Title: Variational Principle for Gravity with Null Boundaries

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Abstract: <p>The fact that the Einstein-Hilbert action, by itself, does not lead to a well-posed variational principle has become textbook knowledge. It can be made well-posed by the addition of suitable boundary terms. There are many boundary terms available in the literature, of which the most famous and most widely used is the Gibbons-Hawking-York (GHY) boundary term. The GHY term is ostensibly defined only for a non-null boundary. There have been very few efforts in the literature to extend its definition to null boundaries. The speaker will present his group's work towards finding a boundary term to render the Einstein-Hilbert action well-posed in the presence of null boundaries. He shall discuss a proposed boundary term, associated boundary conditions, outstanding issues and possible applications.</p>

Variational Principle for Gravity with Null Boundaries

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Quantizing Gravity

Quantum and Gravity

Constructing an Action for GR

Firstly, it's elegant. In fact, it's not just elegant: it's completely gorgeous...
Secondly, it's more powerful...
Finally, and most importantly, it is universal...
and reveals a deep relationship between classical mechanics and quantum mechanics. This is the real reason why it's so important.
—Daniel Baumann, on the least action principle

Constructing an Action for General Relativity

- Assumption:

  Dynamical variable for gravitational field = the metric.

- First attempt: a Lagrangian density made up of the metric and its first derivatives.

- The action should respect the symmetries of the theory
  $\Rightarrow$ Lagrangian density a scalar density.

- Only such scalar density: $\lambda\sqrt{-g}$, where λ is a constant.

- Extreme Machian theory! No Schwarzschild solution.

Lanczos-Lovelock Lagrangians

- ✠ Principle of equivalence+General covariance+Second order field equations⇒ Lanczos-Lovelock Lagrangians

- ✠ D=4⇒ Einstein-Hilbert (with Λ) + Gauss-Bonnet

- ✠ Gauss-Bonnet term is generally thrown away- since total divergence and does not contribute to equations of motion [Yale, Padmanabhan (2011)]

Einstein-Hilbert Action and Its Variation

✠ The Einstein-Hilbert action is varied with the metric and its derivatives as variables:

$$\int_{\mathcal{V}} d^4x\, \delta(\sqrt{-g}R) = -\int_{\mathcal{V}} d^4x\, \sqrt{-g}\, G^{ab}\delta g_{ab} - \int_{\partial\mathcal{V}} d^3x\, n_i\sqrt{h}\left[g^{ab}\delta N^c_{ab}\right]$$

✠ Here, $N^c_{ab} = -\Gamma^c_{ab} + \frac{1}{2}\left(\delta^c_a\Gamma^d_{bd} + \delta^c_b\Gamma^d_{ad}\right)$

✠ Since Γ^c_{ab} contains both metric and its derivatives, the boundary variation contains variations of both the metric and its derivatives.

History of Boundary Terms in Gravity

✠ But not a covariant prescription- Gamma-Gamma Lagrangian not a scalar, zero in local inertial frame

✠ ADM Formalism (1959-1962) introduced the decomposition:

$$\sqrt{-g}R = \sqrt{-g}L_{\rm ADM} - \sqrt{-g}L_{\rm GHY}, \qquad L_{\rm ADM} = {}^{(3)}R + K_{ab}K^{ab} - K^a_a K^b_b; \quad L_{\rm GHY} = 2\nabla_i \left(Kn^i + a^i\right) \ .$$

✠ York (1972) took this decomposition- showed that the last term is effectively the integral of $-2\sqrt{h}K$ on the boundary. Does not mention removing this term.

✠ But there is a paper, Gowdy (1970), cited by York. Has the $-2\sqrt{h}K$ separated out and the statement, "Indeed, these end-point terms must be dropped in order to obtain Einstein's field equations from variations which hold only the initial and final hypersurface three-geometries fixed."

History of Boundary Terms in Gravity

✠ Gibbons and Hawking (1977)- "In order to obtain an action which depends only on the first derivatives of the metric, as is required by the path-integral approach," and "so that for metrics g which satisfy the Einstein equations the action I is an extremum under variations of the metric which vanish on the boundary...but which may have non-zero normal derivatives.", add to the Einstein-Hilbert action the term

$$A_{GHY} = 2\int_{\partial\mathcal{V}} d^3x\sqrt{h}K,$$

plus a term depending only on the induced metric on the boundary (can be used to render the action finite in asymptotically flat spacetimes.)

✠ Myers (1987)- showed that the total action then is precisely the Euler character for a 2-dimensional manifold with a boundary.

Two Ways of Dealing with Dynamics

- Solution of a differential equation that is equation of motion:

For free particle, solve $\ddot{q} = 0$ with q and $\dot{q}$ specified at initial time.

