Title: Spin and evolution in geometric models of matter

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Abstract: <p>In the geometric models of matter, proposed in a joint paper with Michael Atiyah and Nick Manton, static particles like the electron or proton are modelled by Riemannian 4-manifolds. In this talk I will explain how the spin degrees of freedom appear in the geometric framework. I will also discuss a proposal for time evolution in one particular model, namely the Taub-NUT model of the electron.

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Spin and time evolution in a geometric model of a particle

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Perimeter Institute, 21 April 2016

References

1. M Atiyah, N S Manton and B J Schroers, Geometric models of matter, Proc. Roy. Soc. Lond. A468 (2012) 1252–1279
2. R Jante and B J Schroers, Dirac operators on the Taub-NUT space, monopoles and SU(2) representations, JHEP 1401 (2014) 11
3. M Atiyah, G Franchetti and B J Schroers, Time evolution in a geometric model of a particle, JHEP 02 (2015) 062
4. R Jante and B J Schroers, Taub-NUT dynamics with a Maxwell field, J. Geom. Phys. 104 (2016) 305-328
5. R Jante and B J Schroers, Spectral Properties of Schwarzschild Instantons, arXiv:1604.06080

Outline

1. Geometric models of particles
2. Taub-NUT geometry
3. The Dirac operator and spin 1/2
4. Including time
5. Length scales and Dirac's Large Number Hypothesis
6. Comparison with the Schwarzschild instanton
7. Conclusion and outlook

Geometric models of particles

Solitons as particles: Skyrmions

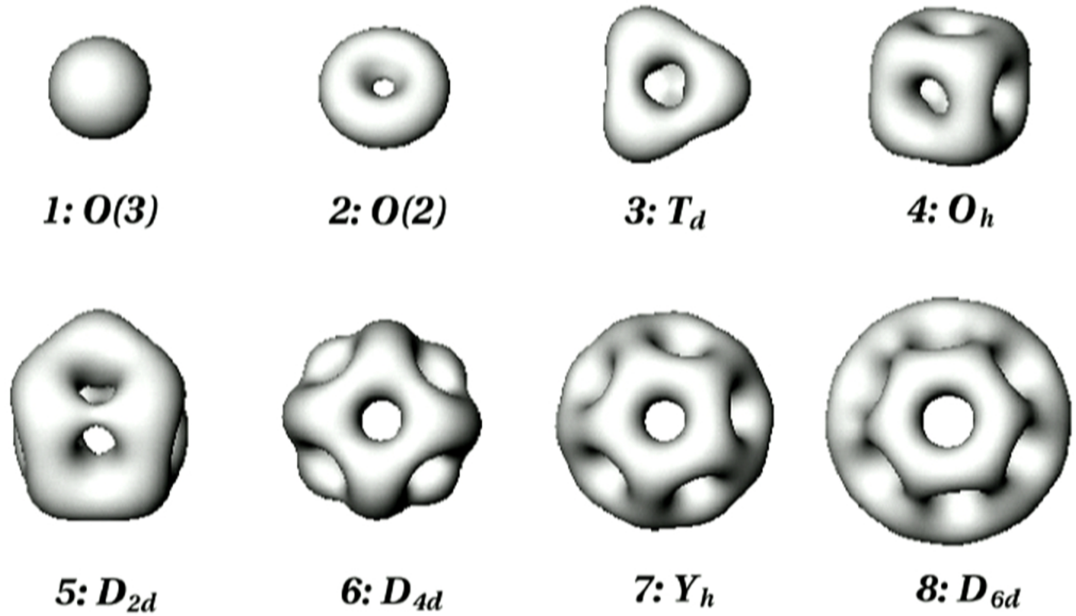
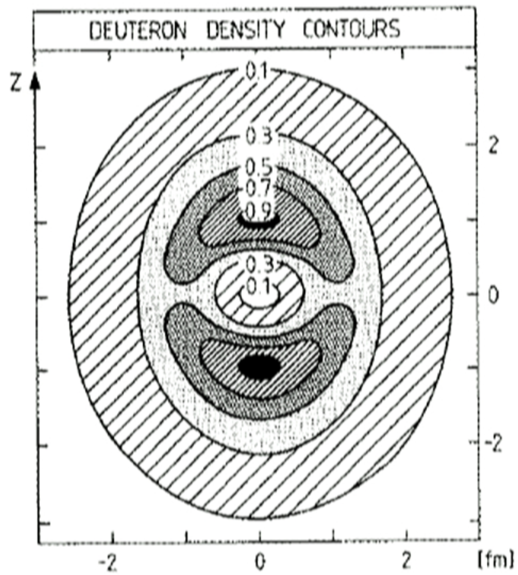
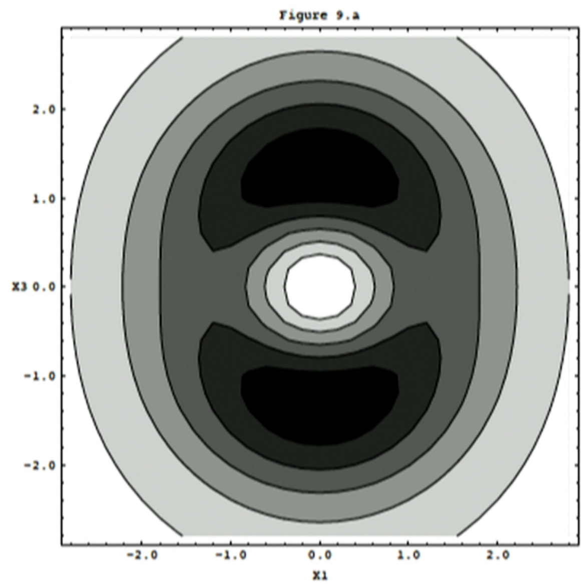


