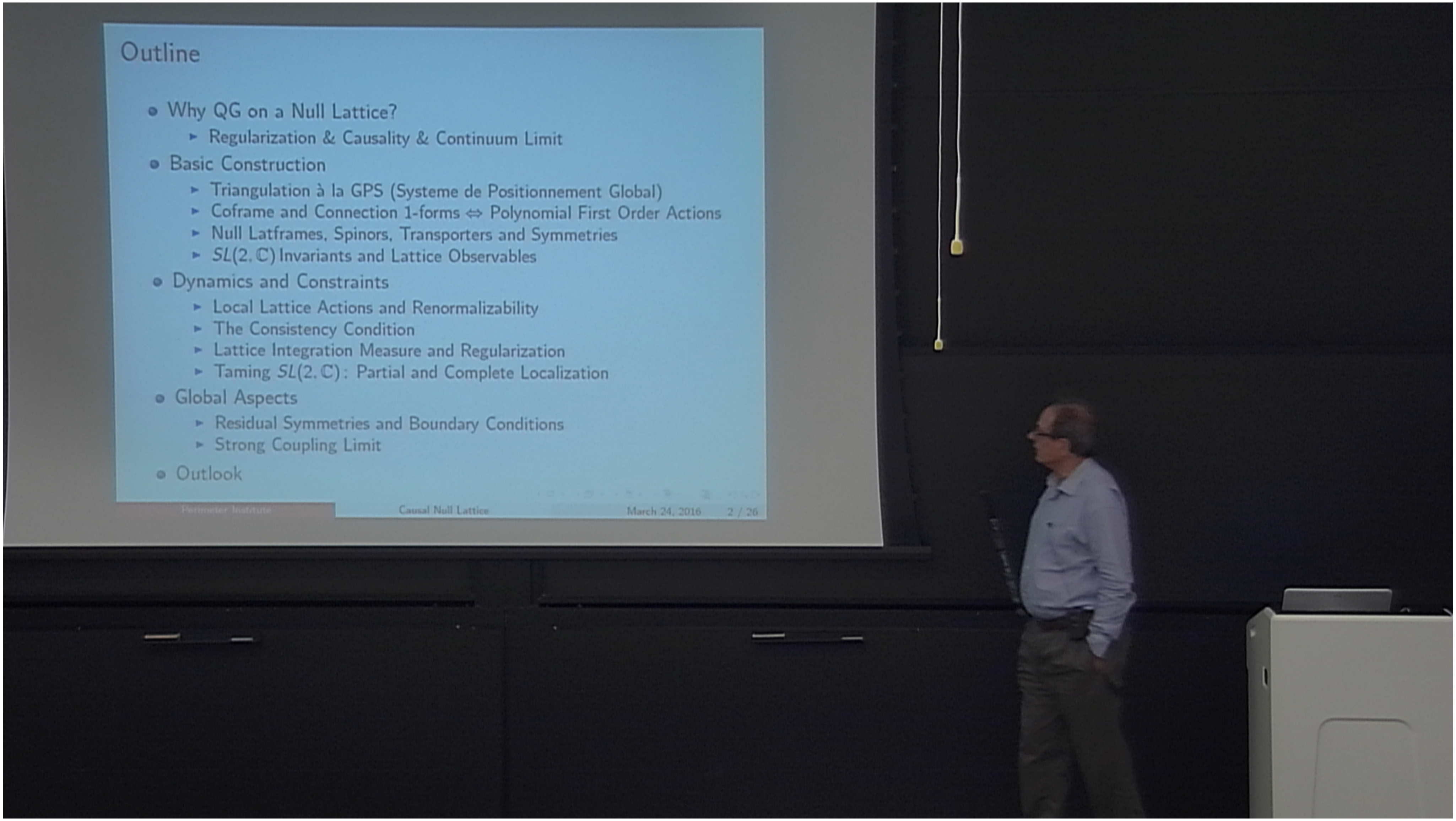
Title: Causal Space-Times on a Null-Lattice

Date: Mar 24, 2016 02:30 PM

URL: http://pirsa.org/16030111

Abstract:


Outline
Why QG on a Null Lattice?
Regularization & Causality & Continuum Limit
Basic Construction
Triangulation à la GPS (Systeme de Positionnement Global)
Coframe and Connection 1-forms ⇔ Polynomial First Order Actions
Null Latframes, Spinors, Transporters and Symmetries
$SL(2,\mathbb{C})$ Invariants and Lattice Observables
Dynamics and Constraints
Local Lattice Actions and Renormalizability
The Consistency Condition
Lattice Integration Measure and Regularization
Taming $SL(2,\mathbb{C})$: Partial and Complete Localization
Global Aspects
Residual Symmetries and Boundary Conditions
Strong Coupling Limit
Outlook
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Causal Null Lattice
March 24, 2016
2 / 26

Why QG on a Null-Lattice?

- No perturbative background.
- Dimension=3+1 makes a difference.
- Hamiltonian QFT is difficult to regularize and control.
  $\Rightarrow$ seek a (regulated) generating functional for causally ordered Greens-functions. "$\langle \mathbf{T}\, A(P)B(Q)\rangle := \int_{\text{CausalWorlds}} e^{iS} A(P)B(Q)$"
- CausalWorlds can be defined by triangulation.
  $\Rightarrow$ Complex of "rigid simplices" + Causality (CDT, Ambjørn & Loll)

But Measure? # dof? Symmetries? Continuum Limit?

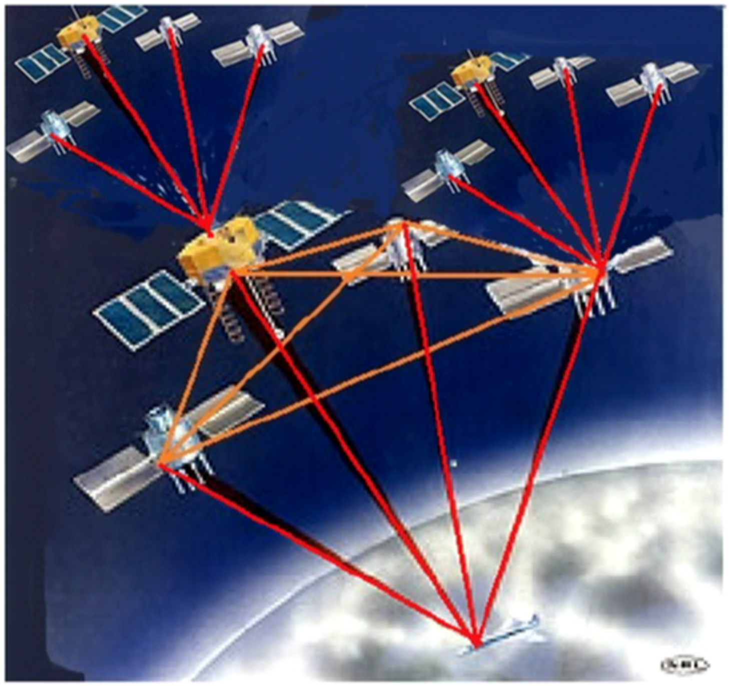
A *deformable* oriented lattice (Regge '61) with causality assured by lightlike separation of neighboring events [$\simeq$ a rigid skeleton of deformable null simplices.]

What is the lattice? Does it regulate?

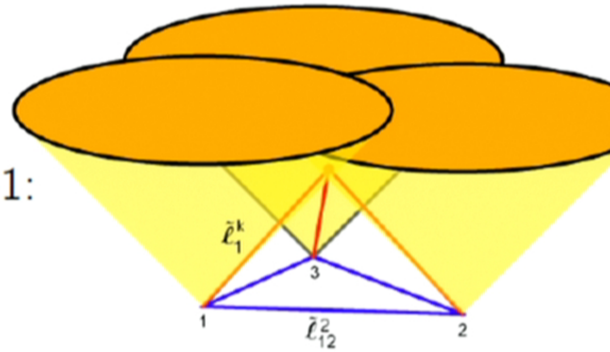
Basic Construction or what is an event?