- Solution of a Variational Problem:

Extremise the action $S = \int_{t1}^{t2} \dot{q}^2$ with boundary conditions. Boundary term is

$$[\dot{q}\delta q]_{t1}^{t2}$$

which is put to zero by fixing $q(t1)$ and $q(t2)$.

Total Derivatives Change the Variational Problem

- Consider adding a total derivative term $-(1/2)\partial_t\,(pq)$ to the Lagrangian. For free particle, this term becomes $-(1/2)\partial_t\,(q\dot{q})$

- The boundary term in the variation now becomes

$$\frac{1}{2}\left[\dot{q}\delta q - q\delta\dot{q}\right]_{t1}^{t2}$$

- $q, \dot{q}$ to be fixed at both boundaries$\Rightarrow$ Four boundary conditions for a second order equation of motion!

- Unless chosen with special care, conflict with equations of motion is assured! Variational problem not well-posed!

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Padmanabhan- Gravitation

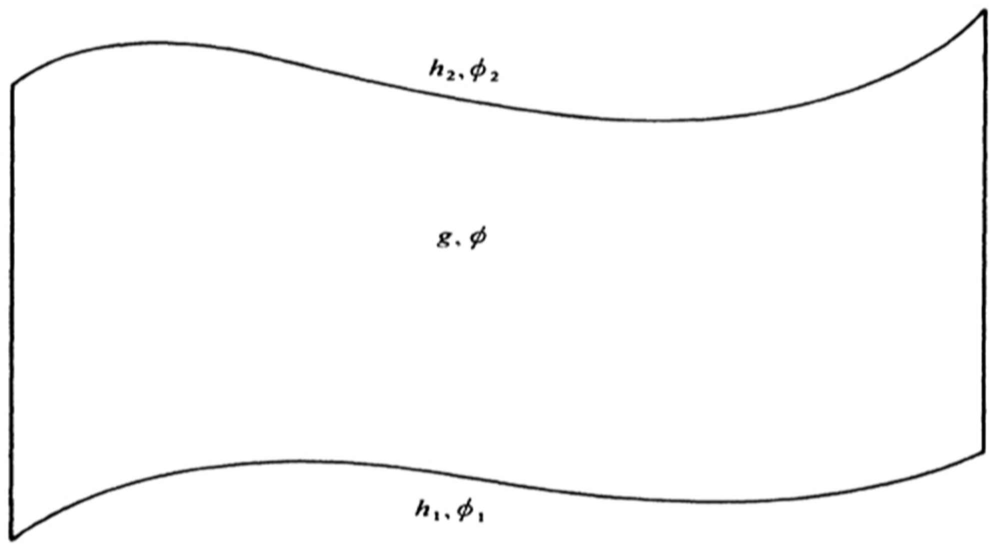
i) The first problem is that, for arbitrary values of q and $\dot{q}$ at both $t = t_1$ and $t = t_2$, we may not have a classical solution satisfying the boundary conditions.

ii) Further, one would like the action principle to obey the composition rule of the following kind. We expect $A(1 \to 2 \to 3) = A(1 \to 2) + A(2 \to 3)$, where the paths connecting (q_1, t_1) and (q_3, t_3) are decomposed at an intermediate time t_2 with $t_1 < t_2 < t_3$. The paths are expected to be continuous at $t = t_2$ but need not be smooth at $t = t_2$. This requires leaving $\dot{q}$ at $t = t_2$ arbitrary in the action principle.

iii) Finally, the action principle has its roots in quantum theory and freezing q and $\dot{q}$ simultaneously will require specifying the values of both coordinate and momentum at a given time, which is inappropriate in the quantum theory.

Hawking's Argument for Path Integrals

- Argument by Hawking- in an Einstein Centenary Survey (1979)



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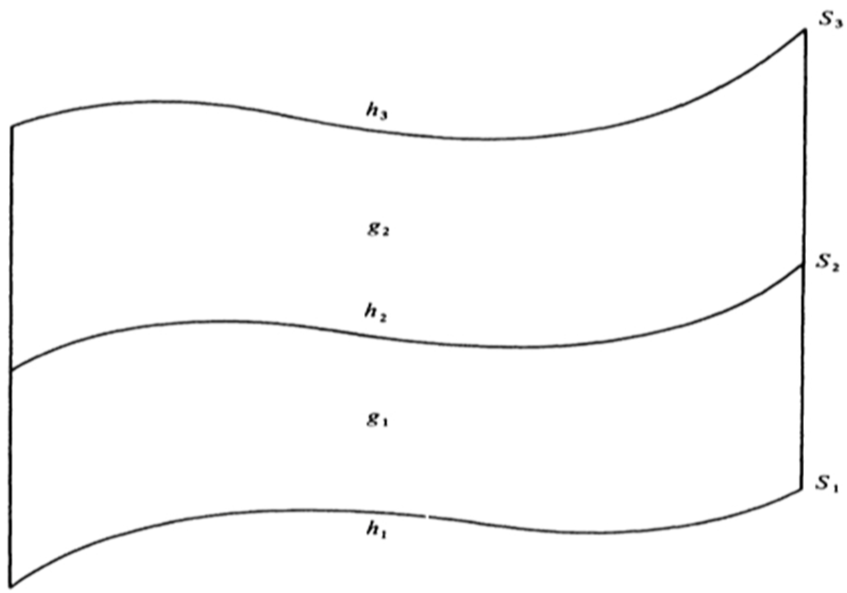

