Title: Deformation quantization and superconformal symmetry in three dimensions

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Abstract: <p>In this talk, I will investigate the structure of certain protected operator algebras that arise in threedimensional N = 4 superconformal field theories. I will show that these algebras can be understood as a quantization of (either of) the half-BPS chiral ring(s). An important feature of this quantization is that it has a preferred basis in which the structure constants of the quantum algebra are equal to the OPE coefficients of the underlying superconformal theory. I will present evidence in examples that for a given choice of quantum algebra (defined up to a certain gauge equivalence), there is at most one choice of canonical basis, and conjecture that this is true in general.</p>

Solvable truncations of conformal bootstrap equations

- extended SCFTs $\longrightarrow$ exactly solvable truncation
- For example,
  - $d = 4$ $\mathcal{N} \geq 2$ SCFTS $\longrightarrow$ chiral algebra [Beem, Lemos, Liendo, WP, Rastelli, van Rees]
  - $d = 6$ $\mathcal{N} = (2, 0)$ SCFTS $\longrightarrow$ chiral algebra [Beem, Rastelli, van Rees]
- This talk:
  - $d = 3$ $\mathcal{N} \geq 4$ SCFTS $\longrightarrow$ one-dimensional topological algebra (see also [Chester, Lee, Pufu, Yacoby])
- Tool: cohomological construction

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Three-dimensional $\mathcal{N} = 4$ SCFTs: symmetries

- $\mathcal{N} = 4$ superconformal algebra

$$\mathfrak{osp}(4|4,\mathbb{R}) \supset \mathfrak{so}(4) \oplus \mathfrak{sp}(4,\mathbb{R}) \cong (\mathfrak{su}(2)_H \oplus \mathfrak{su}(2)_C) \oplus \mathfrak{so}(3,2)$$

- Poincaré supercharges: $Q_\alpha^{a_H \tilde{a}_C}$; conformal supercharges: $S^\alpha_{a_H \tilde{a}_C}$

$$\{Q_\alpha^{a\tilde{a}}, Q_\beta^{b\tilde{b}}\} = 2\epsilon^{ab}\epsilon^{\tilde{a}\tilde{b}} P_{\alpha\beta} \ ,$$
$$\{S^\alpha_{a\tilde{a}}, S^\beta_{b\tilde{b}}\} = 2\epsilon_{ab}\epsilon_{\tilde{a}\tilde{b}} K^{\alpha\beta} \ ,$$
$$\{Q_\alpha^{a\tilde{a}}, S^\beta_{b\tilde{b}}\} = 2\delta^a_b \delta^{\tilde{a}}_{\tilde{b}} \left(M_\alpha{}^\beta + \delta^\beta_\alpha D\right) - 2\delta^\beta_\alpha (R^a{}_b \delta^{\tilde{a}}_{\tilde{b}} + \delta^a_b \tilde{R}^{\tilde{a}}{}_{\tilde{b}}) \ .$$

Three-dimensional $\mathcal{N} = 4$ SCFTs: symmetries

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Three-dimensional $\mathcal{N} = 4$ SCFTs: Higgs branch

- Higgs branch of 3d $\mathcal{N} = 4$ SCFT is *hyperkähler cone*
- hyperkähler:
  - 3 complex structures J_i satisfying $J_i^2 = -1$ and $J_1 J_2 = J_3$ (and cyclic)
  - J_i *compatible* with metric: $g(X, Y) = g(J_i X, J_i Y)$
  - associated Kähler forms ω_i are *closed*
- cone: action of dilatation $\rightarrow$ cone structure

$$ds^2 = dr^2 + r^2 ds^2_{base}$$

Three-dimensional $\mathcal{N} = 4$ SCFTs: Higgs branch

- pick Cartan generator in $\mathfrak{su}(2)_H$
  - $\rightsquigarrow$ *distinguishes* one of the complex structures, say J_3
  - Higgs branch is thought of as Kähler manifold with Kähler form ω_3
  - other two Kähler forms combine into $\Omega = \omega_1 + i\omega_2$, *a closed (2,0)-form*
  - Ω defines holomorphic symplectic structure

