Title: Quantization, reduction mod p, and the Weyl algebra

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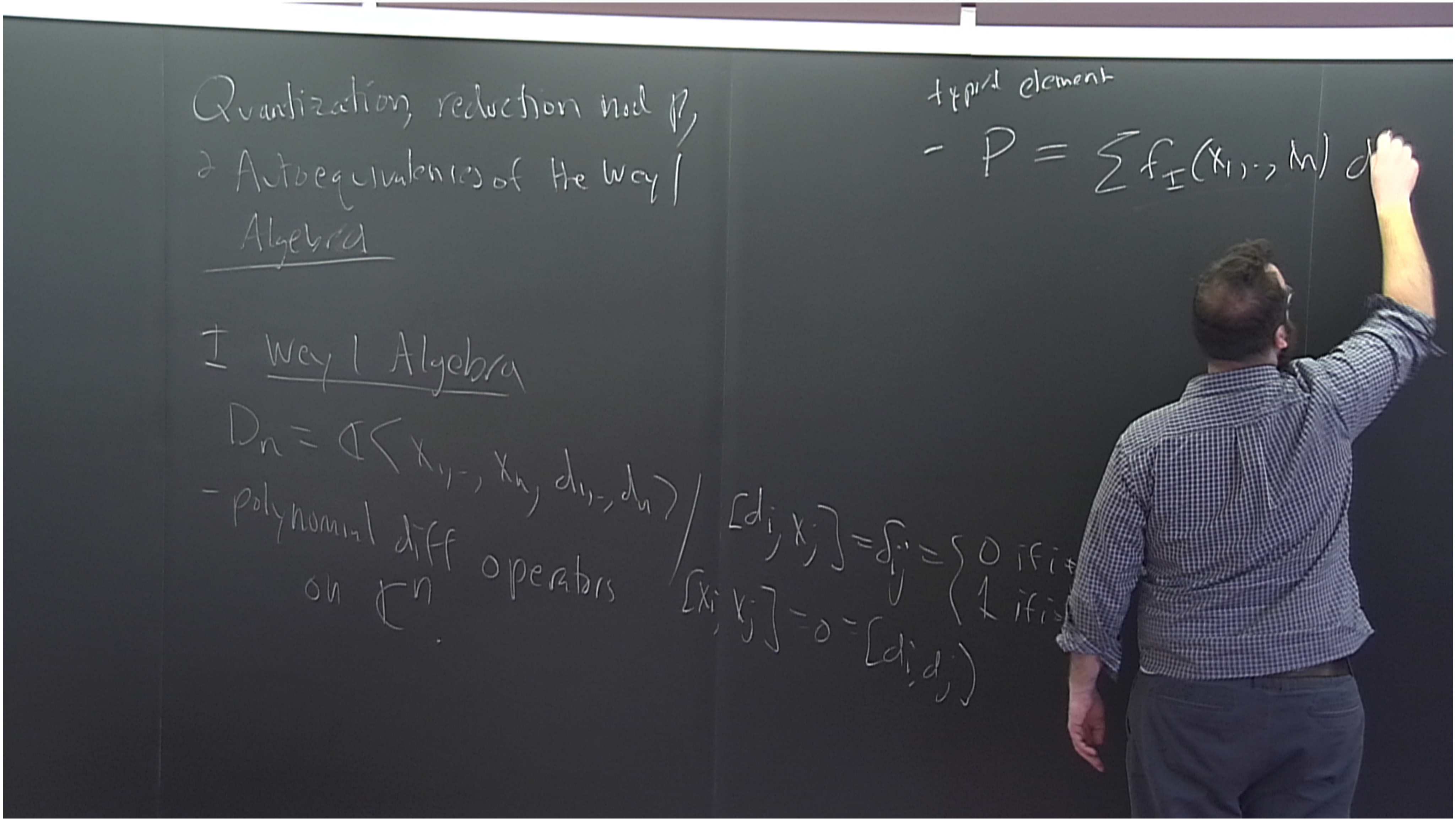
Abstract:

Quantization, reduction mod p, & Autoequivalences of the Weyl Algebra

I Weyl Algebra


Quantization, reduction mod p,
& Autoequivalences of the Weyl Algebra
I Weyl Algebra
$D_n = \mathbb{C}\langle x_1, \dots, x_n, d_1, \dots, d_n \rangle / [d_i, x_j] = \delta_{ij}$
$[x_i, x_j] = 0 = [d_i, d_j]$
- polynomial diff operators on $\mathbb{C}^n$.
typical element
$- P = \sum f_I(x_1, \dots, x_n)$

duction mod p,
s of the way!

typical element

$$- P = \sum_I f_I(x_1,\dots,x_n)\, d_1^{i_1}\cdots d_n^{i_n}$$

module

d_i acts on f via $\frac{\partial}{\partial x_i}$.

bra

$\dots, x_n, d_1,\dots,d_n\rangle / [d_i, x_j] = \delta_{ij} = \begin{cases} 0 & \text{if } i\neq j \\ 1 & \text{if } i=j \end{cases}$

operators $\quad [x_i, x_j] = 0 = [d_i, d_j]$

$\Rightarrow$ to any such P, can associate

$$M_P := D_n / D_n \cdot P$$

$f_I(x_1,\dots,x_n)\, \partial_1^{i_1} \cdots \partial_n^{i_n}$

∂_i acts on f via $\frac{\partial}{\partial x_i}$.

$\Rightarrow$ to any such P, can associate

$$M_P := D_n / D_n \cdot P$$

module M reflects properties of P (as a linear PDE)

– e.g. suppose we have P & Q s.t.

$M_P \cong M_Q$ (over D_n)

$\Rightarrow$ solution spaces for P & Q are "the same".

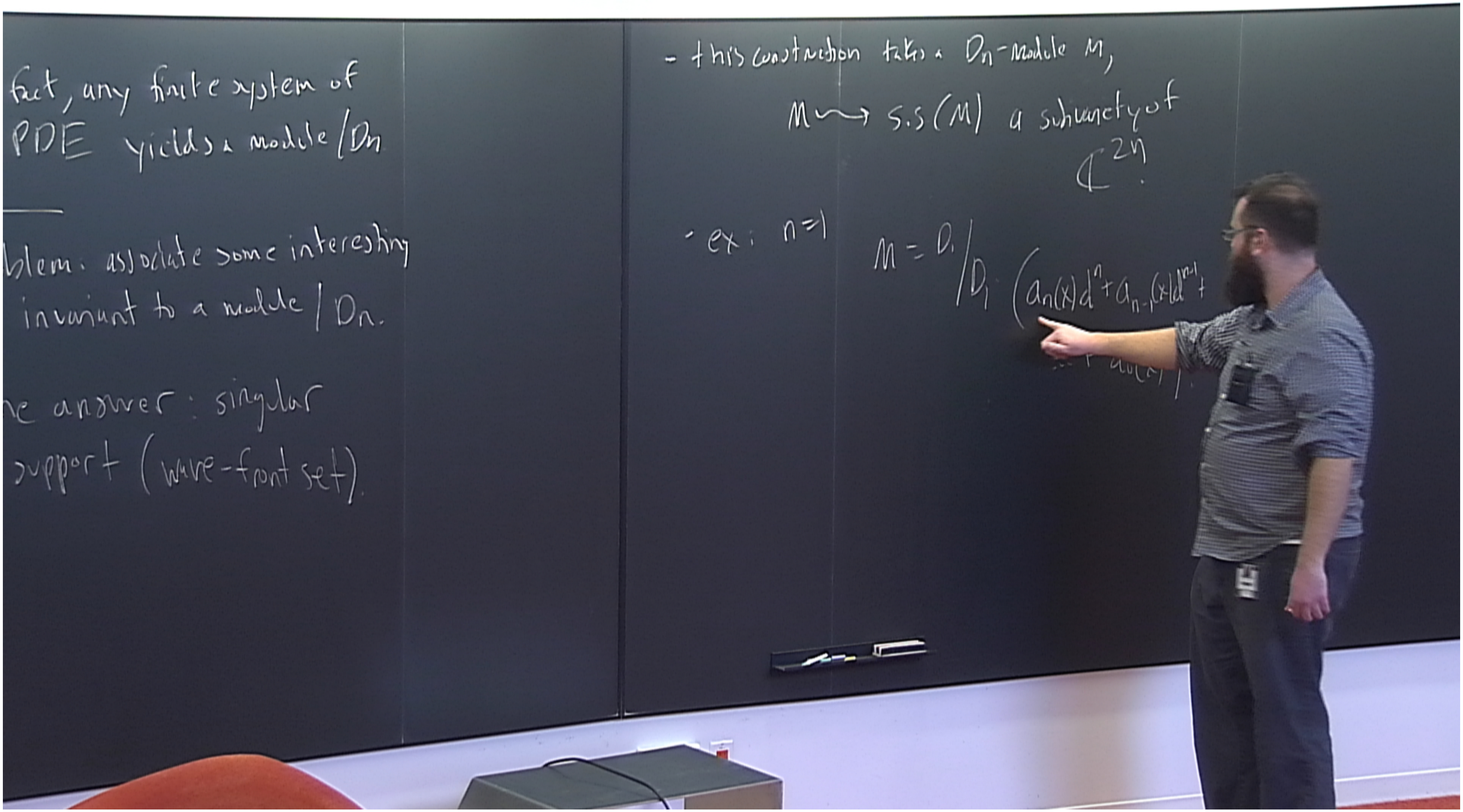
module M reflects properties of
P (as a linear PDE)