Hawking's Argument for Path Integrals

- Hawking's argument hinges on the assumption that the paths which are continuous in the metric but discontinuous in the first derivatives of the metric are to be included in the path integral.

- But why not only consider those paths which are continuous in first derivatives too?

- Or, to go the other way, why even demand continuity in the metric?

Feynman and Hibbs

- The argument makes use of the formula:

$$\left\langle \left(\frac{x_{k+1} - x_k}{\epsilon} \right)^2 \right\rangle = -\frac{\hbar}{im\epsilon} \langle 1 \rangle$$

As $\epsilon \to 0$, transition element of square of velocity diverges. Typical paths do not have a definite slope.

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As $\epsilon \to 0$, transition element of square of velocity diverges. Typical paths do not have a definite slope.

- The formula can also be rewritten as

$$\langle (x_{k+1} - x_k)^2 \rangle = -\frac{\hbar\epsilon}{im} \langle 1 \rangle$$

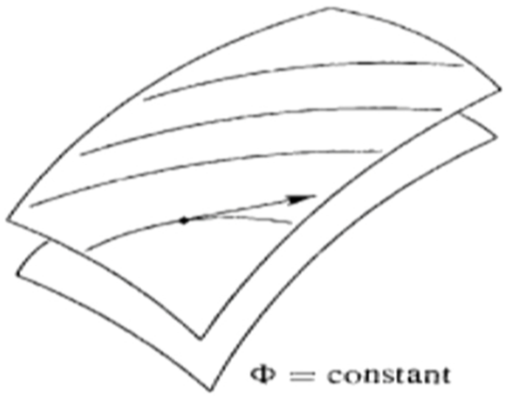
As $\epsilon \to 0$, transition element of displacement goes to zero. Typical paths are continuous.

- These conclusions essentially follow from the equations of motion. So, should be valid even for Einstein-Hilbert action, which is a higher derivative action but with normal second derivative equations of motion. (In general, for higher derivative actions, typical paths are continuous in velocity [Simon (1990)].)

Null Surfaces

What is this Null Surface?

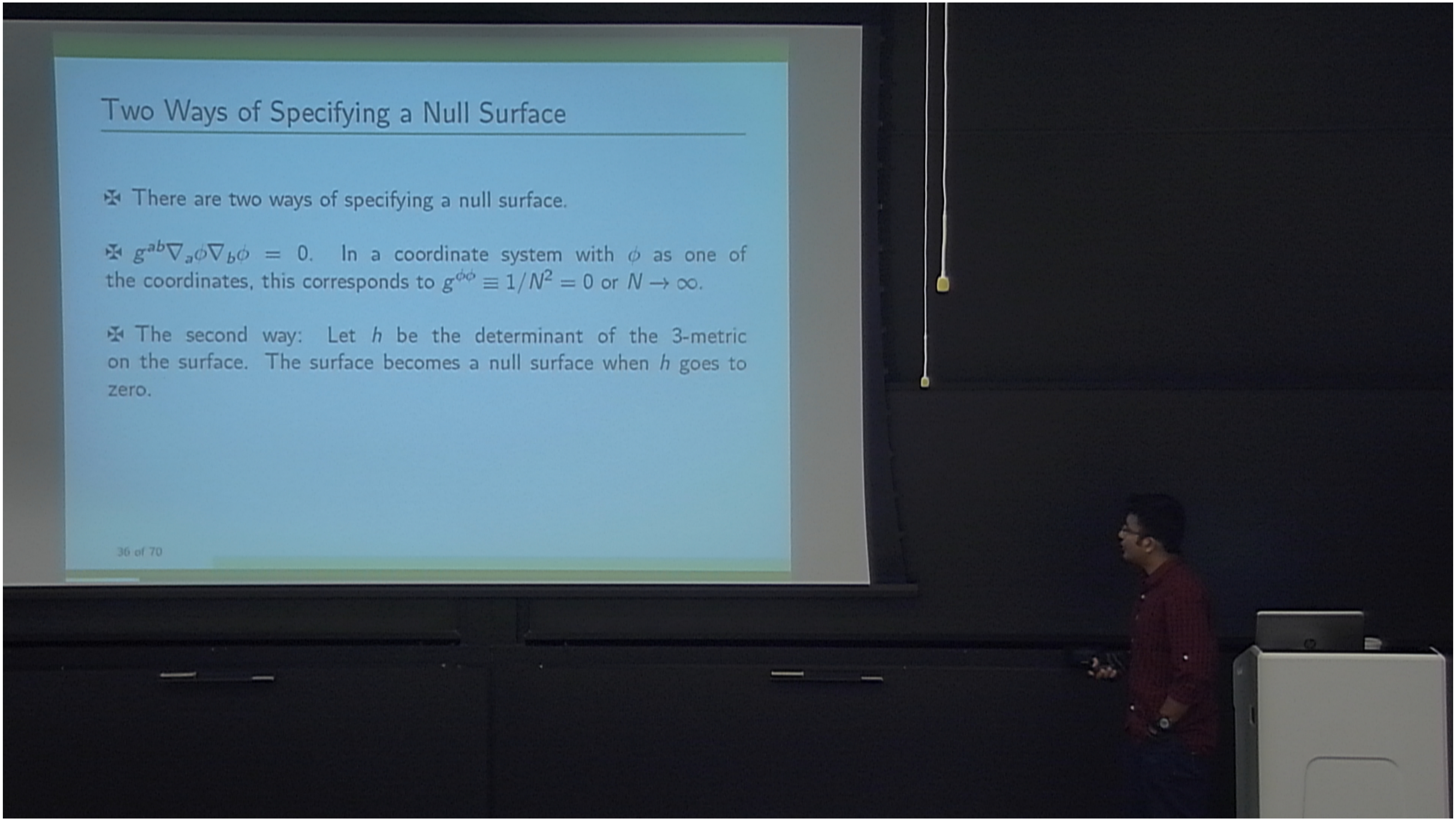
- Normal one-form $\ell_a = \nabla_a \phi$. To normalize, divide by $\sqrt{g^{ab}\ell_a \ell_b}$ to get n_a such that $n_a n^a = \pm 1$.