- ring of holomorphic functions on Higgs branch has natural holomorphic Poisson bracket

$$\{f_1, f_2\}_{PB} = (\Omega^{-1})^{ij}\,\partial_i f_1\,\partial_j f_2$$

Three-dimensional $\mathcal{N} = 4$ SCFTs: Higgs branch chiral ring

- Higgs branch chiral ring operators $\Longleftrightarrow$ superconformal primary operators annihilated by
$$Q_\alpha^{1\tilde{a}}$$
- Higgs branch chiral ring operators $\Longleftrightarrow$ Lorentz scalar operators with
$$\Delta = (R_3)_H = R_H \ , \qquad (\tilde{R}_3)_C = \tilde{R}_C = 0$$
- Higgs branch chiral ring $\equiv$ holomorphic coordinate ring of Higgs branch

Three-dimensional $\mathcal{N} = 4$ SCFTs: free hypermultiplet

- field content: $(q_i^{a_H}, \psi_{\alpha,i}^{\tilde{a}_C})$, where i is an $\mathfrak{su}(2)_F$ index

$$q_i^{a_H} = \begin{pmatrix} \tilde{q} & q \\ -q^* & \tilde{q}^* \end{pmatrix} , \quad \text{with OPE} \quad q_i^a(x) q_j^b(y) \sim \frac{\epsilon^{ab}\epsilon_{ij}}{|x-y|} + (q_i^a q_j^b)(y) .$$

- Wick theorem provides OPEs of all composites of descendants
- Higgs branch: $\mathbb{C}^2$ with $w \leftrightarrow q_1^1 = \tilde{q}$, $z \leftrightarrow q_2^1 = q$

$$\omega_3 = \frac{i}{2}(dw \wedge d\bar{w} + dz \wedge d\bar{z}) , \qquad \Omega = dw \wedge dz .$$

- Higgs branch chiral ring: all composites of q_i^1, *i.e.* of z, w

$$\{w, z\}_{PB} = 1 \qquad \Longleftrightarrow \qquad \{q_i^1, q_j^1\}_{PB} = \epsilon_{ij} .$$

Constructing interesting nilpotent supercharges

- recall algebra $\mathfrak{osp}(4|4,\mathbb{R}) \supset (\mathfrak{su}(2)_H \oplus \mathfrak{su}(2)_C) \oplus \mathfrak{so}(3,2)$
- choose a line $\mathbb{R}_{top.} \subset \mathbb{R}^3$ with coordinate s
- subalgebra of conformal algebra keeping this line fixed set-wise:

$$\mathfrak{su}(1,1) \oplus \mathfrak{so}(2)_\perp \subset \mathfrak{so}(3,2)$$

- inside $\mathfrak{osp}(4|4,\mathbb{R})$ the $\mathfrak{su}(1,1)$ can be supersymmetrized to

$$\mathfrak{su}(1,1|2)_H \qquad \text{or} \qquad \mathfrak{su}(1,1|2)_C$$

Constructing interesting nilpotent supercharges

- $\mathfrak{su}(1,1|2)_H$ has generators

$$\underbrace{L_0,\ L_{\pm 1}}_{\mathfrak{su}(1,1)\cong\mathfrak{sl}(2)}\ ,\ \underbrace{(R_3)_H,\ (R_\pm)_H}_{\mathfrak{su}(2)_H},\ \mathcal{Z};\ \mathcal{Q}^{a_H},\ \tilde{\mathcal{Q}}^{a_H},\ \mathcal{S}_{a_H},\ \tilde{\mathcal{S}}_{a_H},$$

- algebraic fact: there exist two nilpotent supercharges

$$\mathbb{Q}_1 \equiv \mathcal{Q}^1 - \zeta\tilde{\mathcal{S}}_1\ , \qquad \mathbb{Q}_2 \equiv \mathcal{S}_1 + \frac{1}{\zeta}\tilde{\mathcal{Q}}^1$$

such that

$$\{\mathbb{Q}_i, \ldots\} = \text{diag}(\mathfrak{su}(1,1), \mathfrak{su}(2)_H) \equiv \widehat{\mathfrak{sl}(2)}$$

- $\widehat{\mathfrak{sl}(2)}$ is generated by

$$\hat{L}_{-1} = L_{-1} + \zeta(R_-)_H\ , \quad \hat{L}_0 = L_0 - (R_3)_H\ , \quad \hat{L}_{+1} = L_{+1} - \zeta^{-1}(R_+)_H\ .$$

