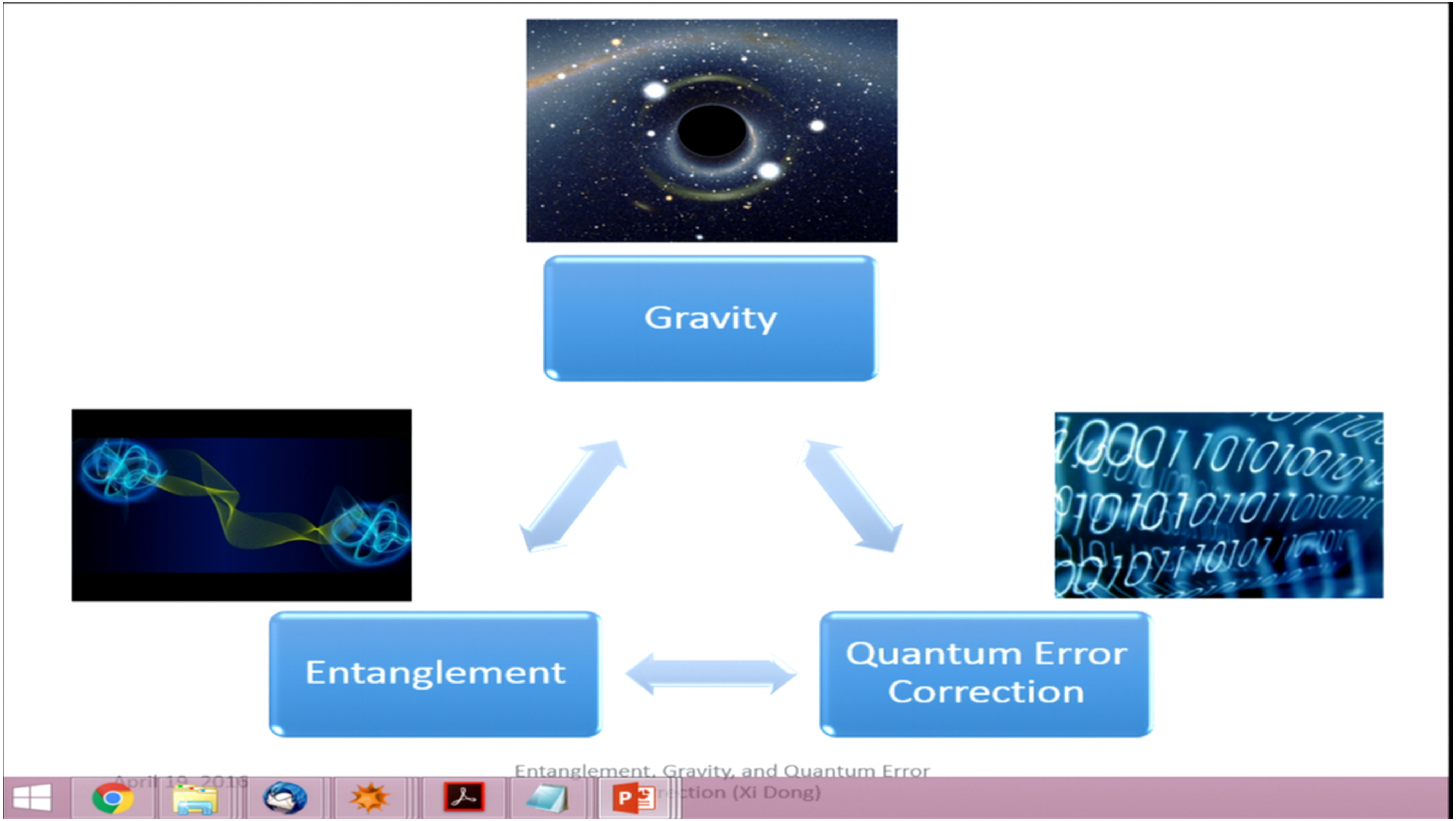
Title: Entanglement, Gravity, and Quantum Error Correction

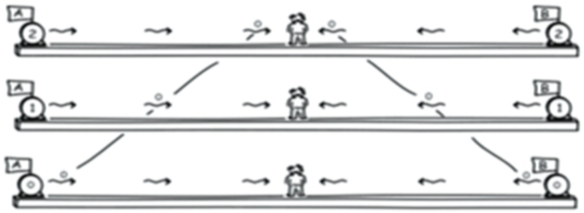
Date: Apr 19, 2016 02:00 PM

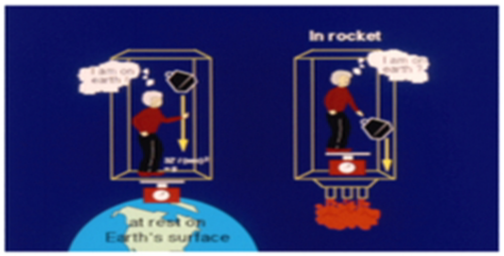
URL: http://pirsa.org/16040057

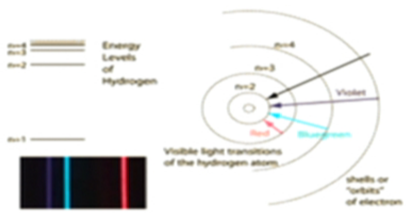
Abstract: <p>Over the last few years it has become increasingly clear that there is a deep connection between quantum gravity and quantum information. The connection goes back to the discovery that black holes carry entropy with an amount given by the horizon area. I will present evidence that this is only the tip of the iceberg, and prove that a similar area law applies to more general Renyi entanglement entropies. To demonstrate the simplicity of this prescription, I will use it to calculate the mutual Renyi information between two disks of arbitrary dimension. Furthermore, I will provide quantum corrections to the area law and use it to solve the following important problem: what region of the dual spacetime is described by a subregion in a holographic theory? The answer to this question lies in a new perspective that I will advocate: holography is a quantum error correcting code.</p>


Gravity
Entanglement
Quantum Error Correction
Entanglement, Gravity, and Quantum Error
ction (Xi Dong)

Breakthroughs inspired by surprises







- **Special Relativity:**

The speed of light is identical for all observers.