- e.g. suppose we have P & Q s.t.
  $M_P \cong M_Q$ (over D_n)
  $\Rightarrow$ solution spaces for P & Q are "the same".

(in fact, any finite system of PDE yields a module $/ D_n$)

Problem: associate some interesting invariant to a module $/ D_n$.

- one answer: singular support (wave-front set)


fact, any finite system of
PDE yields a module / Dn
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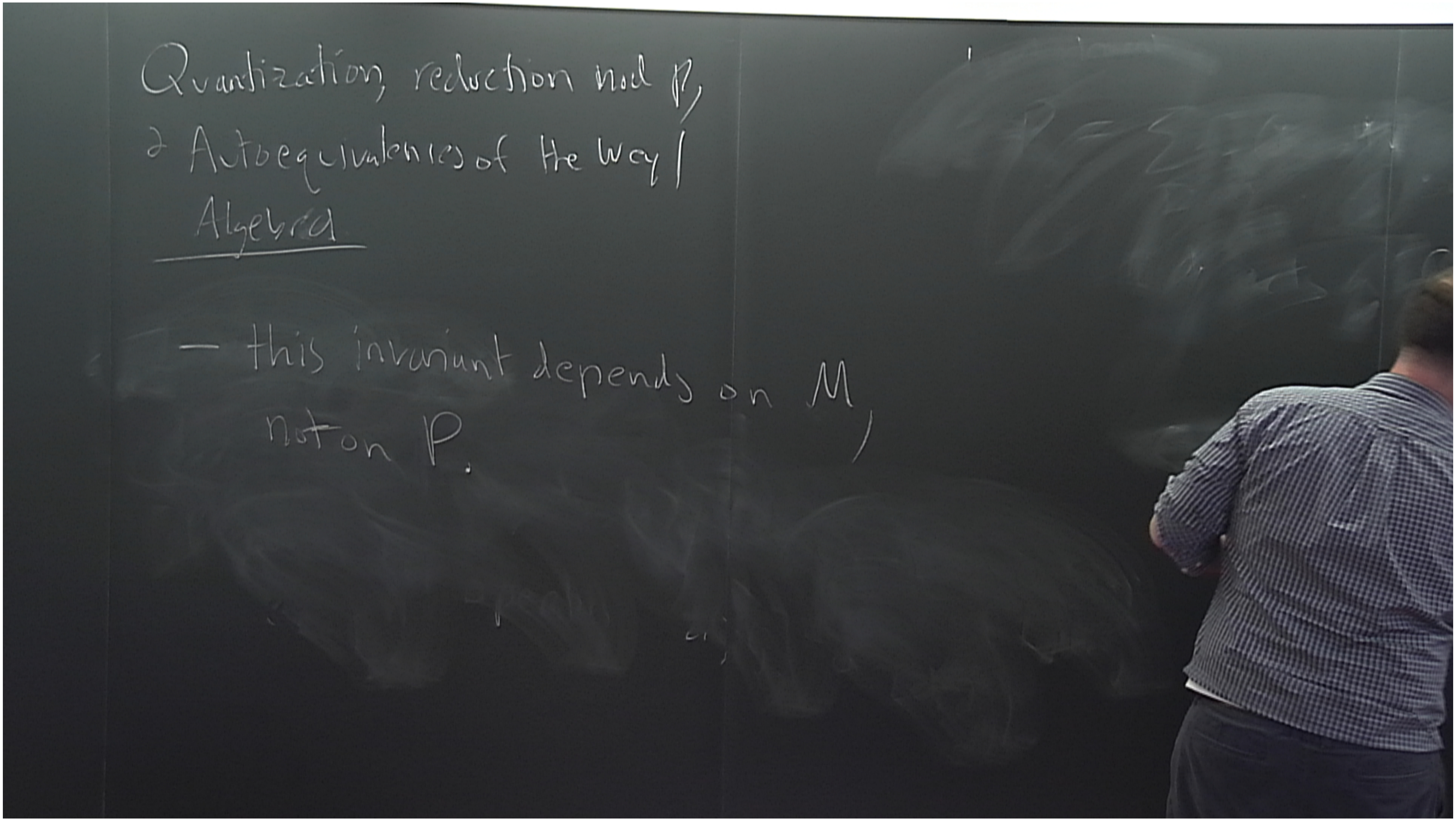
invariant to a module / Dn.
he answer: singular
support (wave-front set).
- this construction takes a Dn-module M,
M ⇝ S.S(M) a subvariety of
$\mathbb{C}^{2n}$!
· ex: n=1
$M = D_1 / D_1 \cdot (a_n(x) d^n + a_{n-1}(x) d^{n-1} + \dots + a_0(x))$.


Quantization, reduction mod p,
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Algebra
– this invariant depends on M,
not on P.

Quantization, reduction mod p, & Autoequivalences of the Weyl Algebra

— this invariant depends on M, not on P.