Grappling with Null Surfaces

- $\ell^a \ell_a = \ell^a \partial_a \phi = 0$ means ϕ does not change along $\ell^a \Rightarrow \ell^a$ lies on the $\phi = \mathrm{constant}$ null surface!

- Projection of tensors onto the surface, etc., which we generally do using normalized normal, can be done for a null surface too. By using an auxiliary null vector k^a such that $k^a \ell_a = -1$.


Two Ways of Specifying a Null Surface
There are two ways of specifying a null surface.
$g^{ab}\nabla_a\phi\nabla_b\phi = 0$. In a coordinate system with ϕ as one of the coordinates, this corresponds to $g^{\phi\phi} \equiv 1/N^2 = 0$ or $N \to \infty$.
The second way: Let h be the determinant of the 3-metric on the surface. The surface becomes a null surface when h goes to zero.
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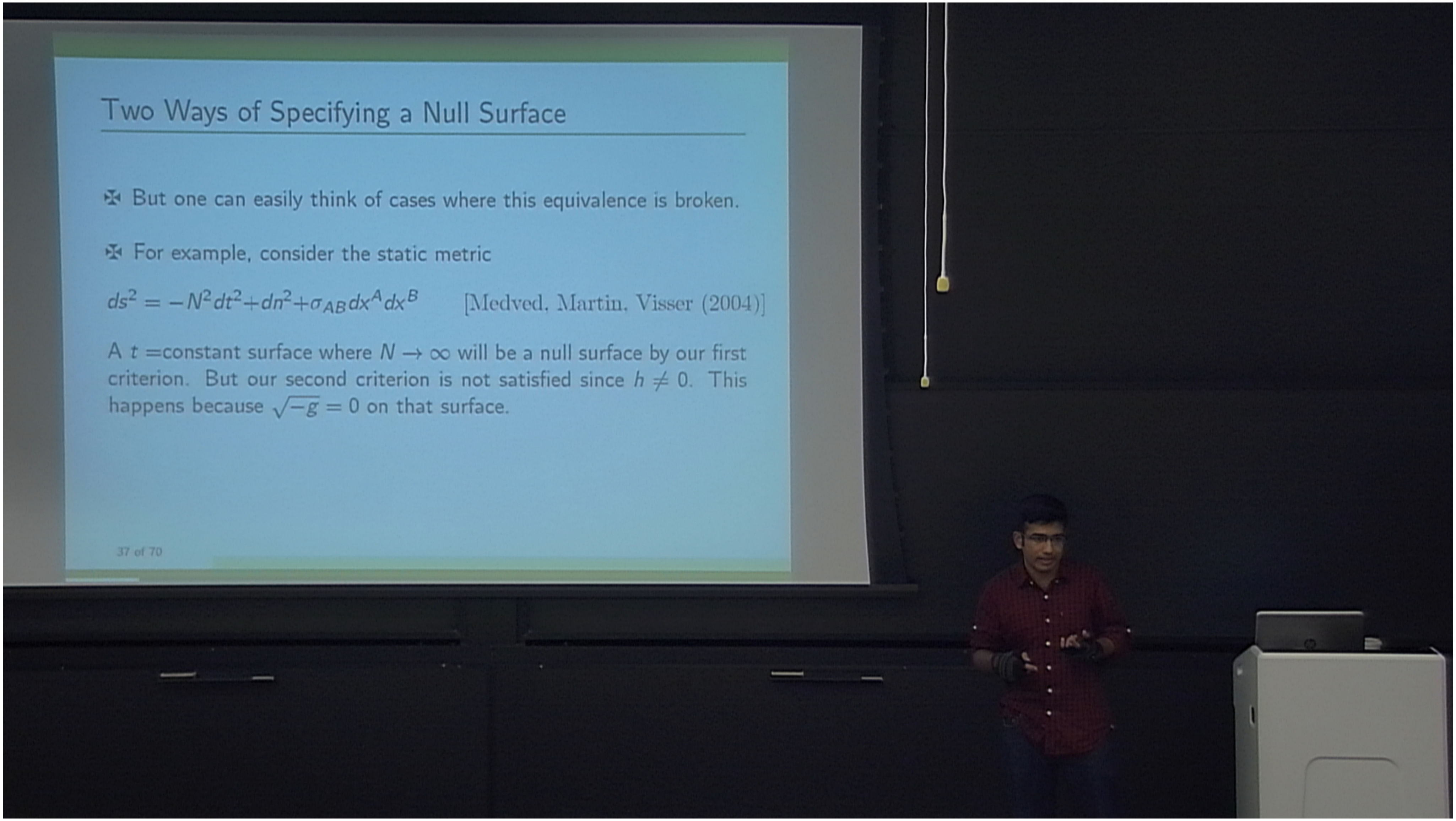
- The second way: Let h be the determinant of the 3-metric on the surface. The surface becomes a null surface when h goes to zero.