Solving the $\mathbb{Q}_i$ cohomology

- at the origin:

$$\{\mathbb{Q}_i, \mathcal{O}(0)] = 0 \quad \text{but} \quad \mathcal{O}(0) \neq \{\mathbb{Q}_i, \ldots]$$

$$\iff \mathcal{O}(0) \text{ is Higgs branch chiral ring operator}$$

- move away from origin while staying in cohomology: use $\mathbb{Q}_i$-exact $\hat{L}_{-1}$
- consider $\mathcal{O}^{(a_1 \ldots a_k)}$ in spin $k/2$ irrep of $\mathfrak{su}(2)_H$ with $\mathcal{O}^{(1\ldots1)}$ in HBCR

$$\mathcal{O}(s) \equiv e^{s\hat{L}_{-1}} \mathcal{O}^{(1\ldots1)}(0) e^{-s\hat{L}_{-1}}$$

$$= u_{a_1}(s) \ldots u_{a_k}(s) \mathcal{O}^{(a_1 \ldots a_k)}(s) \qquad \text{where} \qquad u_a(s) = \begin{pmatrix} 1 \\ \zeta s \end{pmatrix}$$

- s dependence drops out in cohomology, but ordering of operators along line does matter!

$$\mathcal{O}^{(s)} \equiv [\mathcal{O}(s)]_{\mathbb{Q}_i}$$

Example: free hypermultiplet

- One then finds

$$q_i \star q_j = \zeta \epsilon_{ij} + (q_i q_j)$$

- observe that this star-product is
  - noncommutative
  - zeroth order term is product in chiral ring
  - first order term given by holomorphic Poisson bracket

- Product of more general Higgs branch chiral ring elements

$$(q_{i_1} \ldots q_{i_k}) \star (q_{j_1} \ldots q_{j_l}) = (q_{i_1} \ldots q_{i_k}) \exp\left[\zeta \epsilon_{\kappa\lambda} \overleftarrow{\partial}_{q_\kappa} \overrightarrow{\partial}_{q_\lambda}\right] (q_{j_1} \ldots q_{j_l})$$

which is the Moyal star product

General structure

- let $f \in \mathcal{A}_p, g \in \mathcal{A}_q$ be holomorphic functions over the Higgs branch corresponding to operators of conformal dimension $\Delta = \frac{p}{2}$ and $\Delta = \frac{q}{2}$
- one can argue that their star products reads

$$f \star g = f \cdot g + \frac{\zeta}{2}\{f, g\}_{PB} + \sum_{k=2}^{\lfloor \frac{p+q}{2} \rfloor} \zeta^k C^k(f, g) \,,$$

where

- associativity of OPE $\Longrightarrow f \star (g \star h) = (f \star g) \star h$
- $\mathfrak{su}(2)_H$-charge $\Longrightarrow C^k : \mathcal{A}_p \otimes \mathcal{A}_q \to \mathcal{A}_{p+q-2k}$ (equivariance)
- $\mathfrak{su}(2)_H$-charge $\Longrightarrow C^k(f, g) = 0$ for $k > \min(p, q)$ (truncation)
- symmetry properties OPE $\Longrightarrow C^k(f, g) = (-1)^k C^k(g, f)$ (evenness)

General structure

- unitarity demands that for a complex scalar operator $\mathcal{O}$

$$\langle \mathcal{O}(x)\mathcal{O}^\dagger(y)\rangle = \frac{n_\mathcal{O}}{|x-y|^{2\Delta_\mathcal{O}}}\,, \qquad \text{with} \qquad n_\mathcal{O} > 0$$

- let ρ be a rotation over π in $\mathfrak{su}(2)_H$ followed by complex conjugation, then for $f, g \in \mathcal{A}_p$

$$\theta(f,g) \equiv C^p(\rho(f), g) \qquad \text{is a positive definite Hermitian form}$$

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Deformation quantization

- a $\mathbb{C}^*$-equivariant deformation quantization of a commutative $\mathbb{C}^*$-graded Poisson algebra $\mathcal{A}$ in the direction of the Poisson bracket reads

$$f \star g = \sum_{k=0}^{\lfloor \frac{p+q}{2} \rfloor} \hbar^k C^k(f,g)$$

  where

  - $\star$ is associative
  - $C^0(f,g) = f \cdot g$ original commutative product
  - $C^1(f,g) - C^1(g,f) = \{f,g\}$
  - equivariance: $C^k : \mathcal{A}_p \otimes \mathcal{A}_q \to \mathcal{A}_{p+q-2k}$