Figure: Skyrmions from B=1 to B=8 (with $m_\pi = 0$) [R.A. Battye and P.M. Sutcliffe]

The deuteron as a quantised Skyrmion





Ideas (preliminary)

- Inspired by topological solitons like the Skyrme model and 'holographic dual QCD' (Sakai and Sugimoto)
- Quantum numbers from topology, gauge fields and symmetries from geometry.
- Static geometric models: dualised and generalised KK model
- Stable elementary particles modelled by Euclidean 4-manifolds M with asymptotic fibration by circles
  Electric charge= - Chern class of asymptotic circle bundle
- Taub-NUT for electron, Euclidean Schwarzschild for neutron, Taub-Bolt or Atiyah-Hitchin for proton ...
- L^2-cohomology and $U(1)$ instantons on M play a role
- Spin 1/2 from kernel of Dirac operator on M coupled to $U(1)$ instanton

Here: discuss **spin** and include **time** in model of electron.

Taub-NUT geometry

A geometrical soliton

Kaluza-Klein geometrisation of the Dirac monopole

$$\mathbb{R}^4 \simeq \{0\} \cup \left(\underbrace{\mathbb{R}^+}_{\text{radial coordinate}} \times \underbrace{S^3}_{\text{total space of Hopf bundle}} \right)$$

Self-duality

Taub-NUT metric is of Bianchi IX form

$$ds^2 = f^2 dr^2 + a^2\sigma_1^2 + b^2\sigma_2^2 + c^2\sigma_3^2.$$

Here r is transverse coordinate to $SU(2) \simeq S^3$, and

$$h^{-1}dh = t_j\sigma_j, \quad h \in SU(2), \quad [t_1, t_2] = t_3.$$

Self-duality with respect to complex orientation:

$$\frac{2bc}{f}\frac{da}{dr} = (b-c)^2 - a^2, \ \text{+ cycl.},$$

Complex vesus angular coordinates

TN family has $a = b$ and $c = 0 \Rightarrow a = b = 0$. With

$$h = \begin{pmatrix} z_1 & -\bar{z}_2 \\ z_2 & \bar{z}_1 \end{pmatrix}, \quad |z_1|^2 + |z_2|^2 = 1$$

and

$$z_1 = e^{-\frac{i}{2}(\varphi+\psi)} \cos\frac{\theta}{2}, \quad z_2 = e^{\frac{i}{2}(\varphi-\psi)} \cos\frac{\theta}{2},$$