- Both methods are connected by the relation $\sqrt{-g} = N\sqrt{h}$. If we take $\sqrt{-g}$ to be finite and non-zero, both relations are equivalent.


Two Ways of Specifying a Null Surface
But one can easily think of cases where this equivalence is broken.
For example, consider the static metric
$ds^2 = -N^2 dt^2 + dn^2 + \sigma_{AB} dx^A dx^B$ [Medved, Martin, Visser (2004)]
A t =constant surface where $N \to \infty$ will be a null surface by our first criterion. But our second criterion is not satisfied since $h \neq 0$. This happens because $\sqrt{-g} = 0$ on that surface.
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Importance of Null Surfaces

It has generally been customary, in relativity theory, to work largely in terms of spacelike hypersurfaces and timelike directions...I wish to stress another point of view: that null hypersurfaces and null directions are, in fact, much more convenient and may perhaps also be regarded as more fundamental...For the two fields most important to relativity theory, namely, electromagnetism and gravitation... propagate in null directions and along null hypersurfaces...also a strong mathematical reason for considering null directions...This is essentially that the (two component) spinor calculus is more elementary than the tensor calculus...This fact that all the direct large-scale observations of the universe that have been made lie on what is essentially a null cone, has also been emphasized by J. L. Synge and others.
—Roger Penrose

Previous Attempts

✠ Barth (1983)- PhD Thesis- done in the framework of a particular form of the metric

A Direct Limit?

- GHY Term- $\sqrt{h}K$: $\sqrt{h}$ goes to zero, K diverges on null limit- but maybe the combination has a null limit?

- Note: The counterpart of $h_{ab} = g_{ab} - \epsilon n_a n_b$ on a null surface- $q_{ab} = g_{ab} + \ell_a k_b + k_a \ell_b$- cannot be obtained as a limit.

How to (re)discover the GHY Term?

✠ Suppose we did not have the ADM decomposition and GHY had not given us the boundary term. How can you discover it?

✠ Look at the boundary variation of the Einstein-Hilbert action.

$$\delta A_{\partial\mathcal{V}} = \int d^3x \sqrt{h} n_c \left(g^{ab} \delta\Gamma^c_{ab} - g^{cb} \delta\Gamma^a_{ab} \right) = \int d^3x \, \delta\mathcal{L}_{\partial\mathcal{V}}$$

✠ Our purpose- Find some B such that δB added to above eliminates terms with variations of normal derivatives.

How to (re)discover the GHY Term?