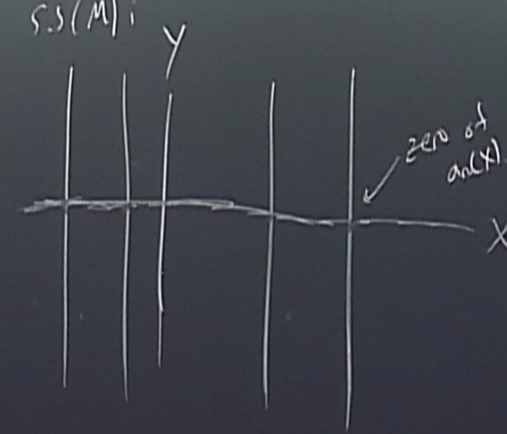
- this construction takes a D_n-module M,

$$M \rightsquigarrow s.s(M) \text{ a subvariety of } \mathbb{C}^{2n}.$$

- ex: $n=1$

$$M = D_1 / D_1\left(a_n(x)\partial^n + a_{n-1}(x)\partial^{n-1} + \dots + a_0(x)\right).$$

s.s(M):



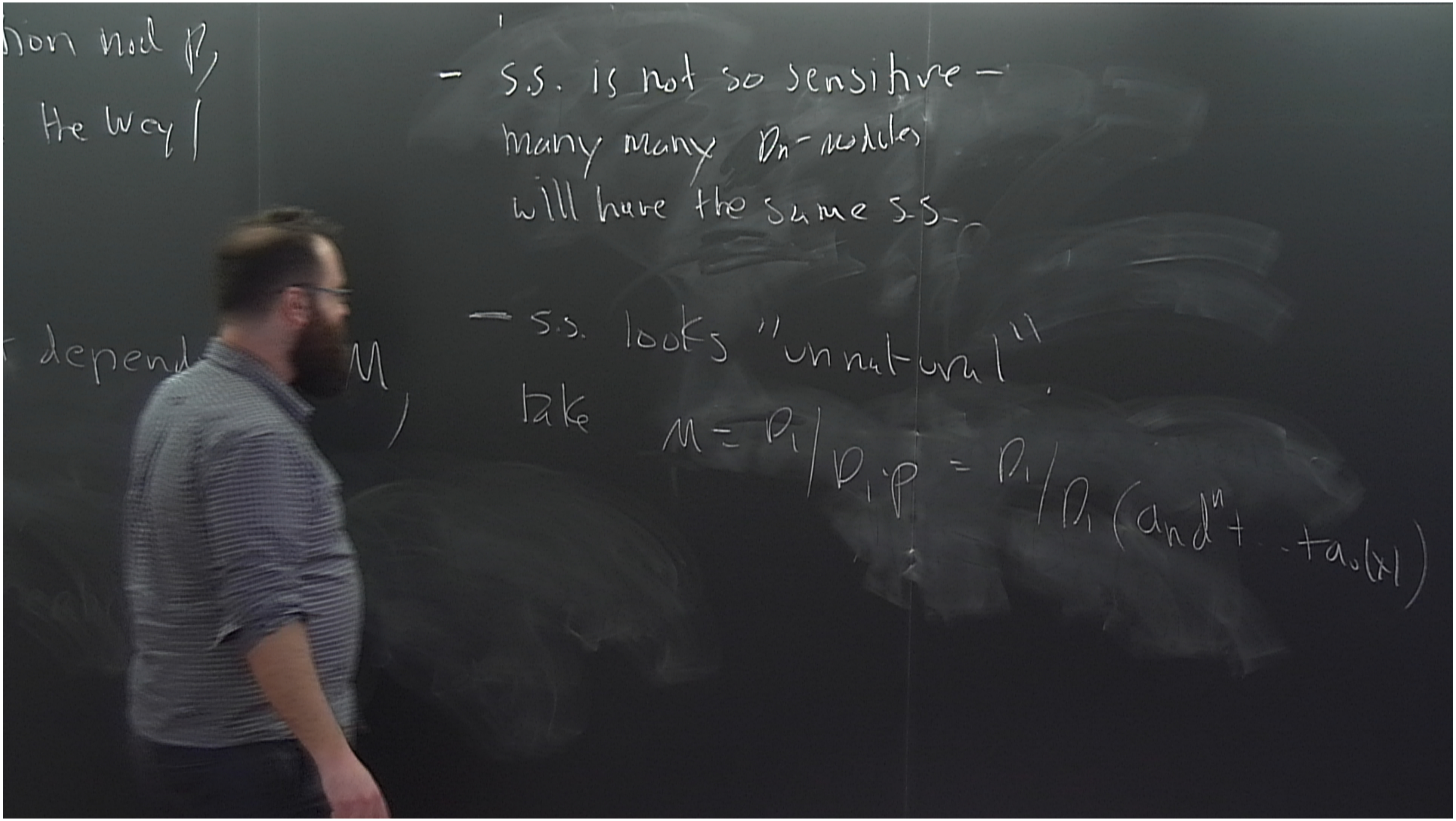
$$Z(a_n(x)\cdot y^n)$$

sion mod p,
the way

- S.S. is not so sensitive -
many many D_n-modules
will have the same S.S.

-

depends on M,



tion mod p,
the way
- s.s. is not so sensitive -
many many D_n-modules
will have the same s.s.
- s.s. looks "unnatural".
depend M
take $M = D_1 / D_1 \cdot P = D_1 / D_1(a_n d^n + \cdots + a_0(x))$

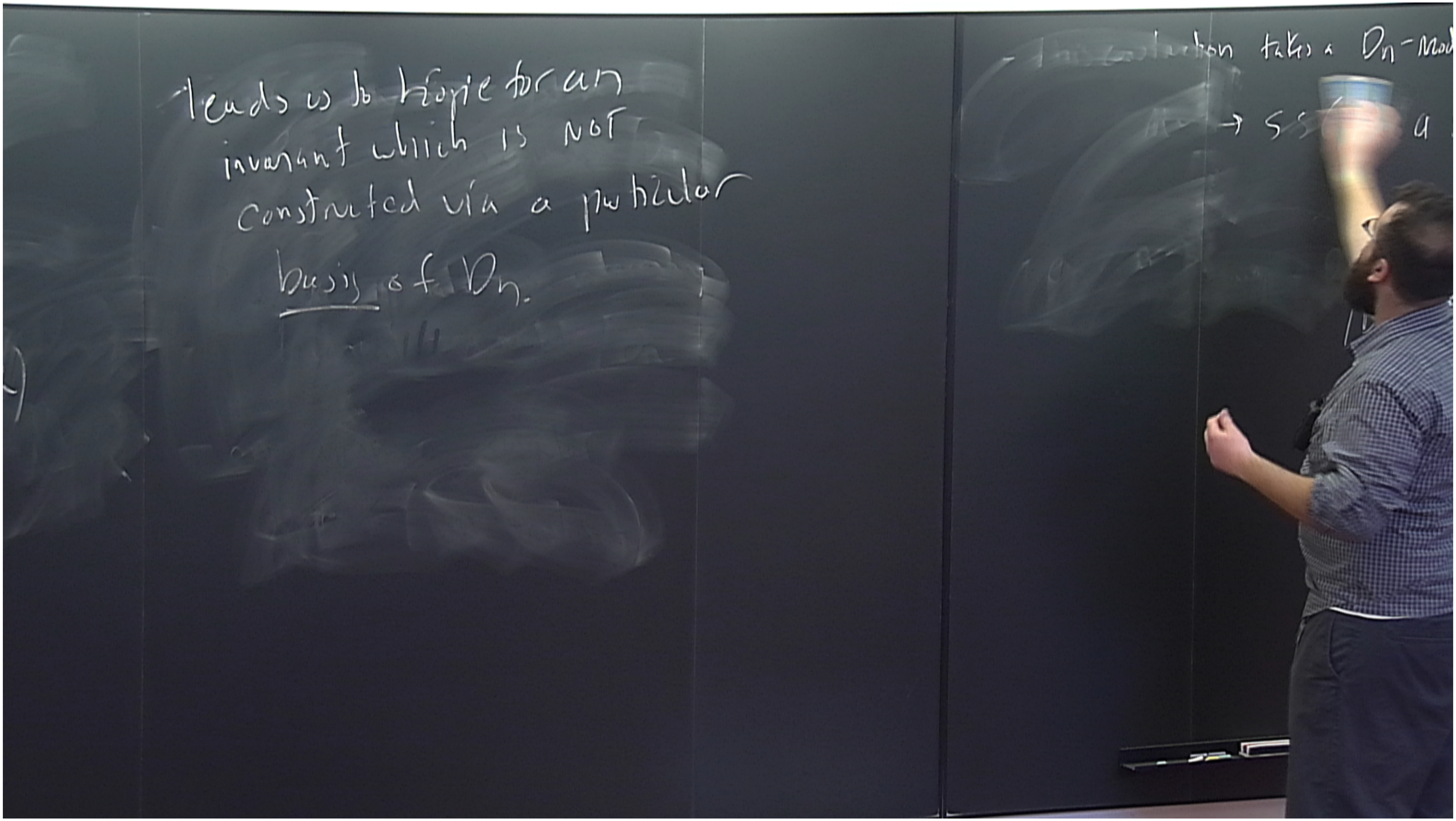
- S.S. is not so sensitive –
many many $\mathcal{D}_n$-modules
will have the same S.S.

