack $\mathbb{D}_{X_{DR}}: (dAff_k/X_{DR})^{o} \to (\varepsilon\text{-}cdga_k^{gr})$
$(\mathrm{Spec} A \to X_{DR}) \mapsto \mathbb{D}(A) = DR(A^{red}/A)$
is called the crystalline structure stack of X
$Y \quad \mathbb{D}_Y: (dAff/Y) \to (\varepsilon\text{-}cdga^{gr})$
$(\mathrm{Spec} A \to Y) \mapsto DR(A^{red}/A)$

$\frac{\epsilon}{2}$

2) $\mathcal{B}_X = \mathbb{D}_{X/X_{DR}}$: $(\operatorname{Spec} A \to X_{DR}) \longmapsto \mathbb{D}(X_A)$

is called the prestack of principal parts of X $\quad X_A = X \times_{X_{DR}} \operatorname{Spec} A$.

$$\mathbb{D}_{X_{DR}} \longrightarrow \mathcal{B}_X$$


E is perfect if E(A) = B_X(A) ⊗_{A^red} E_G
∀ A → B E(B) = B_X(B) ⊗_{B_X(A)} E(A)
(dAff/X_DR) → E-dg_k^gr