the metric is

$$ds^2 = f^2 dr^2 + a^2(d\theta^2 + \sin^2\theta d\varphi^2) + c^2\sigma_3^2,$$

where

$$\sigma_3 = \cos\theta d\varphi + d\psi.$$

U(2) invariance and spin coordinates

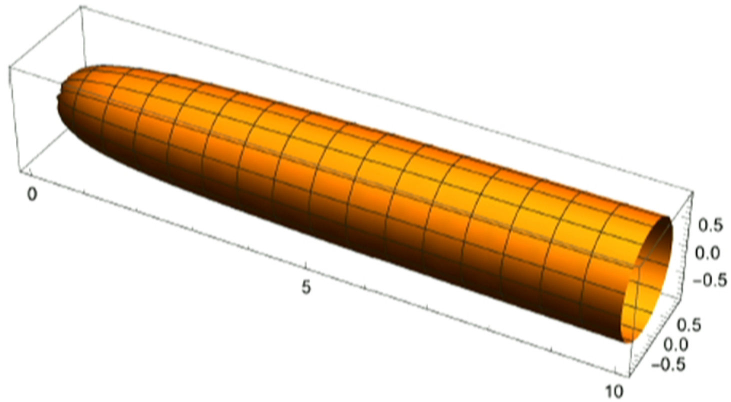
With c as radial coordinate, have

$$c \in [0, \Lambda), \quad \text{and} \quad a = \frac{c}{1 - \frac{c^2}{\Lambda^2}}.$$

Cigar-shaped geodesic submanifolds

$$ds^2 = \frac{4}{(1 - \frac{c^2}{\Lambda^2})^4} dc^2 + c^2 d\psi^2,$$

with asymptotic radius 2Λ and Gauss curvature $K = \frac{1}{\Lambda^2}$ at tip.



U(2) invariance and spin coordinates

TN is naturally $B_\Lambda \subset \mathbb{C}^2$ with global complex coordinates

$$w = 2c\,z \in \{z \in \mathbb{C}^2 |\ |z| < \Lambda\}.$$

The metric near nut is flat:

$$ds^2 \approx |dw_1|^2 + |dw_2|^2.$$

Macroscopic or position coordinates

The Hopf fibration $\pi : S^3 \to S^2, \quad z \mapsto \vec{n} = z^\dagger \vec{\tau} z$ together with

$$f = -a/r, \quad \vec{x} = r\vec{n} = (r \sin\theta\cos\varphi, r\sin\theta\sin\varphi, r\cos\theta)$$

gives the usual isotropic form of the TN metric

$$ds^2 = \frac{a^2}{r^2} d\vec{x}^2 + c^2\sigma_3^2.$$

Solving SD equation introduces macroscopic length scale L as integration constant. With

$$\epsilon = \frac{L^2}{\Lambda^2}, \quad V = \epsilon + \frac{L}{r}, \quad c = L\sqrt{\frac{r}{\epsilon r + L}}$$

obtain Gibbons-Hawking form

$$ds^2 = \left(\epsilon + \frac{L}{r}\right) d\vec{x}^2 + \frac{rL^2}{\epsilon r + L}\sigma_3^2$$

Macroscopic or position coordinates

Can define macroscopic coordinates invariantly via Hyperkähler moment maps

$$\mu_i = Lx_i$$

Have macroscopic identification with $\mathbb{C}^2$

$$R = 2\sqrt{Lr}, \quad W = Rz \in \mathbb{C}^2$$

which agrees with microscopic coordinates near nut.

Scaling properties

The metric in limit $\epsilon = 0$ is flat:

$$ds^2 = |dW_1|^2 + |dW_2|^2$$

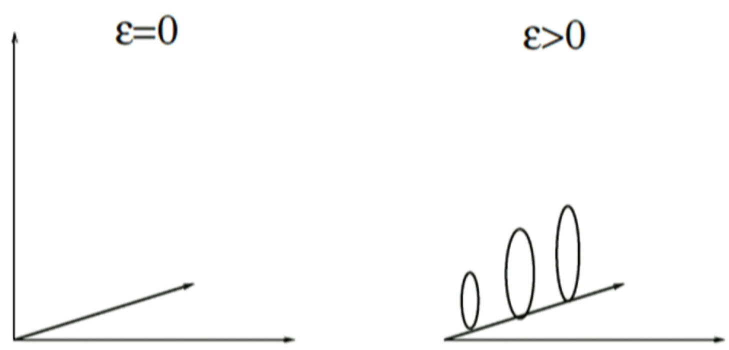


Figure: Taub-NUT from flat space

Abelian Instantons

For arbitrary $p \in \mathbb{R}$ and on static Taub-NUT, consider gauge field

$$A = \frac{ip}{2}\frac{c^2}{\Lambda^2}\sigma_3 = \frac{ip}{2}\frac{\epsilon r}{\epsilon r + L}\sigma_3,$$

Properties:

- A is $U(2)$-invariant
- The curvature

$$F = dA = \frac{i\epsilon p}{2}\left(\frac{L}{(\epsilon r + L)^2} dr \wedge \sigma_3 - \frac{r}{\epsilon r + L}\sin\theta d\theta \wedge d\varphi\right)$$

  is self-dual and hence co-closed
- F generates $L^2 H^2(TN) \simeq \mathbb{R}$

Zero modes in static case

Dirac operator on Taub-NUT minimally coupled to A

$$\not{D}_p = \begin{pmatrix} 0 & T_p^\dagger \\ T_p & 0 \end{pmatrix},$$

has

$$\dim \ker T_p^\dagger = 0, \qquad \dim \ker T_p = \frac{1}{2}[|p|]\,([|p|]+1)$$

where $[x]$ is integer strictly less than x (Pope).

Explicit form of solutions for fixed total angular momentum j (R Jante, BJS)

$$\Psi(r, z_1, z_2) = \begin{pmatrix} cr^{j-\frac{1}{2}} e^{((2j+1)-p)\frac{\epsilon r}{2L}} \sum_{m=-j}^{j} a_m z_1^{j-m} z_2^{j+m} \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

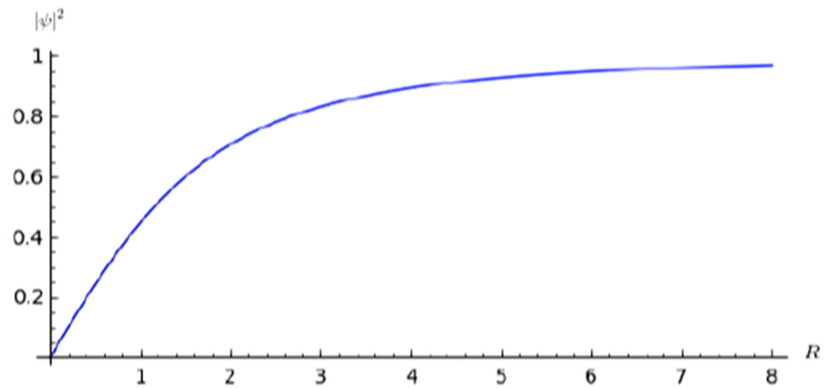
Spin 1/2 and microscopic coordinates

For $p = 2$, $j = 1/2$ doublet as model for spin states:

Non-square integrable ‘vortex’ form is **linear** function of the microscopic coordinates

$$\Psi_{\frac{1}{2}}(r, z_1, z_2) = \begin{pmatrix} a_{-\frac{1}{2}} w_1 + a_{\frac{1}{2}} w_2 \\ 0 \\ 0 \\ 0 \end{pmatrix} .$$

compare Skyrmion spin states!



The limit $\epsilon = 0$

In limit

$$\epsilon \to 0, \quad \epsilon p \to \tilde{p} \neq 0,$$

the curvature is essentially the Kähler form on $\mathbb{C}^2$:

$$F = \frac{\tilde{p}}{4L^2} \left(dW_1 \wedge d\bar{W}_1 + dW_2 \wedge \bar{W}_2 \right) .$$

Constant magnetic field!

For fixed j and m, the non-vanishing spinor component become

$$W_1^{j-m} W_2^{j+m} e^{-\tilde{p}\frac{|W|^2}{2L}} .$$

The limit $\epsilon = 0$

In limit

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Constant magnetic field!