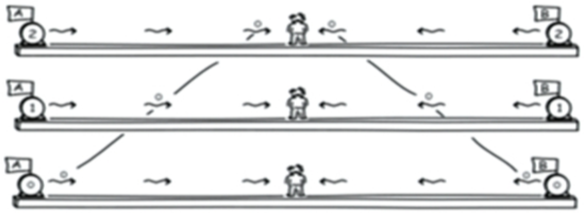
- **General Relativity:**

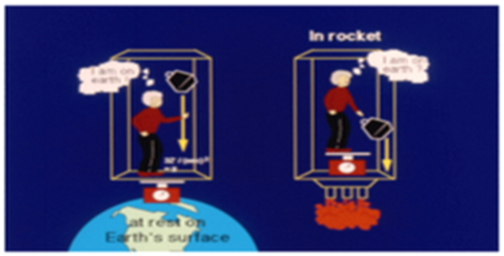
Gravity is indistinguishable from acceleration.

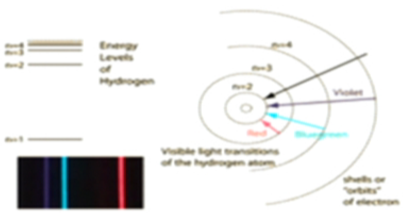
- **Quantum Mechanics:**

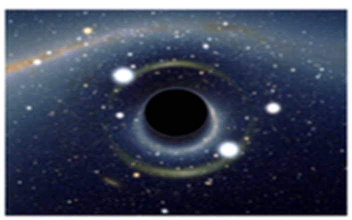
Energy is “quantized.”

Breakthroughs inspired by surprises









- **Special Relativity:**

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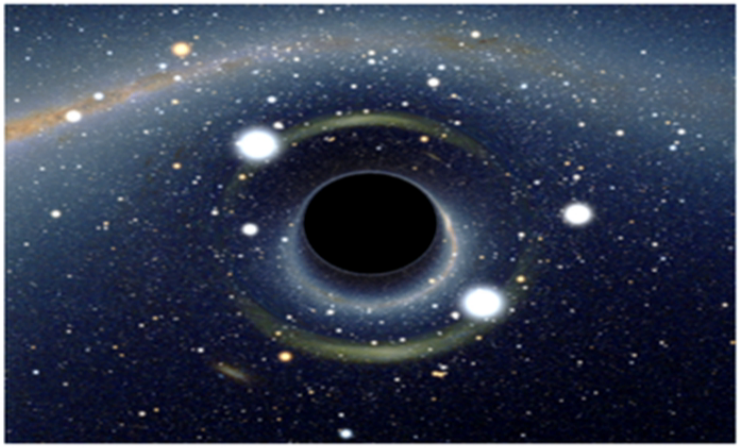
- **Quantum Gravity (?):**

Entropy is given by the area.

Bekenstein-Hawking entropy for black holes ['73, '75]

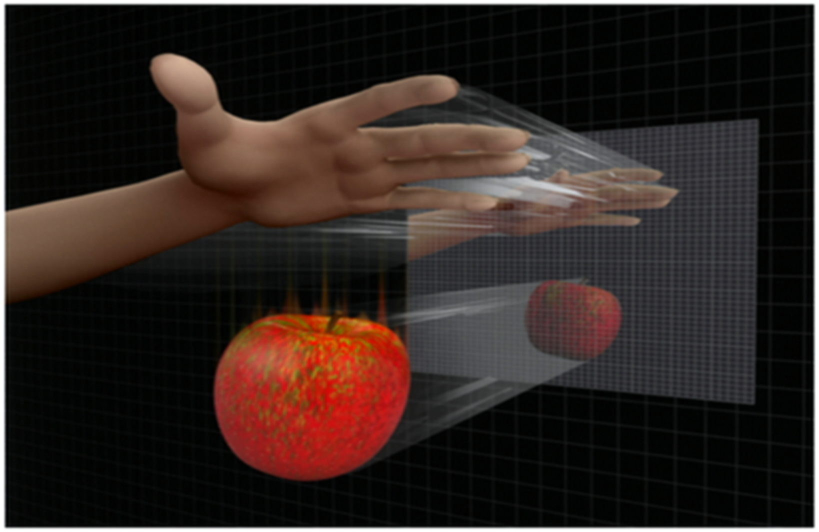


$$S = \frac{k c^3 \,\mathrm{Area(Horizon)}}{4 G_N \hbar}$$

- Incorporates quantum mechanics, gravity, special relativity, holography, and statistical mechanics.
- Led to much progress in understanding quantum gravity, e.g. holographic principle. ['t Hooft '93; Susskind '94]

Anti-de Sitter/Conformal Field Theory Correspondence

[Maldacena '97]

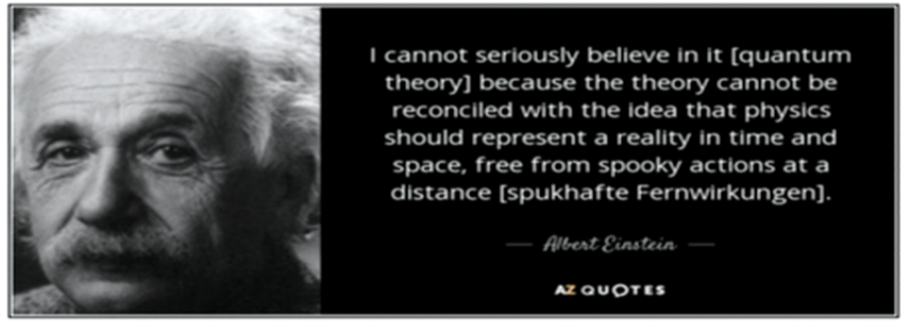


Quantum gravity in AdS_{d+1} (bulk)	Holographic CFTs on ∂AdS_{d+1} (boundary)
Isometry group $O(d,2)$	Conformal group $O(d,2)$
Black hole states	Thermal states
Gauge symmetry	Global symmetry
States and operators	States and operators

- Best-understood model of quantum gravity
- Concrete example of emergent spacetime/gravity
- Also known as gauge/gravity duality
- Framework for generalizing area law beyond black holes

Quantum entanglement

- Correlation between subsystems
- "Spooky action at a distance"



- Simplest example:

$$|\Psi\rangle = |\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle$$

- Entanglement entropy:

$$S \stackrel{\text{def}}{=} -\mathrm{Tr}(\rho \ln \rho)$$

- For example:

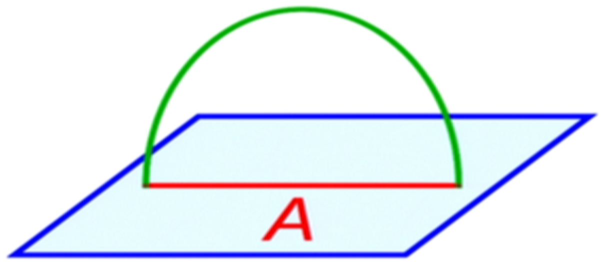
$$\rho = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix} \text{ gives } S = \ln 2$$

- For 200 spins, density matrix ρ has 2^{400} elements, more than the total number of atoms in our universe.