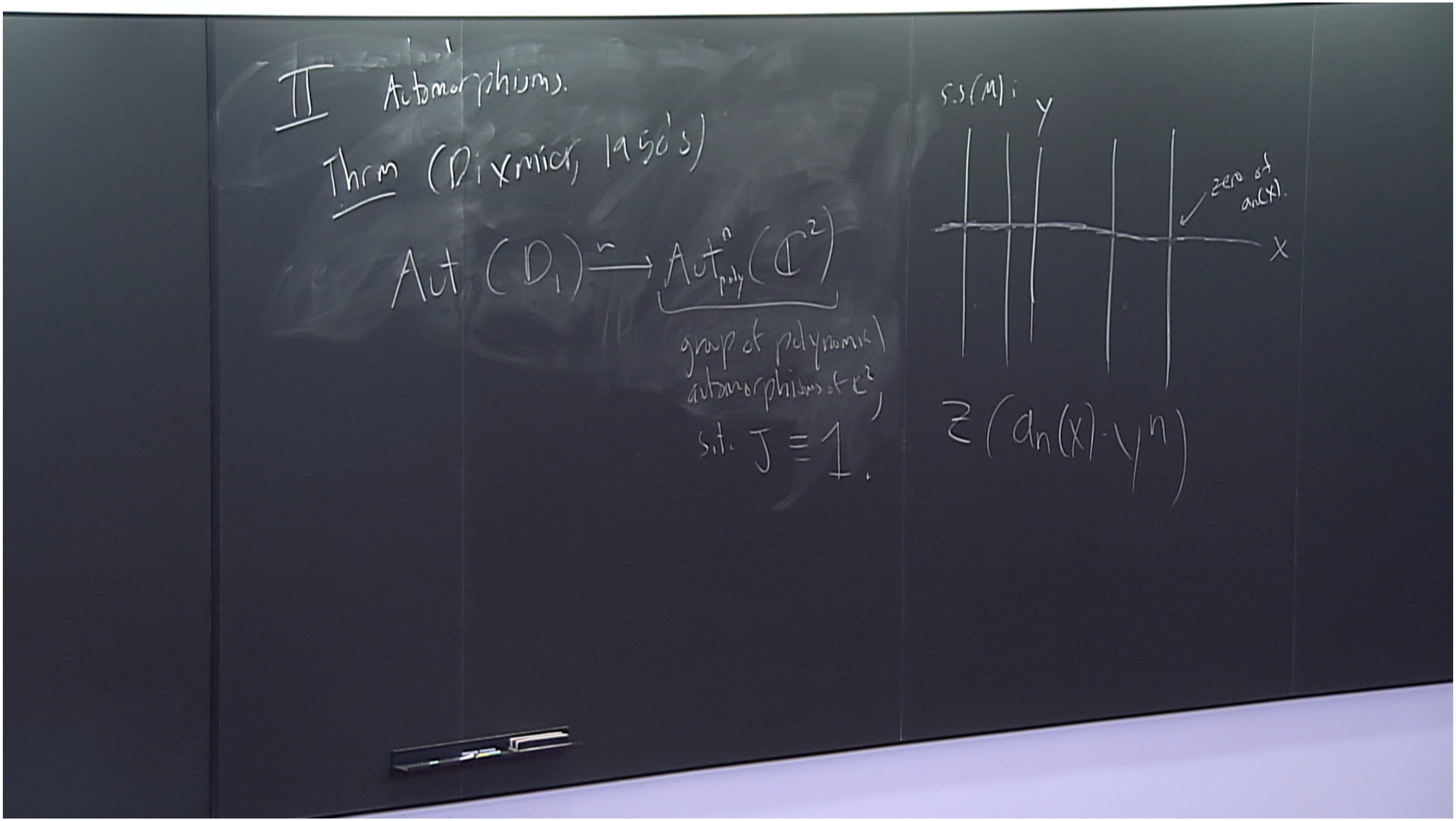
– S.S. looks "unnatural".

...nds on M,

Take $M = \mathcal{D}_1 / \mathcal{D}_1 \cdot P = \mathcal{D}_1 / \mathcal{D}_1 (a_n \partial^n + \ldots + a_0(x))$

the sol^{n}s to this ODE $\xrightarrow{\wedge}$ the sol^{n}s to another ODE


leads us to hope for an
invariant which is NOT
constructed via a particular
basis of D_n.


II Automorphisms.
Thrm (Dixmier, 1950's)
$Aut(D_1) \xrightarrow{\sim} Aut^n_{poly}(\mathbb{C}^2)$
group of polynomial automorphisms of $\mathbb{C}^2$, s.t. $J \equiv 1$.
s.s(M):
y
x
zero of $a_n(x)$.
$Z(a_n(x) \cdot y^n)$

II Automorphisms.

Thrm (Dixmier, 1950's)

$$\mathrm{Aut}(D_1) \xrightarrow{\sim} \underbrace{\mathrm{Aut}^0_{poly}(\mathbb{C}^2)}_{\substack{\text{group of polynomial} \\ \text{automorphisms of } \mathbb{C}^2, \\ \text{s.t. } J \equiv 1.}}$$

$$D_1 = \mathbb{C}\langle x, d\rangle / [d,x]=1$$

$$g \in GL_2, \quad [gd, gx] = \det g\, [d,x]$$

proof goes by describing both sides.

LHS: $\nearrow SL_2(\mathbb{C})$

$\searrow B = \begin{cases} x \to x \\ d \mapsto d + f(x) \\ f(x) \in \mathbb{C}[x] \end{cases}$

$$\mathrm{Aut}(D_1) = SL_2 \ast B,$$

...ection mod β,

of the Way |

$$\beta' = \begin{cases} x \to x \\ y \to y + f(x) \end{cases}$$

- suggests a hope for what a very refined invariant of D_1-modules would do.

$$M \longmapsto \mathrm{Cyc}(M) \subseteq \mathbb{C}^2$$

s.t.

$a^* M$ is the module

$$D_1 \xrightarrow{a} D_1 \curvearrowright M.$$

[e.g. $a = \begin{pmatrix} x \mapsto d \\ d \mapsto -x \end{pmatrix}$

$a^* M = \hat{M}$].

III $\underline{D_n(K)}$

$\mathrm{char}(K) > 0, \quad K = \bar{K}.$

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III. $D_n(K)$

$\mathrm{char}(K) > 0$, $K = \bar{K}$.

$$D_n = K\langle x_1, \dots, x_n, d_1, \dots, d_n \rangle \Big/ \begin{array}{l} [d_i, x_j] = \delta_{ij} \\ [x_i, x_j] = 0 \\ = [d_i, d_j] \end{array}$$

centre of D_n!

$a^* M$ is the module

$$D_1 \xrightarrow{a} D_1 \curvearrowright M.$$

[e.g. $a = \begin{pmatrix} x \mapsto d \\ d \mapsto -x \end{pmatrix}$

$$a^* M = \hat{M}$$].