GPS: The intersection of rays from D spatially separate events is an event

D=3+1



D=2+1:



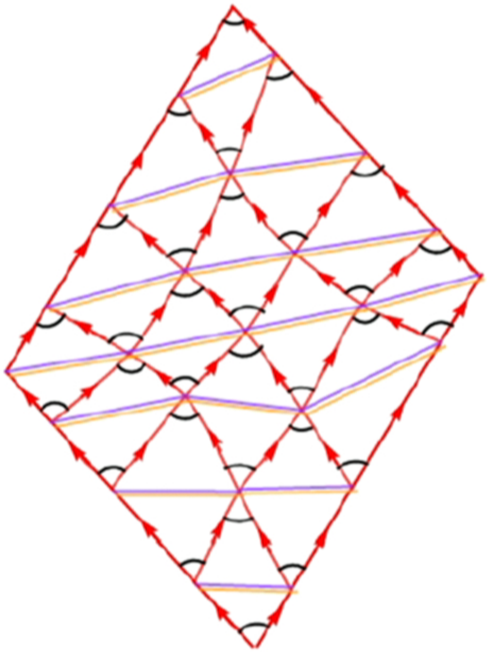
Note:
4(3)-dimensional null B-simplex with 4(3) null edges and a tetrahedal (triangular) face with 6(3) spatial edges $\widetilde{\ell}_{\mu\nu} = \widetilde{\ell}_{\nu\mu}$; $1 \leq \mu < \nu \leq 4(3)$.

Unique apex: $\det \widetilde{\ell}^2_{\mu\nu} < 0$ in D=3+1
No restrictions in D=1+1 & D=2+1.

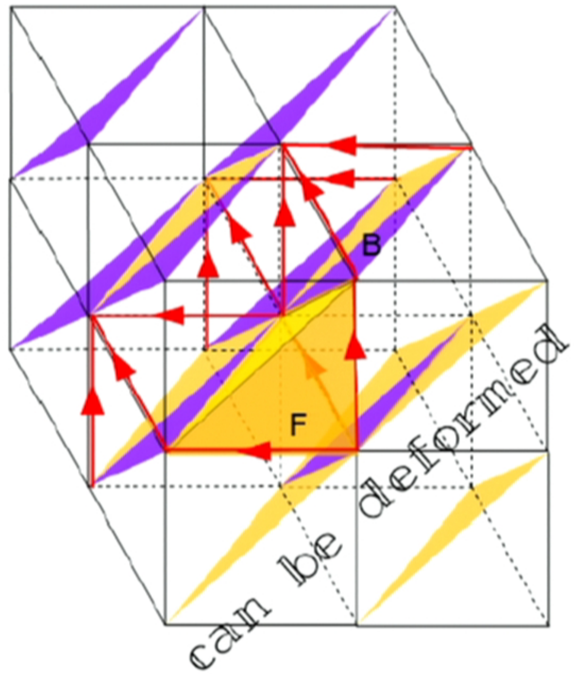
The lattice in,

D=1+1

D=2+1



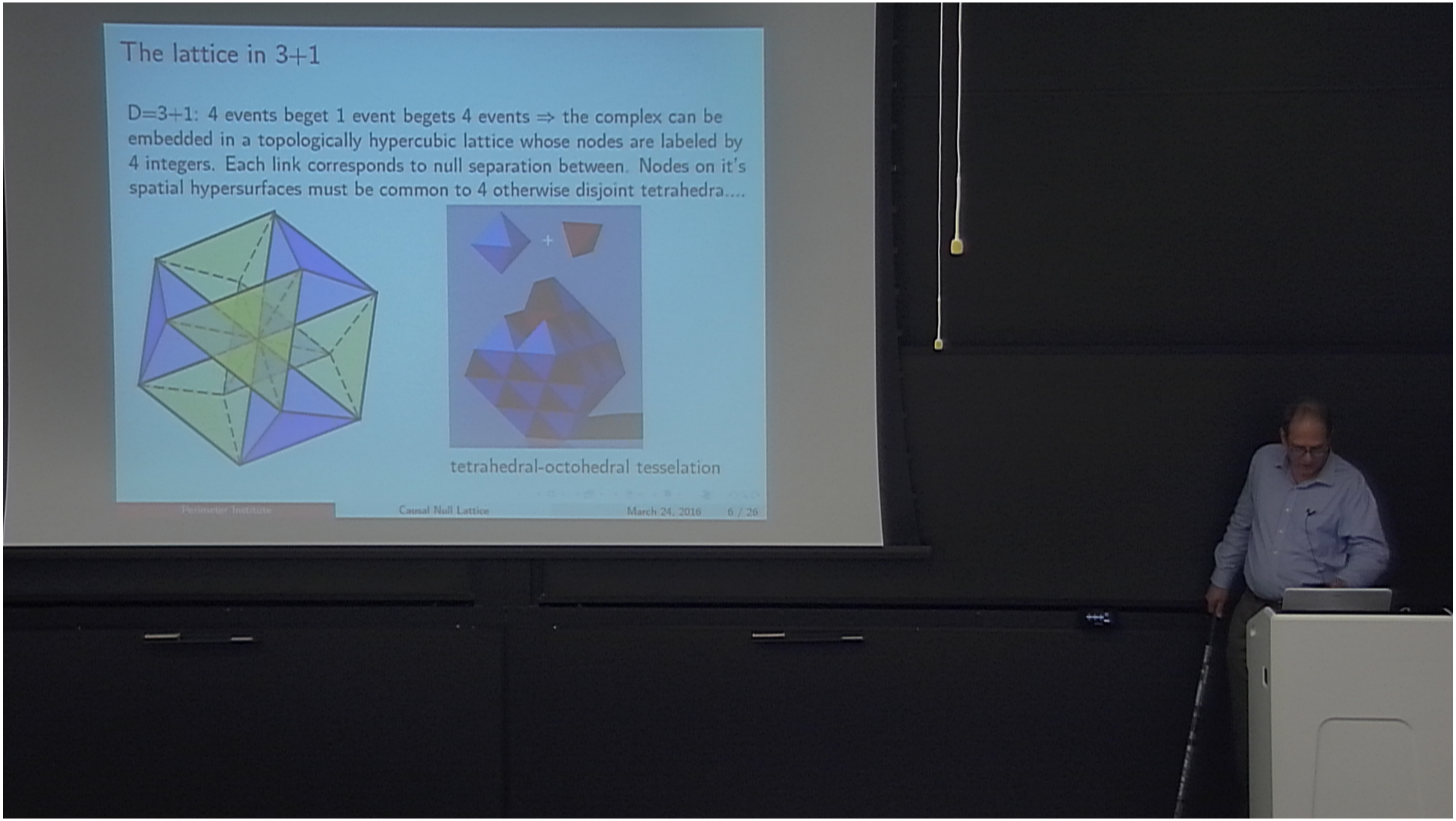
a)



b)

The lattice in 3+1

D=3+1: 4 events beget 1 event begets 4 events ⇒ the complex can be embedded in a topologically hypercubic lattice whose nodes are labeled by 4 integers. Each link corresponds to null separation between. Nodes on it's spatial hypersurfaces must be common to 4 otherwise disjoint tetrahedra....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tetrahedral-octohedral tesselation
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March 24, 2016
6 / 26

First Order Continuum Actions are Polynomial Vol. Forms

coframe 1-form: $e^a = e^a_k dx^k$; Curvature 2-form: $R^a_{\ b}(\omega) = d\omega^a_{\ b} + \omega^a_{\ c} \wedge \omega^c_{\ b}$
YM-curvature 2-form: $\mathcal{F}(A) = dA + A \wedge A$; Auxiliary 0-forms: B^a , B^{ab} .

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Hilbert Palatini with Cosmological Term

$$S_{\rm HP} = \frac{1}{l_P^2}\int_M e^a \wedge e^b \wedge [\frac{\Lambda}{6}e^c \wedge e^d - R^{cd}(\omega)]\varepsilon_{abcd} \ , \qquad l_P = 1$$