Holographic Entanglement Entropy

A remarkably simple prescription for von Neumann entropy:

$$S = \frac{\text{Area}(\text{Minimal Surface})}{4G_N}$$



[Ryu & Takayanagi '06]

Recall the definition: $S \stackrel{\text{def}}{=} -\text{Tr}(\rho_A \ln \rho_A)$

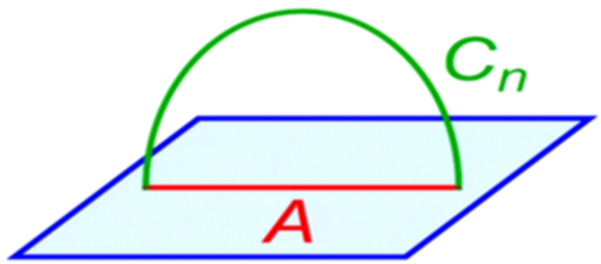
This is only the tip of the iceberg!

Holographic Renyi Entropy

A simple and powerful prescription for Renyi entropy:

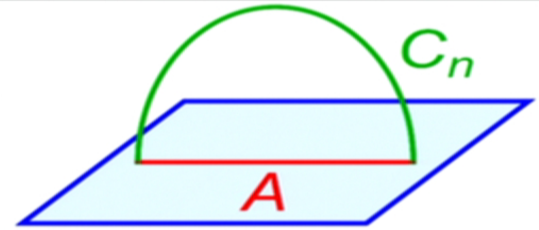
$$n^2\partial_n\left(\frac{n-1}{n}S_n\right) = \frac{\text{Area}(\text{Cosmic Brane}_n)}{4G_N}$$

[XD 1601.06788]



Gravity dual of Renyi entropy is a cosmic brane!

Area Law for Renyi Entropy

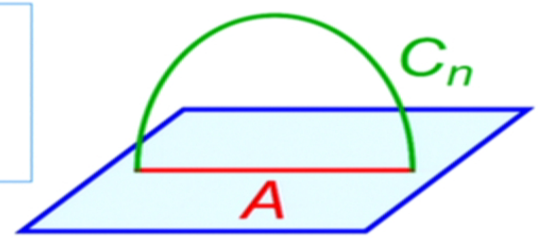


$$n^2 \partial_n \left(\frac{n-1}{n} S_n \right) = \frac{\text{Area(Cosmic Brane}_n)}{4G_N}$$

[XD 1601.06788]

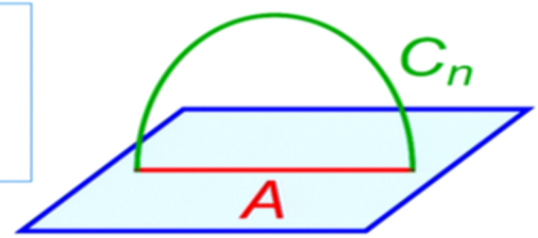
- Renyi entropy $S_n \overset{\text{def}}{=} \frac{1}{1-n} \ln \mathrm{Tr} \rho_A^n$
- For example: $\rho = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$ gives $S_n = \frac{1}{1-n} \ln(\alpha^n + \beta^n)$
- One-parameter generalization of von Neumann entropy
- Is a measure of entanglement containing much richer information about the state than von Neumann entropy
- Possible to measure in experiments or study numerically
- But still very complicated for large systems [Islam et al. '15]

$$n^2\partial_n\left(\frac{n-1}{n}S_n\right)=\frac{\text{Area}(\text{Cosmic Brane}_n)}{4G_N}$$



- Cosmic brane similar to minimal surface; they are both codimension-2 and anchored at edge of A.
- But brane is different in having tension $T_n=\frac{n-1}{4nG_N}$.
- Backreacts on ambient geometry by creating conical deficit angle $2\pi\frac{n-1}{n}$.
- Useful way of getting the geometry: find solution to classical action $I_{\text{total}}=I_{\text{bulk}}+I_{\text{brane}}$.
- As $n\to 1$: probe brane settles at minimal surface.
- One-parameter generation of Ryu-Takayanagi.

$$n^2\partial_n\left(\frac{n-1}{n}S_n\right) = \frac{\text{Area}(\text{Cosmic Brane}_n)}{4G_N}$$



- Why does this area law work?
- Because LHS is a more natural candidate for generalizing von Neumann entropy:

$$\widetilde{S_n} \overset{\text{def}}{=} n^2\partial_n\left(\frac{n-1}{n}S_n\right) = -n^2\partial_n\left(\frac{1}{n}\ln \text{Tr}\rho_A^n\right)$$

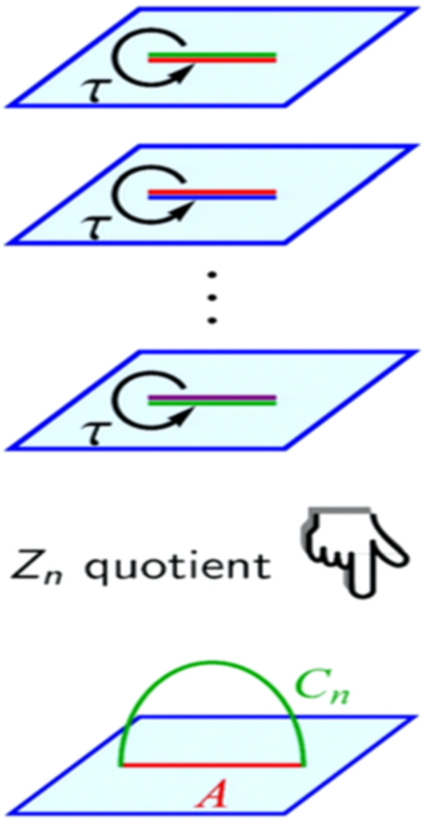
- This is standard thermodynamic relation

$$\widetilde{S_n} = -\frac{\partial F_n}{\partial T}$$ with $F_n = -\frac{1}{n}\ln \text{Tr}\rho_A^n, \quad T = \frac{1}{n}$