III. $D_n(K)$:

$\mathrm{char}(K) > 0$, $K = \bar{K}$.

$$D_n = K\langle x_1, \dots, x_n, d_1, \dots, d_n \rangle \Big/ \begin{matrix} [d_i, x_j] = \delta_{ij} \\ [x_i, x_j] = 0 \\ = [d_i, d_j] \end{matrix}$$

centre of D_n:

$x_1^p, \dots, x_n^p$ are central.

$d_1^p, \dots, d_n^p$

$\ldots, d_n \rangle \Big/ \; [d_i, x_j] = \delta_{ij}$

$[x_i, x_j] = 0 = [d_i, d_j]$

in fact,

$$Z(D_n(k)) = k[x_1^p, \ldots, x_n^p, d_1^p, \ldots, d_n^p]$$

proof goes by describing both sides.

LHS: $\nearrow SL_2(\mathbb{C})$, $\searrow B = \begin{cases} x \mapsto x \\ d \mapsto d + f(x \end{cases}$ $f(x) \in \mathbb{C}[x]$

$\mathrm{Aut}(D_1) = SL_2 * B$

$[d_i, x_j] = \delta_{ij}$

$[x_i, x_j] = 0$

$= [d_i, d_j]$

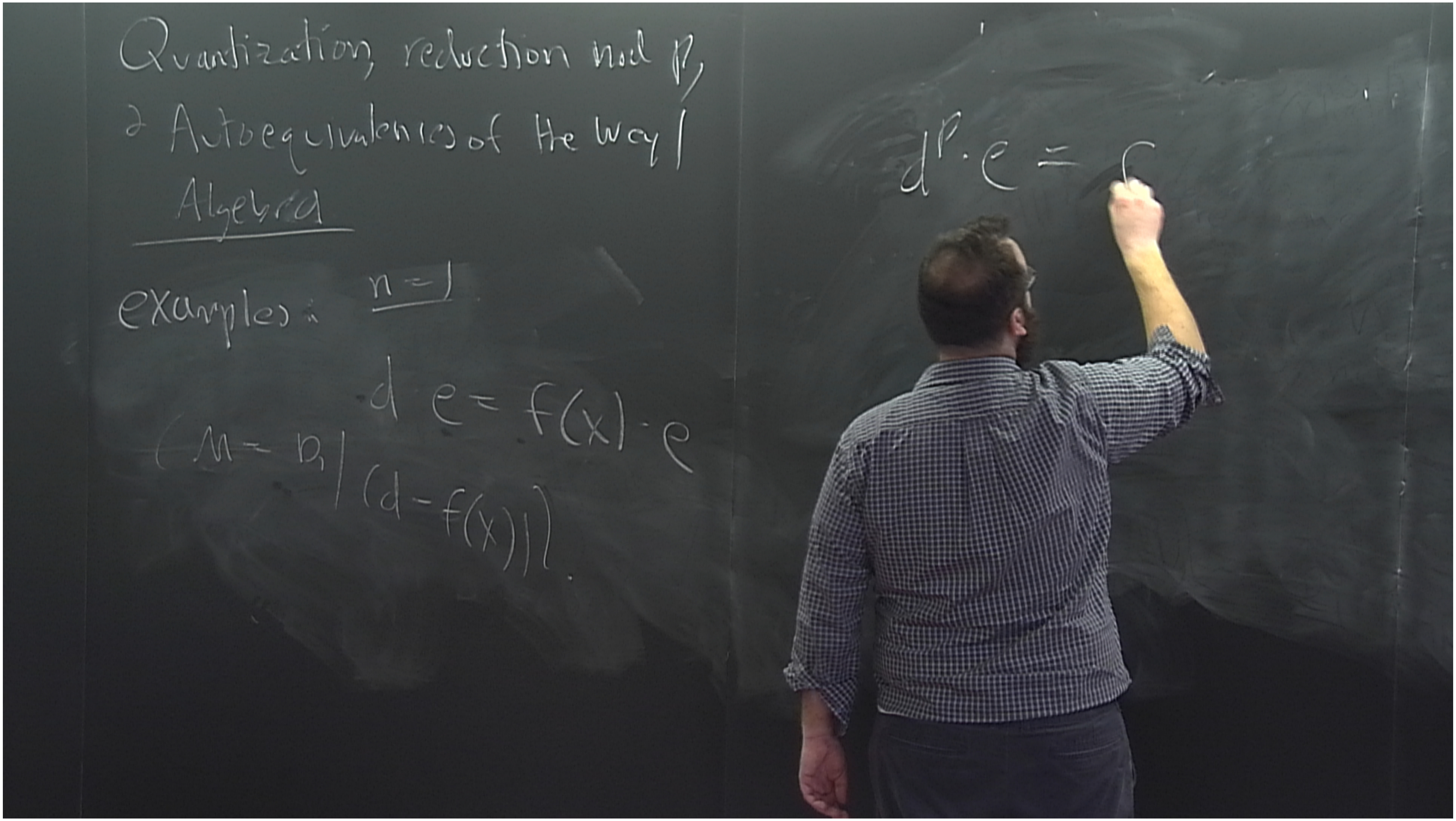
in fact

$$Z(D_n(k)) = k[x_1^p, \dots, x_n^p, d_1^p, \dots, d_n^p]$$

M a $D_n(k)$-module

$$cyc(M) = \mathrm{supp}_{Z(D_n(k))}(M)$$

$$\mathrm{supp} = \mathrm{Zeros}\{f \in Z(D_n(k)) \mid f \cdot M = 0\}.$$


Quantization, reduction mod p, & Autoequivalences of the Weyl Algebra
examples:
n = 1
d · e = f(x) · e
(M = D/(d - f(x)))
d^p · e =

