Title: S-Duality and Helicity Amplitudes

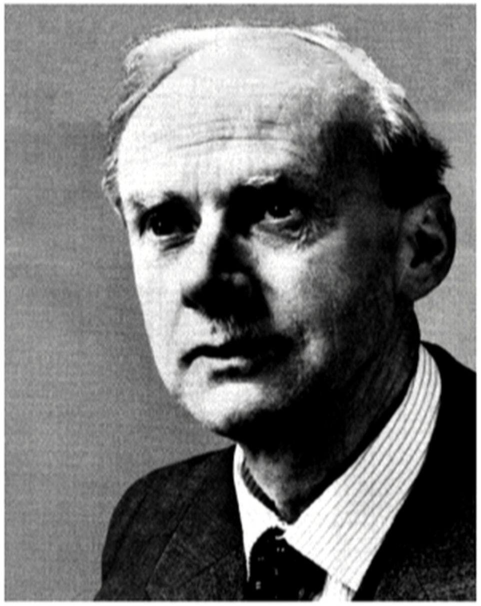
Date: Mar 11, 2016 01:00 PM

URL: http://pirsa.org/16030073

Abstract: <p>I will demonstrate that SL(2,Z) duality is a property of all low-energy effective Abelian theories with electric or magnetic charges The duality will be verified at one loop by comparing the amplitudes in the case of an electron and the dyon that is its SL(2,Z) image, and I will show that it can be extended order by order in perturbation theory. I will discuss how the duality generically breaks down at high energies, and show how the results apply to the Seiberg-Witten theory.</p>


Dirac
g
q
$qg = \frac{n}{2}$
charge quantization
Proc. Roy. Soc. Lond. A133 (1931) 60

Dirac



non-local action?

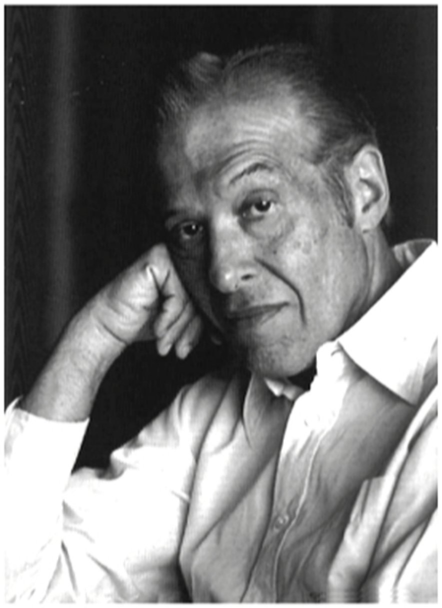
$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + \,^*G_{\mu\nu}$$

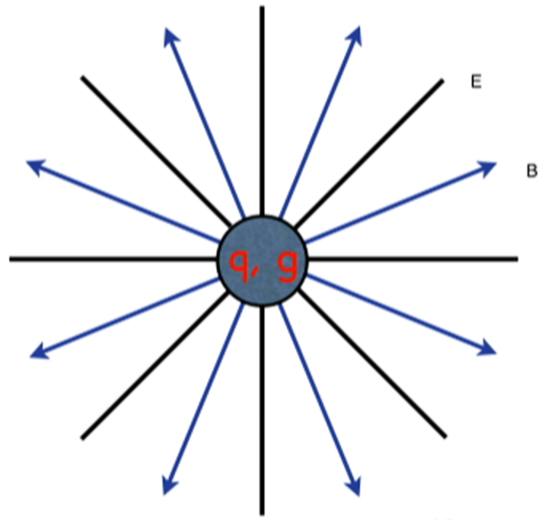
$$\begin{aligned} G_{\mu\nu}(x) &= 4\pi(n\cdot\partial)^{-1}\left[n_\mu K_\nu(x) - n_\nu K_\mu(x)\right] \\ &= \int d^4y\,\left[f_\mu(x-y)K_\nu(y) - f_\nu(x-y)K_\mu(y)\right] \end{aligned}$$

$$\partial_\mu f^\mu(x) = 4\pi\delta(x)$$
$$f^\mu(x) = 4\pi n^\mu\,(n\cdot\partial)^{-1}\,\delta(x)$$

Phys. Rev. 74 (1948) 817

Schwinger



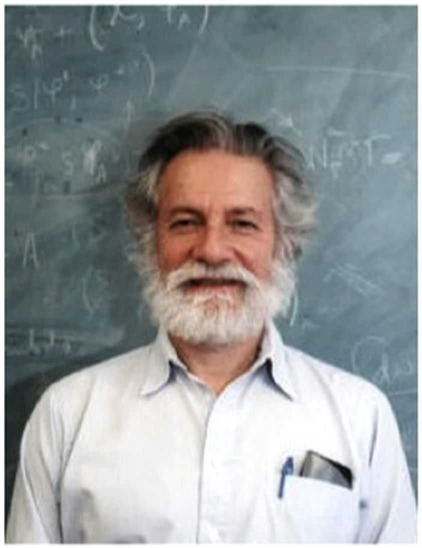


dyons

$$q_1 g_2 - q_2 g_1 = \frac{n}{2}$$

Science 165 (1969) 757

Zwanziger



non-Lorentz invariant, local action?

$$\mathcal{L} = -\frac{1}{2n^2e^2}\{[n\cdot(\partial\wedge A)]\cdot[n\cdot^*(\partial\wedge B)] - [n\cdot(\partial\wedge B)]\cdot[n\cdot^*(\partial\wedge A)] + [n\cdot(\partial\wedge A)]^2 + [n\cdot(\partial\wedge B)]^2\} - J\cdot A - \frac{4\pi}{e^2}K\cdot B.$$

electric magnetic

$$F = \frac{1}{n^2}\left(\{n\wedge[n\cdot(\partial\wedge A)]\} - {}^*\{n\wedge[n\cdot(\partial\wedge B)]\}\right)$$

Phys. Rev. D3 (1971) 880

Witten

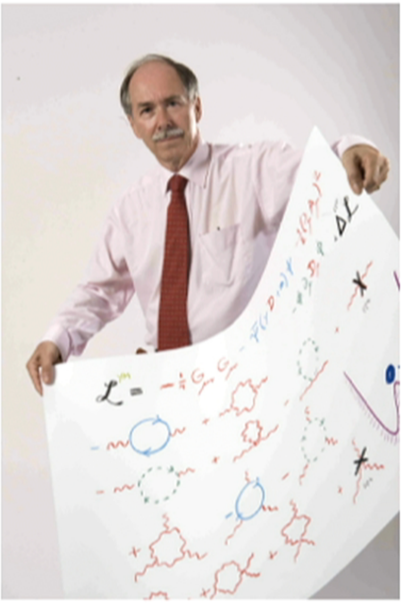
effective charge shifted

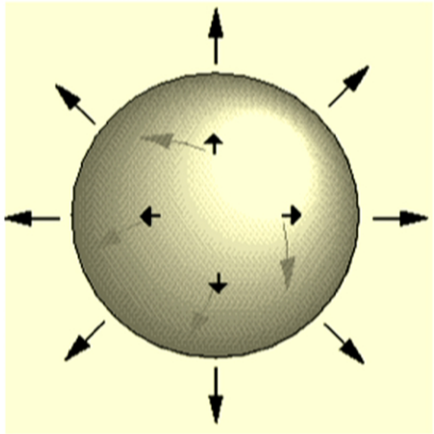
$$\mathcal{L}_{\text{free}} = -\frac{1}{4e^2}F^{\mu\nu}F_{\mu\nu} - \frac{\theta}{32\pi^2}F^{\mu\nu}\,{}^*F_{\mu\nu}$$

$$Q = q + g\frac{\theta}{2\pi}$$

Phys. Lett. B86 (1979) 283

‘t Hooft-Polyakov

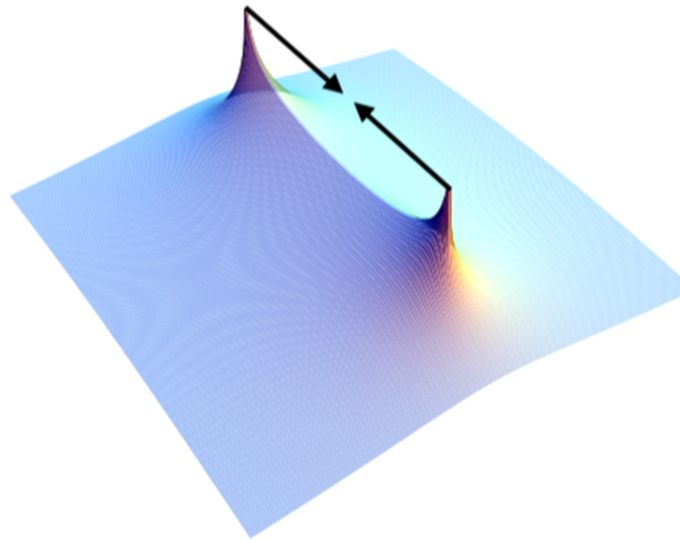




topological monopoles

Nucl. Phys., B79 1974, 276
JETP Lett., 20 1974, 194

Argyres-Douglas



CFT with massless electric and magnetic charges

hep-th/9505062

E-M Duality

$$\vec{E} \rightarrow \vec{B}$$
$$\vec{B} \rightarrow -\vec{E}$$

$${}^*F^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}$$
$$F^{\mu\nu} \rightarrow {}^*F^{\mu\nu}$$