$\text{Tr}\rho_A^n$ is partition function w/ modular Hamiltonian $-\ln\rho_A$

- $\widetilde{S_n} \geq 0$ generally. [Beck & Schögl '93] Automatic by area law!

$$n^2 \partial_n \left(\frac{n-1}{n} S_n \right) = \frac{\text{Area}(\text{Cosmic Brane}_n)}{4G_N}$$



Can be derived from gauge/gravity duality:

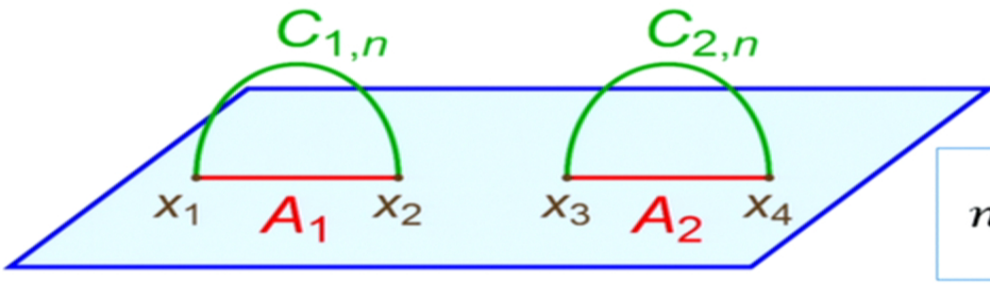
[XD 1601.06788]

- Replica Trick
- Gauge/Gravity Duality
- Quotient by Z_n Symmetry

→ Classical Mechanics

→ Area Law for Renyi Entropy

Example: Renyi entropy for two disks in holographic CFT

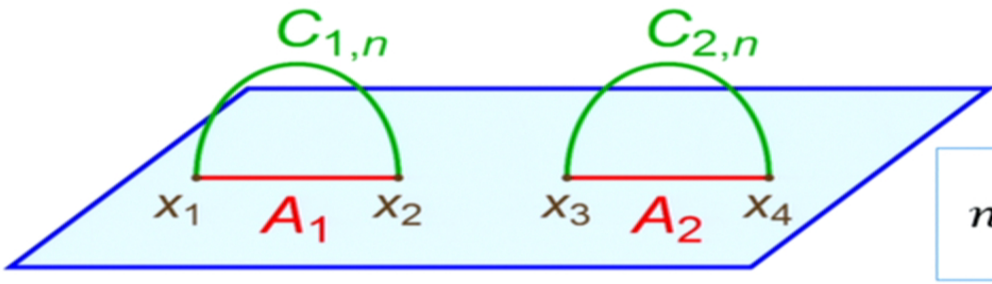


$$n^2 \partial_n \left(\frac{n-1}{n} S_n \right) = \frac{\text{Area}(\text{Cosmic Brane}_n)}{4G_N}$$

- For one disk, it was calculated by exploiting a symmetry and finding hyperbolic black hole solutions. [Hung, Myers, Smolkin & Yale '11]
- Area-law prescription is more powerful: does not need symmetry; applies to arbitrary regions.
- For two disks, we study mutual Renyi information

$$I_n(A_1, A_2) \stackrel{\text{def}}{=} S_n(A_1) + S_n(A_2) - S_n(A_1 \cup A_2)$$

Example: Renyi entropy for two disks in holographic CFT



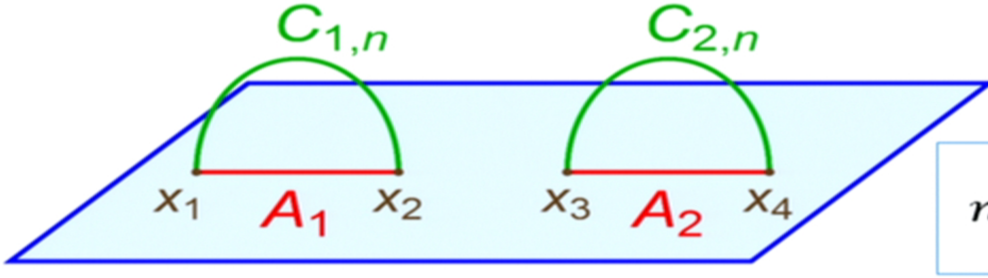
$$n^2 \partial_n \left(\frac{n-1}{n} S_n \right) = \frac{\text{Area(Cosmic Brane}_n)}{4G_N}$$

- Two phases depending on the separation.
- Mutual information $I_{n=1}$ vanishes in large-distance phase, but I_n is generally nonzero.
- Two cosmic branes feel backreaction of each other.
- To linear order in $\delta n = n - 1$, backreaction is weak:

$$I_n = \frac{2^{3-d}\pi^{d+1} C_T \delta n}{d(d^2-1)\,\Gamma\left(\frac{d-1}{2}\right)^2} \frac{2-x}{x} B\left(\left(\frac{x}{2-x}\right)^2; \frac{d+1}{2}, \frac{2-d}{2}\right) + O(\delta n^2)$$

[XD 1601.06788]

Example: Renyi entropy for two disks in holographic CFT



$$n^2 \partial_n \left(\frac{n-1}{n} S_n \right) = \frac{\text{Area}(\text{Cosmic Brane}_n)}{4G_N}$$

$$I_n = \frac{2^{3-d} \pi^{d+1} C_T \delta n}{d(d^2-1)\, \Gamma\left(\frac{d-1}{2}\right)^2} \frac{2-x}{x} B\left(\left(\frac{x}{2-x}\right)^2 ; \frac{d+1}{2}, \frac{2-d}{2} \right) + O(\delta n^2)$$

[XD 1601.06788]

- Here $x = \frac{(x_1 - x_2)(x_3 - x_4)}{(x_1 - x_3)(x_2 - x_4)}$ is a cross ratio.
- C_T is a central charge appearing in stress tensor two-point function.
- Applies to any dimension; agrees with a CFT calculation for the special case of $d = 2$ in e.g. [Perlmutter '15].