- our $\star$-product is clearly of this type

Deformation quantization

- equivariant deformation quantizations are typically organized in huge equivalence classes

- let $T(f) = f + \sum_{k=1}^{\lfloor \frac{p}{2} \rfloor} \hbar^k f^{(k)}$ where $f \in \mathcal{A}_p$ and $f^{(k)} \in \mathcal{A}_{p-2k}$

- let $T(\hbar) = \hbar$

- then

$$f \tilde{\star} g = T^{-1}(T(f) \star T(g))$$

  is an equivalent equivariant deformation quantization

Deformation quantization

- classification theorem for equivariant deformation quantizations of hyperkähler cones (up to equivalences)[Braden, Proudfoot, Webster], [Braden, Licata, Proudfoot, Webster]:
- in physics language:
  - consider space of FI parameters that resolve the Higgs branch into a smooth variety
  - mod out by Weyl group of global symmetries that act on Coulomb branch as hyperkähler isometries
- finite space!

Deformation quantization

- a $\mathbb{C}^*$-equivariant deformation quantization of a commutative $\mathbb{C}^*$-graded Poisson algebra $\mathcal{A}$ in the direction of the Poisson bracket reads

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Deformation quantization

- the bootstrap problem gives rise to an even quantization:

$$C^k(f,g) = (-1)^k C^k(g,f)$$

- refined classification:
  - the FI parameter must lie on same Weyl orbit as its negative

- Conjecture: for *even* quantization deformations, the truncation condition and the unitarity constraints are perfect gauge fixing conditions

General strategy

Given a Higgs branch and its holomorphic coordinate ring, there are two strategies to construct the $\star$-product:

- first strategy (bootstrap philosophy):
  1. write down most general Ansatz for the $\star$-product satisfying all requirements except for the unitarity constraints
  2. solve associativity constraints for all triples of increasing total $\mathbb{C}^*$-grading
  3. impose unitarity constraints after the fact

General strategy

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General strategy

- second strategy (deformation quantization philosophy):
  1. identify the quantum algebra (up to gauge equivalence)
  2. fix the parameters to the appropriate values
  3. solve the gauge fixing conditions; we conjecture they have a unique solution

Minimal nilpotent orbits

- Given a group G, consider dimension one adjoint valued generators μ^A satisfying the relations

$$(\mu \otimes \mu)|_{\mathcal{I}_2} = 0 \,, \qquad \mathrm{Sym}^2(\mathbf{adj}) = (2\mathbf{adj}) \oplus \mathcal{I}_2 \,.$$

- arise as Higgs branch of SCFTs obtained as IR fixed point of UV Lagrangian quiver gauge theories

A_n : 

D_n : 

- the minimal nilpotent orbit thus contains

$$(\mu \otimes \ldots \otimes \mu)|_{(n\mathbf{adj})} \qquad \text{of dimension } n \text{ for any } n$$

- G-equivariant quantization deformation do not have gauge ambiguities

Minimal nilpotent orbits

- even, $\mathbb{C}^*$-equivariant quantization deformation depends on

$\mathfrak{g}$	# parameters
$\mathfrak{sl}(2)$	1
$\mathfrak{sl}(n \geq 3)$	0
$\mathfrak{g} \neq \mathfrak{sl}(n)$	0

- for $\mathfrak{g} \neq \mathfrak{sl}(2)$ the truncated conformal bootstrap problem is completely determined

- for $\mathfrak{g} = \mathfrak{sl}(2)$ we can compute the single parameter through localization (from flavor current two-point function coefficient [Closset, Dumitrescu, Festuccia, Komargodski, Seiberg])

Minimal nilpotent orbits

- precisely these algebras appear as (generalized) higher spin algebras
- closed form expressions for the structure constants for the classical groups have been worked out [Joung, Mkrtchyan]
- for the case of $\mathfrak{sl}(2)$, *i.e.*, Higgs branch $\mathcal{O}_{min}(\mathfrak{sl}(2)) = \mathbb{C}^2/\mathbb{Z}_2$

$$\text{higher spin algebra:} \quad hs[\lambda] = \mathcal{U}(\mathfrak{sl}(2))/\{C_2 = \frac{1}{4}(\lambda^2 - 1)\}$$

a.k.a. "lone star product" [Pope, Romans, Shen]