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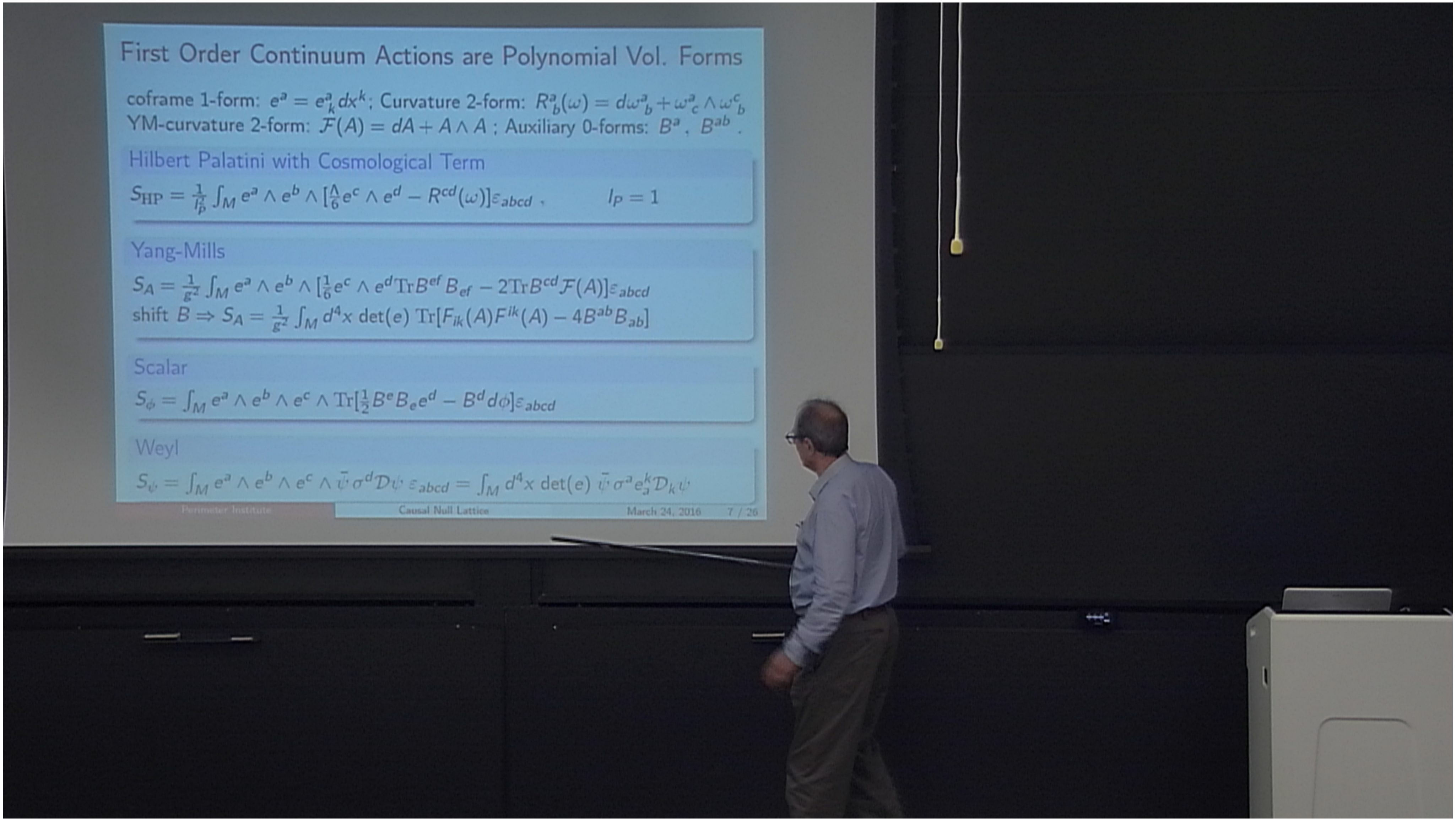
Yang-Mills

$$S_A = \frac{1}{g^2} \int_M e^a \wedge e^b \wedge [\frac{1}{6} e^c \wedge e^d \mathrm{Tr} B^{ef} B_{ef} - 2 \mathrm{Tr} B^{cd} \mathcal{F}(A)]\varepsilon_{abcd}$$

shift $B \Rightarrow S_A = \frac{1}{g^2} \int_M d^4x \det(e)\, \mathrm{Tr}[F_{ik}(A) F^{ik}(A) - 4 B^{ab} B_{ab}]$

Scalar

$$S_\phi = \int_M e^a \wedge e^b \wedge e^c \wedge \mathrm{Tr}[\frac{1}{2} B^e B_e e^d - B^d d\phi]\varepsilon_{abcd}$$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Weyl
$S_\psi = \int_M e^a \wedge e^b \wedge e^c \wedge \bar{\psi} \sigma^d \mathcal{D}\psi\, \varepsilon_{abcd} = \int_M d^4x \det(e)\, \bar{\psi} \sigma^a e^k_a \mathcal{D}_k \psi$
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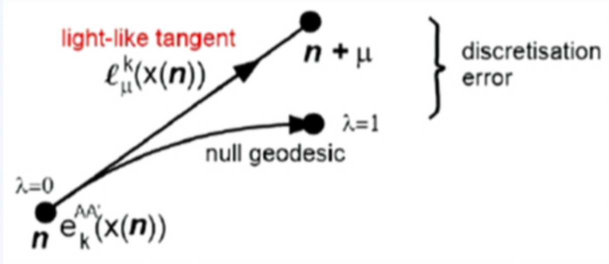
March 24, 2016
7 / 26

Lattice Variables

Anti-hermitian Latframe $E_\mu(\mathbf{n})$

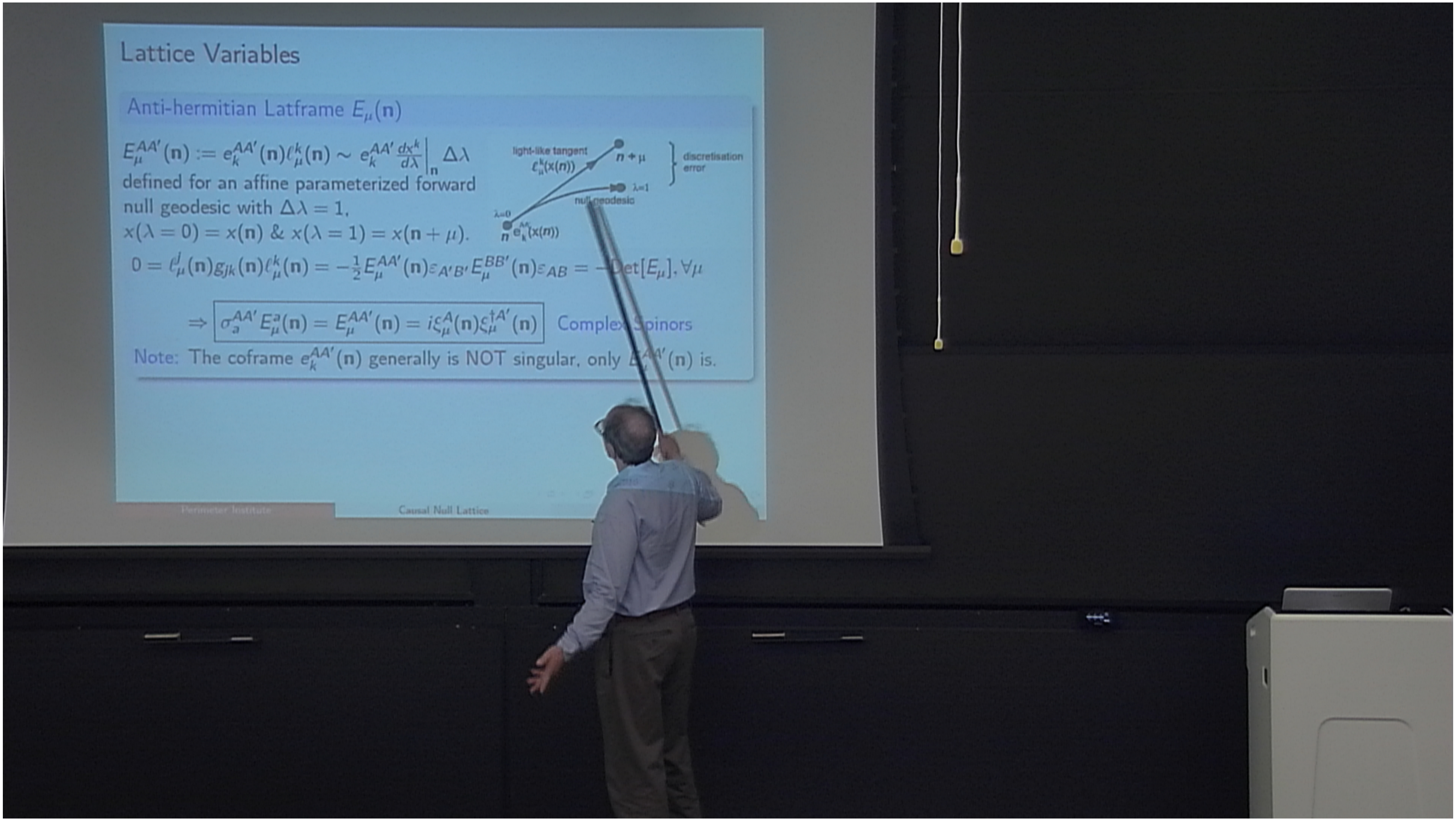
$E_\mu^{AA'}(\mathbf{n}) := e_k^{AA'}(\mathbf{n})\ell_\mu^k(\mathbf{n}) \sim e_k^{AA'} \frac{dx^k}{d\lambda}\Big|_{\mathbf{n}} \Delta\lambda$
defined for an affine parameterized forward null geodesic with $\Delta\lambda = 1$,
$x(\lambda = 0) = x(\mathbf{n})$ & $x(\lambda = 1) = x(\mathbf{n} + \mu)$.