Shift Symmetry

$$\mathcal{L}_{\text{free}} = -\frac{1}{4e^2} F^{\mu\nu} F_{\mu\nu} - \frac{\theta}{32\pi^2} F^{\mu\nu}\, {}^*F_{\mu\nu}$$

$$\theta \to \theta + 2\pi$$

$$\tau \equiv \frac{\theta}{2\pi} + \frac{4\pi i}{e^2}$$

S Duality

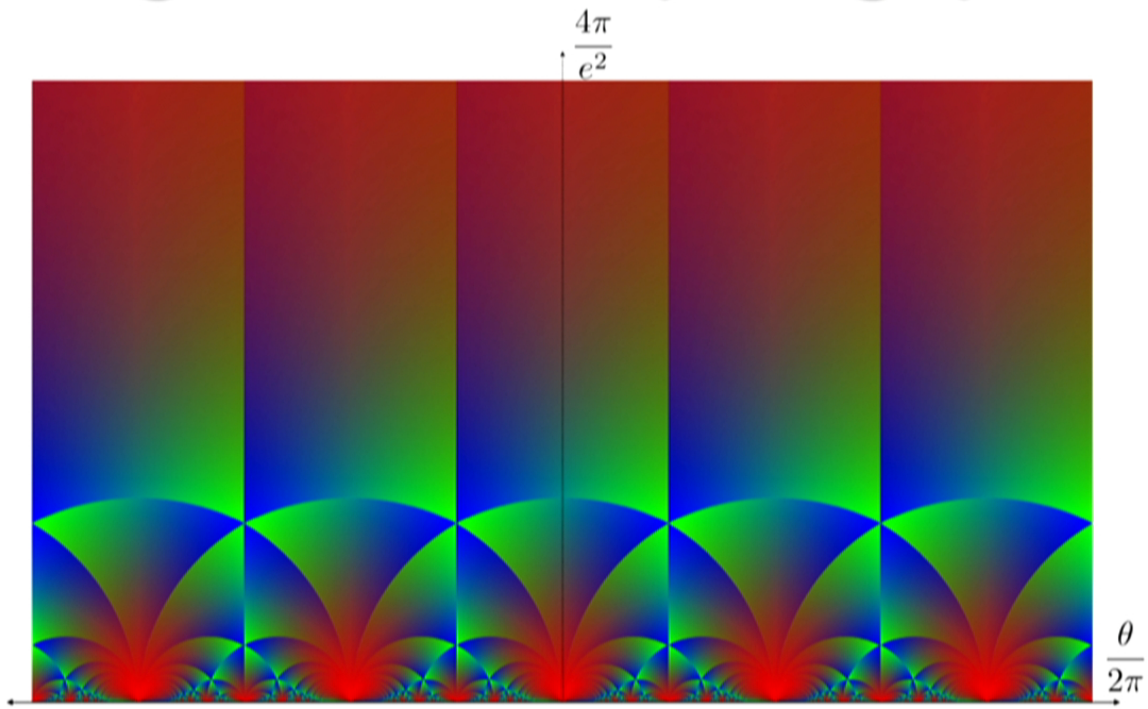
$$\mathcal{L}_{\text{free}} = -\text{Im}\,\frac{\tau}{32\pi}\left(F^{\mu\nu} + i\,{}^*F^{\mu\nu}\right)^2$$

$$\mathcal{L}_c = \frac{1}{4\pi} B^\mu \partial\,{}^*F_{\mu\nu}$$

$$\tilde{\mathcal{L}} = \text{Im}\,\frac{1}{32\pi\tau}\left(\tilde{F}^{\mu\nu} + i\,{}^*\tilde{F}^{\mu\nu}\right)^2$$

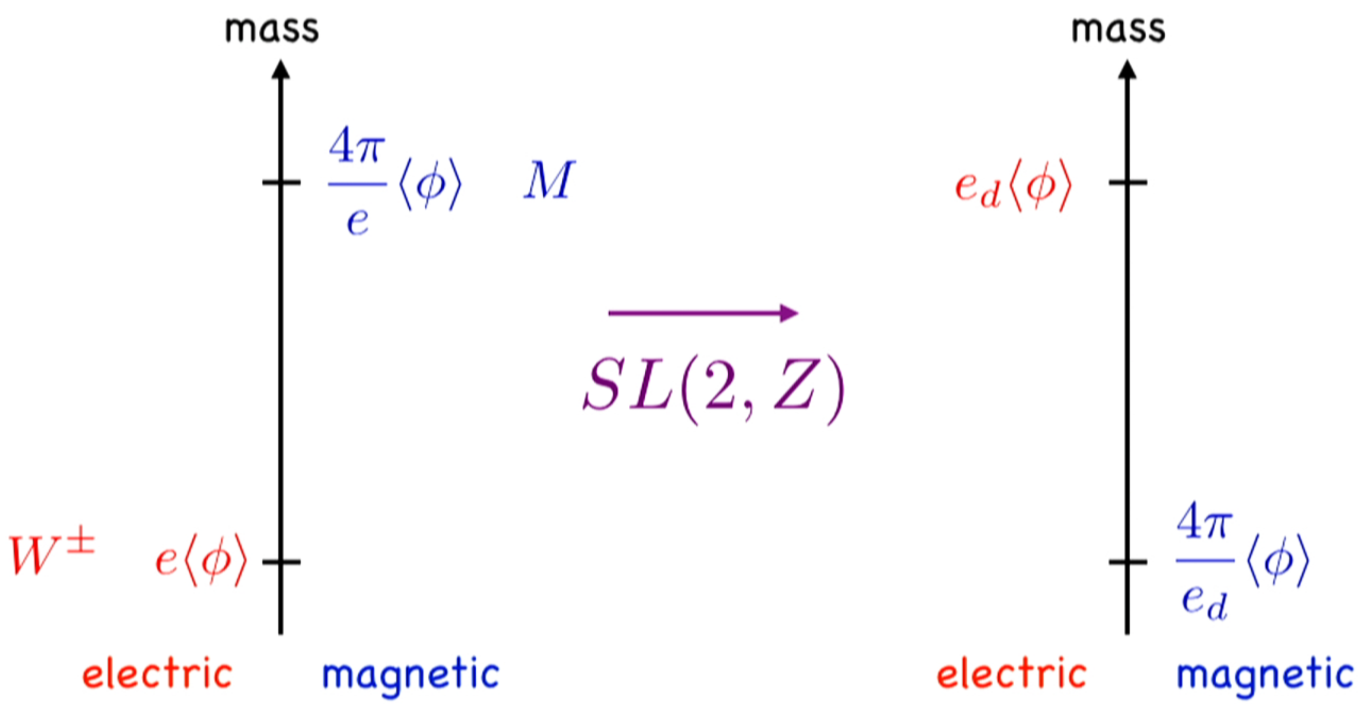
$$\tilde{F}_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$$

Tiling the coupling plane



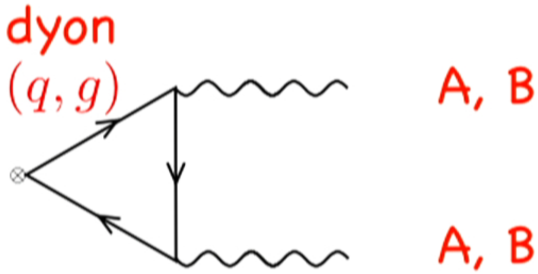
Cardy and Rabinovici Nucl. Phys. B**205** (1982) 1