Shape dependence of Renyi entropy

vs

- Universal part of Renyi entropy in 4D CFT depends simply on shape of entangling surface Σ:

$$S_n^{\text{univ}} = \left[\frac{f_a(n)}{2\pi}\int_\Sigma R_\Sigma + \frac{f_b(n)}{2\pi}\int_\Sigma \operatorname{tr} K^2_{\text{traceless}} - \frac{f_c(n)}{2\pi}\int_\Sigma C^{ab}{}_{ab}\right] \ln \epsilon$$

[Fursaev '12]

- For von Neumann entropy ($n = 1$):

$$f_a(1) = a\,,\ \ f_b(1) = f_c(1) = c$$

[Solodukhin '08]

- A conjecture:

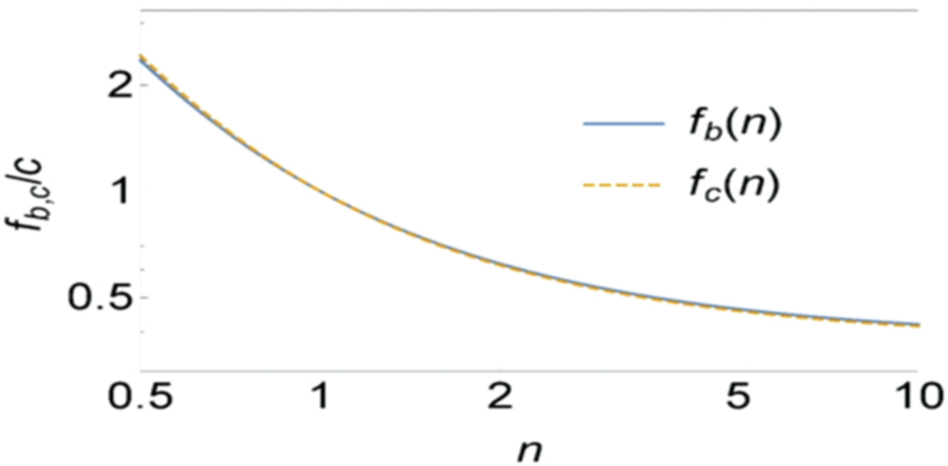
$$f_b(n) = f_c(n)$$

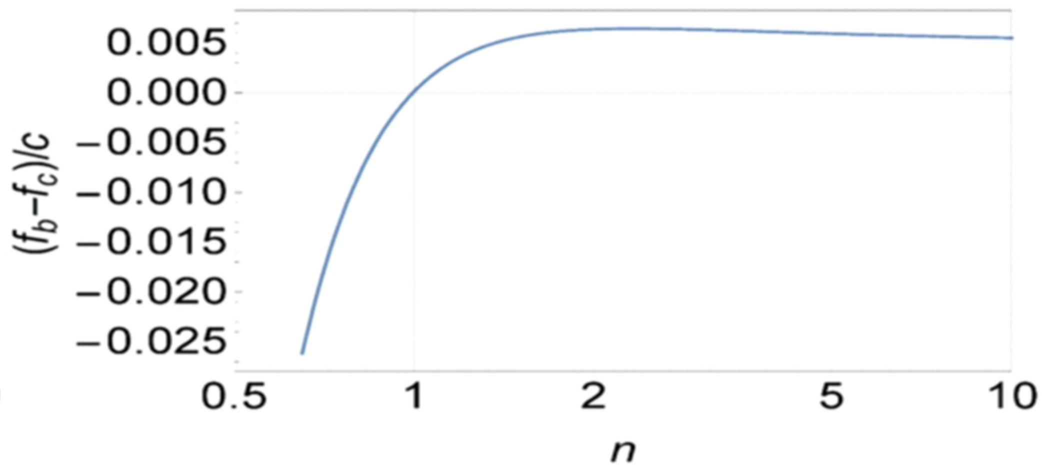
[Lee, McGough & Safdi '14]
[Lewkowycz & Perlmutter '14]
[Bueno & Myers '15]
[Bianchi, Meineri, Myers & Smolkin '15]

Shape dependence of Renyi entropy

$$S_n^{\text{univ}} = \left[\frac{f_a(n)}{2\pi}\int_\Sigma R_\Sigma + \frac{f_b(n)}{2\pi}\int_\Sigma \operatorname{tr} K_{\text{traceless}}^2 - \frac{f_c(n)}{2\pi}\int_\Sigma C^{ab}{}_{ab}\right]\ln\epsilon$$

- Unfortunately the $f_b(n) = f_c(n)$ conjecture fails in holographic CFTs: [XD 1602.08493]





$$f_b'(1) = -\frac{11}{12}c\,, \quad f_c'(1) = -\frac{17}{18}c$$

So far:

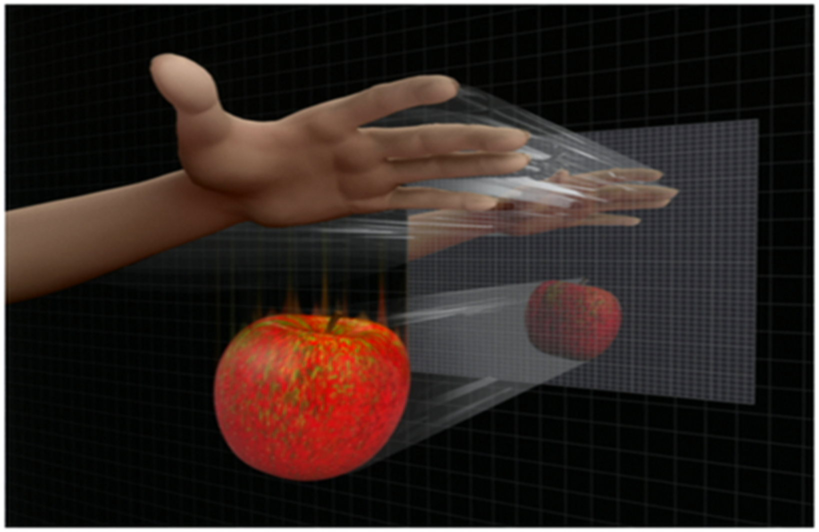
We used the area law to understand structure of quantum entanglement and to efficiently study Renyi entropies.

