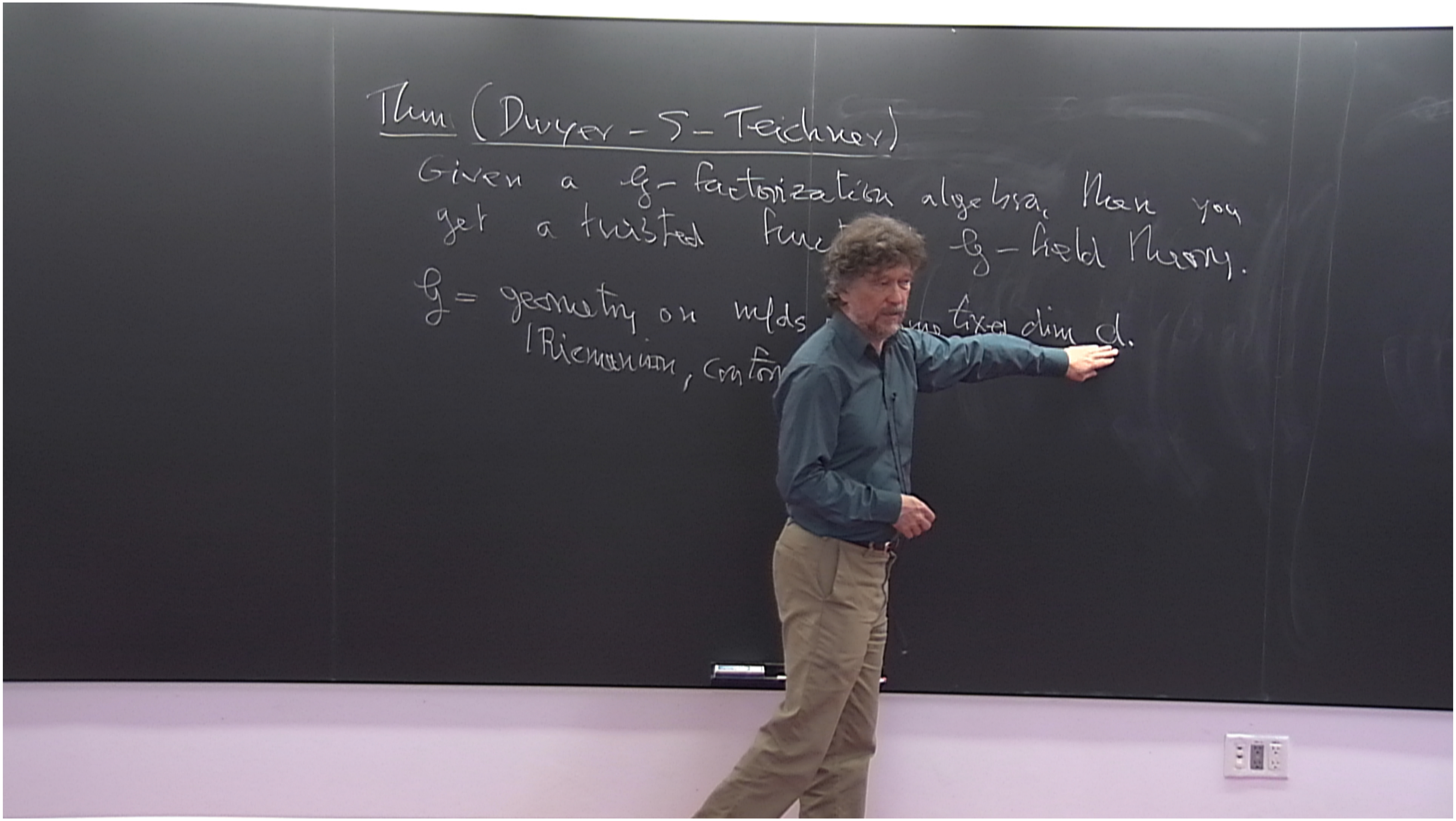
Title: Functorial field theories from factorization algebras

Date: Mar 10, 2016 02:00 PM

URL: http://pirsa.org/16030115

Abstract:

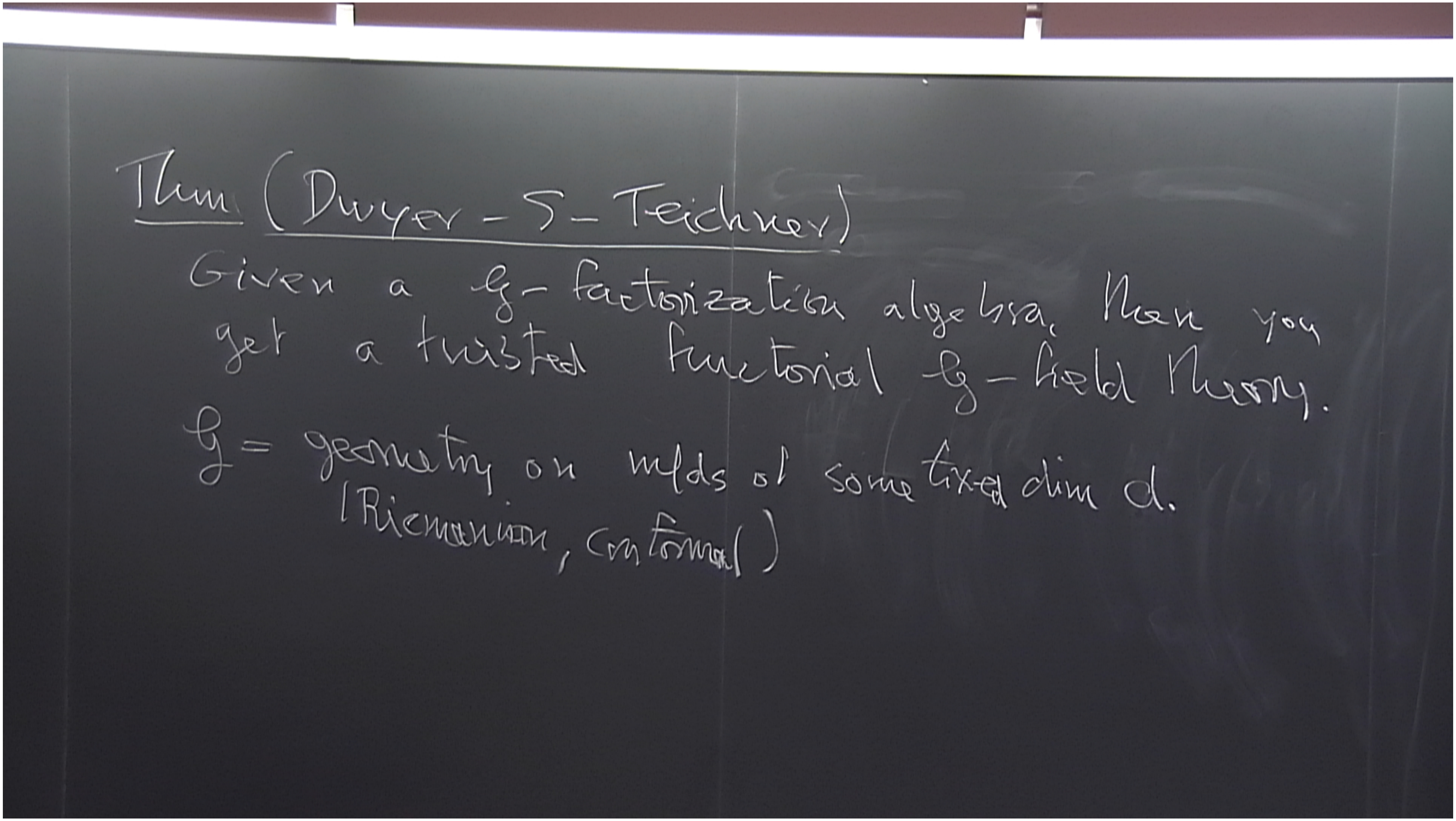


Thm (Dwyer – S – Teichner)

Given a $\mathcal{G}$-factorization algebra, then you get a twisted funct $\mathcal{G}$-field theory.

$\mathcal{G}$ = geometry on mflds fixed dim d.