$\mathcal{N} = 4$ SUSY



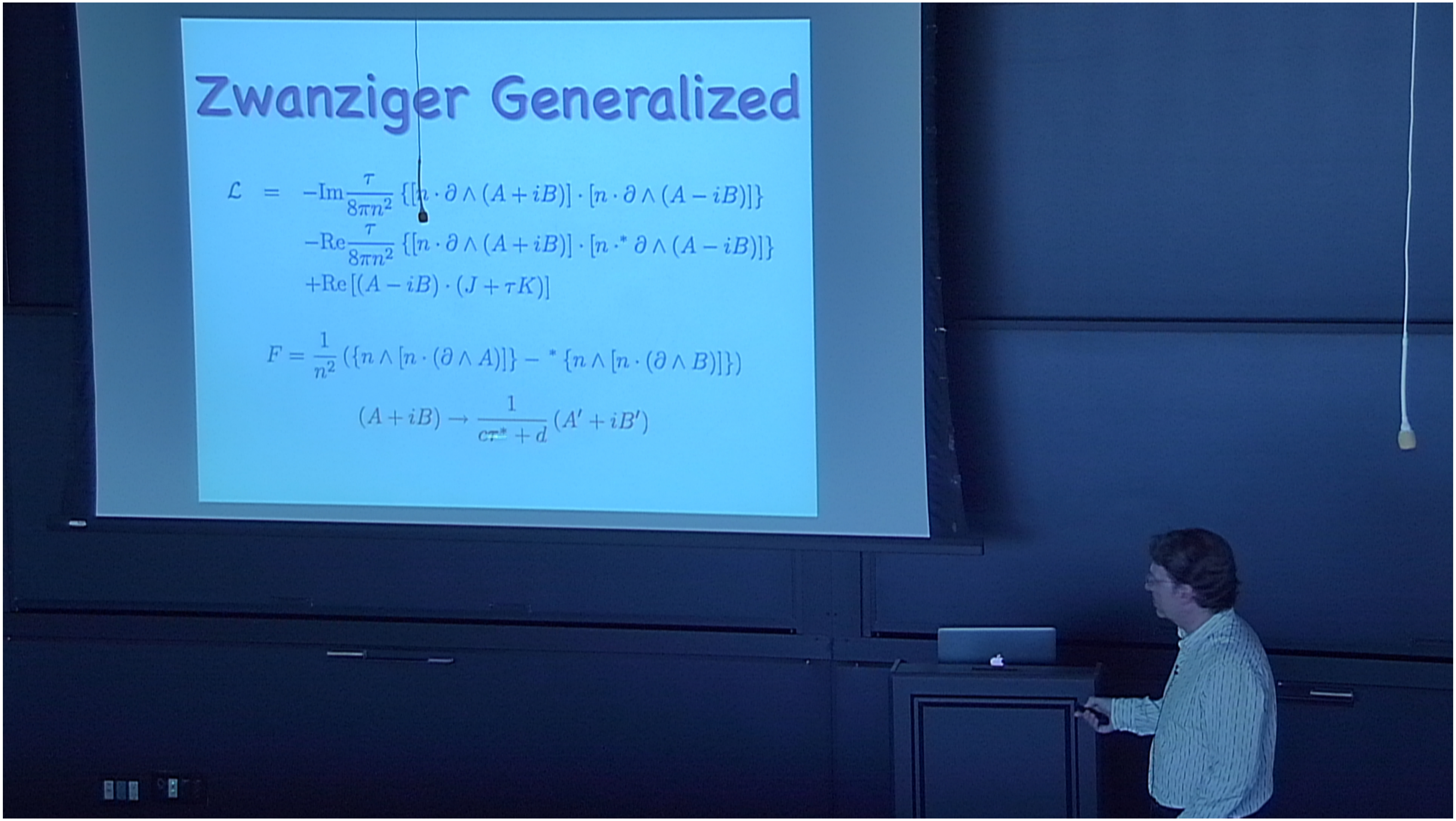
weak <-> weak duality

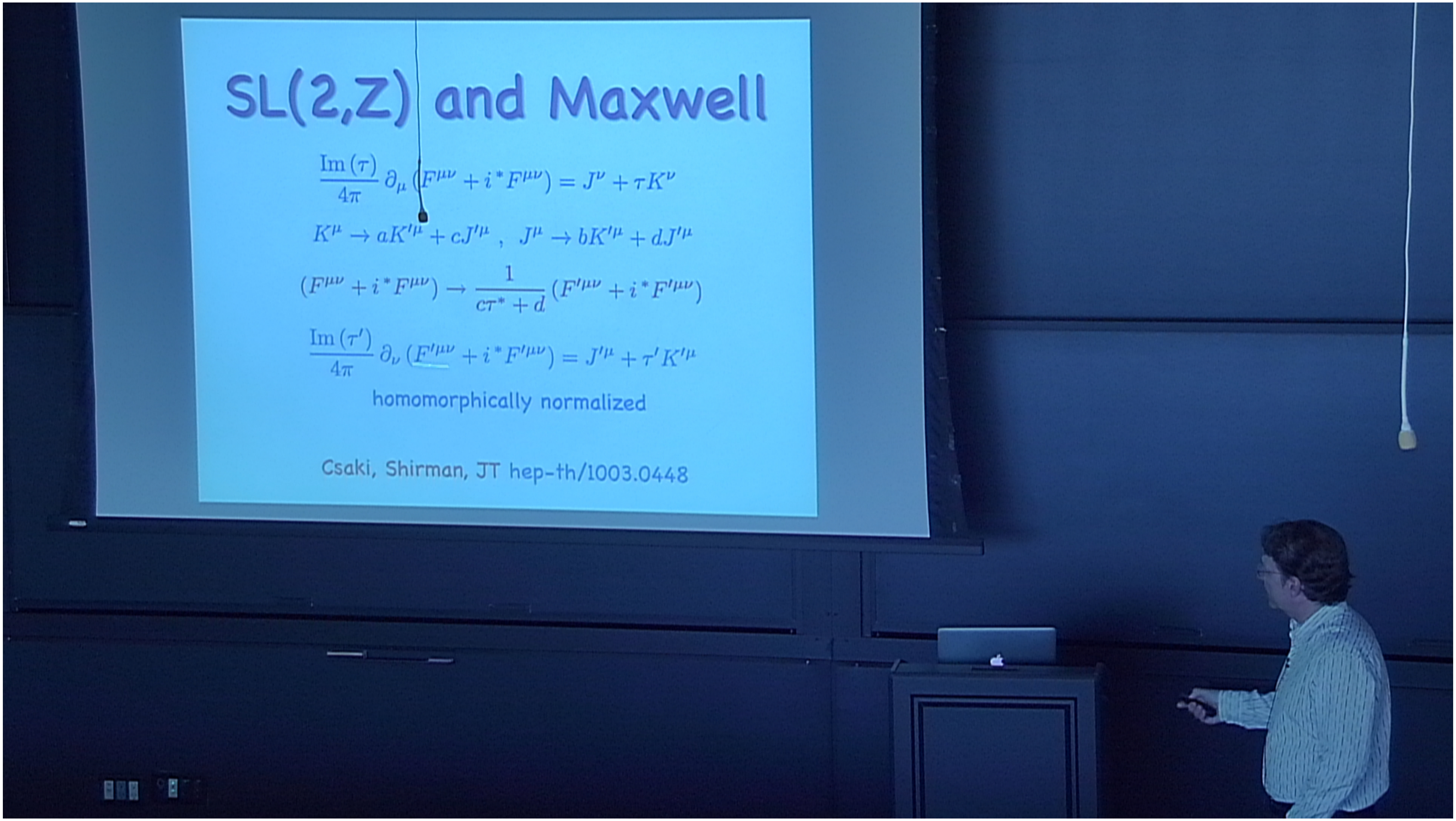
Anomalies



$$\mathcal{L} = -\frac{1}{2n^2e^2}\{[n\cdot(\partial\wedge A)]\cdot[n\cdot^*(\partial\wedge B)] - [n\cdot(\partial\wedge B)]\cdot[n\cdot^*(\partial\wedge A)] + [n\cdot(\partial\wedge A)]^2 + [n\cdot(\partial\wedge B)]^2\} - J\cdot A - \frac{4\pi}{e^2}K\cdot B.$$

electric magnetic


Zwanziger Generalized
$\mathcal{L} = -\mathrm{Im}\frac{\tau}{8\pi n^2}\{[n\cdot\partial\wedge(A+iB)]\cdot[n\cdot\partial\wedge(A-iB)]\}$
$-\mathrm{Re}\frac{\tau}{8\pi n^2}\{[n\cdot\partial\wedge(A+iB)]\cdot[n\cdot^{*}\partial\wedge(A-iB)]\}$
$+\mathrm{Re}[(A-iB)\cdot(J+\tau K)]$
$F = \frac{1}{n^2}(\{n\wedge[n\cdot(\partial\wedge A)]\} - {}^{*}\{n\wedge[n\cdot(\partial\wedge B)]\})$
$(A+iB)\rightarrow\frac{1}{c\tau^{*}+d}(A'+iB')$


SL(2,Z) and Maxwell
$\frac{\mathrm{Im}\,(\tau)}{4\pi}\,\partial_\mu\left(F^{\mu\nu}+i\,{}^*F^{\mu\nu}\right)=J^\nu+\tau K^\nu$
$K^\mu\to aK'^\mu+cJ'^\mu\;,\;\;J^\mu\to bK'^\mu+dJ'^\mu$
$\left(F^{\mu\nu}+i\,{}^*F^{\mu\nu}\right)\to\frac{1}{c\tau^*+d}\left(F'^{\mu\nu}+i\,{}^*F'^{\mu\nu}\right)$
$\frac{\mathrm{Im}\,(\tau')}{4\pi}\,\partial_\nu\left(F'^{\mu\nu}+i\,{}^*F'^{\mu\nu}\right)=J'^\mu+\tau' K'^\mu$
homomorphically normalized
Csaki, Shirman, JT hep-th/1003.0448

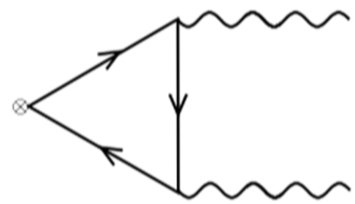
Mapping the Charge

$$\begin{pmatrix} a & -b \\ -c & d \end{pmatrix} \begin{pmatrix} q \\ g \end{pmatrix} = \begin{pmatrix} q' \\ 0 \end{pmatrix} \qquad q' = \gcd(q, g)$$