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Landau ground state!

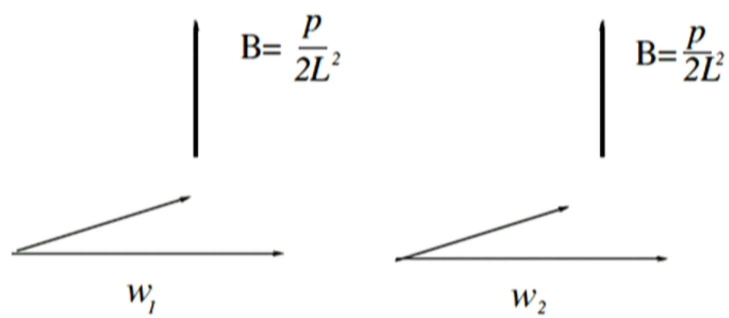


Figure: Landau levels in constant magnetic field

Full spectrum of Dirac operator on Taub-NUT and Euclidean Schwarzschild coupled to self-dual Maxwell field

(with R Jante)

Taub-NUT spectrum is exactly computable via dynamical symmetries

- For fixed '$U(1)$ charge' s satisfying

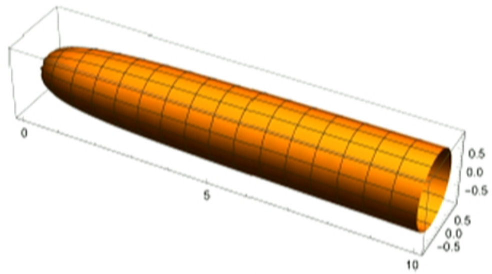
$$s^2 < \frac{p^2}{4}$$

  there are infinitely many Coulomb-like bound states with binding energies

$$E = \frac{2}{L^2}\left[-n^2 + s\left(s - \frac{p}{2}\right)\right] + \frac{2n}{L^2}\sqrt{n^2 - s^2 + \frac{p^2}{4}}, n = |s|+1, |s|+2, \ldots$$

- Modified p-dependent expression for Runge-Lenz vectors
- Combination of '**Landau problem in fibre**' and '**Coulomb problem in base**'

Bound states and Clairaut's theorem



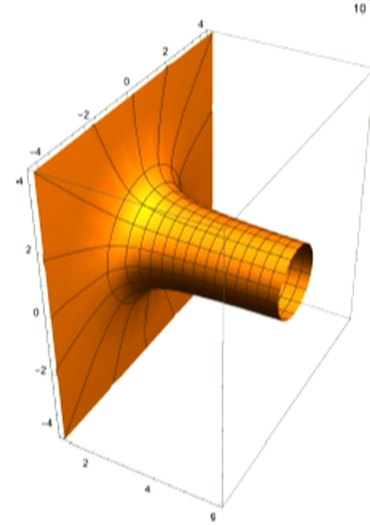


Figure: Positive and negative mass TN

Including Time

Time-dependent Taub-NUT

Allow ϵ to vary with time and consider $ds^2 = -dt^2 + g_{TN}(t)$.
With

$$V = \epsilon(t) + \frac{L}{r},$$

the Ricci scalar is

$$S = \frac{2r}{(\epsilon(t)r + L)} \frac{d^2\epsilon}{dt^2}$$

and the Ricci tensor is

$$\mathrm{Ric}_{\mu\nu} = \mathrm{diag}(-2, 1, 1, -1, 1)\frac{S}{4}$$

Time-dependent Taub-NUT

Solution of 4+1 vacuum Einstein equation:

$$\epsilon = \alpha t + \beta, \quad \alpha, \beta \in \mathbb{R}.$$

(First in Gibbons, Lü and Pope, BraneWorlds in Collision, Phys. Rev. Lett. 94 (2005) 131602)