Rest of the talk:

How to use it to understand quantum gravity?

To motivate the question, we need to take a step back.

AdS/CFT: our best-understood model of quantum gravity [Maldacena '97]

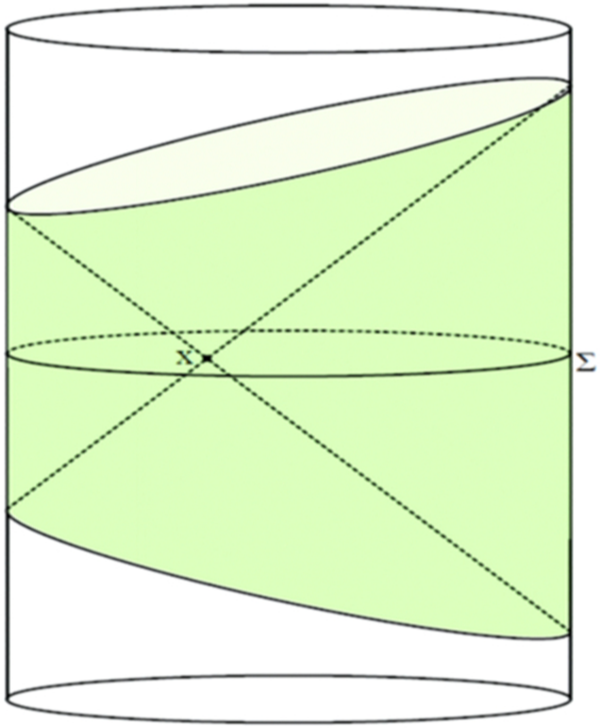


Quantum gravity in AdS_{d+1}	Holographic CFTs on ∂AdS_{d+1}
Isometry group $O(d,2)$	Conformal group $O(d,2)$
Black hole states	Thermal states
Gauge symmetry	Global symmetry
States and operators	States and operators

$$\lim_{r\to\infty} r^{\Delta}\,\phi(r,x) = O(x)$$
$$\phi(r,x) = ?$$

- What operator in CFT represents a local bulk operator?
- Answering this question helps us reconstruct the bulk.

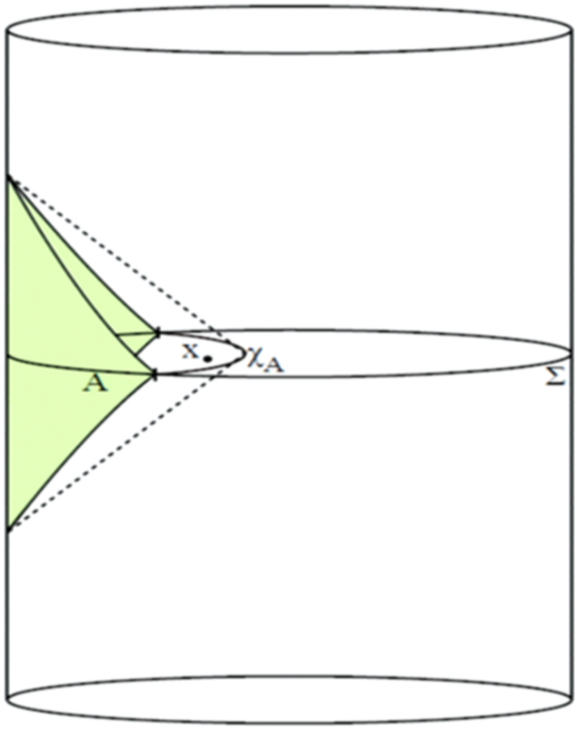
Global AdS reconstruction



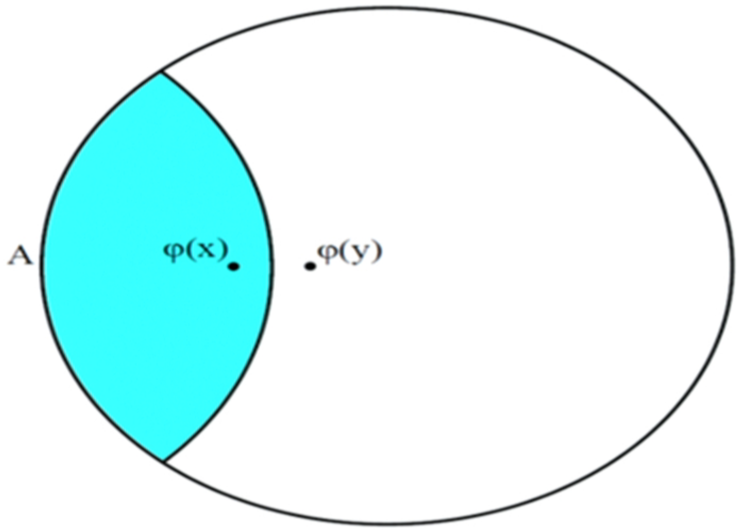
$$\phi(x) = \int_{\mathbb{S}^{d-1}\times\mathbb{R}} dY\, K(x;Y)\mathcal{O}(Y)$$

[Hamilton, Kabat, Lifschytz, Lowe '06]

AdS-Rindler reconstruction for disk A



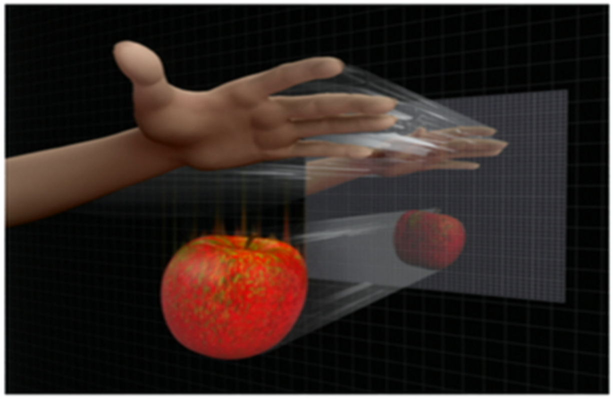
$$\phi(x) \sim \int_{D[A]} dY\, K_A(x;Y)\mathcal{O}(Y)$$



April 19, 2016

Entanglement, Gravity, and Quantum Error Correction (Xi Dong)