  - SQED with $N_f = 2$: $\lambda = 0$
  - $\mathbb{Z}_2$ gauge theory of free hypermultiplet: $\lambda = \frac{1}{2}$
    indeed known that $hs[\frac{1}{2}]$ can be expressed in terms of Moyal product

A_n-type Kleinian singularities

- three generators X, Y, Z

	2Δ	q_F
Z	2	0
X	$n+1$	$+1$
Y	$n+1$	-1

  satisfying the relation

$$XY = Z^{n+1}$$

- occur for example as Higgs branch of
  - circular quiver of Abelian gauge groups of $n+1$ nodes
  - $\mathbb{Z}_{n+1}$ gauge theory of the free hypermultiplet
- holomorphic symplectic Poisson bracket

$$\{X, Y\} = -(n+1)Z^n\,, \qquad \{Z, X\} = X\,, \qquad \{Z, Y\} = -Y\,.$$

A_2 Kleinian singularities

- we find a one-parameter gauge slice respecting the evenness and truncation constraints
- the parameter appears most easily in the definition of Z^2

$$Z^2 = \hat{Z}^2 + \alpha\zeta^2$$

- some explicit star-products

$$Z \star Z = Z^2 - \alpha\zeta^2 ,$$

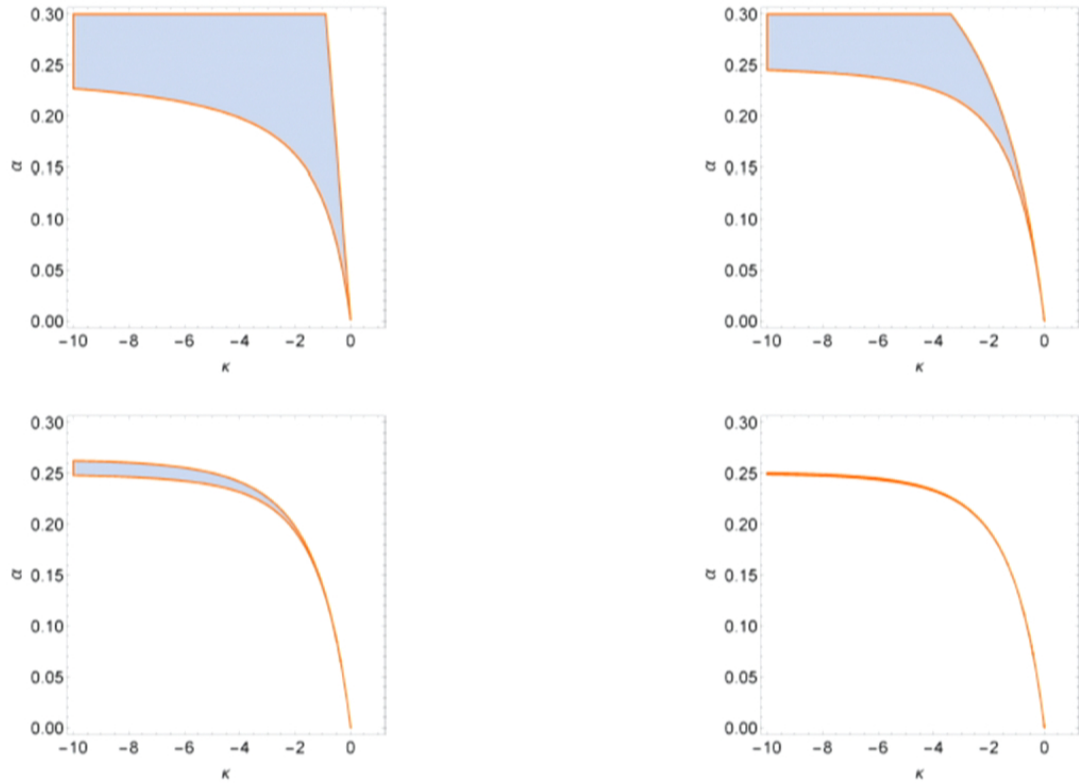
$$Z \star X = ZX + \frac{1}{2}\zeta X ,$$