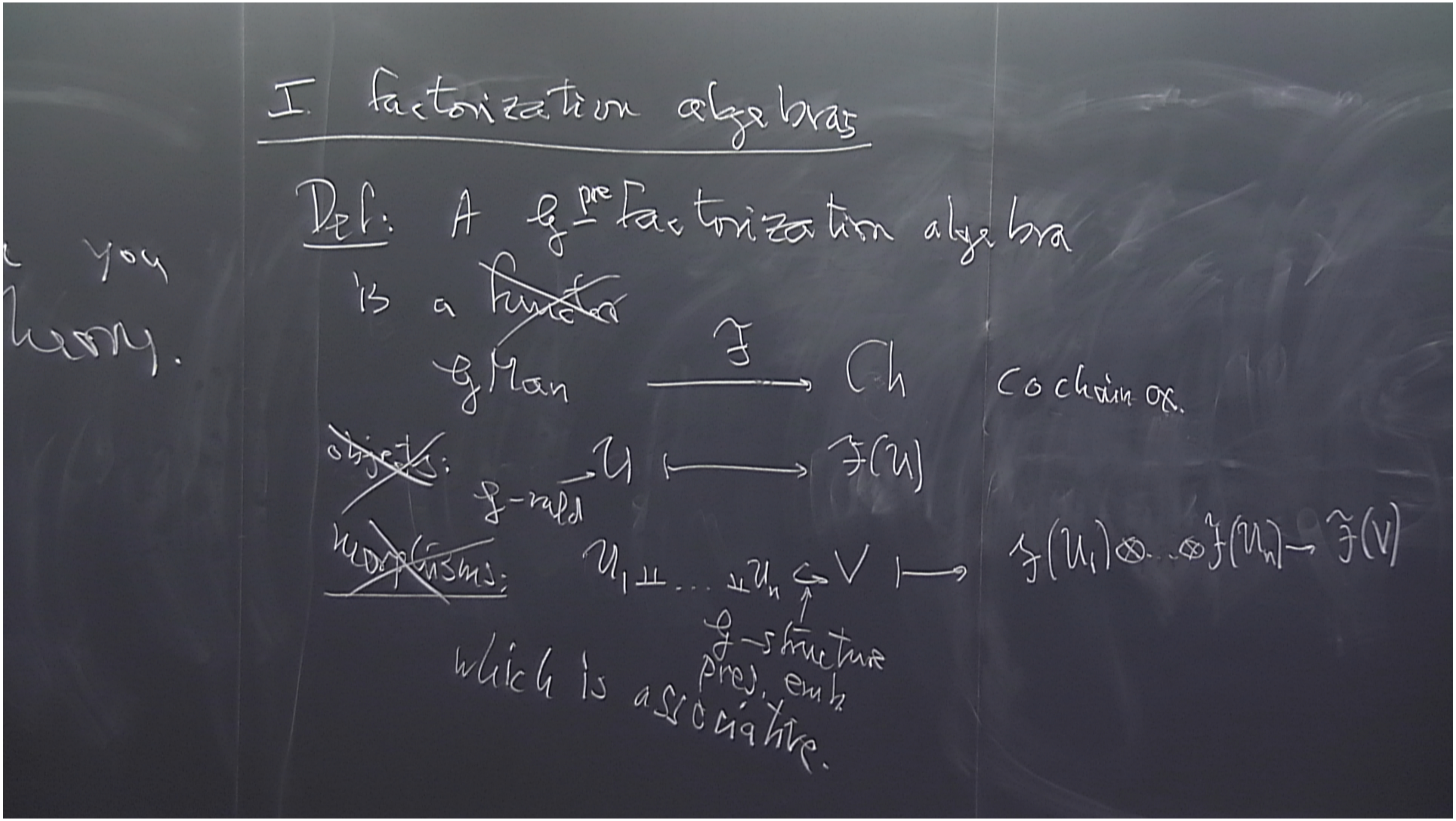
(Riemannian, confor



Thm (Dwyer - S - Teichner)

Given a $\mathcal{G}$-factorization algebra, then you get a twisted functorial $\mathcal{G}$-field theory.

$\mathcal{G}$ = geometry on mflds of some fixed dim d.
(Riemannian, conformal)



you

heory.

I. factorization algebras

Def: A $\mathcal{G}$-$^{\text{pre}}$factorization algebra is a ~~functor~~

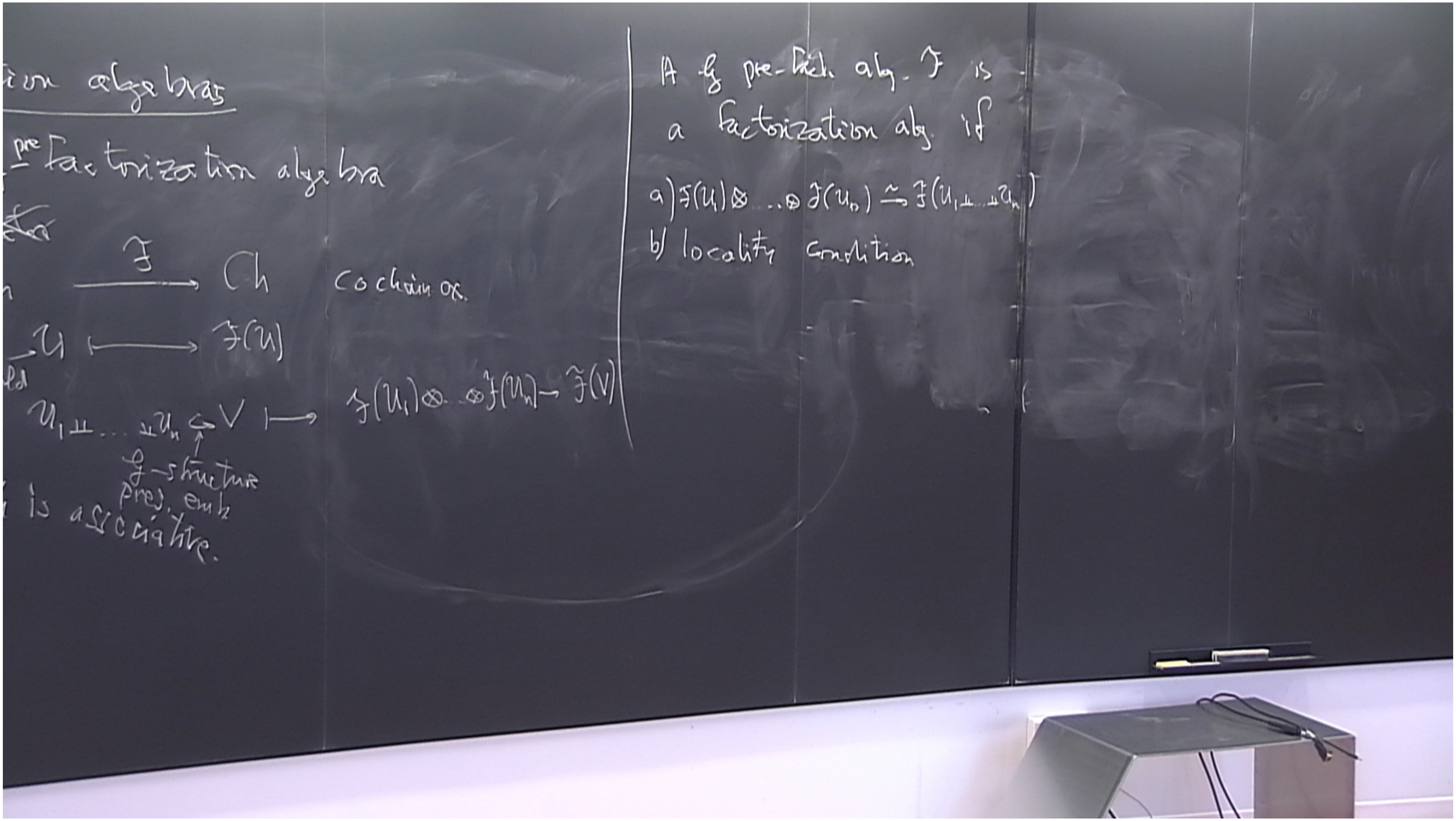
$$\mathcal{G}\mathrm{Man} \xrightarrow{\quad \mathcal{F} \quad} \mathrm{Ch} \qquad \text{cochain cx.}$$

~~objects~~: $\mathcal{G}$-mfld $\to U \longmapsto \mathcal{F}(U)$

~~morphisms~~: $U_1 \amalg \dots \amalg U_n \hookrightarrow V \longmapsto \mathcal{F}(U_1)\otimes \dots \otimes \mathcal{F}(U_n) \to \mathcal{F}(V)$

$\uparrow$ $\mathcal{G}$-structure pres. emb.

which is associative.



ion algebras

pre Factorization algebra

n $\xrightarrow{\mathcal{F}}$ Ch cochain cx.

$\mathcal{U} \longmapsto \mathcal{F}(\mathcal{U})$

ed

$\mathcal{U}_1 \sqcup \ldots \sqcup \mathcal{U}_n \hookrightarrow V \longmapsto \mathcal{F}(\mathcal{U}_1) \otimes \ldots \otimes \mathcal{F}(\mathcal{U}_n) \to \mathcal{F}(V)$

$\mathcal{G}$-structure pres. emb

is associative.

A $\mathcal{G}$ pre-Fact. alg. $\mathcal{F}$ is a factorization alg. if

a) $\mathcal{F}(\mathcal{U}_1) \otimes \ldots \otimes \mathcal{F}(\mathcal{U}_n) \xrightarrow{\sim} \mathcal{F}(\mathcal{U}_1 \sqcup \ldots \sqcup \mathcal{U}_n)$

b) locality condition

A $\mathcal{G}$ pre-fact. alg. $\mathcal{F}$ is a factorization alg. if

a) $\mathcal{F}(U_1) \otimes \dots \otimes \mathcal{F}(U_n) \xrightarrow{\simeq} \mathcal{F}(U_1 \sqcup \dots \sqcup U_n)$

b) locality condition

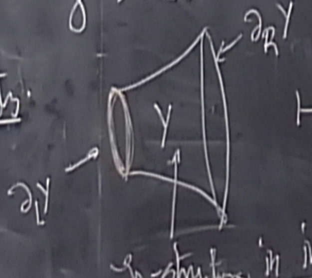
(V)

II. Functorial field theories

Def.: A $\mathcal{G}$-field theory is a functor (symmetric monoidal)

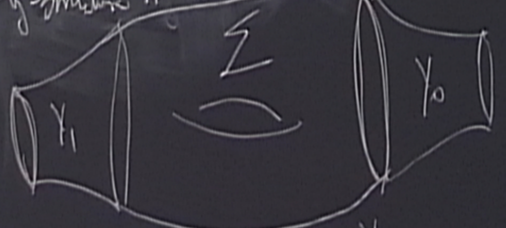
$$\mathcal{G}\text{Bord} \longrightarrow \text{Vect}$$

objects:



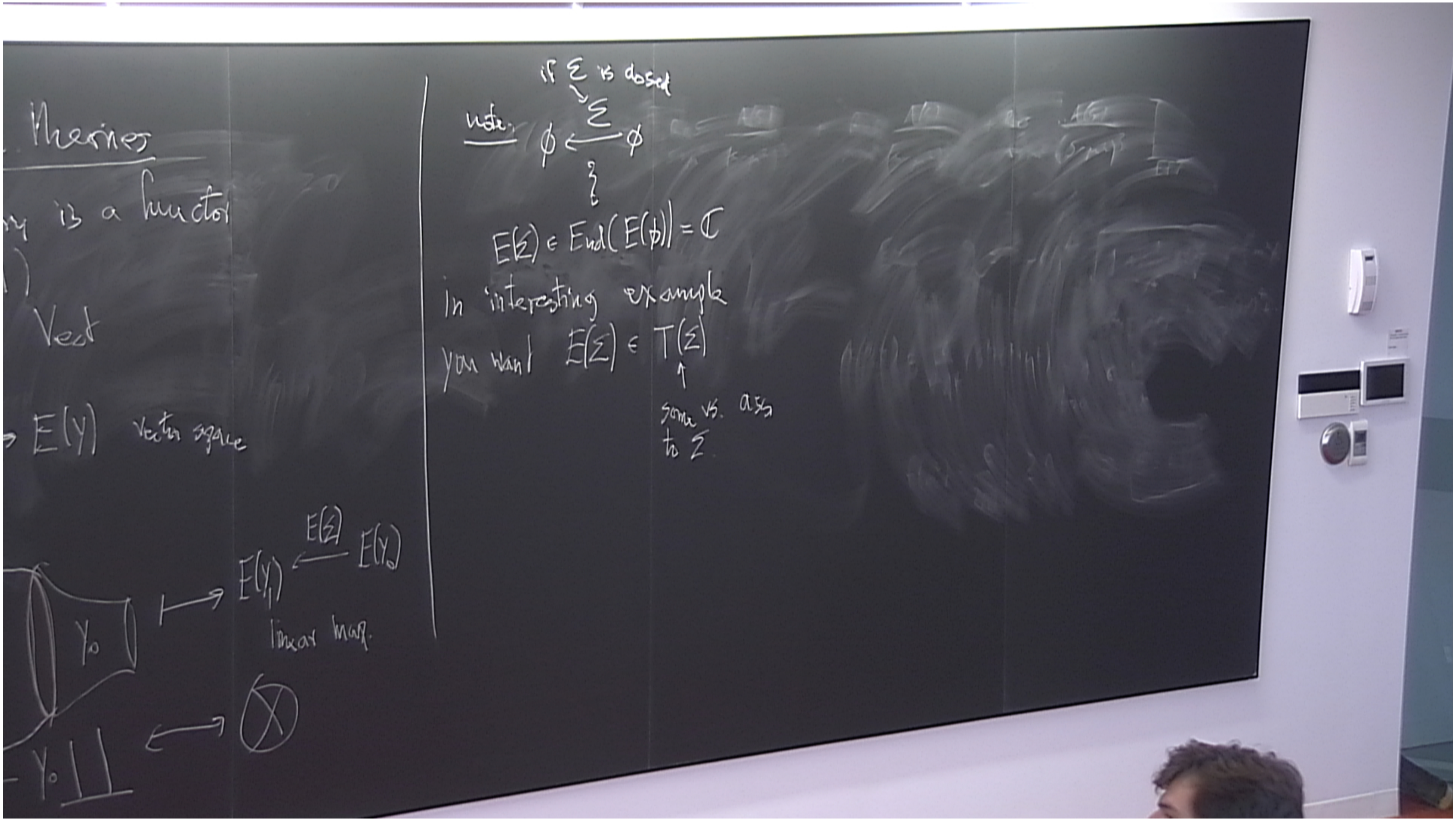
$\longmapsto E(Y)$ vector space

morphisms:



$\longmapsto$

$Y_1 \longleftarrow Y_0$



theories

y is a functor

)

Vect

$\to E(Y)$ vector space

$$\begin{tikzcd} E(Y_1) & E(Y_2) \arrow[l, "E(\Sigma)"'] \end{tikzcd}$$

linear map.

Y_0 $\longmapsto$ $E(Y_1)$

$Y_0 \sqcup$ $\longleftrightarrow$ $\otimes$

note: if Σ is closed

$$\begin{tikzcd} \emptyset & \emptyset \arrow[l, "\Sigma"'] \end{tikzcd}$$

$\wr$

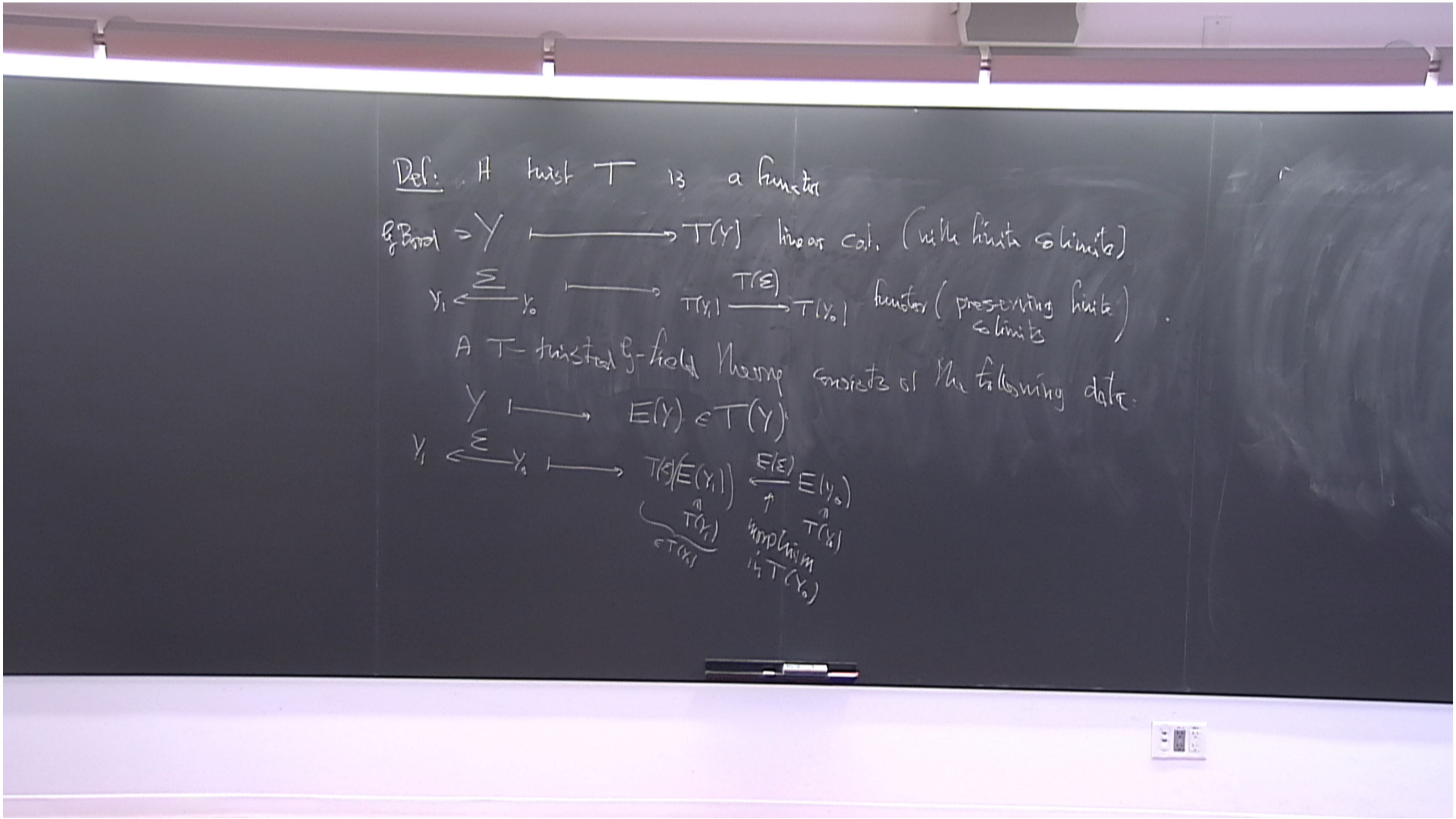
$E(\Sigma) \in End(E(\emptyset)) = \mathbb{C}$

in interesting example

you want $E(\Sigma) \in T(\Sigma)$

$\uparrow$

some vs. ass to Σ



Def: A twist T is a functor

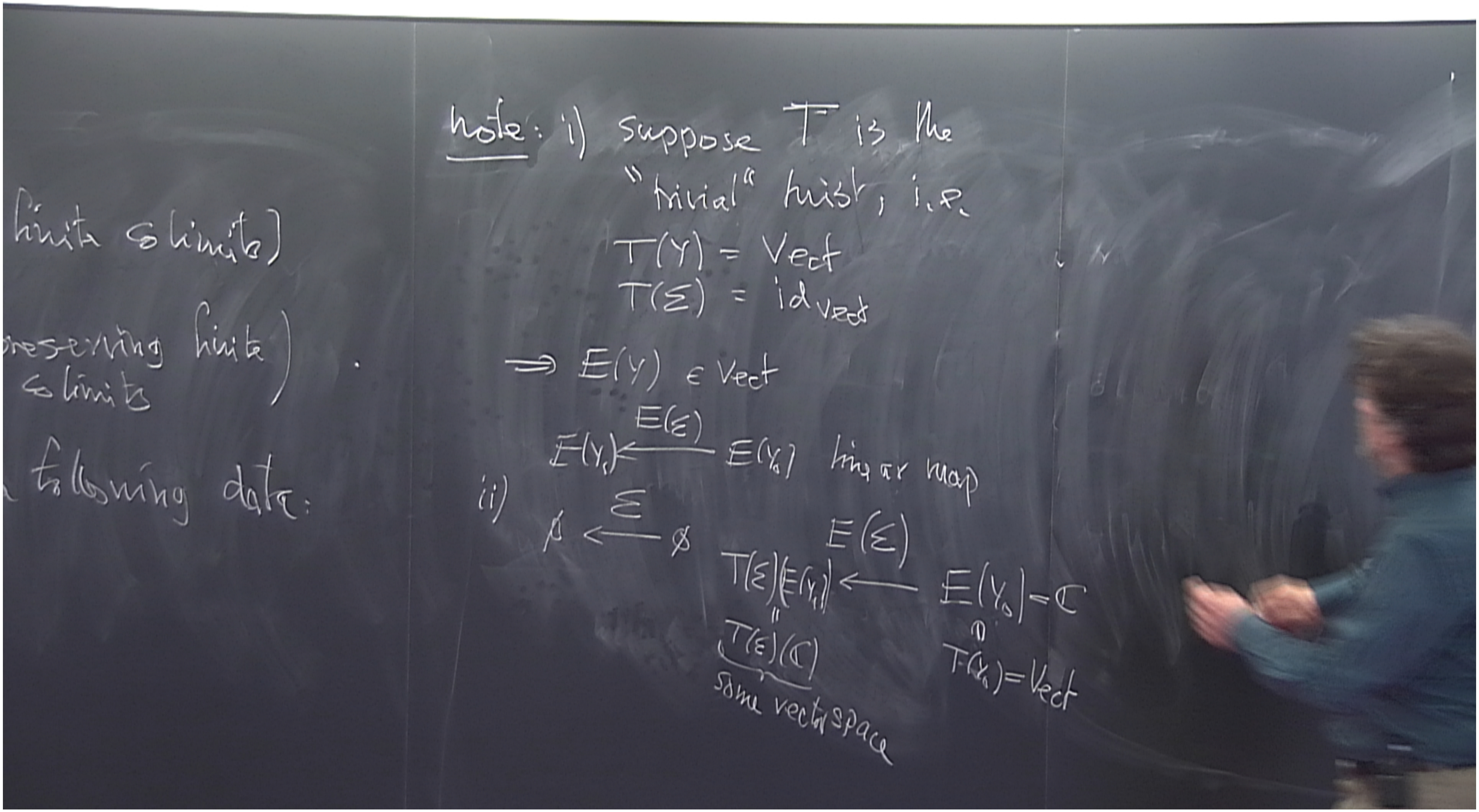
$\mathfrak{g}\text{Bord} \ni Y \longmapsto T(Y)$ linear cat. (with finite colimits)