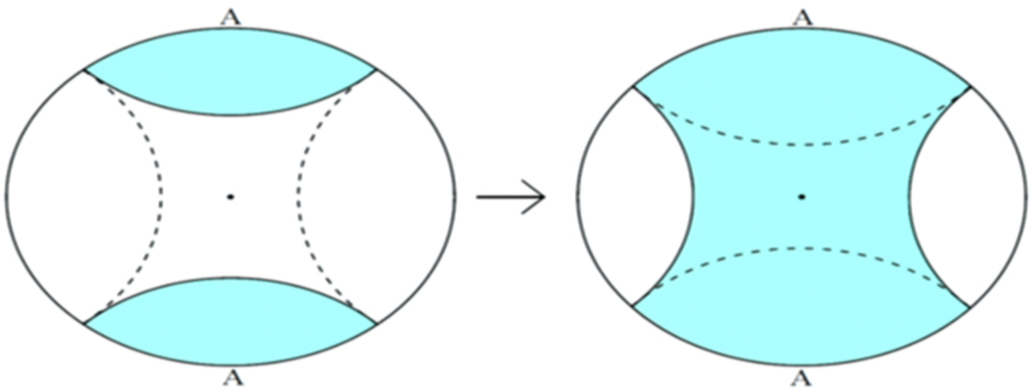
What region of the dual spacetime is described by a general subregion in a holographic CFT?



So far the discussion does not seem to involve any concept in quantum information theory, but that is about to change.

Reconstruction conjecture for entanglement wedge

- **Entanglement wedge** is defined as a bulk region bounded by the Ryu-Takayanagi minimal surface.
- It may change discontinuously.
- Conjecture: Any bulk operator in entanglement wedge of A may be represented as a CFT operator on A.



[Czech, Karczmarek, Nogueira & Van Raamsdonk '12]
[Wall '12]
[Headrick, Hubeny, Lawrence & Rangamani '14]

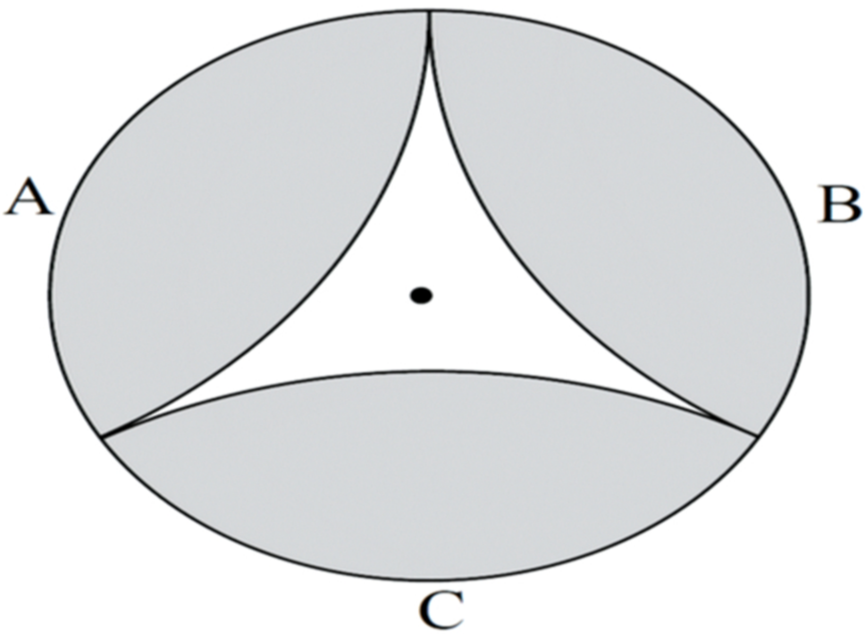
Conjecture:
Any bulk operator in entanglement wedge of A may be represented as a CFT operator on A.

Ingredients for proving the conjecture:

- ➢Holography as a quantum error correcting code
- ➢CFT relative entropy = bulk relative entropy

⇒Reconstruction theorem for entanglement wedge

Puzzle about AdS-Rindler reconstruction



- $\phi(x)$ can be represented on $A \cup B$, $B \cup C$, or $A \cup C$.
- Obviously they cannot be the same CFT operator.
- Defining feature for quantum error correction.
- Holography is a quantum error correcting code.

[Almheiri, XD, Harlow '14]

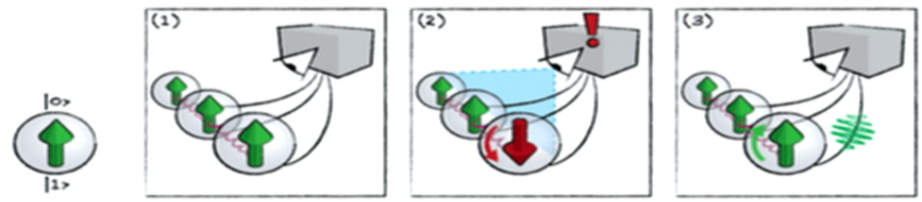
Quantum error correction

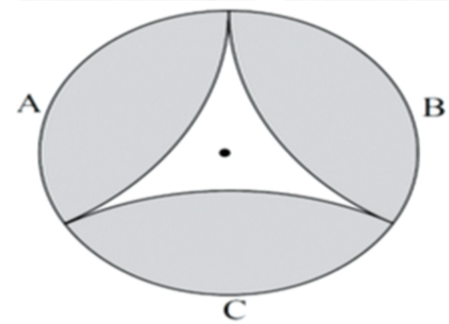
- Alice wants to send a qutrit by mail.
- She encodes it into the Hilbert space of 3 qutrits.

$$|\widetilde{0}\rangle = \frac{1}{\sqrt{3}}\left(|000\rangle + |111\rangle + |222\rangle\right)$$

$$|\widetilde{1}\rangle = \frac{1}{\sqrt{3}}\left(|012\rangle + |120\rangle + |201\rangle\right)$$

$$|\widetilde{2}\rangle = \frac{1}{\sqrt{3}}\left(|021\rangle + |102\rangle + |210\rangle\right)$$



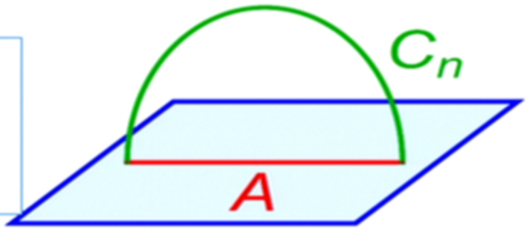


- These states span the code subspace.
- In holography, code subspace contains bulk states.