$$Z \star Y = ZY - \frac{1}{2}\zeta Y ,$$

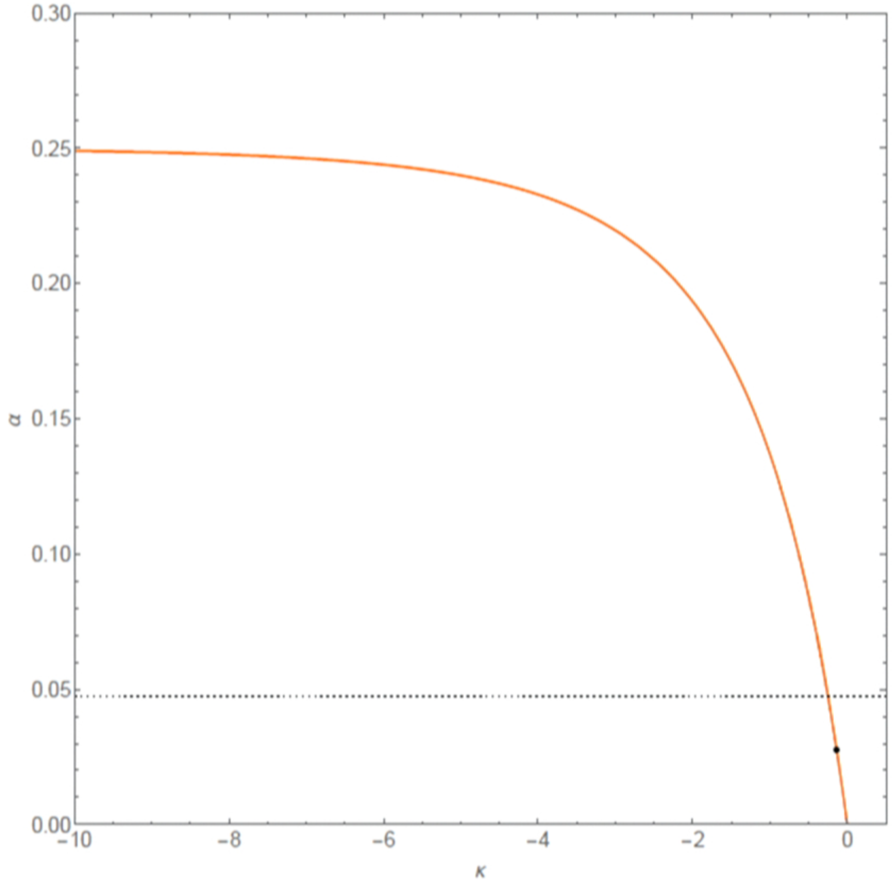
$$X \star Y = Z^3 - \frac{3}{2}\zeta Z^2 - \frac{3\alpha + \kappa}{4\alpha}\zeta^2 Z + \frac{3\alpha + \kappa}{2}\zeta^3 .$$

A_2 Kleinian singularities

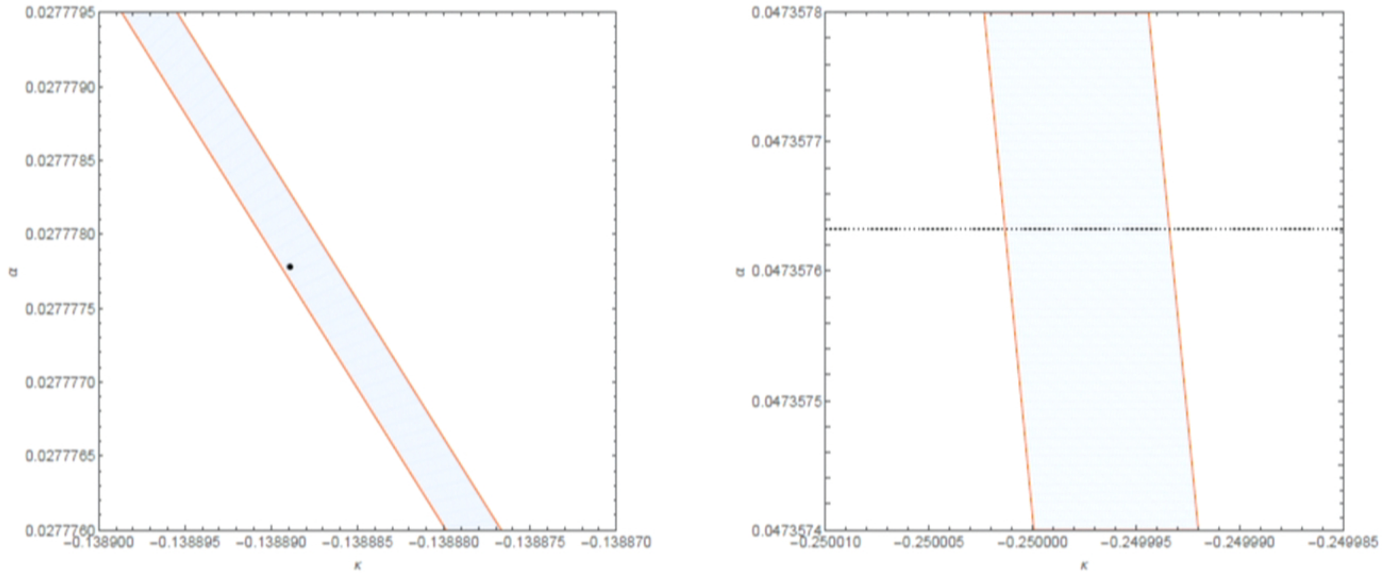
- imposing the unitarity constraints excludes many values of (κ, α)