$Y_1 \xleftarrow{\varepsilon} Y_0 \longmapsto T(Y_1) \xrightarrow{T(\varepsilon)} T(Y_0)$ functor (preserving finite colimits).

A T-twisted $\mathfrak{g}$-field theory consists of the following data:

$Y \longmapsto E(Y) \in T(Y)$

$$Y_1 \xleftarrow{\varepsilon} Y_0 \longmapsto \underbrace{T(\varepsilon)\big(\underset{T(Y_1)}{\overset{\cap}{E(Y_1)}}\big)}_{\in T(Y_0)} \xleftarrow[\substack{\uparrow \\ \text{morphism} \\ \text{in } T(Y_0)}]{E(\varepsilon)} \underset{T(Y_0)}{\overset{\cap}{E(Y_0)}}$$



limits & limits]

preserving limits & limits).

following data:

note: i) suppose T is the "trivial" hist, i.e.

$T(Y) = \text{Vect}$

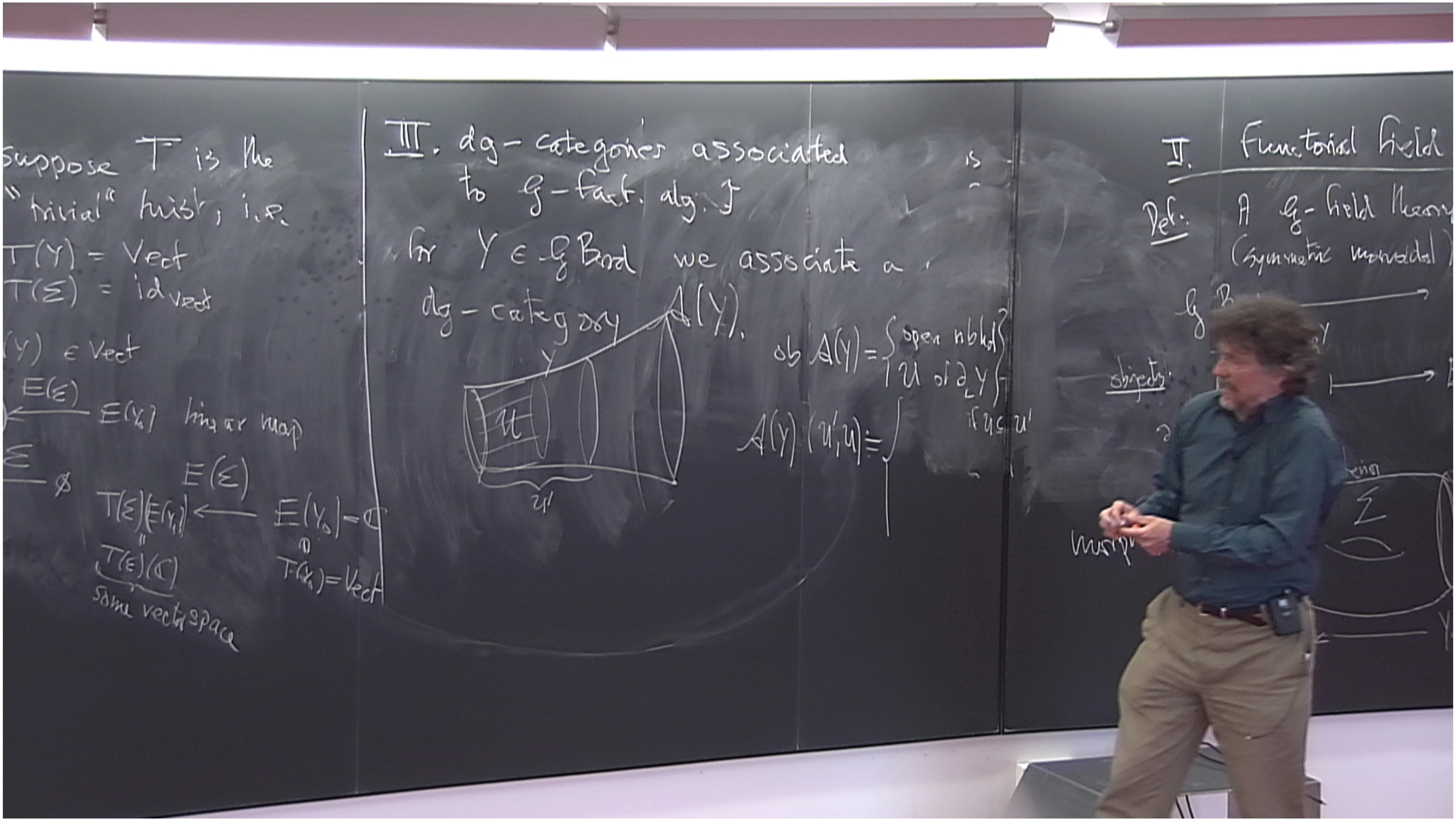
$T(\Sigma) = \text{id}_{\text{Vect}}$

$\Longrightarrow E(Y) \in \text{Vect}$

$E(Y_1) \xleftarrow{E(\Sigma)} E(Y_0)$ linear map

ii) $\emptyset \xleftarrow{\Sigma} \emptyset$

$$\begin{array}{ccc} T(\Sigma)(E(Y_1)) & \xleftarrow{E(\Sigma)} & E(Y_0) = \mathbb{C} \\ \parallel & & \rotatebox{90}{$\in$} \\ \underbrace{T(\Sigma)(\mathbb{C})}_{\text{some vectr space}} & & T(Y_0) = \text{Vect} \end{array}$$



Suppose T is the "trivial" twist, i.e.

$T(Y) = \mathrm{Vect}$

$T(\Sigma) = \mathrm{id}_{\mathrm{Vect}}$

$(Y) \in \mathrm{Vect}$

$E(\Sigma)$

$E(Y_0)$ linear map

$T(\Sigma)(E(Y_1)) \leftarrow E(Y_0) = \mathbb{C}$

$T(\Sigma)(\mathbb{C})$ some vectorspace

$T(Y_0) = \mathrm{Vect}$

III. dg-categories associated to $\mathcal{G}$-fact. alg. $\mathcal{F}$

For $Y \in \mathcal{G}\mathrm{Bord}$ we associate a dg-category $\mathcal{A}(Y)$.

$\mathrm{ob}\, \mathcal{A}(Y) = \{\text{open nbhd } \mathcal{U} \text{ of } \partial_{in} Y\}$