$$0 = \ell_\mu^j(\mathbf{n}) g_{jk}(\mathbf{n}) \ell_\mu^k(\mathbf{n}) = -\tfrac{1}{2} E_\mu^{AA'}(\mathbf{n}) \varepsilon_{A'B'} E_\mu^{BB'}(\mathbf{n}) \varepsilon_{AB} = -\mathrm{Det}[E_\mu], \forall \mu$$

$$\Rightarrow \boxed{\sigma_a^{AA'} E_\mu^a(\mathbf{n}) = E_\mu^{AA'}(\mathbf{n}) = i\xi_\mu^A(\mathbf{n})\xi_\mu^{\dagger A'}(\mathbf{n})} \quad \text{Complex Spinors}$$

Note: The coframe $e_k^{AA'}(\mathbf{n})$ generally is NOT singular, only $E_\mu^{AA'}(\mathbf{n})$ is.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light-like tangent
$\ell^k_\mu(x(n))$
$n + \mu$
discretisation error
$\lambda{=}1$
null geodesic
$\lambda{=}0$
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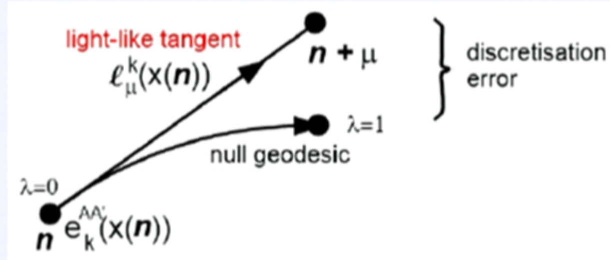

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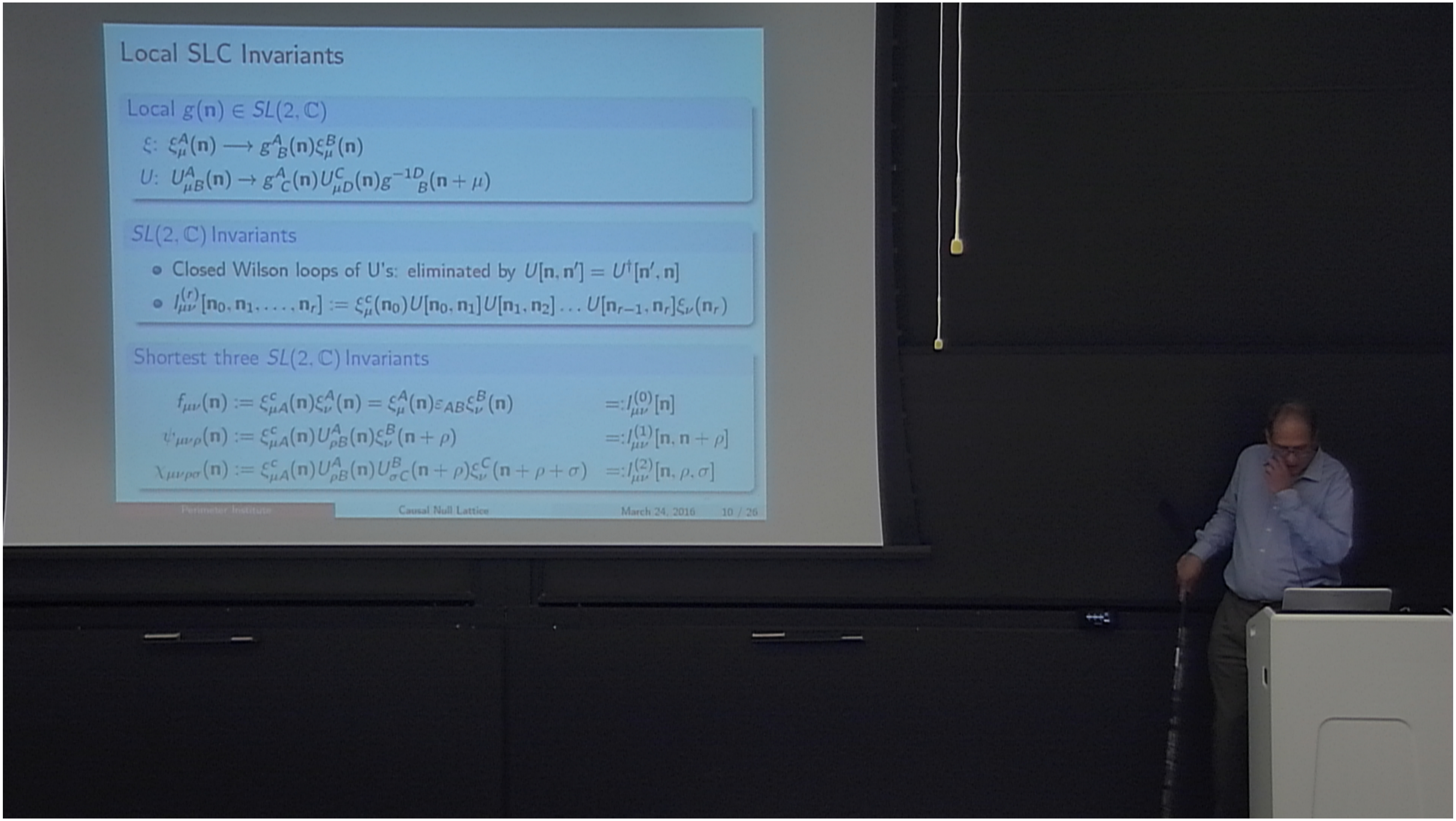
$SL(2,\mathbb{C})$ Transporters $U_\mu(\mathbf{n})$

Spinors are parallel transported from site $\mathbf{n} + \mu$ to $\mathbf{n}$ by $SL(2,\mathbb{C})$-matrices
$U^A_{\mu B}(\mathbf{n})$: $\xi^A_{\|}(\mathbf{n}) = U^A_{\mu B}(\mathbf{n})\xi^B(\mathbf{n} + \mu)$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Shortest three $SL(2,\mathbb{C})$ Invariants
$f_{\mu\nu}(\mathbf{n}) := \xi^c_{\mu A}(\mathbf{n})\xi^A_\nu(\mathbf{n}) = \xi^A_\mu(\mathbf{n})\varepsilon_{AB}\xi^B_\nu(\mathbf{n}) \quad =: I^{(0)}_{\mu\nu}[\mathbf{n}]$
$\psi_{\mu\nu\rho}(\mathbf{n}) := \xi^c_{\mu A}(\mathbf{n})U^A_{\rho B}(\mathbf{n})\xi^B_\nu(\mathbf{n}+\rho) \quad =: I^{(1)}_{\mu\nu}[\mathbf{n},\mathbf{n}+\rho]$
$\chi_{\mu\nu\rho\sigma}(\mathbf{n}) := \xi^c_{\mu A}(\mathbf{n})U^A_{\rho B}(\mathbf{n})U^B_{\sigma C}(\mathbf{n}+\rho)\xi^C_\nu(\mathbf{n}+\rho+\sigma) \quad =: I^{(2)}_{\mu\nu}[\mathbf{n},\rho,\sigma]$
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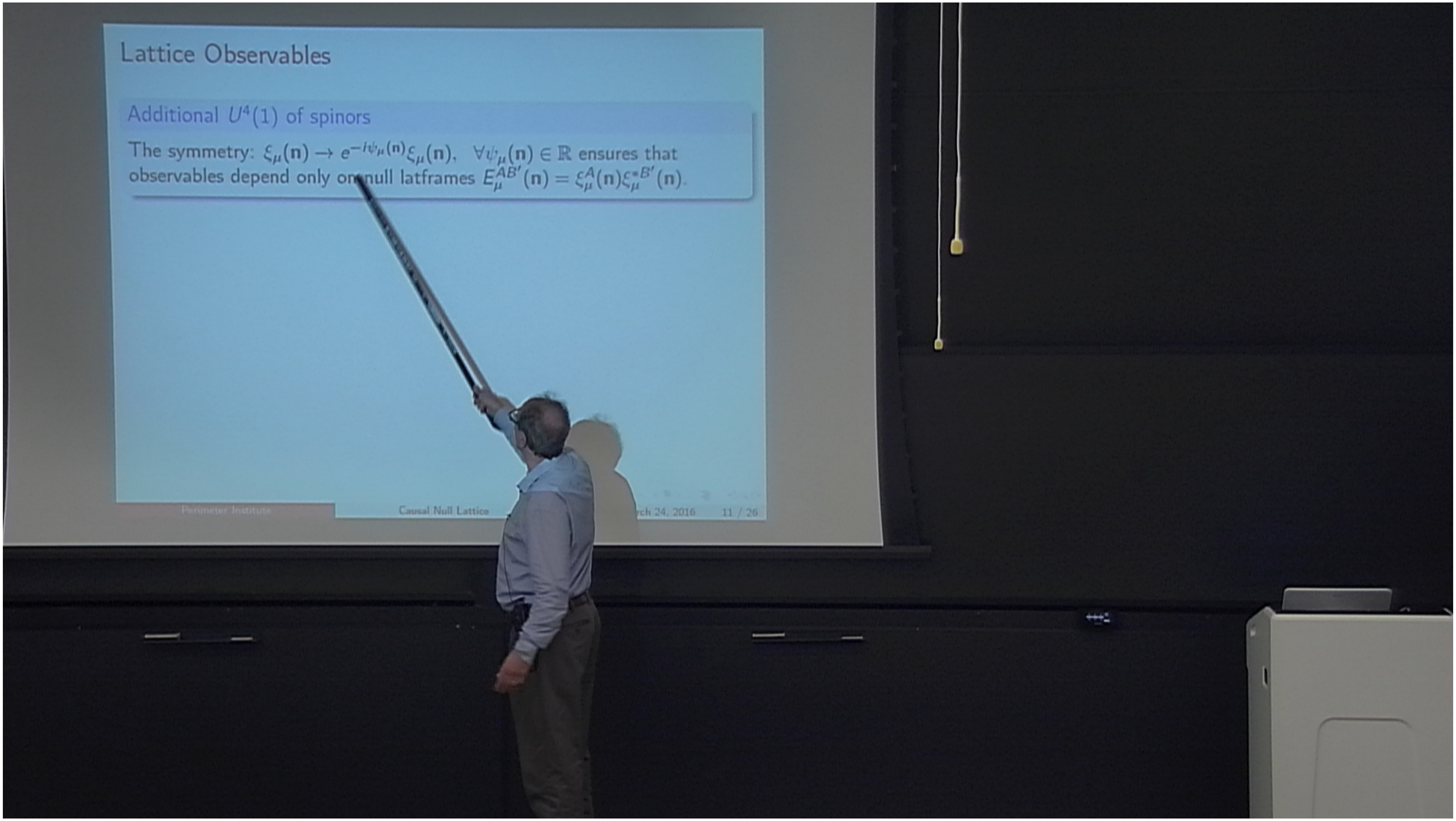
March 24, 2016
10 / 26