...n mod p,

...the way |

...what

$$d^p \cdot e = \left(f(x)^p + \left(\frac{d}{dx}\right)^{p-1} f\right) e$$

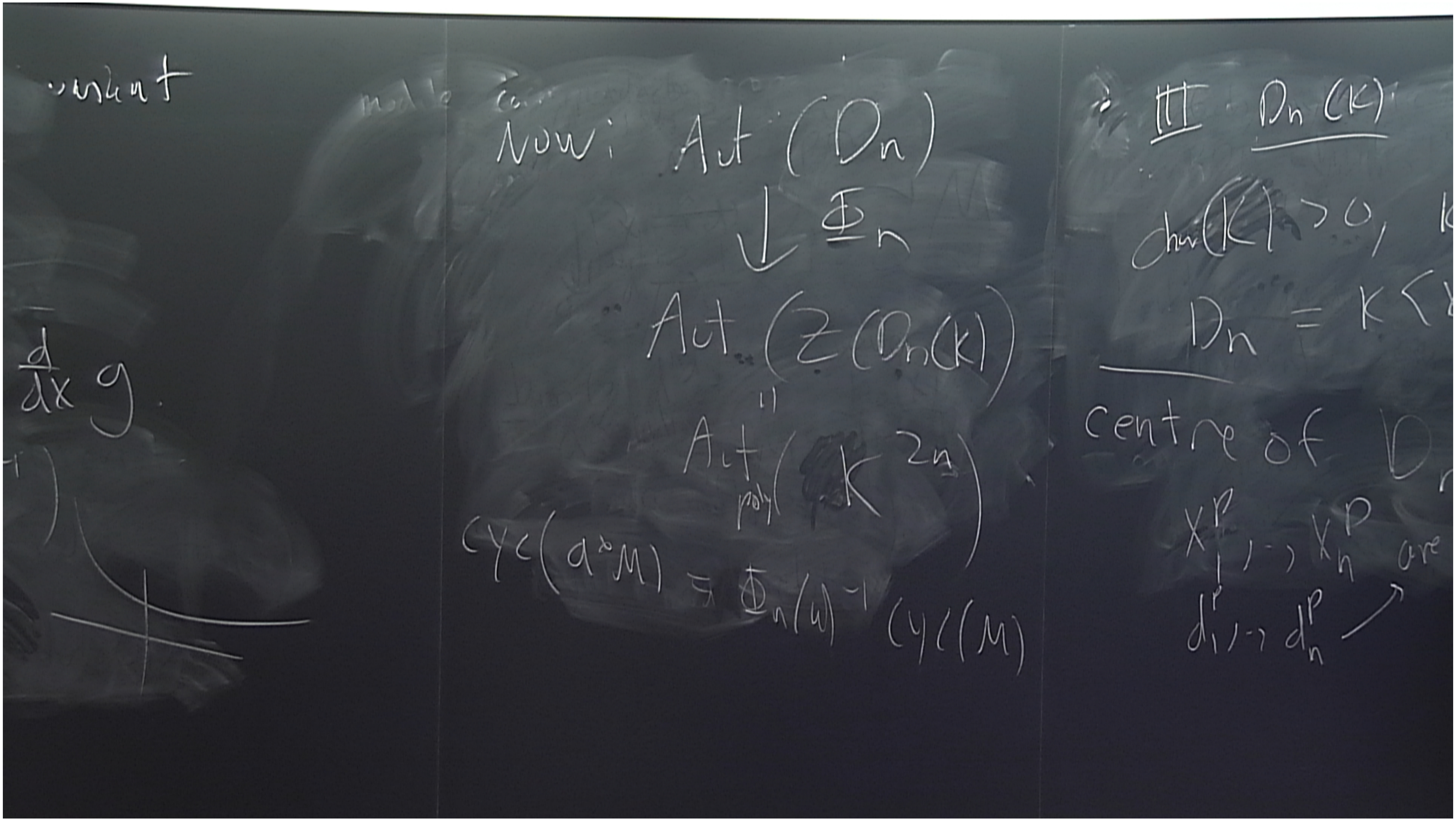
$= 0$ if $f = \frac{d}{dx} g$.

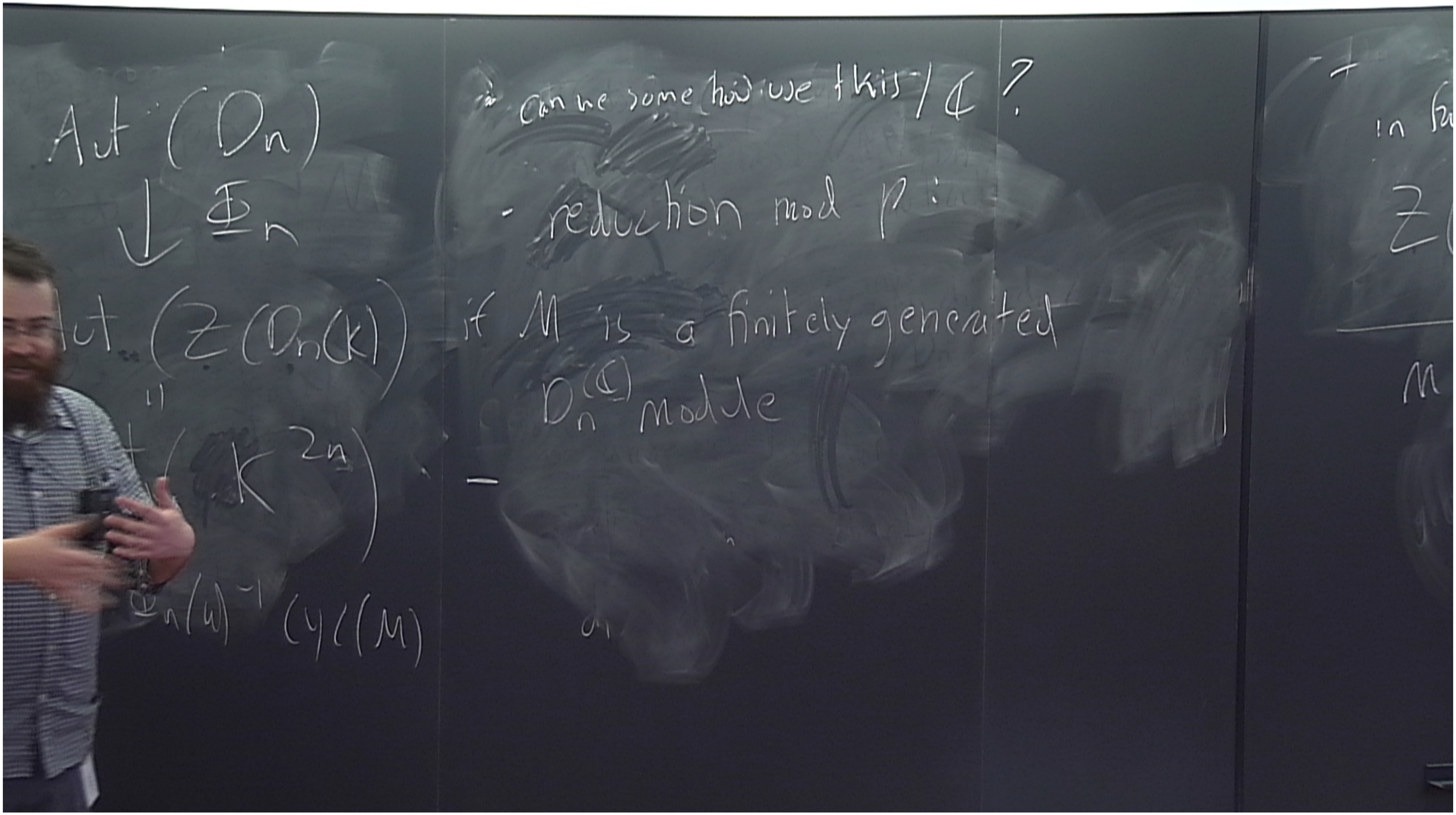
$(\neq 0$ if $f = x^{p-1})$

$f(x) \cdot e$

$$\mathrm{cyc}(m) = \sum \left(d^p - f^p - \left(\frac{d}{dx}\right)^{p-1} f\right)$$

...).


Now: Aut(D_n)
Φ_n
Aut(Z(D_n(k)))
Aut_poly(k^2n)
Cyc(a·M) = Φ_n(a)^-1 Cyc(M)
III. D_n(k)
char(k) > 0,
D_n = k
centre of D
are


Aut (Dn)
Φn
Aut (Z(Dn(k))
(k 2n)
Φn(ω)⁻¹ (Cyl(M))
Can we somehow use this / ℂ ?
- reduction mod p :
if M is a finitely generated
Dn(ℂ) module
in
M

$\mathrm{Aut}(D_n)$

$\downarrow \Phi_n$

$\mathrm{Aut}(Z(D_n(k)))$

$\shortparallel$

$\mathrm{Aut}_{poly}(k^{2n})$

$M) = \Phi_n(\omega)^{-1}(\mathrm{Cyc}(M))$

- can we somehow use this $/\mathbb{C}$?