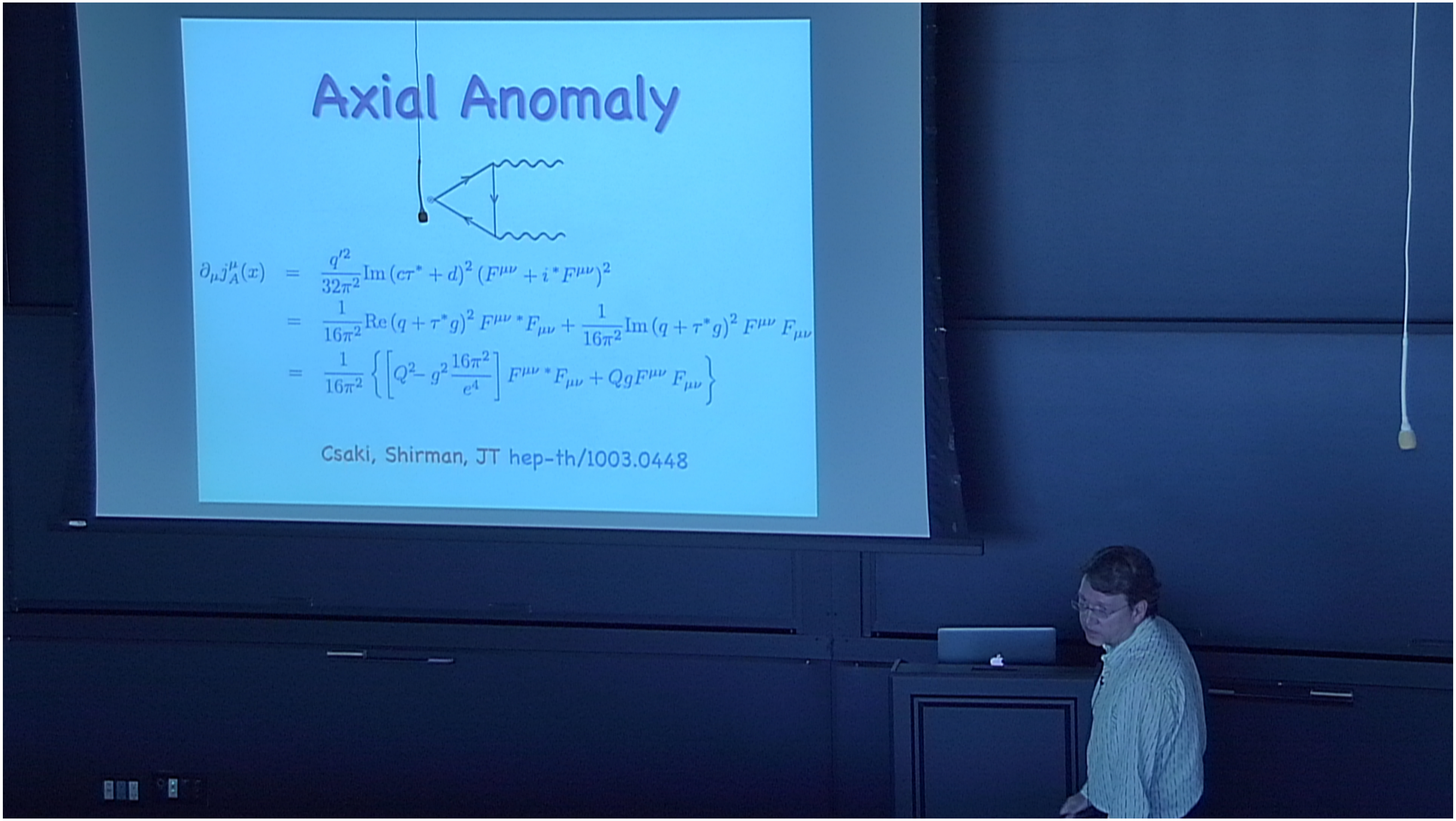
$$c = g/q',\, d = q/q' \quad aq - bg = q'$$

Axial Anomaly from SL(2,Z)

$$(q, g) \to (q', 0)$$



$$\begin{aligned} \partial_\mu j^\mu_A(x) &= \frac{q'^2}{16\pi^2} F'^{\mu\nu} \, {}^*F'_{\mu\nu} \\ &= \frac{q'^2}{32\pi^2} \mathrm{Im} \left(F'^{\mu\nu} + i \, {}^*F'^{\mu\nu}\right)^2 \end{aligned}$$


Axial Anomaly
$\partial_\mu j^\mu_A(x) = \frac{q'^2}{32\pi^2}\mathrm{Im}\,(c\tau^* + d)^2\,(F^{\mu\nu} + i\,{}^*F^{\mu\nu})^2$
$= \frac{1}{16\pi^2}\mathrm{Re}\,(q + \tau^* g)^2\,F^{\mu\nu}\,{}^*F_{\mu\nu} + \frac{1}{16\pi^2}\mathrm{Im}\,(q + \tau^* g)^2\,F^{\mu\nu}\,F_{\mu\nu}$
$= \frac{1}{16\pi^2}\left\{\left[Q^2 - g^2\frac{16\pi^2}{e^4}\right]F^{\mu\nu}\,{}^*F_{\mu\nu} + QgF^{\mu\nu}\,F_{\mu\nu}\right\}$
Csaki, Shirman, JT hep-th/1003.0448

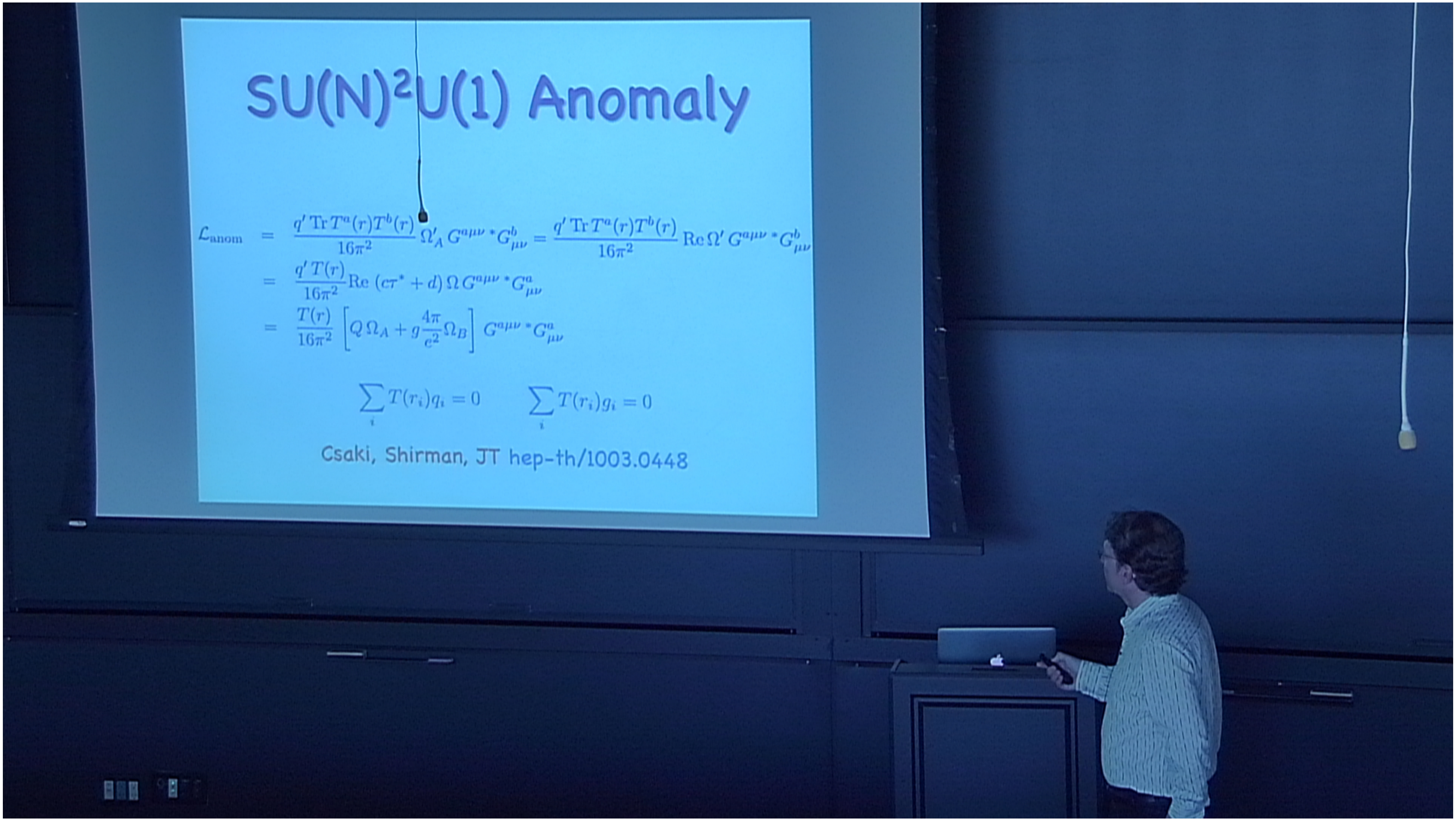
SU(N)²U(1) Anomaly

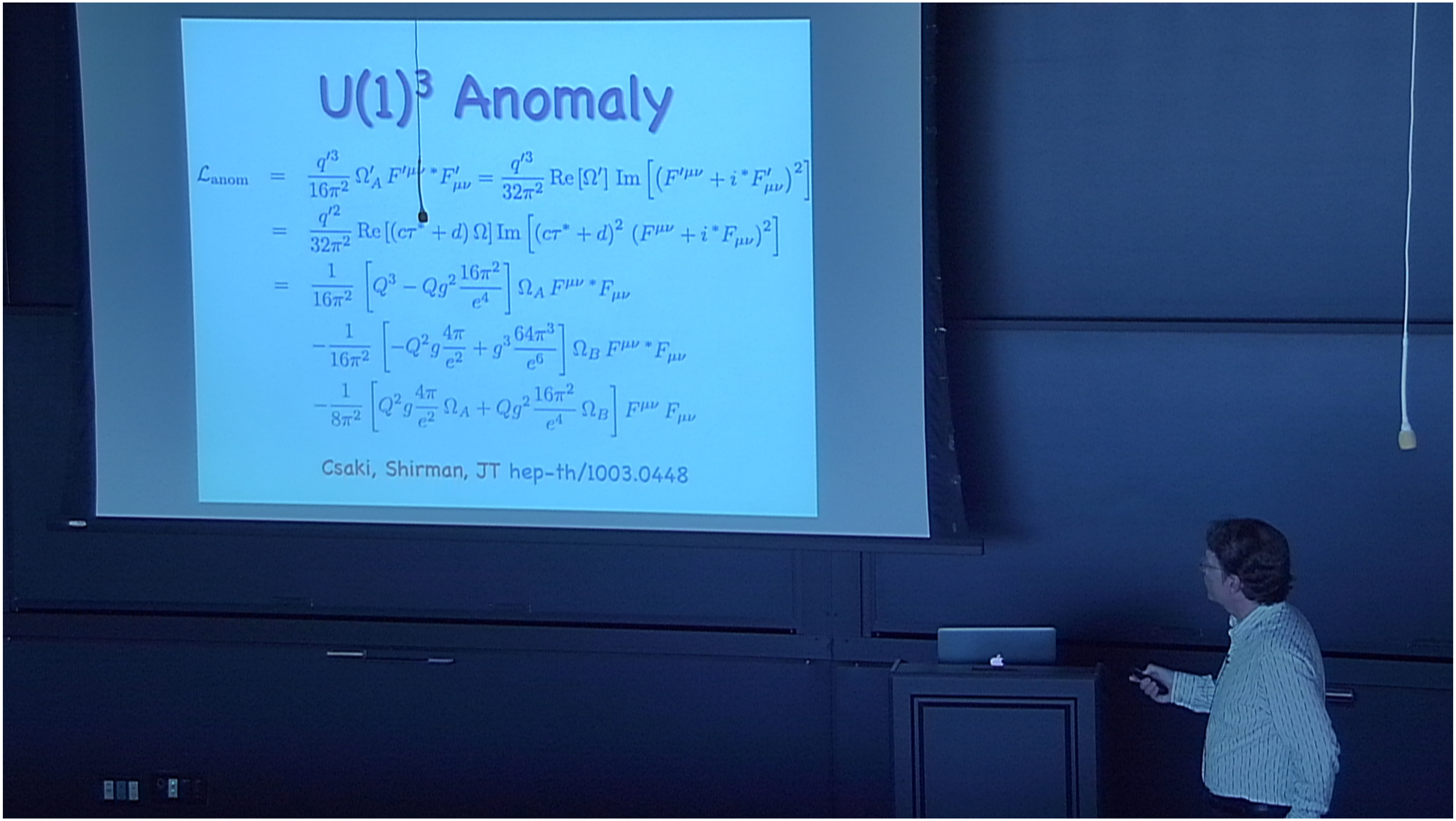
$$\mathcal{L}_{\rm anom} \quad = \quad c\,\Omega\, G^{a\mu\nu}\, {}^*G^{a}_{\mu\nu}$$