Lattice Observables
Additional $U^4(1)$ of spinors
The symmetry: $\xi_\mu(\mathbf{n}) \to e^{-i\psi_\mu(\mathbf{n})}\xi_\mu(\mathbf{n}), \quad \forall \psi_\mu(\mathbf{n}) \in \mathbb{R}$ ensures that observables depend only on null latframes $E_\mu^{AB'}(\mathbf{n}) = \xi_\mu^A(\mathbf{n})\xi_\mu^{*B'}(\mathbf{n})$.
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rch 24, 2016
11 / 26

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Observables correspond to closed loops, that reverse at spinor "bookends"

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Local SLC Invariants

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$SL(2,\mathbb{C})$ Invariants

- Closed Wilson loops of U's: eliminated by $U[\mathbf{n},\mathbf{n}'] = U^\dagger[\mathbf{n}',\mathbf{n}]$
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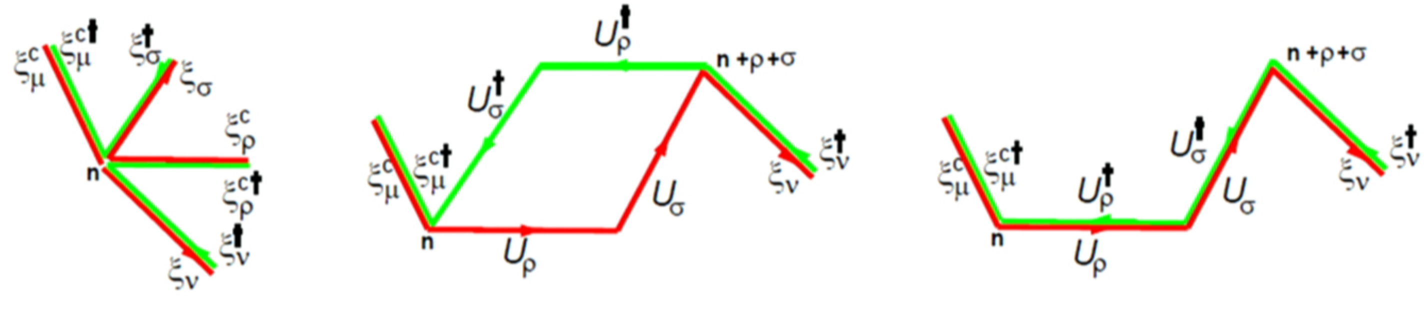
  $= \mathrm{Tr}E_\mu(\mathbf{n})\varepsilon E_\nu^T(\mathbf{n})\varepsilon E_\rho(\mathbf{n})\varepsilon E_\sigma^T(\mathbf{n})\varepsilon$
- $\mathcal{P}_{\mu\nu\rho\sigma}(\mathbf{n}) := \chi_{\mu\nu\rho\sigma}(\mathbf{n})\chi^*_{\mu\nu\sigma\rho}(\mathbf{n})$
  $= \mathrm{Tr}\varepsilon E_\mu^T(\mathbf{n})\varepsilon U_\rho(\mathbf{n})U_\sigma(\mathbf{n}+\rho)E_\nu(\mathbf{n}+\rho+\sigma)U_\rho^\dagger(\mathbf{n}+\sigma)U_\sigma^\dagger(\mathbf{n})$
- $\mathcal{Q}_{\mu\nu\rho\sigma}(\mathbf{n}) := |\chi_{\mu\nu\rho\sigma}(\mathbf{n})|^2$
  $= \mathrm{Tr}\varepsilon E_\mu^T(\mathbf{n})\varepsilon U_\rho(\mathbf{n})U_\sigma(\mathbf{n}+\rho)E_\nu(\mathbf{n}+\rho+\sigma)U_\sigma^\dagger(\mathbf{n}+\rho)U_\rho^\dagger(\mathbf{n})$

Lattice Observables and Actions

Lattice Actions are local anti-symmetric observables. There are only 3:



a) $V(\mathbf{n}) := \frac{-i}{48}\varepsilon(\mu\nu\rho\sigma)\mathcal{V}_{\mu\nu\rho\sigma} = \det[E] = \det[\ell^k_\mu]\det[e] = \frac{1}{4}\sqrt{-\det[\ell^2_{\mu\nu}]}$

b) $P(\mathbf{n}) := i\varepsilon(\mu\nu\rho\sigma)\mathcal{P}_{\mu\nu\rho\sigma}(\mathbf{n}) \rightarrow$ P&T even Hilbert-Palatini + (Euler)

c) $Q(\mathbf{n}) := \varepsilon(\mu\nu\rho\sigma)\mathcal{Q}_{\mu\nu\rho\sigma}(\mathbf{n}) \rightarrow$ P&T odd Barbero-Immirzi + (Pontryagin) topological invariants.

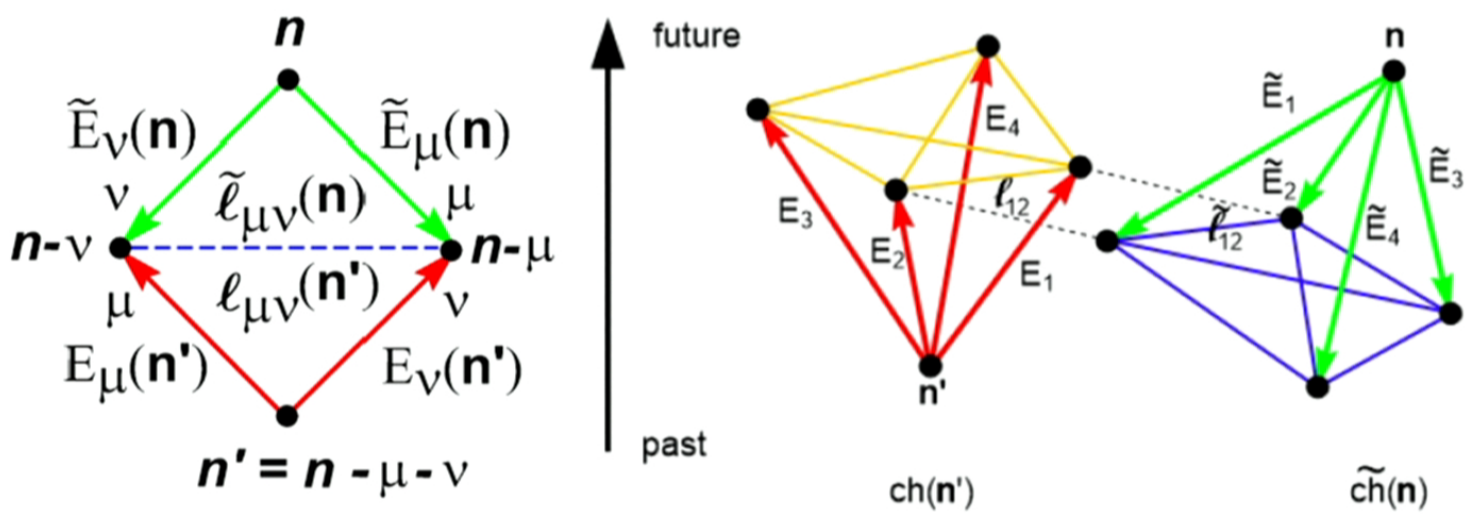
P & T even lattice action: $S_{\mathsf{L}} = \sum_{\mathbf{n}}[P(\mathbf{n}) - 4\lambda V(\mathbf{n})]$