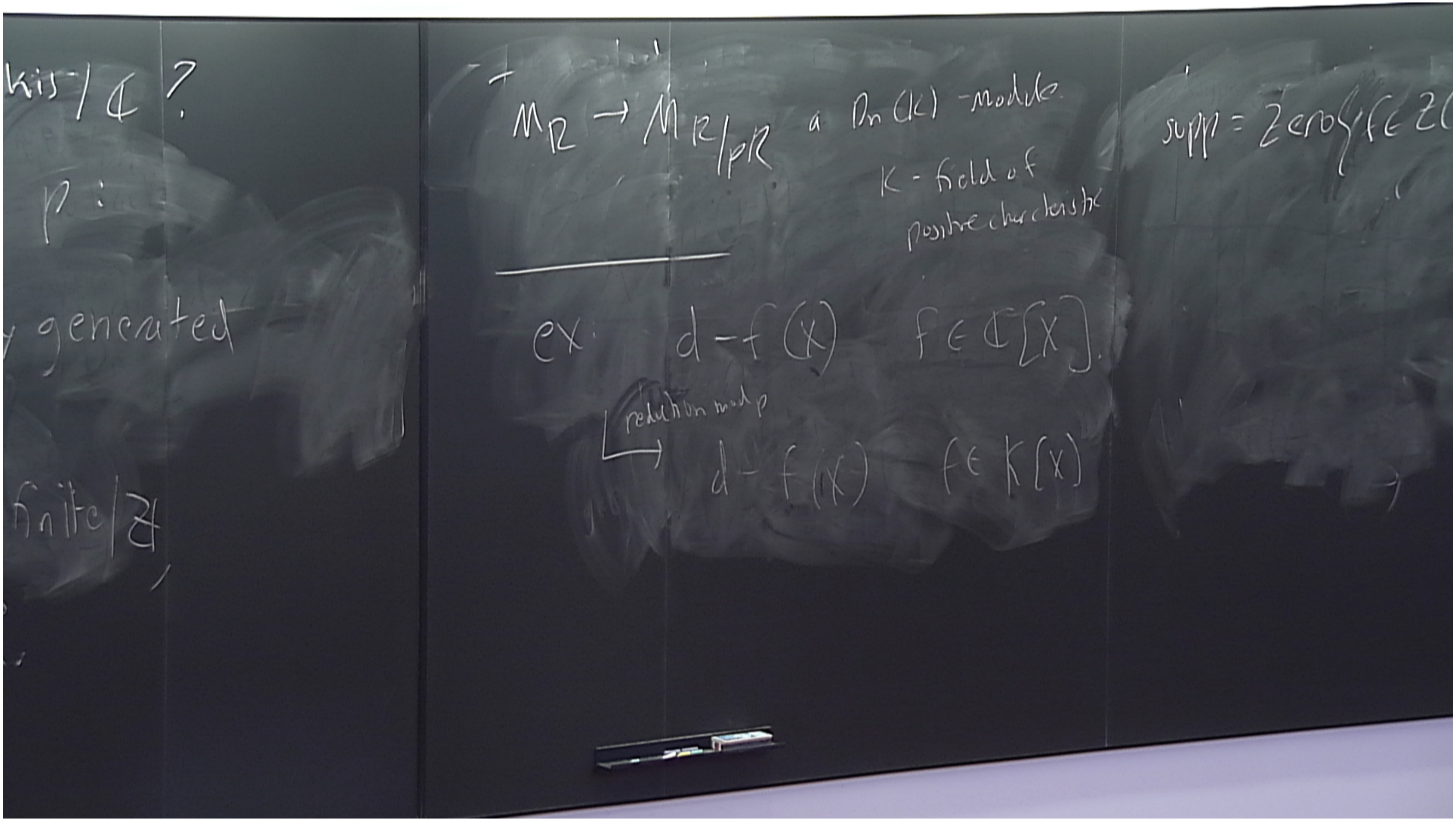
- reduction mod p:

if M is a finitely generated $D_n(\mathbb{C})$ module

- $\exists$ a ring $R \subseteq \mathbb{C}$, finite$/\mathbb{Z}$ s.t. M is defined $/R$.

in fi

M


kis / ℂ ?
p :
generated
finite/ℤ
M_R → M_R/pR a D_n(K)-module.
K - field of positive characteristic
ex: d - f(x) f ∈ ℂ[x].
reduction mod p
d - f(x) f ∈ K[x]
supp = zeros{f ∈ Z(

$\to M_{R/pR}$ a $D_n(K)$-module

K - field of positive characteristic

$d - f(x)$ $\quad f \in \mathbb{C}[x]$.

reduction mod p

$d - f(x)$ $\quad f \in K[x]$

$\mathrm{supp} = \mathrm{Zeros}\{f \in Z(D_n(K)) \mid f \cdot M = 0\}$.

$c_{\chi}(d - f(x))$

$= d^p - f^p$ in $K[x^p, d^p]$

$\downarrow 1/p$

$y = f(x)$ inside $\mathbb{A}^2$.

Quantization, reduction mod p,
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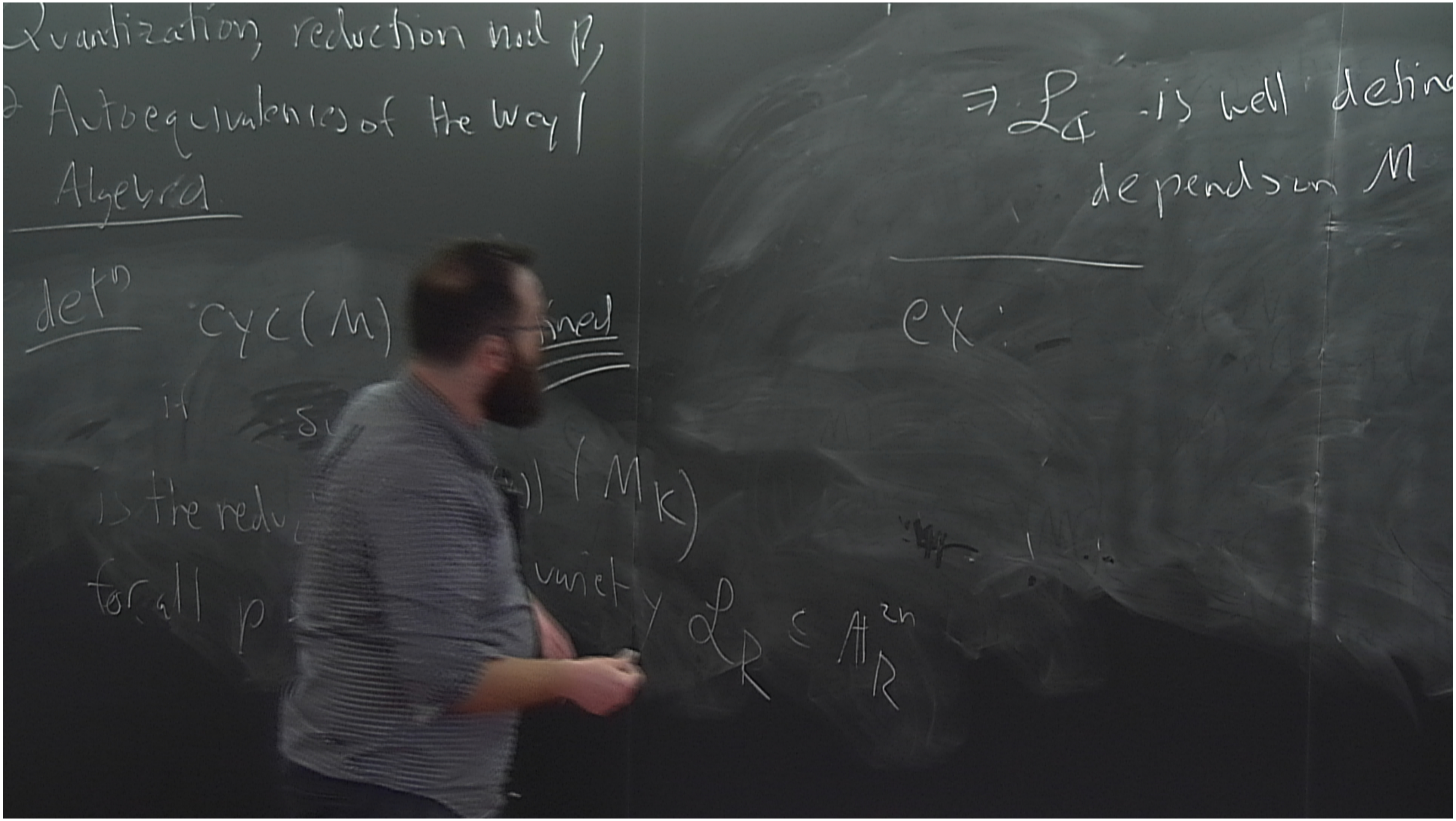
Algebra

defn $Cyc(M)$ is defined

if $supp_{Z(D_h(k))}(M_k)$
is the reduction of some variety $\mathcal{L}_R$
for all $p >> 0$.