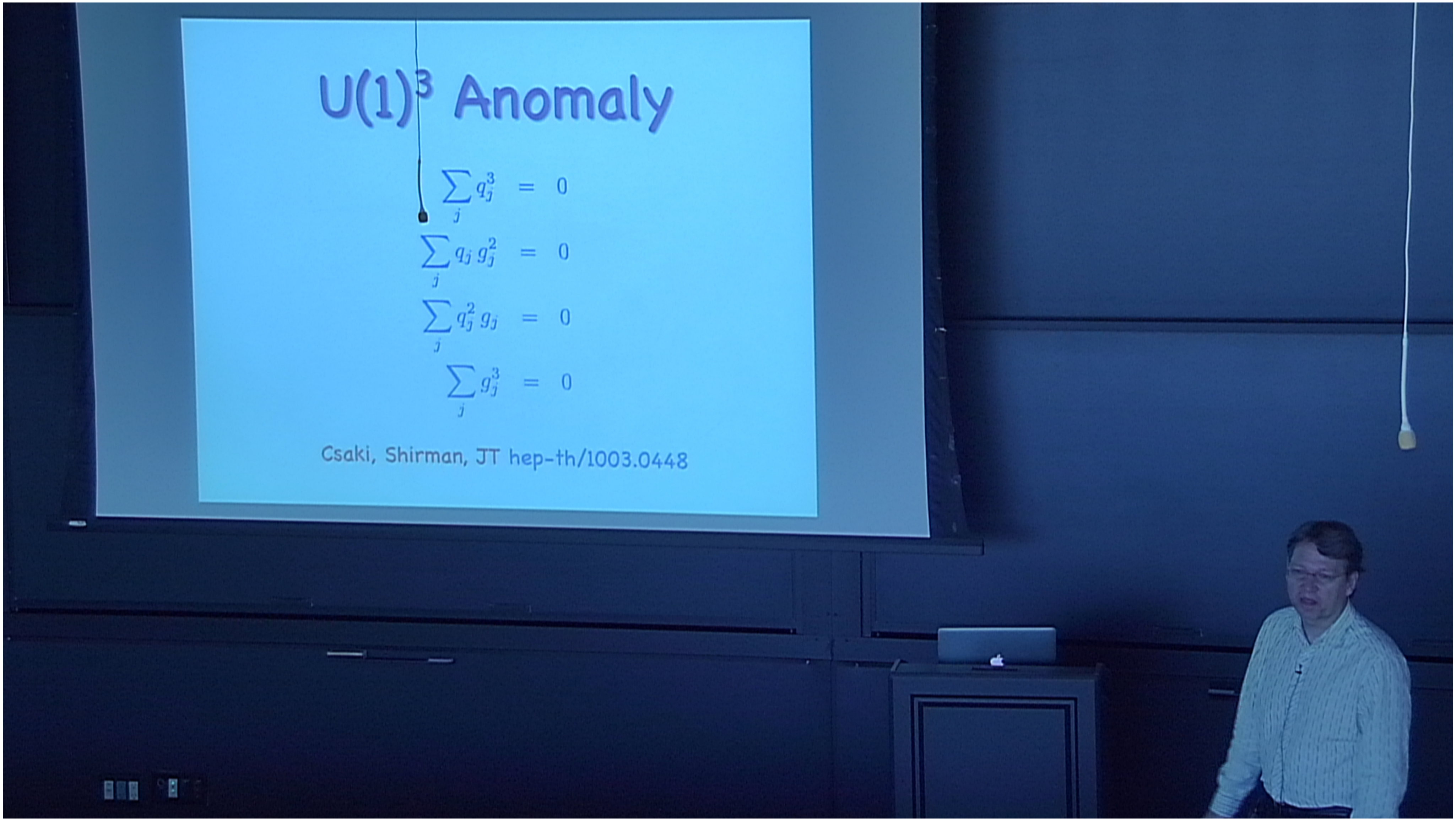
gauge parameter

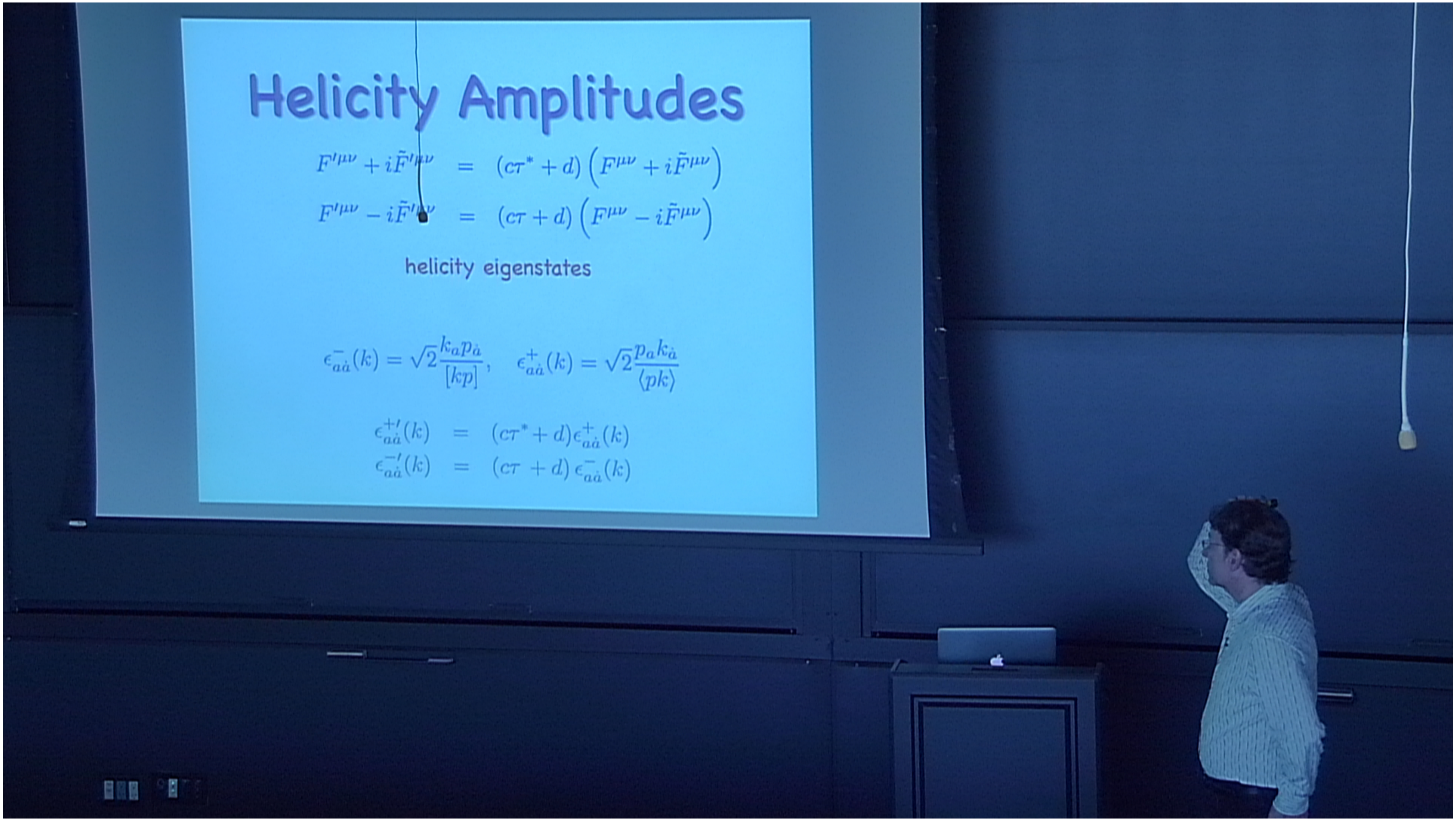
$$\Omega = \Omega_A + i\,\Omega_B$$

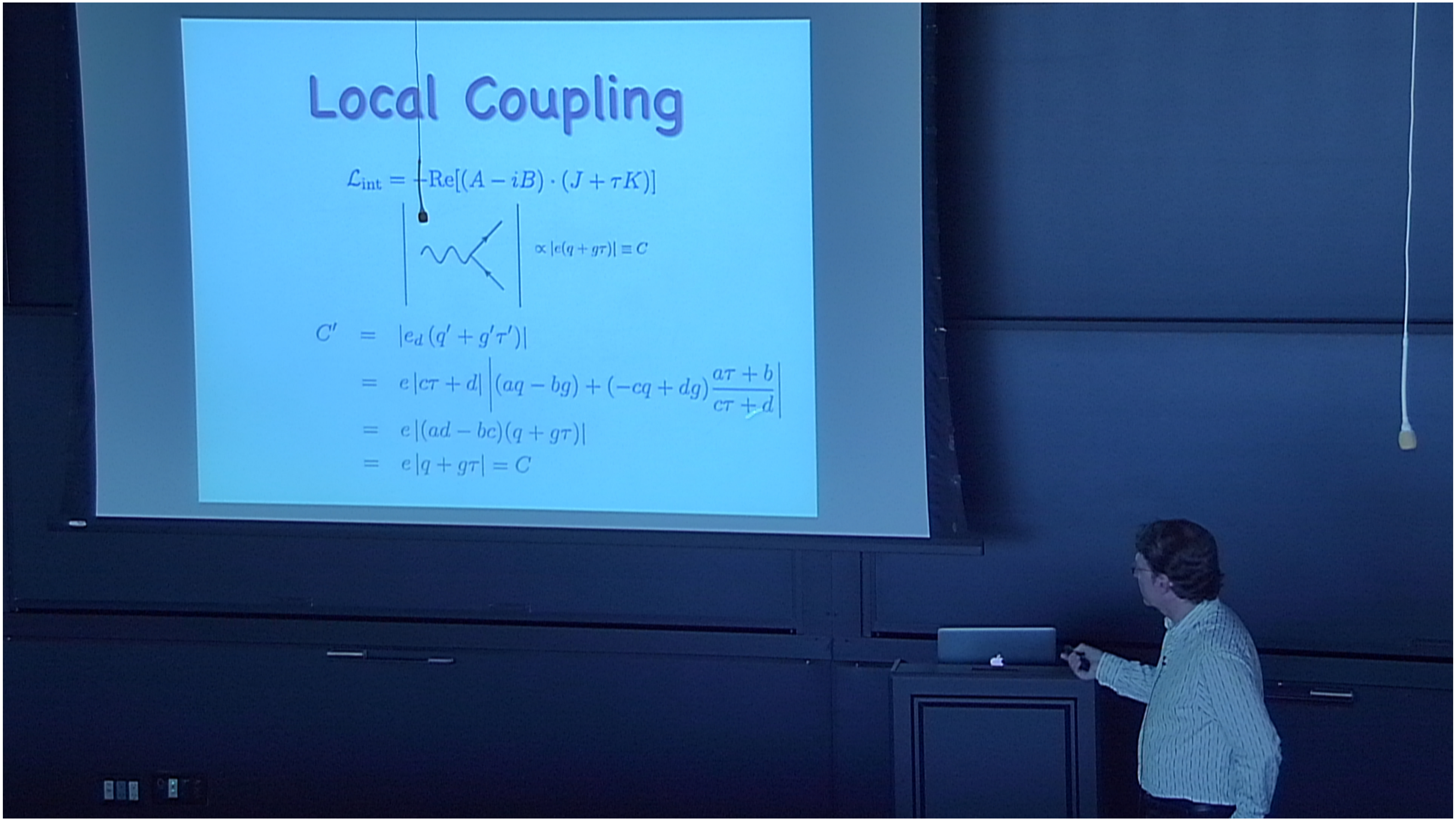
$$\Omega \to \frac{1}{c\tau^* + d}\,\Omega'$$

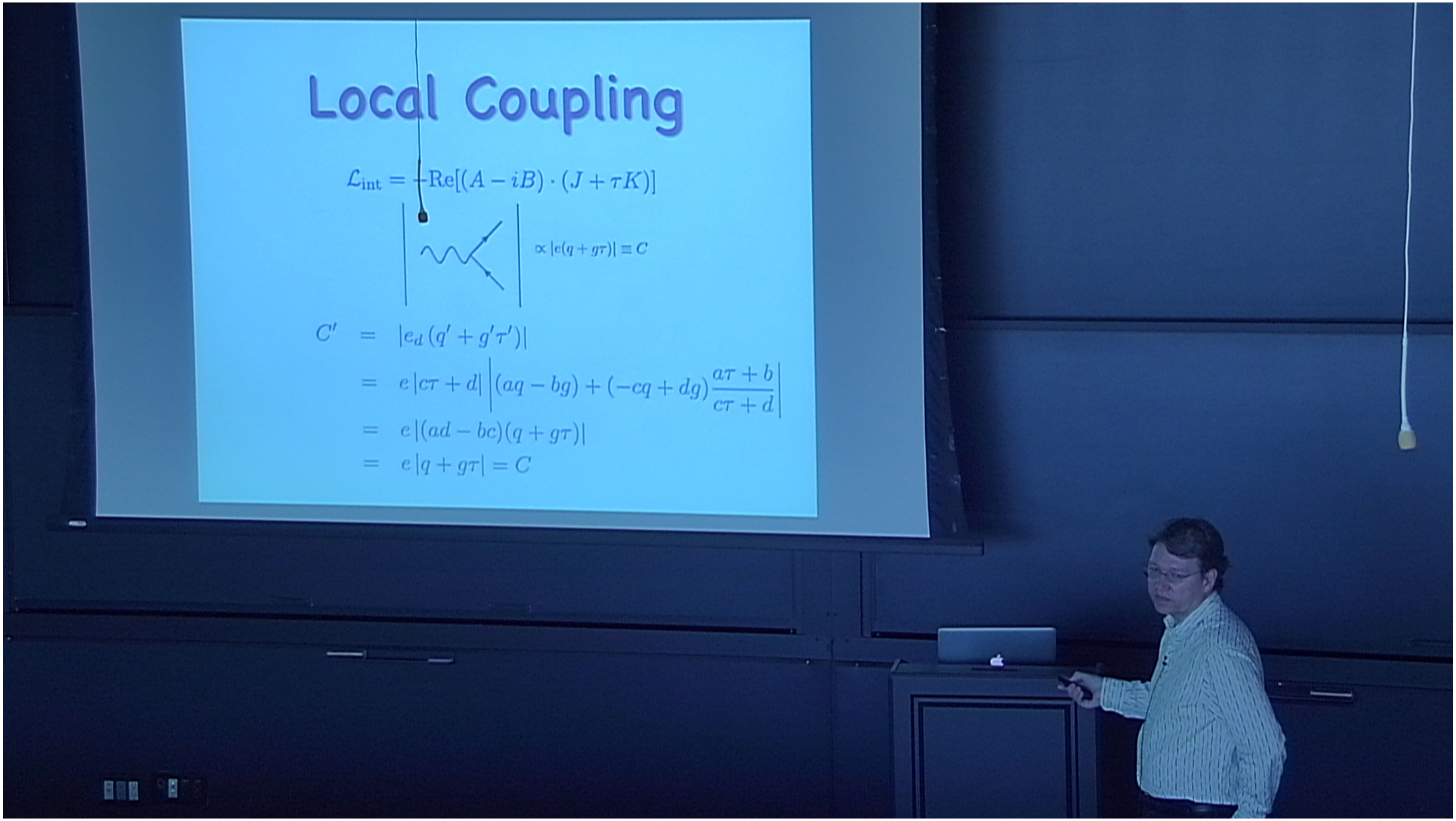

SU(N)²U(1) Anomaly
$\mathcal{L}_{\rm anom} = \frac{q'\,{\rm Tr}\,T^a(r)T^b(r)}{16\pi^2}\,\Omega'_A\,G^{a\mu\nu}\,{}^*G^b_{\mu\nu} = \frac{q'\,{\rm Tr}\,T^a(r)T^b(r)}{16\pi^2}\,{\rm Re}\,\Omega'\,G^{a\mu\nu}\,{}^*G^b_{\mu\nu}$
$= \frac{q'\,T(r)}{16\pi^2}{\rm Re}\,(c\tau^* + d)\,\Omega\,G^{a\mu\nu}\,{}^*G^a_{\mu\nu}$
$= \frac{T(r)}{16\pi^2}\left[Q\,\Omega_A + g\frac{4\pi}{e^2}\Omega_B\right]G^{a\mu\nu}\,{}^*G^a_{\mu\nu}$
$\sum_i T(r_i)q_i = 0 \qquad \sum_i T(r_i)g_i = 0$