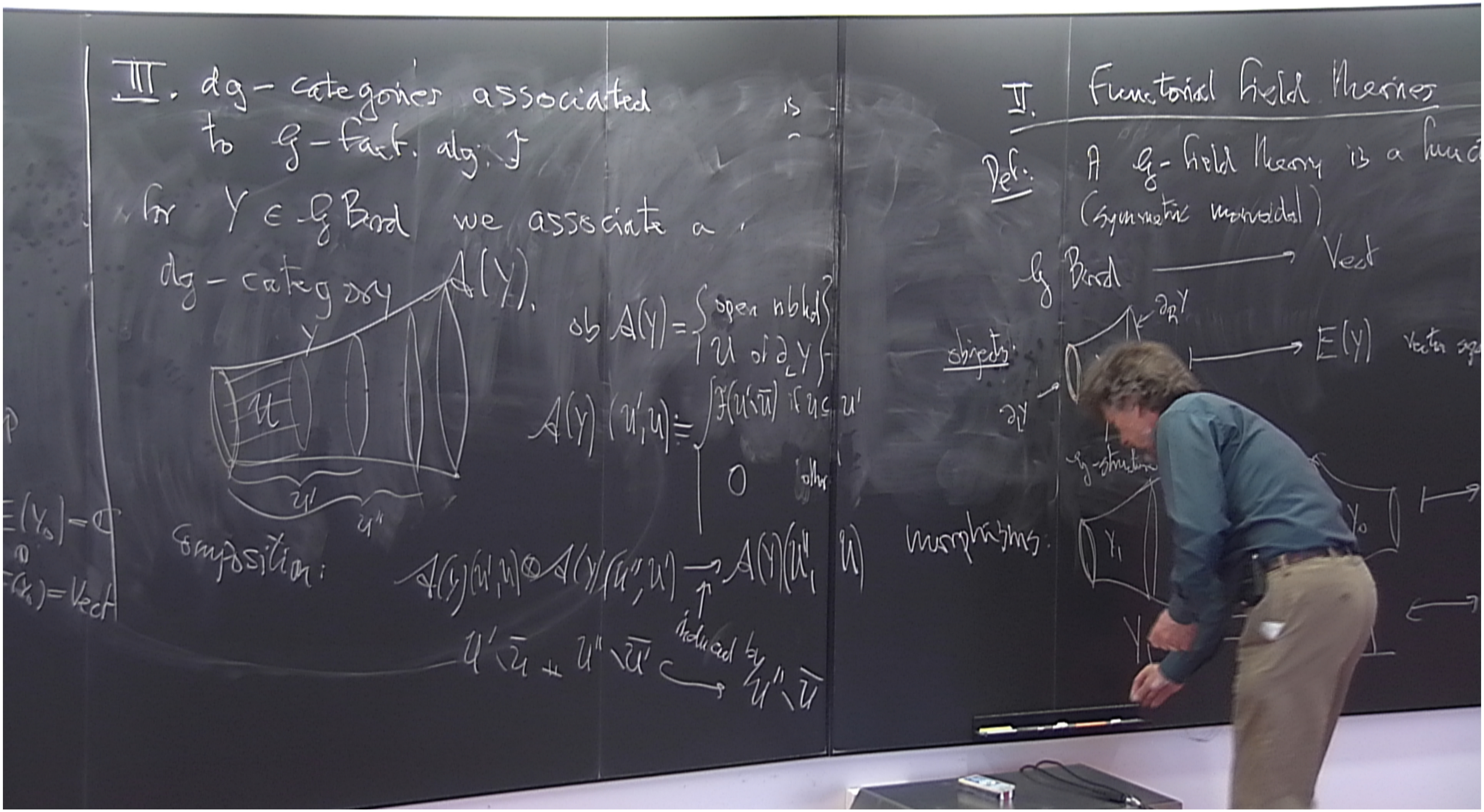
$\mathcal{A}(Y)(\mathcal{U}', \mathcal{U}) = \{$ if $\mathcal{U} \subseteq \mathcal{U}'$

II. Functorial field

Def. A $\mathcal{G}$-field theory (symmetric monoidal)

objects

morphisms



E(Y_0) = ℂ

F(Y_0) = Vect

III. dg-categories associated to 𝒢-fact. alg. 𝓕

For $Y \in \mathcal{G}\text{Bord}$ we associate a dg-category $\mathcal{A}(Y)$.

$\text{ob}\,\mathcal{A}(Y) = \{\text{open nbhd } \mathcal{U} \text{ of } \partial_L Y\}$

$\mathcal{A}(Y)(\mathcal{U}', \mathcal{U}) = \begin{cases} \mathcal{F}(\mathcal{U}' \setminus \overline{\mathcal{U}}) & \text{if } \mathcal{U} \subseteq \mathcal{U}' \\ 0 & \text{other} \end{cases}$

Composition: $\mathcal{A}(Y)(\mathcal{U}', \mathcal{U}) \otimes \mathcal{A}(Y)(\mathcal{U}'', \mathcal{U}') \to \mathcal{A}(Y)(\mathcal{U}'', \mathcal{U})$

induced by

$\mathcal{U}' \setminus \overline{\mathcal{U}} \amalg \mathcal{U}'' \setminus \overline{\mathcal{U}'} \hookrightarrow \mathcal{U}'' \setminus \overline{\mathcal{U}}$

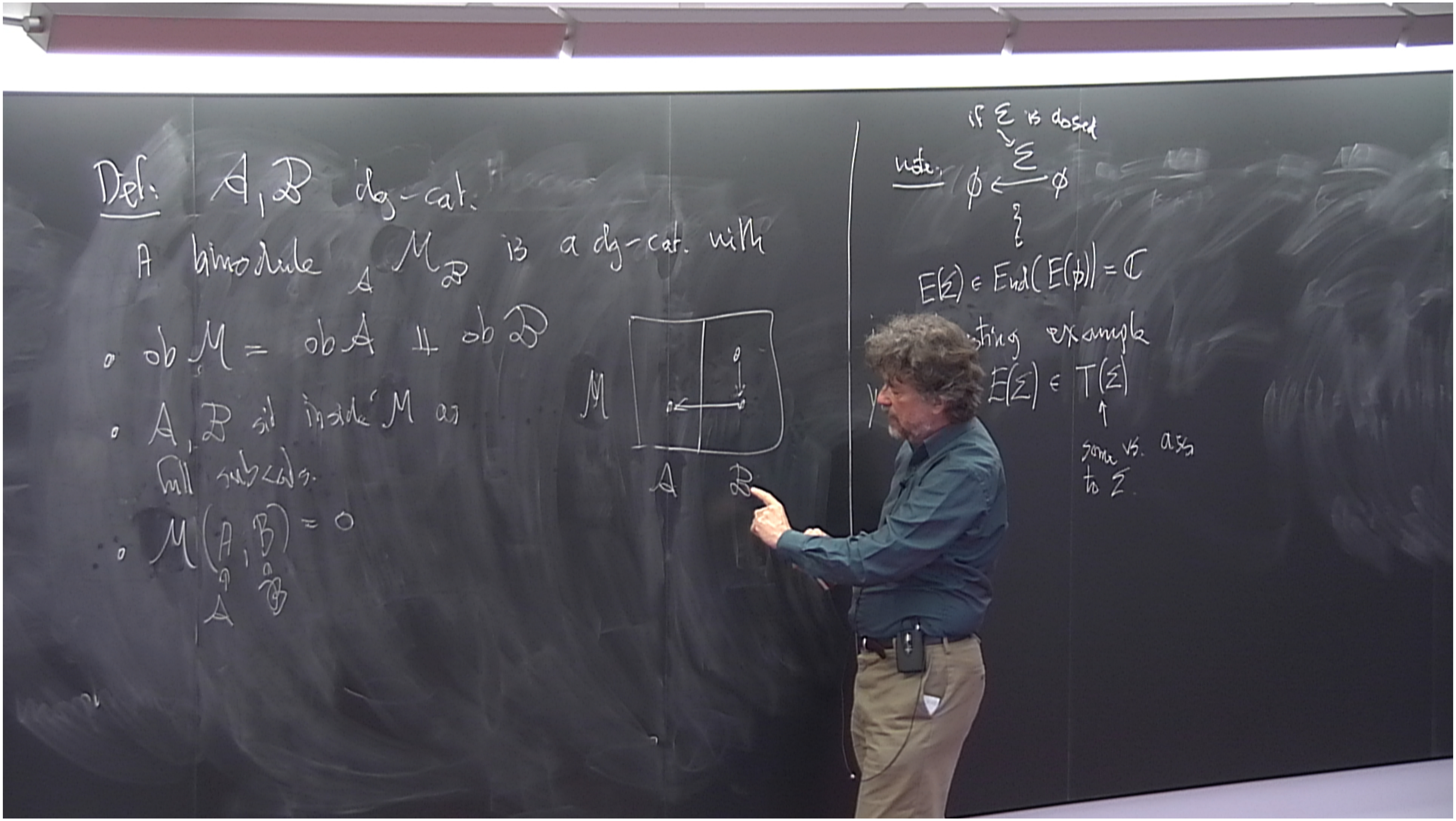
II. Functorial field theories

Def. A 𝒢-field theory is a func… (symmetric monoidal)

$\mathcal{G}\text{Bord} \longrightarrow \text{Vect}$

objects: $\longmapsto E(Y)$ vector spa…

morphisms:



Def: $\mathcal{A}, \mathcal{B}$ dg-cat.

A bimodule ${}_{\mathcal{A}}\mathcal{M}_{\mathcal{B}}$ is a dg-cat. with

- ob $\mathcal{M}$ = ob $\mathcal{A} \amalg$ ob $\mathcal{B}$
- $\mathcal{A}, \mathcal{B}$ sit inside $\mathcal{M}$ as full subcats.
- $\mathcal{M}(\underset{\underset{\mathcal{A}}{\in}}{A}, \underset{\underset{\mathcal{B}}{\in}}{B}) = 0$