- First observation:

$$\delta\mathcal{L}_{\partial\mathcal{V}} = \sqrt{h}\left\{\nabla_a(\delta n_\perp^a) - \delta(2\nabla_a n^a) + \nabla_a n_b\ \delta g^{ab}\right\}; \qquad \delta n_\perp^a \equiv \delta n^a + g^{ab}\delta n_b$$

- Separate out the normal derivatives from the surface derivatives:

$$\nabla_a(\delta n_\perp^a) = D_a(\delta n_\perp^a) - \epsilon n_i a_j \delta g^{ij}; \qquad D_a A_b = h_a^m h_b^n \nabla_m A_n; \quad a_j = n^i \nabla_i n_j$$

-

$$\delta\mathcal{L}_{\partial\mathcal{V}} = \sqrt{h}\left\{D_a(\delta n_\perp^a) - \delta(2\nabla_a n^a) + (\nabla_a n_b - \epsilon n_i a_j)\ \delta g^{ab}\right\};$$

Analyzing the Boundary Variation

- Consider a 4-dimensional manifold.

- Take a 3-dimensional manifold in the 4-dimensional manifold as our boundary. We shall be considering the Einstein-Hilbert action on one side of this boundary

Analyzing the Boundary Variation

- Consider a 4-dimensional manifold.
- Take a 3-dimensional manifold in the 4-dimensional manifold as our boundary. We shall be considering the Einstein-Hilbert action on one side of this boundary
- We need to specify the position of this boundary somehow. Simplest method: Introduce a scalar field ϕ such that $\phi = \phi_0$ corresponds to the position of the boundary.
- When we consider variations of the metric, we shall not vary ϕ. Note that if we define $s_c = \nabla_c \phi = \partial_c \phi$, we will have $\delta s_c = 0$.

Analyzing the Boundary Variation

- It is easiest to think of in terms of the coordinates (ϕ, x^1, x^2, x^3) for some (x^1, x^2, x^3).

- In these coordinates, the total derivative in the variation of the Einstein-Hilbert Lagrangian can then be converted to a boundary term, which is the integral over the boundary of

$$\delta\mathcal{L}_{\partial\mathcal{V}} = \sqrt{-g}s_c\left(g^{ab}\delta\Gamma^c_{ab} - g^{cb}\delta\Gamma^a_{ab}\right); \quad s_c = \nabla_c\phi \ .$$

- By algebraic manipulations, we can convert the boundary term into

$$\delta\mathcal{L}_{\partial\mathcal{V}} = \sqrt{-g}\nabla_c[\delta u^c_{(s)}] - 2\delta(\sqrt{-g}\nabla_a s^a) + \sqrt{-g}(\nabla_a s_b - g_{ab}\nabla_c s^c)\delta g^{ab} \ ;$$
$$\delta u^c_{(s)} = \delta s^c + g^{cb}\delta s_b \ .$$

Constructing a Projector

- More explicitly, $\Pi^a{}_b \Pi^b{}_c = \Pi^a{}_c$, and hence a projector.
- $\Pi^a{}_b A^b = (0, A^\alpha + \overline{t}^\alpha A^\phi)$.
- Of course, given a coordinate system, you might want to choose $t^\alpha = 0$. But note that this is a coordinate-dependent statement. That is, it depends on the choice of (x^1, x^2, x^3), since we are demanding

$$t^a \propto \left.\frac{\partial x^a}{\partial \phi}\right|_{x^1, x^2, x^3}$$

.

Invariant Volume Element on the Surface

✠ How to give a coordinate-invariant definition of the volume element $\sqrt{-g}s_a$, written in coordinates (ϕ, x^1, x^2, x^3) on the boundary?

✠ For a non-null surface, $\sqrt{-g} = N\sqrt{h}$, where N is the lapse function. Hence,

$$\sqrt{-g}s_c = \sqrt{h}N\nabla_c\phi = \sqrt{h}n_c \ .$$

✠ Can we have a similar prescription without restricting to non-null?

✠ We have the two vectors t^a and s^a. (I am assuming that we have an invertible 4-metric. So every one-form has an associated vector and vice-versa.)

Invariant Volume Element on the Surface

✠ For a non-null surface, we can choose $t^a \propto s^a$ respecting the condition $t^a s_a \neq 0$. Then, one can show that $\sqrt{-g}s_a = \sqrt{h}n_a$.

✠ For a null surface, we cannot make this choice. Even for a non-null surface, we need not necessarily make the above choice. You might not necessarily want to evolve along the normal direction.

✠ So consider the case that t^a and s^a are linearly independent. Choose t^a to be the basis vector along the ϕ direction, since it is off-the-surface. Actually, since $s_a = \nabla_a \phi = (1, 0, 0, 0)$ in these coordinates, we should choose

$$\frac{\partial}{\partial \phi} = -\overline{t} \ .$$

✠ Choose any two vectors e^a_A, $A = 1, 2$, which are orthogonal to $-\overline{t}^a$ and s^a. These lie on the boundary surface.