Csaki, Shirman, JT hep-th/1003.0448


U(1)³ Anomaly
$\mathcal{L}_{\rm anom} = \frac{q'^3}{16\pi^2}\,\Omega'_A\,F'^{\mu\nu}\,{}^*F'_{\mu\nu} = \frac{q'^3}{32\pi^2}\,{\rm Re}\,[\Omega']\,{\rm Im}\left[\left(F'^{\mu\nu} + i\,{}^*F'_{\mu\nu}\right)^2\right]$
$= \frac{q'^2}{32\pi^2}\,{\rm Re}\,[(c\tau^* + d)\,\Omega]\,{\rm Im}\left[(c\tau^* + d)^2\,\left(F^{\mu\nu} + i\,{}^*F_{\mu\nu}\right)^2\right]$
$= \frac{1}{16\pi^2}\left[Q^3 - Qg^2\frac{16\pi^2}{e^4}\right]\Omega_A\,F^{\mu\nu}\,{}^*F_{\mu\nu}$
$-\frac{1}{16\pi^2}\left[-Q^2 g\frac{4\pi}{e^2} + g^3\frac{64\pi^3}{e^6}\right]\Omega_B\,F^{\mu\nu}\,{}^*F_{\mu\nu}$
$-\frac{1}{8\pi^2}\left[Q^2 g\frac{4\pi}{e^2}\,\Omega_A + Qg^2\frac{16\pi^2}{e^4}\,\Omega_B\right]F^{\mu\nu}\,F_{\mu\nu}$
Csaki, Shirman, JT hep-th/1003.0448