Consistency Condition (Triangle Inequalities)

NOT ALL configurations correspond to triangulated causal manifolds. The vertices of the tetrahedron with spatial lengths

$$\tilde{\ell}_{\mu\nu}(\mathbf{n}) := \ell_{\mu\nu}(\mathbf{n}-\mu-\nu) \geq 0$$

must lie on the backward light cone of $\mathbf{n}$.



Consistency Condition

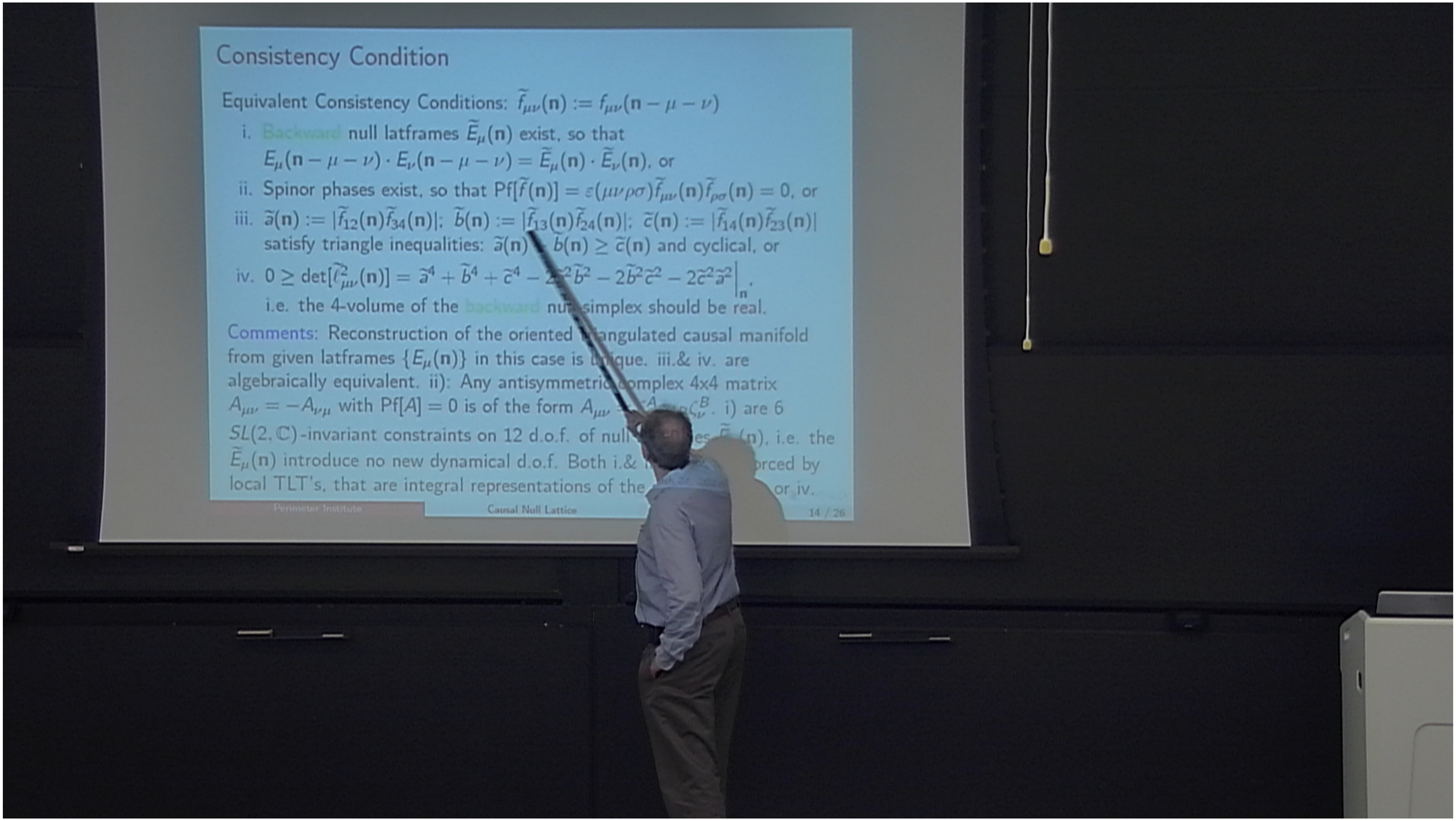
Equivalent Consistency Conditions: $\widetilde{f}_{\mu\nu}(\mathbf{n}) := f_{\mu\nu}(\mathbf{n}-\mu-\nu)$

i. Backward null latframes $\widetilde{E}_\mu(\mathbf{n})$ exist, so that $E_\mu(\mathbf{n}-\mu-\nu)\cdot E_\nu(\mathbf{n}-\mu-\nu) = \widetilde{E}_\mu(\mathbf{n})\cdot\widetilde{E}_\nu(\mathbf{n})$, or

ii. Spinor phases exist, so that $\mathrm{Pf}[\widetilde{f}(\mathbf{n})] = \varepsilon(\mu\nu\rho\sigma)\widetilde{f}_{\mu\nu}(\mathbf{n})\widetilde{f}_{\rho\sigma}(\mathbf{n}) = 0$, or

iii. $\widetilde{a}(\mathbf{n}) := |\widetilde{f}_{12}(\mathbf{n})\widetilde{f}_{34}(\mathbf{n})|$; $\widetilde{b}(\mathbf{n}) := |\widetilde{f}_{13}(\mathbf{n})\widetilde{f}_{24}(\mathbf{n})|$; $\widetilde{c}(\mathbf{n}) := |\widetilde{f}_{14}(\mathbf{n})\widetilde{f}_{23}(\mathbf{n})|$ satisfy triangle inequalities: $\widetilde{a}(\mathbf{n}) + \widetilde{b}(\mathbf{n}) \geq \widetilde{c}(\mathbf{n})$ and cyclical, or

iv. $0 \geq \det[\widetilde{\ell}^2_{\mu\nu}(\mathbf{n})] = \widetilde{a}^4 + \widetilde{b}^4 + \widetilde{c}^4 - 2\widetilde{a}^2\widetilde{b}^2 - 2\widetilde{b}^2\widetilde{c}^2 - 2\widetilde{c}^2\widetilde{a}^2\Big|_{\mathbf{n}}$,
i.e. the 4-volume of the backward null-simplex should be real.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Comments: Reconstruction of the oriented angulated causal manifold
from given latframes $\{E_\mu(\mathbf{n})\}$ in this case is ique. iii.& iv. are
algebraically equivalent. ii): Any antisymmetric complex 4x4 matrix
$A_{\mu\nu} = -A_{\nu\mu}$ with $\mathrm{Pf}[A] = 0$ is of the form $A_{\mu\nu}$. i) are 6
$SL(2,\mathbb{C})$-invariant constraints on 12 d.o.f. of null , i.e. the
$\widetilde{E}_\mu(\mathbf{n})$ introduce no new dynamical d.o.f. Both i.& rced by
local TLT's, that are integral representations of the or iv.
Perimeter Institute
Causal Null Lattice
14 / 26

Lattice Integration Measure

The local lattice integration measure is (almost) uniquely determined by invariance under the $SL(2,\mathbb{C}) \times U^4(1)$ structure group.

$SL(2,\mathbb{C})$ Haar measure of transport U's

For $U \in SL(2,\mathbb{C})$ parametrized by 5 compact Euler angles $0 \le \Psi < 2\pi, 0 \le \vartheta, \bar\vartheta < \pi, 0 \le \phi, \bar\phi < 4\pi$ and a single non-compact variable $0 \le \bar\Psi < \infty$: $U = e^{\sigma_3\Psi/2}\, e^{\sigma_2\vartheta/2}\, e^{\sigma_3\phi/2}\, e^{i\sigma_3\bar\Psi/2}\, e^{\sigma_2\bar\vartheta/2}\, e^{\sigma_3\bar\phi/2}$

the invariant Haar measure is: $\boxed{d\mu[U] = d\Psi \sin\vartheta d\vartheta d\phi d\bar\Psi \sin\bar\vartheta d\bar\vartheta d\bar\phi}$

Note: analytic continuation of $\bar\psi$ gives the $SU_L(2) \times SU_R(2)$ Haar measure.