[Almheiri, XD, Harlow '14]

Quantum corrections to area law

$$n^2 \partial_n \left(\frac{n-1}{n} S_n \right) = \frac{\text{Area}(\text{Cosmic Brane}_n)}{4G_N}$$



- Area law for Renyi entropy receives both higher derivative corrections and quantum corrections.
- This is analogous to case of entanglement entropy.

[XD '13; XD & Miao '15] [Barrella, XD, Hartnoll & Martin '13; Faulkner, Lewkowycz & Maldacena '13]

- Quantum corrections in terms of bulk Renyi entropy:

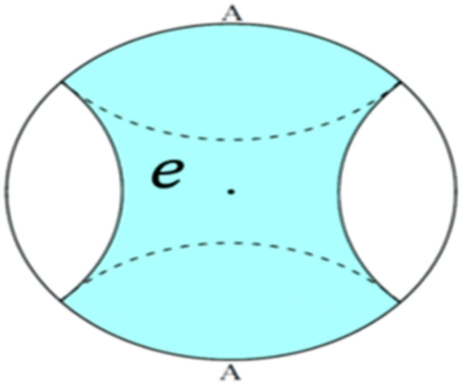
$$\widetilde{S_n} = \frac{\text{Area}(\text{Cosmic Brane}_n)}{4G_N} + \widetilde{S_n}_{,\text{bulk}}$$

[XD 1601.06788]

Reconstruction theorem for entanglement wedge

[XD, Harlow & Wall 1601.05416]

Any bulk operator O_e in the entanglement wedge e of A may be reconstructed as a CFT operator O_A on A via quantum error correction, as long as the relative entropy of any two bulk states satisfies $S(\rho_A|\sigma_A) = S_{\text{bulk}}(\rho_e|\sigma_e)$



Intuitive proof:

- WLG assume O_e is Hermitian.
- Take any bulk state ρ, let σ be $e^{i\lambda O_e}$ acting on ρ.
- Use $S(\rho_{\bar{A}}|\sigma_{\bar{A}}) = S_{\text{bulk}}(\rho_{\bar{e}}|\sigma_{\bar{e}}) = 0$ to conclude $\rho_{\bar{A}} = \sigma_{\bar{A}}$.
- No measurements on $\bar{A}$ distinguish the two states $\Rightarrow [O_e, X_{\bar{A}}] = 0$ under state ρ for any $X_{\bar{A}}$.
- O_e must have a CFT realization as O_A on A.

Via theorem in [Almheiri, XD, Harlow '14]

What We Learned

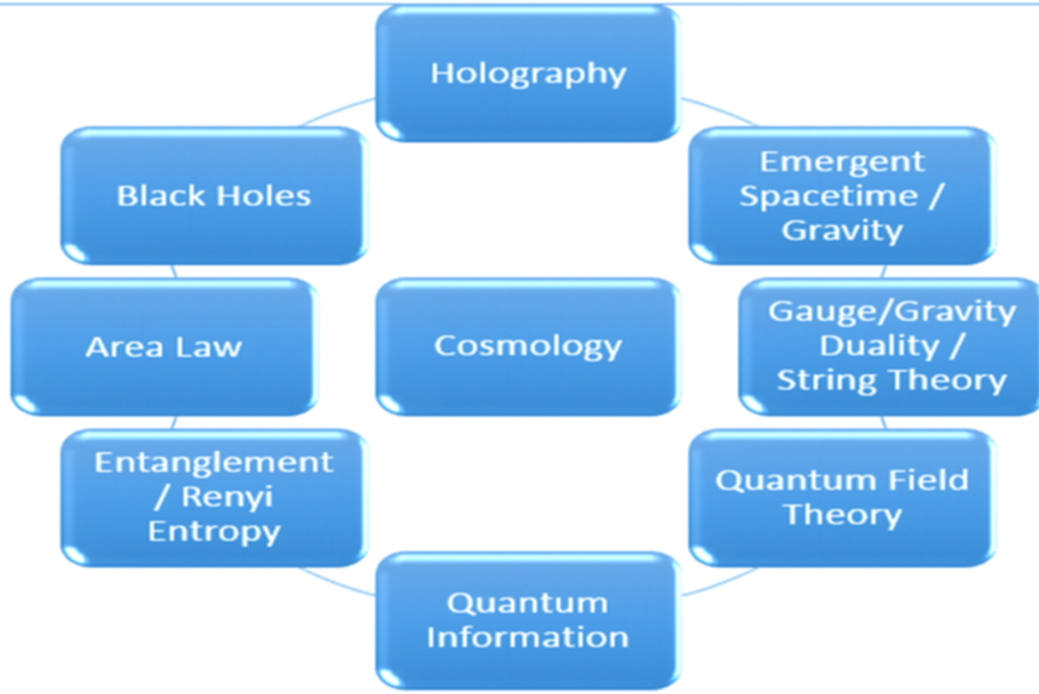
- Area law is universal in quantum gravity and is not restricted to black hole or entanglement entropy.
- It is a powerful statement for Renyi entropy, generalizing the Ryu-Takayanagi prescription.
- Quantum information theory enables us to understand the basic dictionary of quantum gravity.
- Viewing holography as a quantum error correcting code, we can analyze how to "build spacetime from entanglement".

Near-Future Directions

- Renyi entropy has until now been more difficult to study than entanglement entropy. The area law opens a new window for efficiently studying Renyi entropy in strongly coupled systems.
- Is the natural entropy $\widetilde{S_n}$ a more useful measure of quantum entanglement and information?
- Simple explicit reconstruction of operators in the entanglement wedge?
- Derive the Engelhardt-Wall conjecture for quantum correction to all orders in $1/N$ from first principle. [XD & Lewkowycz, to appear]

Long-Term Outlook

- What quantum field theories give rise to emergent, weakly coupled gravity?
- What are their universal features that allow us to eventually solve quantum gravity in our universe?



Computing in the Cloud

Computing in the Bulk (or Black Hole)