U(1)^3 Anomaly
$\sum_j q_j^3 = 0$
$\sum_j q_j g_j^2 = 0$
$\sum_j q_j^2 g_j = 0$
$\sum_j g_j^3 = 0$
Csaki, Shirman, JT hep-th/1003.0448


Helicity Amplitudes
$F'^{\mu\nu} + i\tilde{F}'^{\mu\nu} = (c\tau^* + d)\left(F^{\mu\nu} + i\tilde{F}^{\mu\nu}\right)$
$F'^{\mu\nu} - i\tilde{F}'^{\mu\nu} = (c\tau + d)\left(F^{\mu\nu} - i\tilde{F}^{\mu\nu}\right)$
helicity eigenstates
$\epsilon^-_{a\dot{a}}(k) = \sqrt{2}\frac{k_a p_{\dot{a}}}{[kp]}, \quad \epsilon^+_{a\dot{a}}(k) = \sqrt{2}\frac{p_a k_{\dot{a}}}{\langle pk \rangle}$
$\epsilon^{+\prime}_{a\dot{a}}(k) = (c\tau^* + d)\epsilon^+_{a\dot{a}}(k)$
$\epsilon^{-\prime}_{a\dot{a}}(k) = (c\tau + d)\,\epsilon^-_{a\dot{a}}(k)$


Local Coupling
$\mathcal{L}_{\rm int} = -{\rm Re}[(A - iB)\cdot(J + \tau K)]$
$\propto |e(q + g\tau)| \equiv C$
$C' = |e_d\,(q' + g'\tau')|$
$= e\,|c\tau + d|\,\left|(aq - bg) + (-cq + dg)\frac{a\tau + b}{c\tau + d}\right|$
$= e\,|(ad - bc)(q + g\tau)|$
$= e\,|q + g\tau| = C$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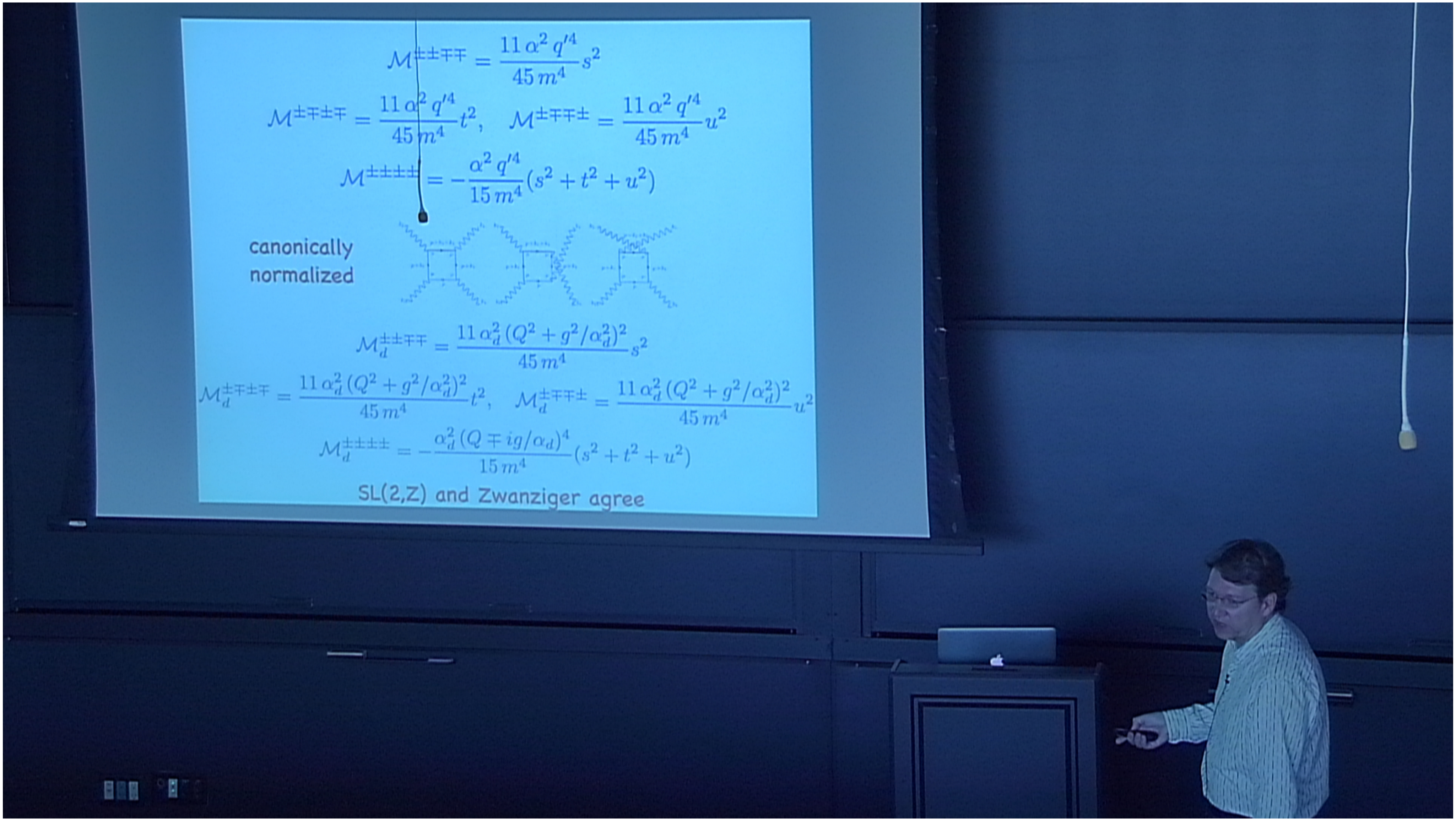
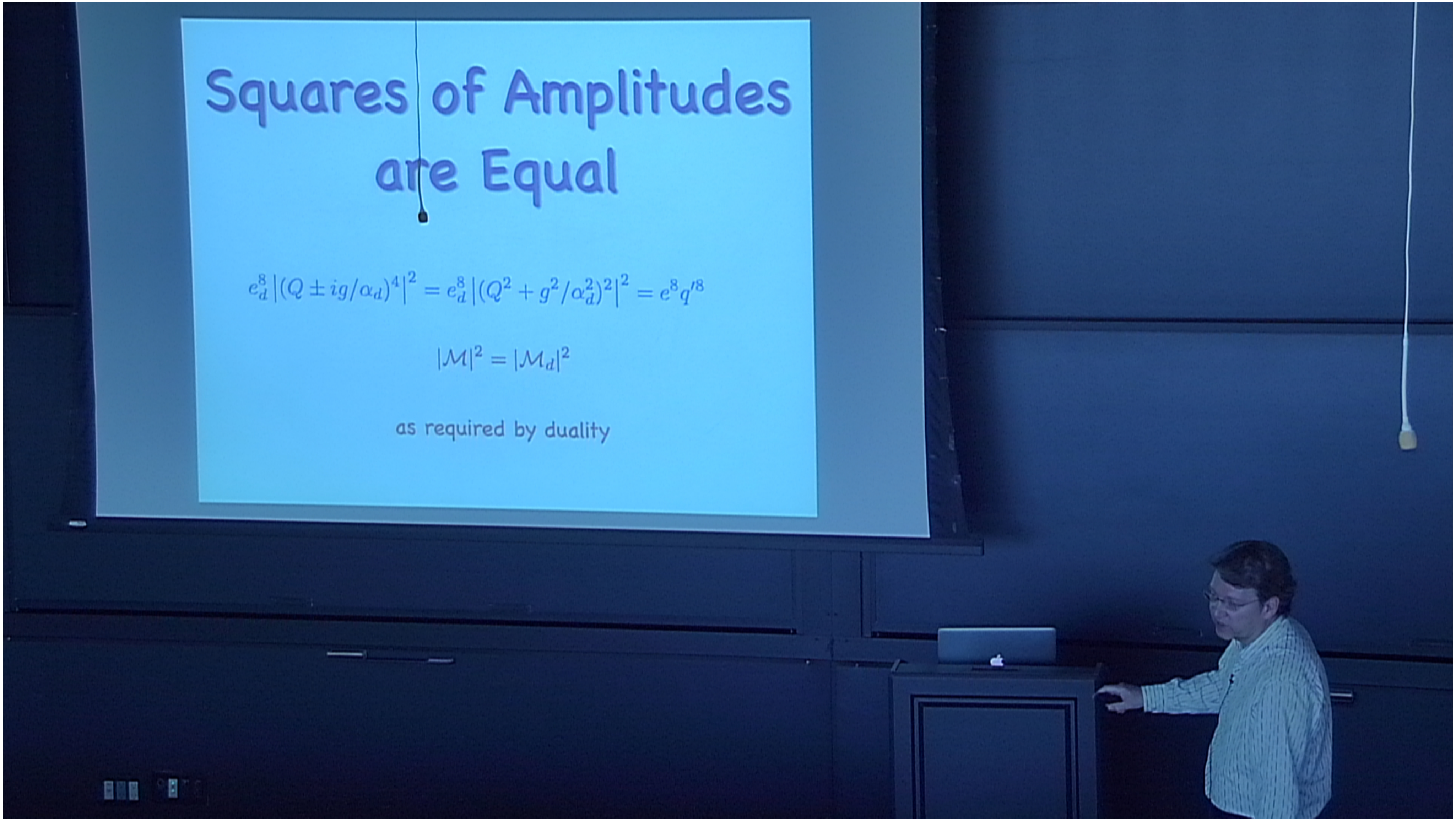
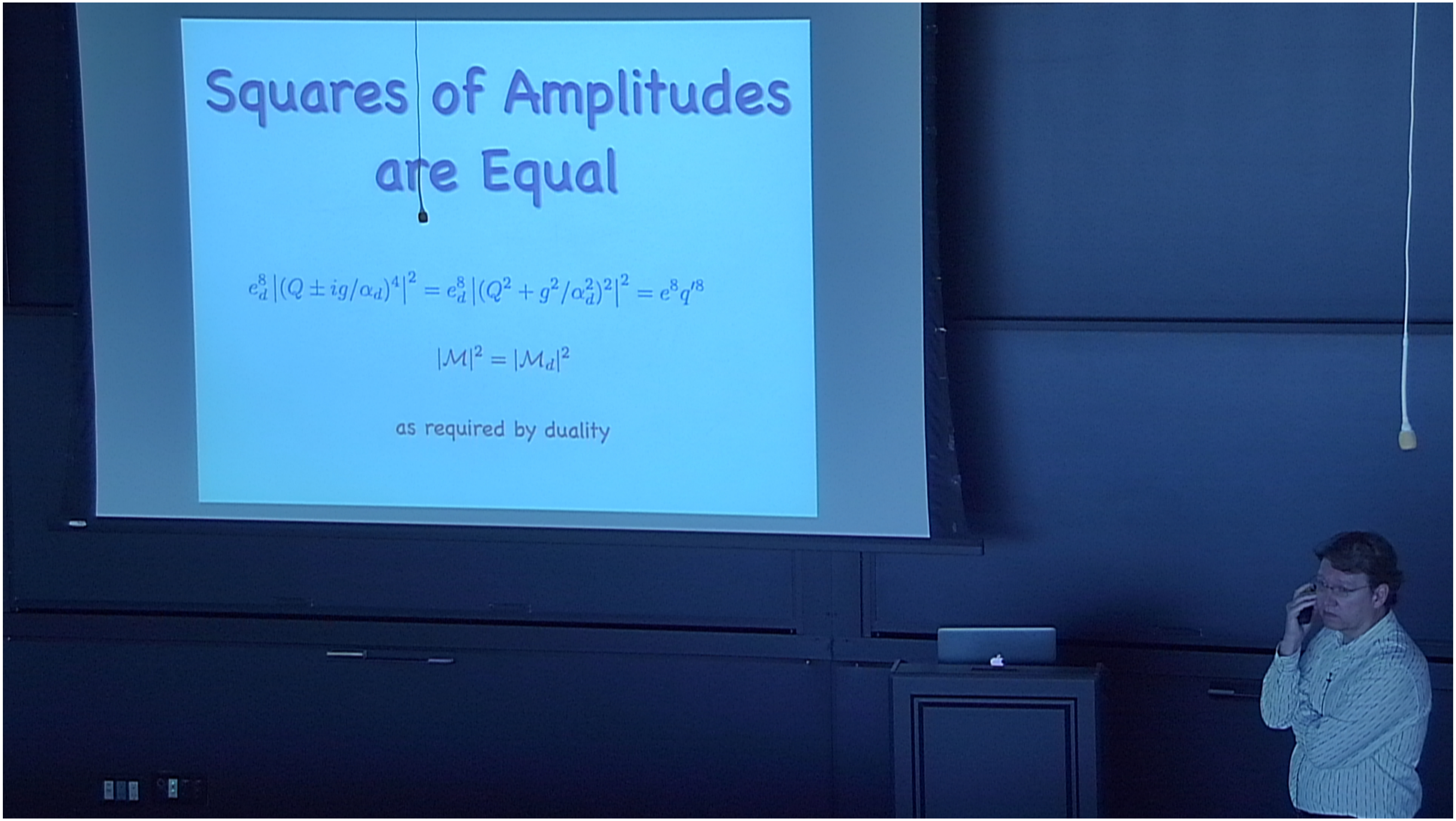