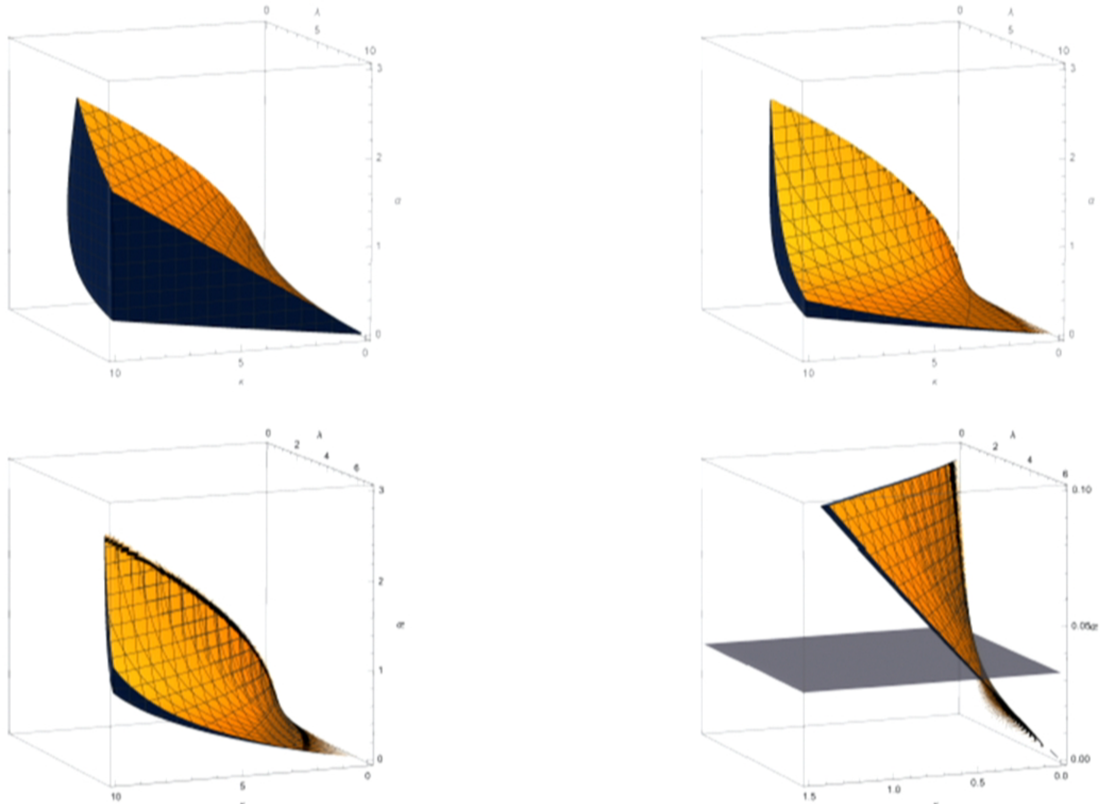
A_2 Kleinian singularities



A_2 Kleinian singularities



A_3 Kleinian singularities



Future directions

- are the algebras studied here accessible through localization?
- connect to quantization deformation obtained from Ω-deformation [Yagi], [Bullimore, Dimofte, Gaiotto]?
- inclusion of conformal defects
- study deformation quantization problem with additional requirements and prove conjecture