$SL(2,\mathbb{C}) \times U(1)$ invariant Spinor Measure

For a parametrization of a spinor by magnitude τ and angles $0 \le \psi < 2\pi$, $0 \le \varphi < 2\pi$, $0 \le \theta \le \pi$: $\xi(\mathbf{n}) = \sqrt{\tau(\mathbf{n})}e^{i\psi(\mathbf{n})} \begin{pmatrix} e^{-i\varphi(\mathbf{n})/2} \cos(\theta(\mathbf{n})/2) \\ e^{i\varphi(\mathbf{n})/2} \sin(\theta(\mathbf{n})/2) \end{pmatrix}$

the invariant measure is: $\boxed{d\mu[\xi] \propto \tau d\tau d\psi \sin\theta d\theta d\varphi}$

Invariant Measure & Regularization

The lattice generating function formally is: $Z_L = \int D[\xi, U] e^{iS_L[\xi, U]}$ with lattice action S_L and integration measure:

$$D[\xi, U] = \prod_{\mathbf{n}} \rho(\mathbf{n}) \prod_{\mu} d[\xi_\mu(\mathbf{n})] d[U_\mu(\mathbf{n})]$$

where the a priori undetermined local density $\rho(\mathbf{n})$ is a real, positive, isotropic and $SL(2, \mathbb{C})$ invariant local function of the fields.
Since the basic local $SL(2, \mathbb{C})$-invariant is $f_{\mu\nu}(\mathbf{n})$, positivity, reality and isotropy suggest that

$$\rho(\mathbf{n}) = \rho(V(\mathbf{n}))\ .$$

And thus for small local 4-Volumes, $\boxed{\rho(V \sim 0^+) \propto V^\gamma (1 + O(V))}$.

The exponent $\gamma > \gamma^*$ regulates the UV-behavior of the finite lattice model and determines its critical limit.
Note: Auxiliary fields B generally change the critical exponent γ^*.

Taming $SL(2,\mathbb{C})$

However, the $SL(2,\mathbb{C})$ structure group of this lattice model is not compact. The generating function is a divergent integral whose definition is formal: the infinite volume of the structure group apparently cancels in the expectation of observables. To make this precise, the $SL(2,\mathbb{C})$ structure group should (at least) be partially "localized" to a compact subgroup. Physically this amounts to considering only a subset of inertial systems, say those in which events 3&4 appear simultaneous and have opposite spatial displacement. This restriction is possible if we consider expectations of $SL(2,\mathbb{C})$-invariant observables only.

Mathematically it amounts to localizing (or (partially) "gauge-fixing") $SL(2,\mathbb{C})$ to the compact $SU(2)$ subgoup of rotations.

This is analogous to computing the average of rotationally invariant "observables" $O(r)$, by a radial integral:

$$\langle O\rangle_\alpha := \frac{\int d^3x\ O(r)e^{-\alpha r^2}}{\int d^3x\ e^{-\alpha r^2}} = \frac{4\pi\int r^2dr\ O(r)e^{-\alpha r^2}}{4\pi\int r^2dr\ e^{-\alpha r^2}}$$

For $SL(2,\mathbb{C})$ the canceled volume is infinite rather than 4π.

Localizing $SL(2,\mathbb{C})$

The $SU(2)$-invariant Morse function

Positive weights $\{w_\mu(\mathbf{n}) \geq 0\}$, define a $SU(2)$-invariant Morse function:
$W_\xi[g(\mathbf{n})] = \sum_\mu w_\mu(\mathbf{n})\xi^\dagger_\mu(\mathbf{n})g^\dagger(\mathbf{n})g(\mathbf{n})\xi_\mu(\mathbf{n}) = \sum_\mu w_\mu(\mathbf{n})\tau^{(g)}_\mu(\mathbf{n}) \geq 0$

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which has $SU(2)$-invariant critical "points", $0 = \sum_\mu w_\mu(\mathbf{n})\xi_\mu^\dagger(\mathbf{n})\,\vec{\sigma}\,\xi_\mu(\mathbf{n})$
and positive Hessian $H(\mathbf{n}) = \mathbf{1}\sum_\mu w_\mu(\mathbf{n})\tau_\mu^{(g)}(\mathbf{n}) = \mathbf{1}W_\xi[g(\mathbf{n})] > 0$.