$Cyc($


Quantization, reduction mod p,
Autoequivalences of the Weyl
Algebra
defn
Cyc(M)
is the redu
for all p
variety $\mathcal{L}_R \subseteq \mathbb{A}_R^{2n}$
$\Rightarrow \mathcal{L}_{\mathbb{C}}$ is well defin
depends on M
ex

show

$\mathrm{cyc}(a^* \mathcal{O}) = \underline{\Phi_1(a)}^{-1}\{x\text{-axis}\}$

$\Rightarrow$ any $\mathbb{A}^1 \subseteq \mathbb{A}^2$ is a cycle of some $\mathcal{D}_1$-module

polynomial embedding

Higher dim'l version:

Thm [D] take any $\mathcal{L}_{\mathbb{C}} \subseteq \mathbb{C}^{2n}$

smooth, algebraic, Lagrangian

$\mathcal{L} \cong \mathbb{C}^n$ as a variety.

then $\exists!$ $\mathcal{D}_n$-module M, whose cycle is defined and $= \mathcal{L}_{\mathbb{C}}$.

show

$$\mathrm{cyc}(a^* \mathcal{O}) = \underline{\Phi_1(a)}^{-1}\{x\text{-axis}\}$$

any $\mathbb{A}^1 \subseteq \mathbb{A}^2$ is a cycle of some D_1-module

polynomial embedding

Higher dim'l version:

Thm [D] take any $\mathcal{L}_{\mathbb{C}} \subseteq \mathbb{C}^{2n}$

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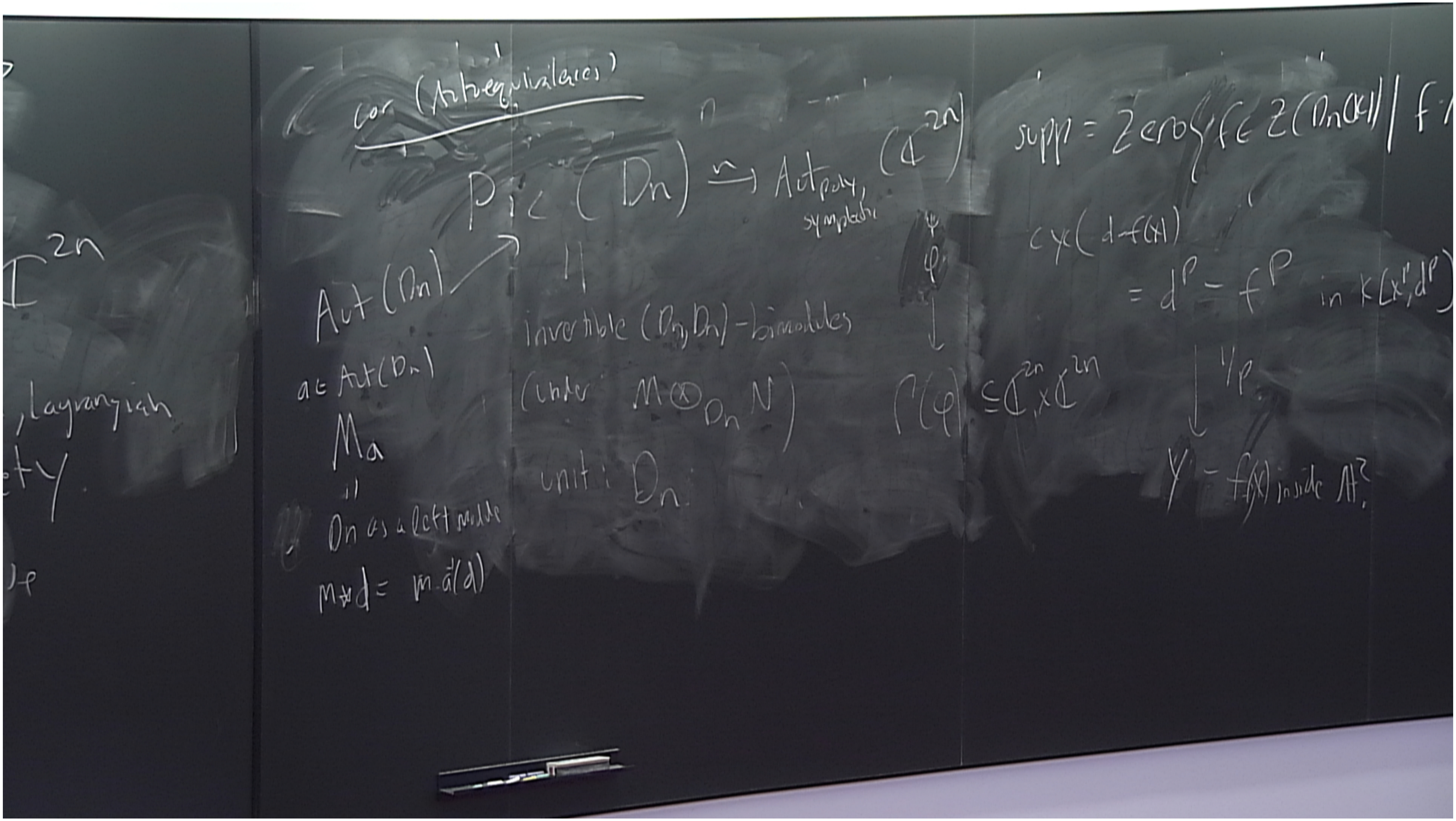

Pic(Dn) ≅ Aut poly, symplectic (C^2n)
Aut(Dn)
invertible (Dn, Dn)-bimodules
(under M ⊗Dn N)
unit: Dn
a ∈ Aut(Dn)
Ma
Dn as a left module
m*d = m·a(d)
supp = Zeros{f ∈ Z(Dn(X)) | f
cyc(d - f(x))
= d^p - f^p in k[x^p, d^p]
Γ(φ) ⊆ C^2n × C^2n
1/p
y = f(x) inside A^2
lagrangian

$\mathrm{supp} = \mathrm{Zeros}\{ f \in Z(D_n(K)) \mid f \cdot M = 0 \}.$

$(\mathbb{C}^{2n})$ poly, symplectic

ψ

φ

$\downarrow$

des

$\Gamma(\varphi) \subseteq \mathbb{C}^{2n} \times \mathbb{C}^{2n}$

$c\, y (d - f(x))$

$= d^p - f^p \quad \text{in } k[x^p, d^p]$

$\downarrow {}^{1/p}$

$y = f(x)$ inside $\mathbb{A}^2$.

$xd - \lambda$

$\downarrow$

$d \cdot e = \frac{\lambda}{x} e \quad \text{on } \mathbb{G}_m.$

$d^p e = \left(\frac{\lambda^p}{x^p} - \frac{\lambda}{x^p} \right) e$

$\lambda^p - \lambda$