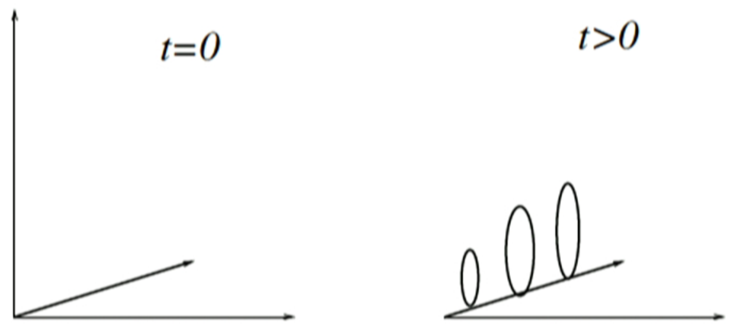


Figure: Taub-NUT from flat space: $\epsilon = t$

NB: This interpolates between
smooth 4+1 Taub-NUT ($t > 0$) $\leftrightarrow \mathbb{R}^5 \leftrightarrow$ singular 5 Taub-NUT ($t < 0$)

Time-dependent Maxwell fields

The gauge field

$$A = \frac{ip(t)}{2}\frac{\epsilon(t)r}{\epsilon(t)r + L}\sigma_3$$

satisfies the Maxwell equation $d \star dA = 0$ iff

$$\ddot{p} = 0, \qquad \ddot{\epsilon} = 0.$$

Conclusion: Adiabatic time evolution

$$\epsilon(t) = \alpha t + \beta, \qquad p(t) = \gamma t + \delta$$

solves Einstein and Maxwell - but **not** coupled Einstein-Maxwell.

Time-dependent Dirac zero modes

Allowing ϵ and p to vary linearly, the Dirac equation for time-dependent spinors is

$$\gamma^0 \left(\frac{\partial}{\partial t} + \frac{1}{2} \frac{\dot{\epsilon} r}{\epsilon r + L} \right) \Psi + \not{D}_p \Psi = 0$$

Obtain **exact adiabatic** solutions if j is fixed and p is constant and quantised:

$$p(t) = (2j + 1)$$

Solution have non-intergrable adiabatic form

$$\frac{r^j}{\sqrt{\epsilon(t) r + L}} e^{((2j+1) - p(t)) \frac{\epsilon(t) r}{2L}} \sum_{m=-j}^{j} a_m z_1^{j-m} z_2^{j+m}$$

$$= \frac{r^j}{\sqrt{\epsilon(t) r + L}} \sum_{m=-j}^{j} a_m z_1^{j-m} z_2^{j+m}$$

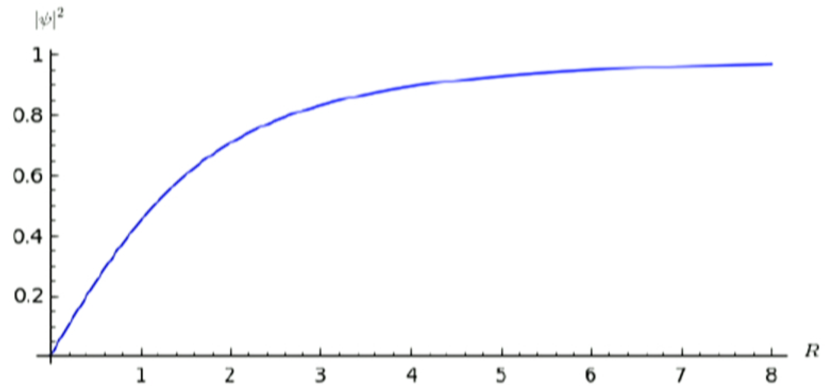
Time dependent spin 1/2 states

The picture for $p = 2$, $j = 1/2$ doublet carries over to time-dependent case: with

$$c(t,r) = \frac{L}{\sqrt{\epsilon(t) + \frac{L}{r}}}, \qquad w(t,r,z) = c(t,r)z,$$

the spin 1/2 states are again

$$\Psi_{\frac{1}{2}}(r, z_1, z_2) = \begin{pmatrix} a_{-\frac{1}{2}} w_1 + a_{\frac{1}{2}} w_2 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$



Length scales and Dirac's Large Number Hypothesis

Atomic units

Set $\epsilon = t$.

so

$$ds^2 = \left(t + \frac{L}{r}\right) d\vec{x}^2 + \frac{L^2 r}{tr + L}\sigma_3^2$$

Asymptotic radius of the circle (= curvature radius of $U(1)$ invariant surface at NUT) is

$$L_m = \frac{L}{\sqrt{t}}$$

In the geometric model of the electron this is the classical electron radius

$$L_m = \frac{e^2}{m_e c^2} \approx 3 \times 10^{-15}\text{m}$$

The Large Number Hypothesis

The ratio

$$\frac{L_M}{L_m} \approx 10^{41}.$$

is one of Dirac's large numbers and should related to age of the universe in atomic units (**LNH**):

$$L_M = tL_m.$$

This is what our model predicts!