Localizing $SL(2,\mathbb{C})$

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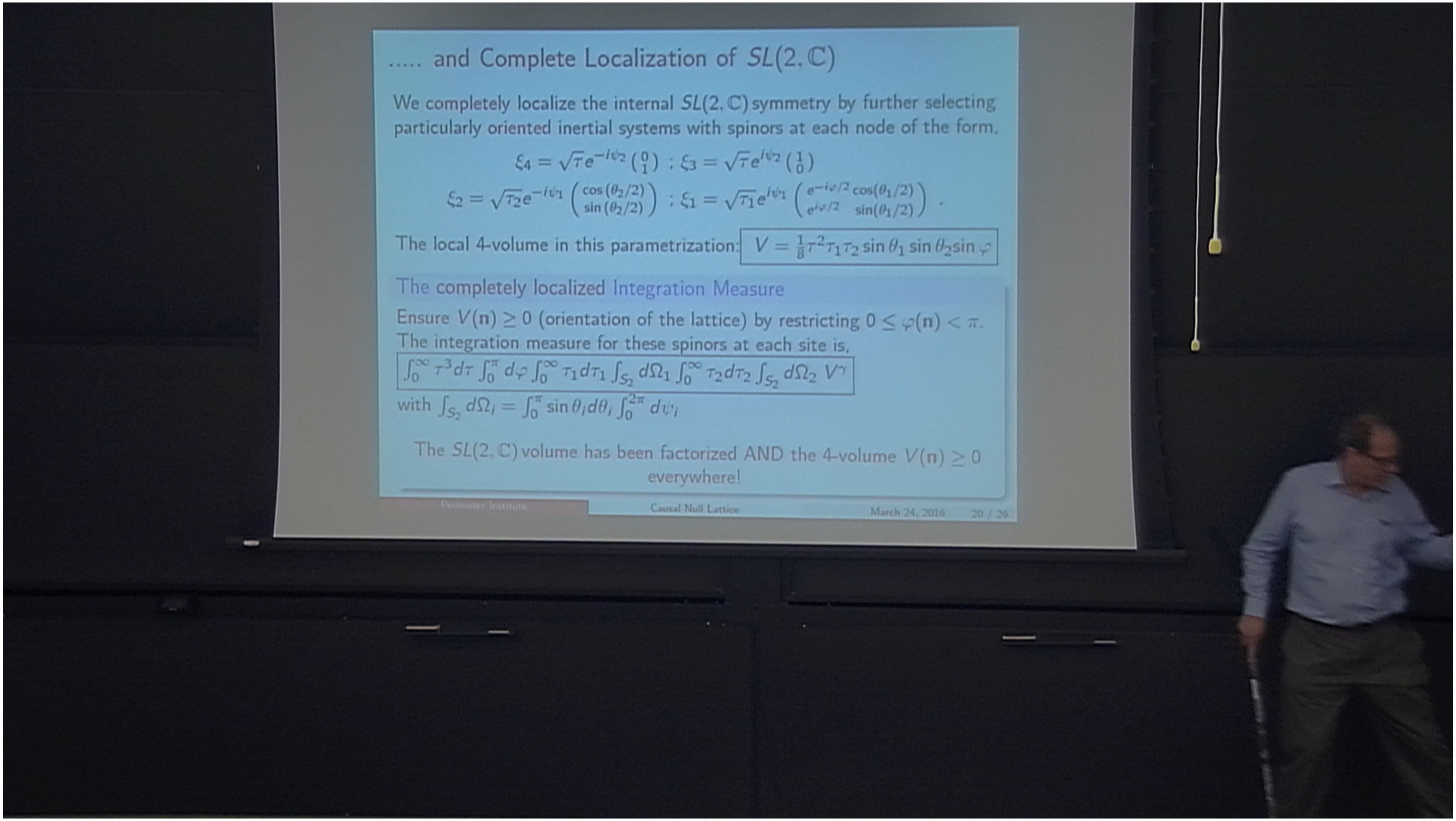
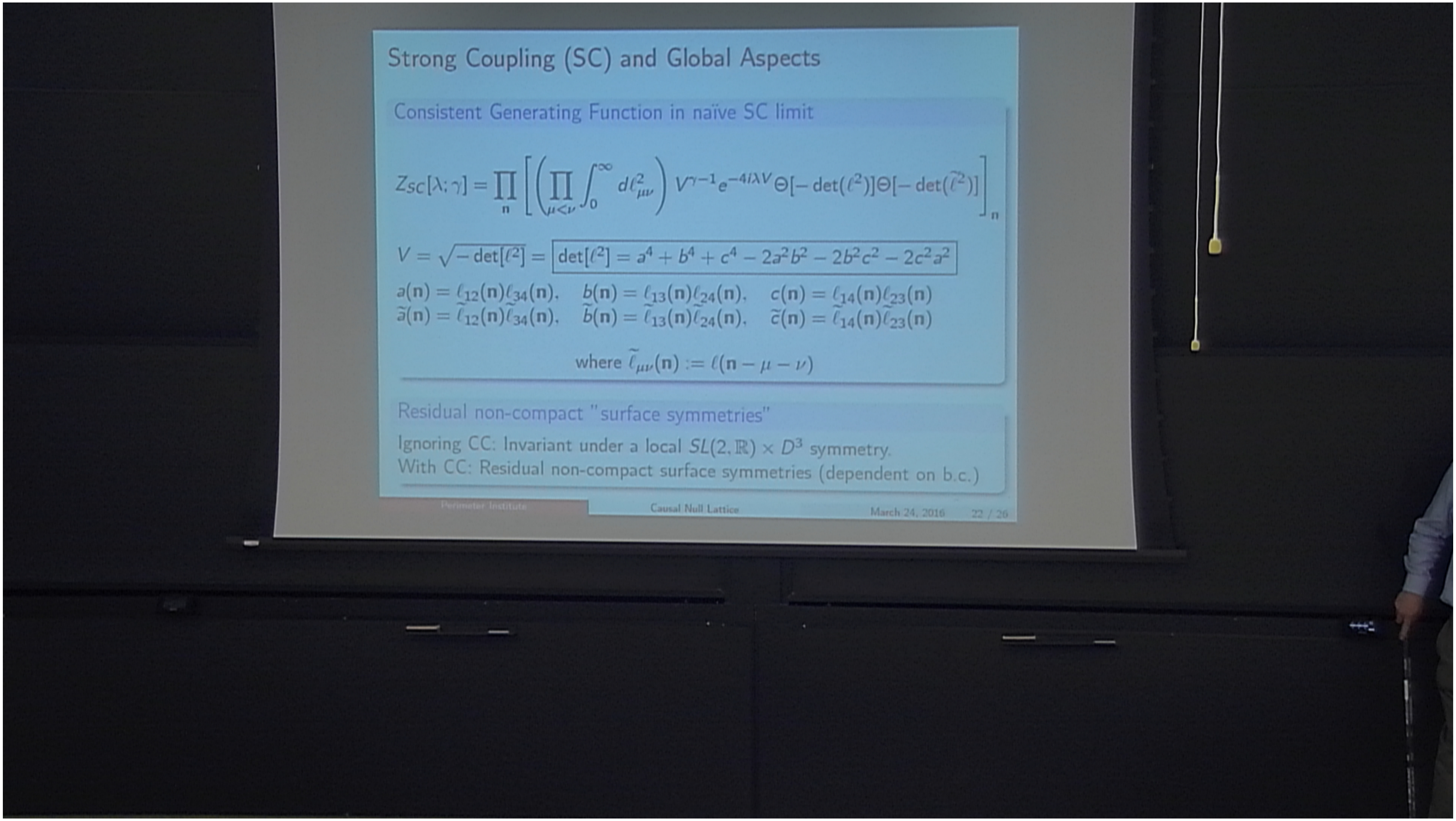
$\Rightarrow$ Exists unique (modulo $SU(2)$) solution to $\boxed{0 = \sum_\mu w_\mu(\mathbf{n})\xi_\mu^\dagger(\mathbf{n})\,\vec{\sigma}\,\xi_\mu(\mathbf{n})}$
$\Rightarrow \int_{SL(2,\mathbb{C})} d\mu[g]\left(\sum_\mu w_\mu(g\xi_\mu)^\dagger(g\xi_\mu)\right)^3\delta^3\left(\sum_\mu w_\mu(g\xi_\mu)^\dagger\,\vec{\sigma}\,(g\xi_\mu)\right) = 2\pi^2$
is a topological integral that does not depend on ξ_μ. It can be inserted in the $SL(2,\mathbb{C})$ invariant partition function and the (infinite) $SL(2,\mathbb{C})$ group volume factorized (Faddeev-Popov trick), resulting in a for observables equivalent finite generating function with $SU(2)$ structure group.


..... and Complete Localization of $SL(2,\mathbb{C})$
We completely localize the internal $SL(2,\mathbb{C})$ symmetry by further selecting particularly oriented inertial systems with spinors at each node of the form,
$\xi_4 = \sqrt{\tau} e^{-i\psi_2} \binom{0}{1}$; $\xi_3 = \sqrt{\tau} e^{i\psi_2} \binom{1}{0}$
$\xi_2 = \sqrt{\tau_2} e^{-i\psi_1} \begin{pmatrix} \cos(\theta_2/2) \\ \sin(\theta_2/2) \end{pmatrix}$; $\xi_1 = \sqrt{\tau_1} e^{i\psi_1} \begin{pmatrix} e^{-i\varphi/2} \cos(\theta_1/2) \\ e^{i\varphi/2} \sin(\theta_1/2) \end{pmatrix}$.
The local 4-volume in this parametrization: $V = \frac{1}{8}\tau^2\tau_1\tau_2 \sin\theta_1 \sin\theta_2 \sin\varphi$
The completely localized Integration Measure
Ensure $V(\mathbf{n}) \geq 0$ (orientation of the lattice) by restricting $0 \leq \varphi(\mathbf{n}) < \pi$.
The integration measure for these spinors at each site is,
$\int_0^\infty \tau^3 d\tau \int_0^\pi d\varphi \int_0^\infty \tau_1 d\tau_1 \int_{S_2} d\Omega_1 \int_0^\infty \tau_2 d\tau_2 \int_{S_2} d\Omega_2 \, V^\gamma$
with $\int_{S_2} d\Omega_i = \int_0^\pi \sin\theta_i d\theta_i \int_0^{2\pi} d\psi_i$
The $SL(2,\mathbb{C})$ volume has been factorized AND the 4-volume $V(\mathbf{n}) \geq 0$ everywhere!
Perimeter Institute
Causal Null Lattice
March 24, 2016
20 / 26


Strong Coupling (SC) and Global Aspects
Consistent Generating Function in naïve SC limit
$Z_{SC}[\lambda;\gamma] = \prod_{\mathbf{n}} \left[\left(\prod_{\mu<\nu} \int_0^\infty d\ell^2_{\mu\nu} \right) V^{\gamma-1} e^{-4i\lambda V} \Theta[-\det(\ell^2)] \Theta[-\det(\widetilde{\ell}^2)] \right]_{\mathbf{n}}$
$V = \sqrt{-\det[\ell^2]} = \boxed{\det[\ell^2] = a^4 + b^4 + c^4 - 2a^2b^2 - 2b^2c^2 - 2c^2a^2}$
$a(\mathbf{n}) = \ell_{12}(\mathbf{n})\ell_{34}(\mathbf{n}), \quad b(\mathbf{n}) = \ell_{13}(\mathbf{n})\ell_{24}(\mathbf{n}), \quad c(\mathbf{n}) = \ell_{14}(\mathbf{n})\ell_{23}(\mathbf{n})$
$\widetilde{a}(\mathbf{n}) = \widetilde{\ell}_{12}(\mathbf{n})\widetilde{\ell}_{34}(\mathbf{n}), \quad \widetilde{b}(\mathbf{n}) = \widetilde{\ell}_{13}(\mathbf{n})\widetilde{\ell}_{24}(\mathbf{n}), \quad \widetilde{c}(\mathbf{n}) = \widetilde{\ell}_{14}(\mathbf{n})\widetilde{\ell}_{23}(\mathbf{n})$
where $\widetilde{\ell}_{\mu\nu}(\mathbf{n}) := \ell(\mathbf{n} - \mu - \nu)$
Residual non-compact "surface symmetries"
Ignoring CC: Invariant under a local $SL(2,\mathbb{R}) \times D^3$ symmetry.
With CC: Residual non-compact surface symmetries (dependent on b.c.)
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Causal Null Lattice
March 24, 2016
22 / 26

Global Aspects of a Conic Lattice

Null triangulation of flat 1+1 dimensional space-time