Invariant Volume Element on the Surface

- The metric in the coordinate system with $(-\overline{t}^a, s^a, e^a_A)$ as basis vectors is given by

$$g_{ab} = \begin{pmatrix} \overline{t}^2 & 1 & 0 & 0 \\ 1 & s^2 & 0 & 0 \\ 0 & 0 & q_{11} & q_{12} \\ 0 & 0 & q_{12} & q_{22} \end{pmatrix}$$

The Boundary Term

✠ The off-the-surface derivatives of the metric can be eliminated by adding the integral on the boundary of

$$C_0 = 2\sqrt{-g}\Pi^a{}_b\nabla_a s^b$$

✠ For a non-null surface, the choice $\overline{t}^a = -\frac{s^a}{s^2}$ reduces this to the GHY term.

✠ In fact, $t^a = -\frac{s^a}{s^2}$ reduces $\Pi^a{}_b$ to h^a_b.

The Boundary Term for a Null Surface

- For a null surface, one may choose $\overline{t}^a = k^a$, the auxiliary null vector. Also, we shall write s_a as ℓ_a.

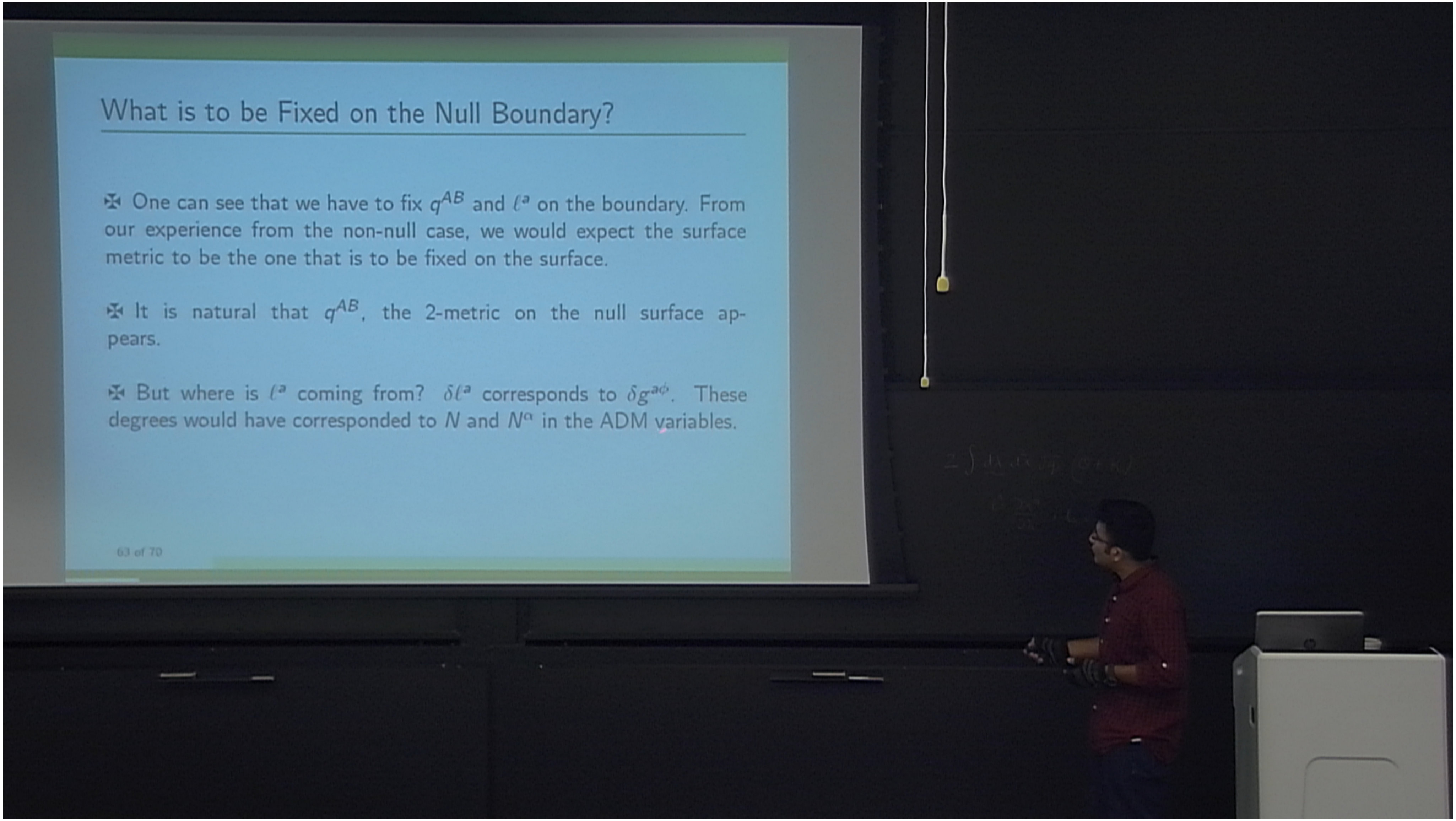
- Then, the boundary term becomes the integral of

$$C_n = 2\sqrt{q}\,(\Theta + \kappa)\,,$$

where Θ is the expansion of the null generators and κ is the non-affinity parameter given by the equation $\ell^a \nabla_a \ell^b = \kappa \ell^b$.