Comparison with the Schwarzschild instanton

Schwarzschild geometry revisited

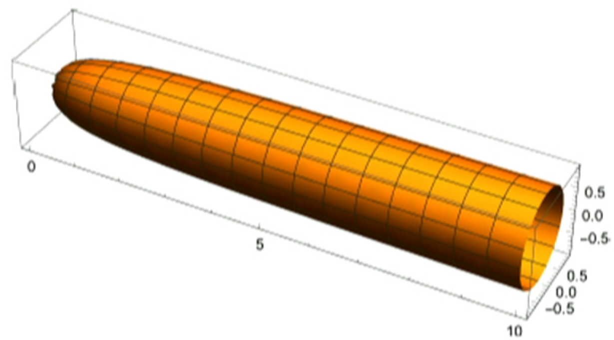
In standard Schwarzschild coordinates

$$ds^2 = r^2(d\theta^2 + \sin^2\theta d\phi^2) + \frac{1}{V}dr^2 + 4L^2 V d\chi^2, \quad V = 1 - \frac{L}{r},$$

Use 'fibre radius' as radial coordinate instead:

$$ds^2 = \frac{L^2}{\left(1 - \frac{c^2}{L^2}\right)^2}(d\theta^2 + \sin^2\theta d\phi^2) + 4\frac{dc^2}{\left(1 - \frac{c^2}{L^2}\right)^4} + 4c^2 d\chi^2.$$

Fibre geometry is that of Taub-NUT!



Gauged Dirac operator

Twisting Dirac operator by abelian instanton

$$F = -\frac{ip}{2}\sin\theta d\theta \wedge d\phi + \frac{ipL}{r^2} dr \wedge d\chi, \quad p \in \mathbb{Z},$$

leads to p^2-dimensional kernel: $|p|$ copies of $|p|$-dimensional irrep of $SU(2)$.

Zero-modes in complex coordinates on S^2 ($q = 1 + |z|^2$):

$$\tilde{\Psi} = \begin{pmatrix} 0 \\ 0 \\ e^{-in\chi} c^n a^{\frac{n}{2}-\frac{3}{4}} e^{(-p+n+\frac{1}{2})\frac{a}{2L}} q^{\frac{1}{2}(1-p)} \sum_{k=0}^{p-1} a_k z^k \\ 0 \end{pmatrix}, \; p \geq 1, \; 0 \leq n \leq p-1.$$

Spectrum of gauged Laplace operator

N	$E(10, 4, N)$	λ for $j = 5$	λ for $j = 6$	λ for $j = 7$
5	0.3095	0.3133	0.5107	0.6371
6	0.4984	0.5008	0.6290	0.7153
7	0.6208	0.6223	0.7097	0.7711
8	0.7041	0.7051	0.7670	0.8122
9	0.7630	0.7637	0.8091	0.8432
10	0.8061	0.8066	0.8409	0.8672
11	0.8386	0.8390	0.8654	0.8861
12	0.8636	0.8639	0.8846	0.9013
13	0.8833	0.8835	0.9001	0.9136
14	0.8990	0.8992	0.9127	0.9238
15	0.9118	0.9119	0.9230	0.9323

Table: TN approximation and numerically computed eigenvalues for ES Laplacian for $p = 10$, $n = -8$ and $j = 5, 6, 7$.

- Taub-NUT is naturally a smooth model of a unit-charge and fermonic particle
- Exact adiabatic solution of Einstein, Maxwell and Dirac equation
- Quantisation of parameters from dynamics
- Time-dependent model of the electron in spirit of Dirac's LHN
- Generalisation to multi-center Taub-NUT would allow study of multi-electron states