note: if Σ is closed

$$\phi \xleftarrow{\Sigma} \phi$$

$\wr$

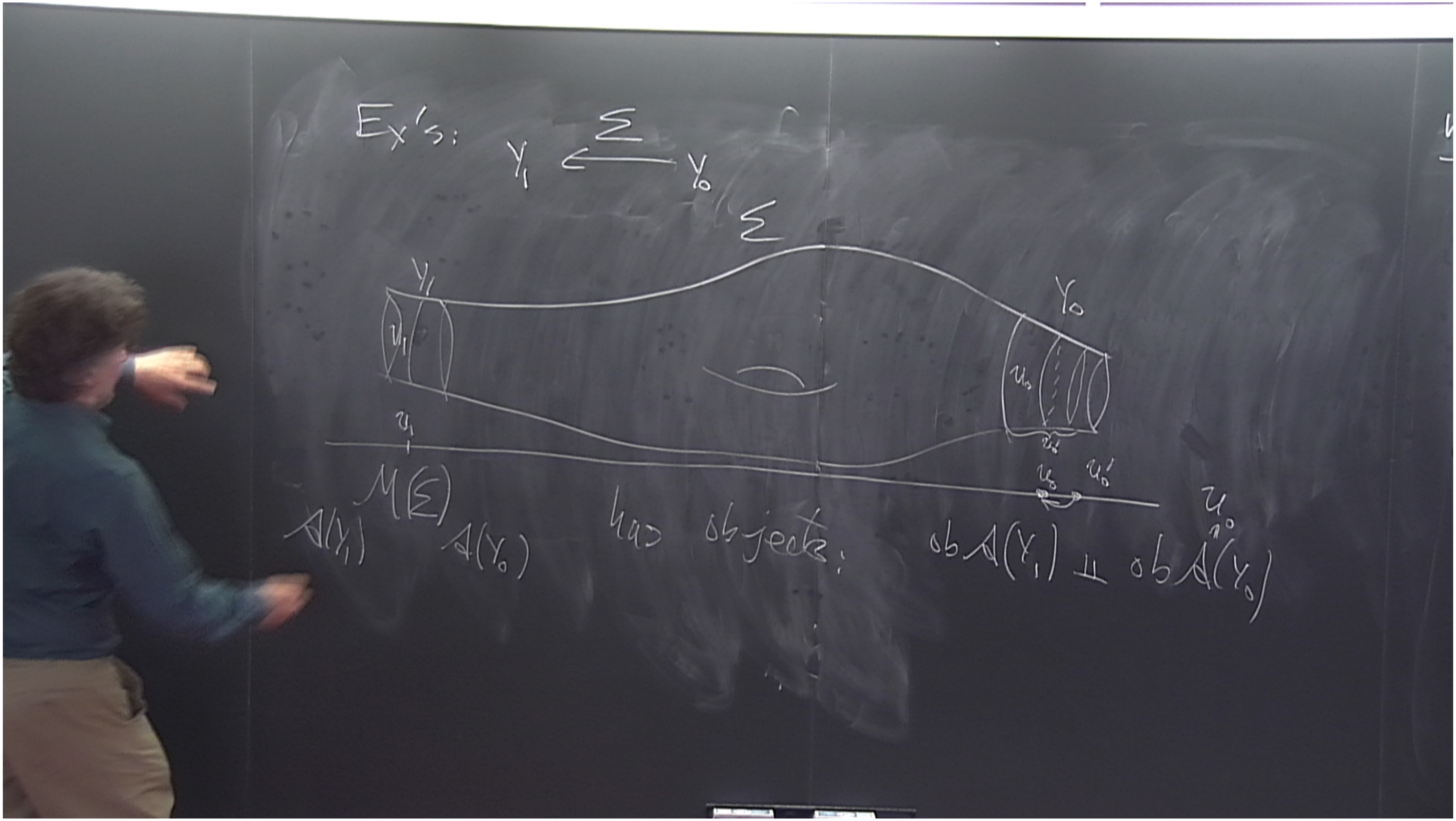
$E(\Sigma) \in \mathrm{End}(E(\phi)) = \mathbb{C}$

sting example

$E(\Sigma) \in T(\Sigma)$

$\uparrow$

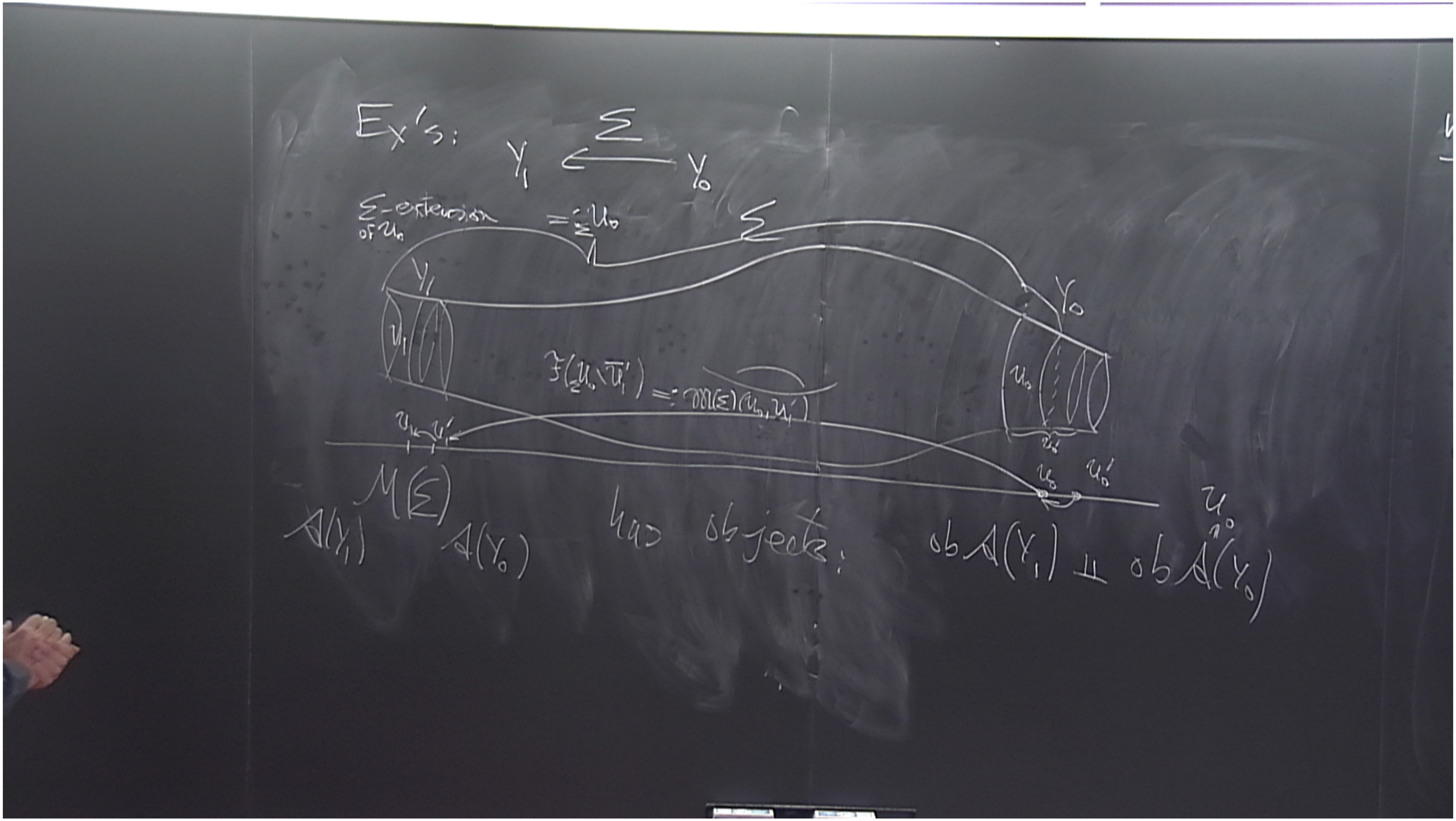
some v.s. ass. to Σ.



Ex's: $Y_1 \xleftarrow{\Sigma} Y_0$

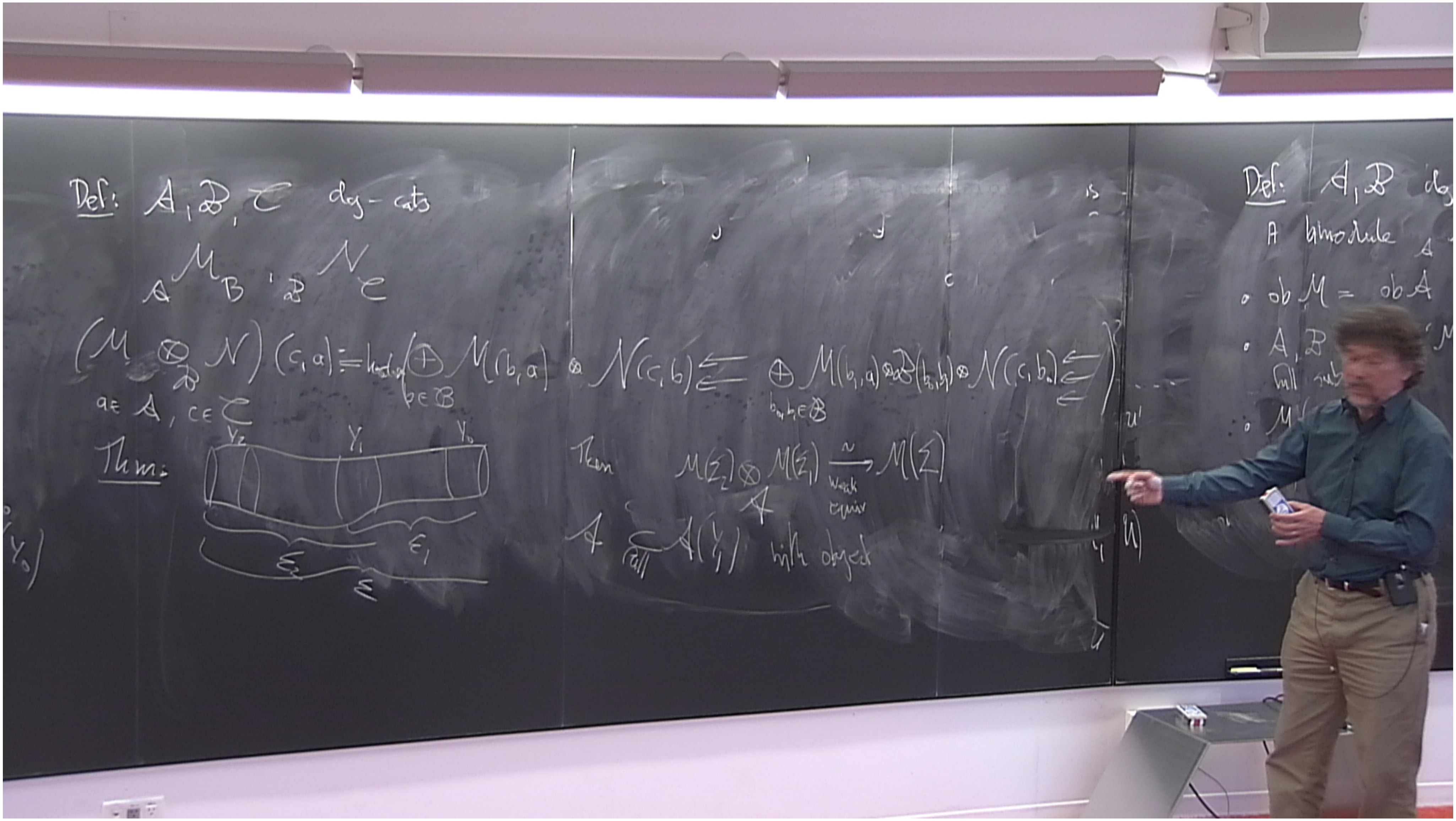
$\mathcal{M}(\Sigma)$ has objects: ob $\mathcal{A}(Y_1) \amalg$ ob $\mathcal{A}(Y_0)$