- The full boundary variation has the form

$$\begin{aligned}\delta\mathcal{L}_{\partial\mathcal{V}} =& \partial_a[\sqrt{q}\Pi^a{}_b\delta\ell^b] - 2\delta\left[\sqrt{q}\,(\Theta+\kappa)\right] + \sqrt{q}\left[\Theta_{ab} - (\Theta+\kappa)\,q_{ab}\right]\delta q^{ab} \\ &+ \sqrt{q}\left[2k_a\,(\Theta+\kappa) - k^b\,(\nabla_a\ell_b + \nabla_b\ell_a)\right]\delta\ell^a\end{aligned}$$


What is to be Fixed on the Null Boundary?
One can see that we have to fix q^{AB} and ℓ^a on the boundary. From our experience from the non-null case, we would expect the surface metric to be the one that is to be fixed on the surface.
It is natural that q^{AB}, the 2-metric on the null surface appears.
But where is ℓ^a coming from? $\delta\ell^a$ corresponds to $\delta g^{a\phi}$. These degrees would have corresponded to N and N^α in the ADM variables.
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What is to be Fixed on the Null Boundary?

- In order to understand this result better, consider the surface metric in the coordinates with basis vectors $(-k^a, \ell^a, e^a_A)$. The original metric, before variation is

$$h_{\alpha\beta} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & q_{11} & q_{12} \\ 0 & q_{12} & q_{22} \end{pmatrix}$$

- It can be proved that the variation of the surface metric is

$$\delta h_{\alpha\beta} = \begin{pmatrix} \delta\ell^\phi & \delta\ell^1 & \delta\ell^2 \\ \delta\ell^1 & \delta q_{11} & \delta q_{12} \\ \delta\ell^2 & \delta q_{12} & \delta q_{22} \end{pmatrix}$$

- $\delta\ell^\phi$ should vanish if the surface is to remain null. The component of $\delta\ell^a$ that does not appear above is $\delta\ell^a k_a$. This corresponds to scaling of the original null vector on the surface and hence does not change the surface metric.

On Non-uniqueness

- Let us now examine the uniqueness of our results.
- GHY term has the nice feature that it not only eliminates the variations of the normal derivatives of the metric, but also allows only variations of the surface metric to appear on the boundary.
- These features will be preserved even if we add a term of the form $F[h_{\alpha\beta}, \partial\gamma h_{\alpha\beta}]$ to the GHY term.

On Non-uniqueness

✠ Also, we have generally been throwing away total divergence terms on the surface.

✠ Any term in the total variation term with only surface derivatives can be converted to a total divergence term and terms with variations of the metric by the formula

$$\delta\left(A\partial_\alpha B\right) = \partial_\alpha\left(A\delta B\right) - \left[\partial_\alpha A\delta B - \partial_\alpha B\delta As\right]$$

On Non-uniqueness

✠ $\sqrt{q}\,(\Theta + \kappa)$- Θ has only surface derivatives. May not contribute on smooth boundaries. All the normal derivatives are in κ. In our framework, if we choose the level surfaces of ϕ to be null, then κ vanishes. No boundary term??

Conclusions

✠ We have provided a decomposition of the boundary variation of EH action on a null boundary:

$$\delta\mathcal{L}_{\partial\mathcal{V}} = \partial_a[\sqrt{q}\Pi^a{}_b\delta\ell^b] - 2\delta\left[\sqrt{q}\left(\Theta+\kappa\right)\right] + \sqrt{q}\left[\Theta_{ab} - \left(\Theta+\kappa\right)q_{ab}\right]\delta q^{ab} + \sqrt{q}\left[2k_a\left(\Theta+\kappa\right) - k^b\left(\nabla_a\ell_b + \nabla_b\ell_a\right)\right]\delta\ell^a$$

"Theoretical physics is fun. Most of us indulge in it for the same reason a painter paints or a dancer dances – the process itself is so enjoyable! Occasionally, there are additional benefits like fame and glory and even practical uses; but most good theoretical physicists will agree that these are not the primary reasons why they are doing it. The fun in figuring out the solutions to Nature's brain teasers is a reward in itself."

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- Padmanabhan, In preface to "Sleeping Beauties in Theoretical Physics."