$\mathcal{A}(Y_1)$ $\mathcal{A}(Y_0)$



Ex's: $Y_1 \xleftarrow{\Sigma} Y_0$

has objects: ob $\mathcal{A}(Y_1) \amalg$ ob $\mathcal{A}(Y_0)$



Def: $\mathcal{A}, \mathcal{B}, \mathcal{C}$ dg-cats

${}_{\mathcal{A}}\mathcal{M}_{\mathcal{B}}, \; {}_{\mathcal{B}}\mathcal{N}_{\mathcal{C}}$

$$(\mathcal{M} \otimes_{\mathcal{B}} \mathcal{N})(c,a) := \mathrm{hocolim}\Big(\bigoplus_{b\in\mathcal{B}} \mathcal{M}(b,a) \otimes \mathcal{N}(c,b) \leftleftarrows \bigoplus_{b_0,b_1\in\mathcal{B}} \mathcal{M}(b_1,a)\otimes\mathcal{B}(b_0,b_1)\otimes\mathcal{N}(c,b_0) \substack{\leftarrow\\ \leftarrow\\ \leftarrow} \cdots\Big)$$

$a \in \mathcal{A}, \; c \in \mathcal{C}$

Thm:

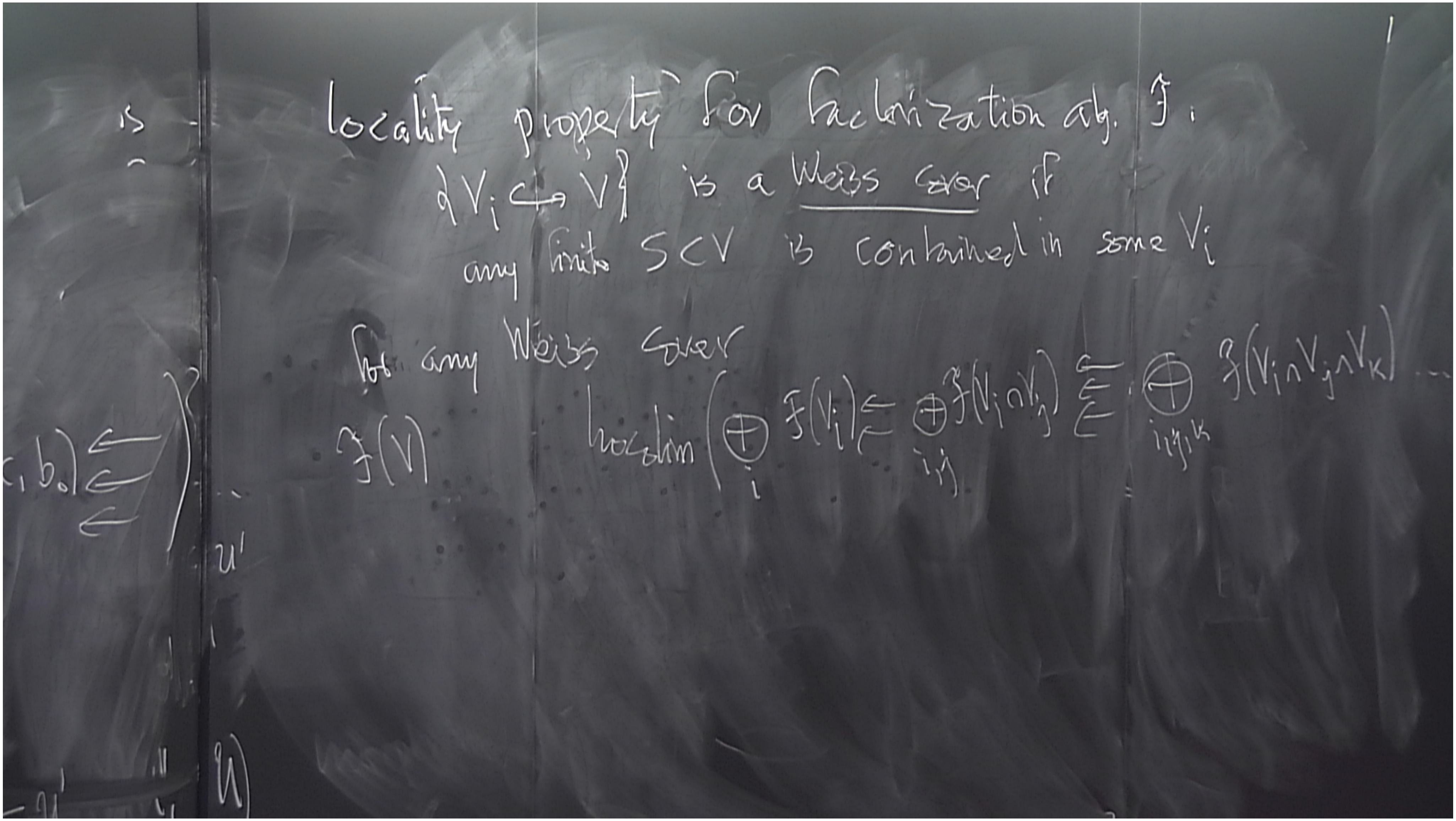
Then $\mathcal{M}(\Sigma_2) \otimes_{\mathcal{A}} \mathcal{M}(\Sigma_1) \xrightarrow[\text{weak equiv}]{\sim} \mathcal{M}(\Sigma)$

$\mathcal{A} \underset{\text{full}}{\subset} \mathcal{A}(Y_1)$ with object

Def. $\mathcal{A}, \mathcal{B}$ dg

A bimodule $\mathcal{A}$

- ob $\mathcal{M}$ = ob $\mathcal{A}$
- $\mathcal{A}, \mathcal{B}$ full sub
- $\mathcal{M}$



is

locality property for factorization alg. $\mathcal{F}$:

$\{V_i \hookrightarrow V\}$ is a Weiss cover if any finite $S \subset V$ is contained in some V_i

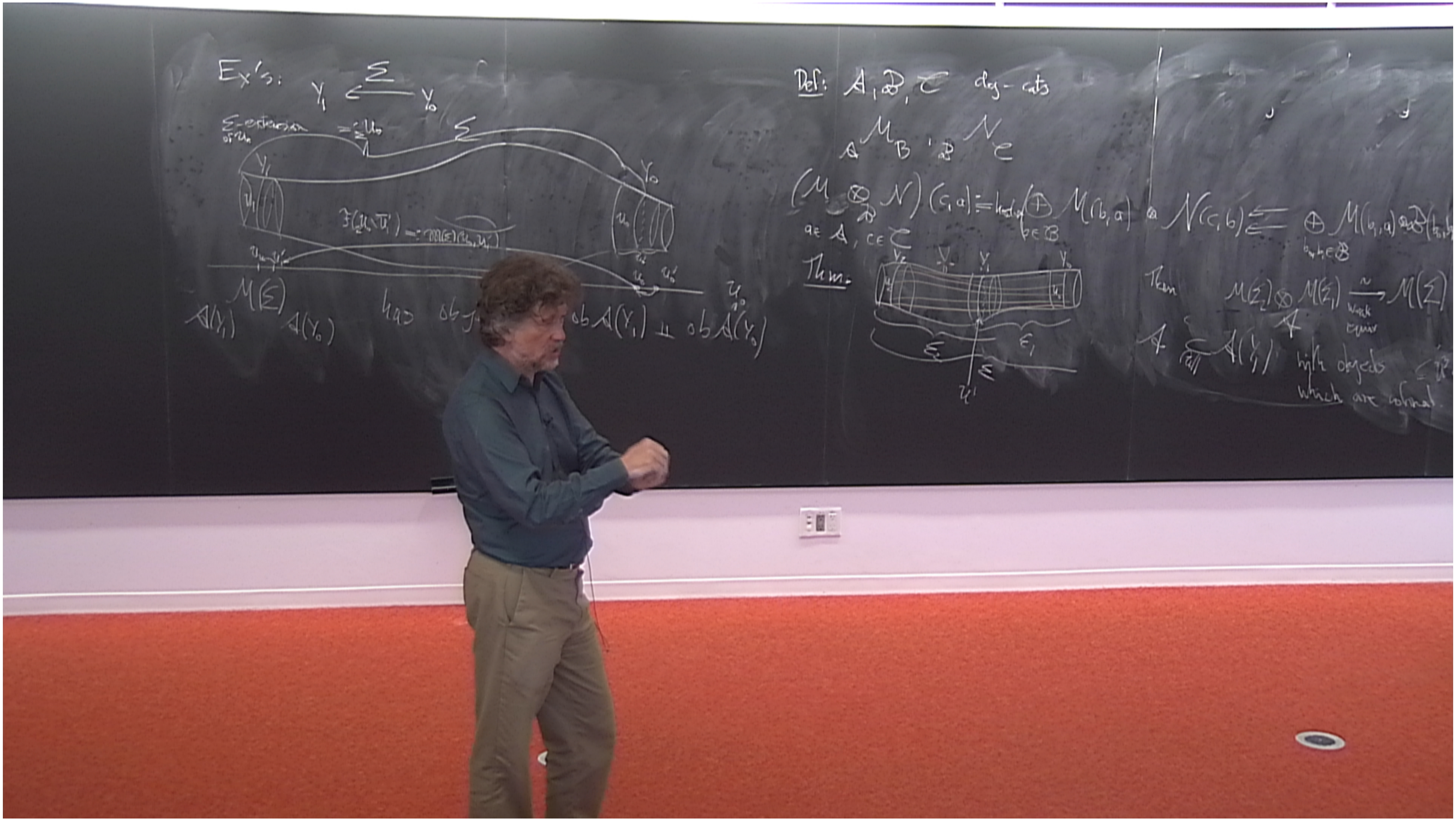
for any Weiss cover

$$\mathcal{F}(V) \quad \operatorname{hocolim}\Big(\bigoplus_i \mathcal{F}(V_i) \leftleftarrows \bigoplus_{i,j} \mathcal{F}(V_i \cap V_j) \overset{\leftarrow}{\leftleftarrows} \bigoplus_{i,j,k} \mathcal{F}(V_i \cap V_j \cap V_k) \cdots$$

$(\ ,b_0) \leftleftarrows$

u'

U



Ex's: $Y_1 \xleftarrow{\Sigma} Y_0$

Σ-extension of u_0 $=: \Sigma u_0$

$\mathcal{F}(\Sigma u_0, \overline{u}_1') =: \mathcal{M}(\Sigma)(u_0, u_1')$

$\mathcal{M}(\Sigma)$ has ob ... ob $\mathcal{A}(Y_1) \amalg$ ob $\mathcal{A}(Y_0)$

$\mathcal{A}(Y_1)$ $\mathcal{A}(Y_0)$

Def: $\mathcal{A}, \mathcal{B}, \mathcal{C}$ dg-cats

${}_{\mathcal{A}}\mathcal{M}_{\mathcal{B}}, \; {}_{\mathcal{B}}\mathcal{N}_{\mathcal{C}}$

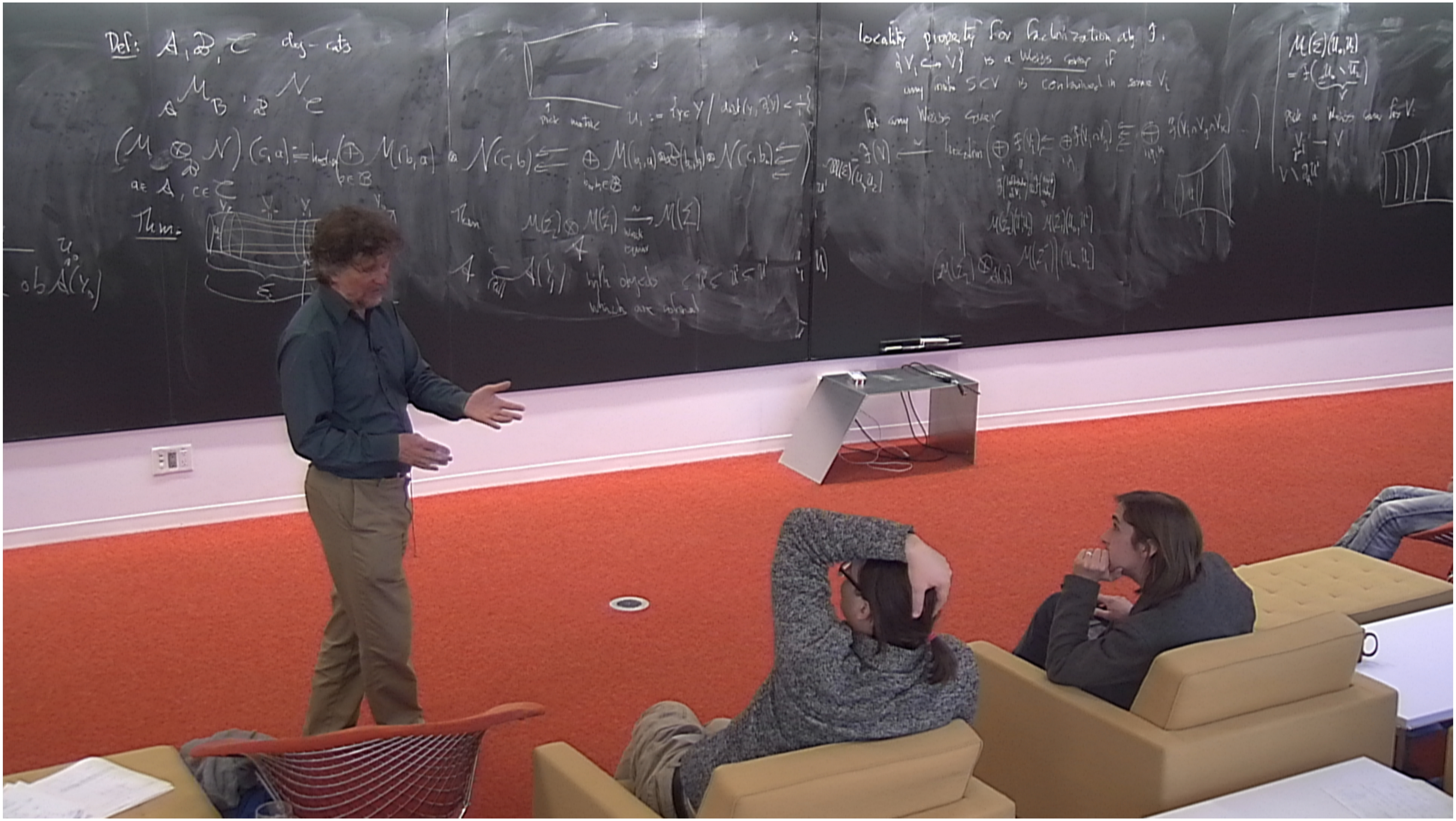
$(\mathcal{M} \otimes_{\mathcal{B}} \mathcal{N})(c,a) := \mathrm{hocolim}\big(\bigoplus_{b \in \mathcal{B}} \mathcal{M}(b,a) \otimes \mathcal{N}(c,b) \leftleftarrows \bigoplus_{b_0, b_1 \in \mathcal{B}} \mathcal{M}(b_1,a) \otimes \mathcal{B}(b_0,b_1$

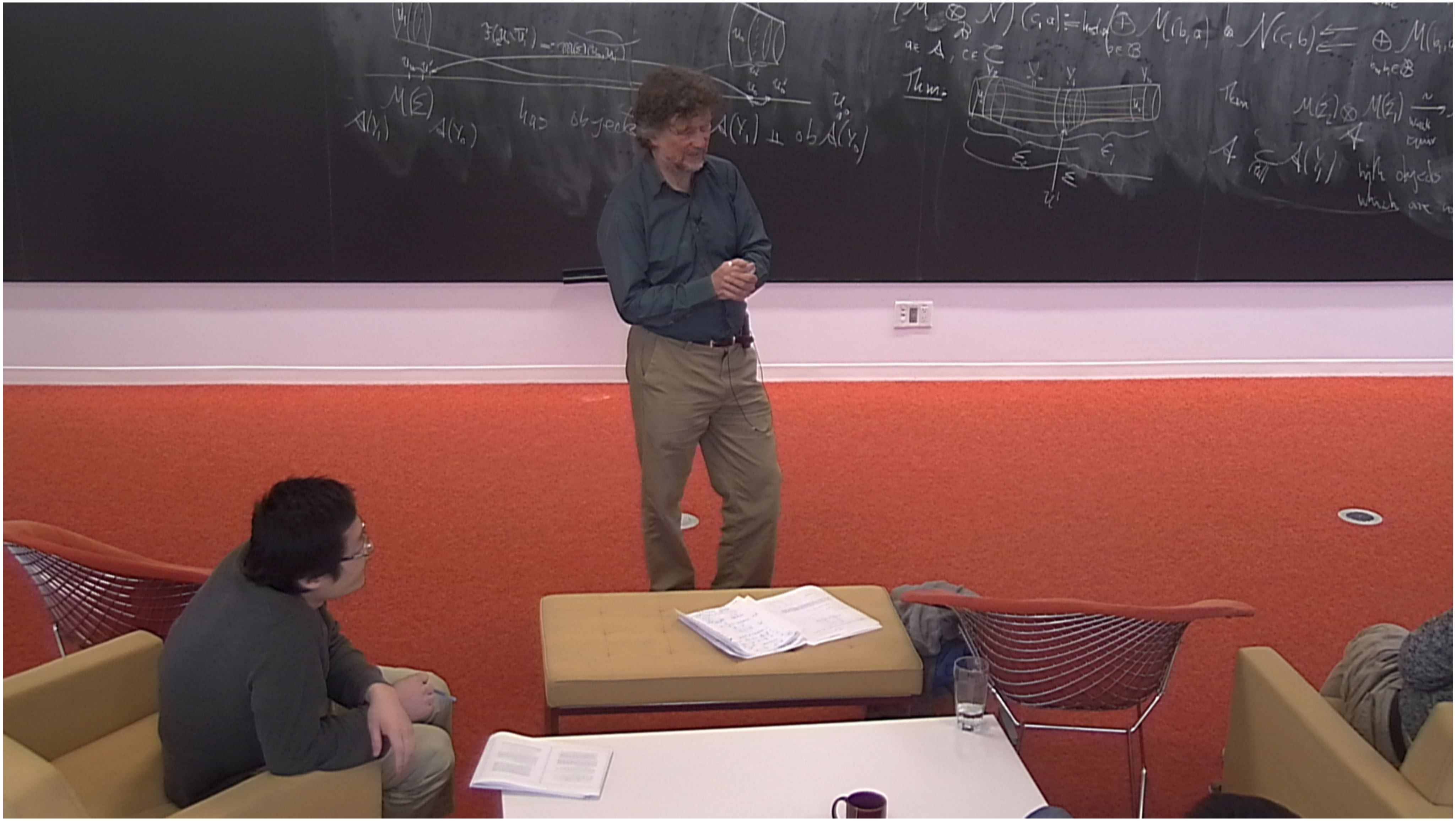
$a \in \mathcal{A}, \; c \in \mathcal{C}$

Thm.

Thm $\mathcal{M}(\Sigma_2) \otimes_{\mathcal{A}} \mathcal{M}(\Sigma_1) \xrightarrow[\text{weak equiv}]{\sim} \mathcal{M}(\Sigma)$

$\mathcal{A} \underset{\text{full}}{\subset} \mathcal{A}(Y_1)$ with objects which are cofinal.


Def: A, B, C dg-cats
locality property for factorization alg. F.
{V_i ↪ V} is a Weiss cover if
any finite S⊂V is contained in some V_i
for any Weiss cover
pick metric
Thm.
Then
weak equiv
Pick a Weiss cover for V


Thm.
has objects
Thm