$\mathcal{M}^{\pm\pm\mp\mp} = \frac{11\,\alpha^2\,q'^4}{45\,m^4}s^2$
$\mathcal{M}^{\pm\mp\pm\mp} = \frac{11\,\alpha^2\,q'^4}{45\,m^4}t^2, \quad \mathcal{M}^{\pm\mp\mp\pm} = \frac{11\,\alpha^2\,q'^4}{45\,m^4}u^2$
$\mathcal{M}^{\pm\pm\pm\pm} = -\frac{\alpha^2\,q'^4}{15\,m^4}(s^2+t^2+u^2)$
canonically normalized
$\mathcal{M}_d^{\pm\pm\mp\mp} = \frac{11\,\alpha_d^2\,(Q^2+g^2/\alpha_d^2)^2}{45\,m^4}s^2$
$\mathcal{M}_d^{\pm\mp\pm\mp} = \frac{11\,\alpha_d^2\,(Q^2+g^2/\alpha_d^2)^2}{45\,m^4}t^2, \quad \mathcal{M}_d^{\pm\mp\mp\pm} = \frac{11\,\alpha_d^2\,(Q^2+g^2/\alpha_d^2)^2}{45\,m^4}u^2$
$\mathcal{M}_d^{\pm\pm\pm\pm} = -\frac{\alpha_d^2\,(Q\mp ig/\alpha_d)^4}{15\,m^4}(s^2+t^2+u^2)$
SL(2,Z) and Zwanziger agree


Squares of Amplitudes are Equal
$e_d^8 \left|(Q \pm ig/\alpha_d)^4\right|^2 = e_d^8 \left|(Q^2 + g^2/\alpha_d^2)^2\right|^2 = e^8 q'^8$
$|\mathcal{M}|^2 = |\mathcal{M}_d|^2$
as required by duality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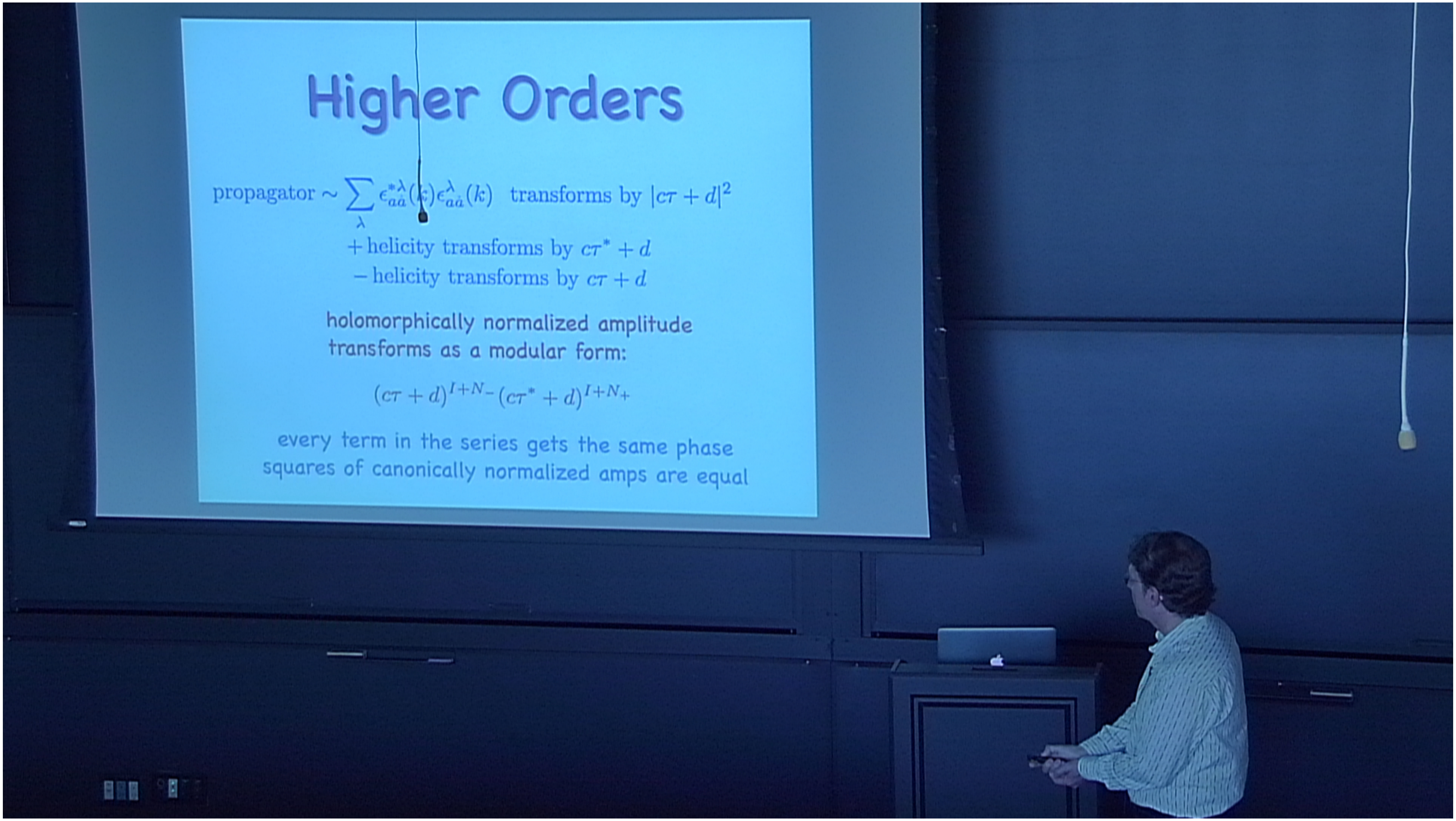
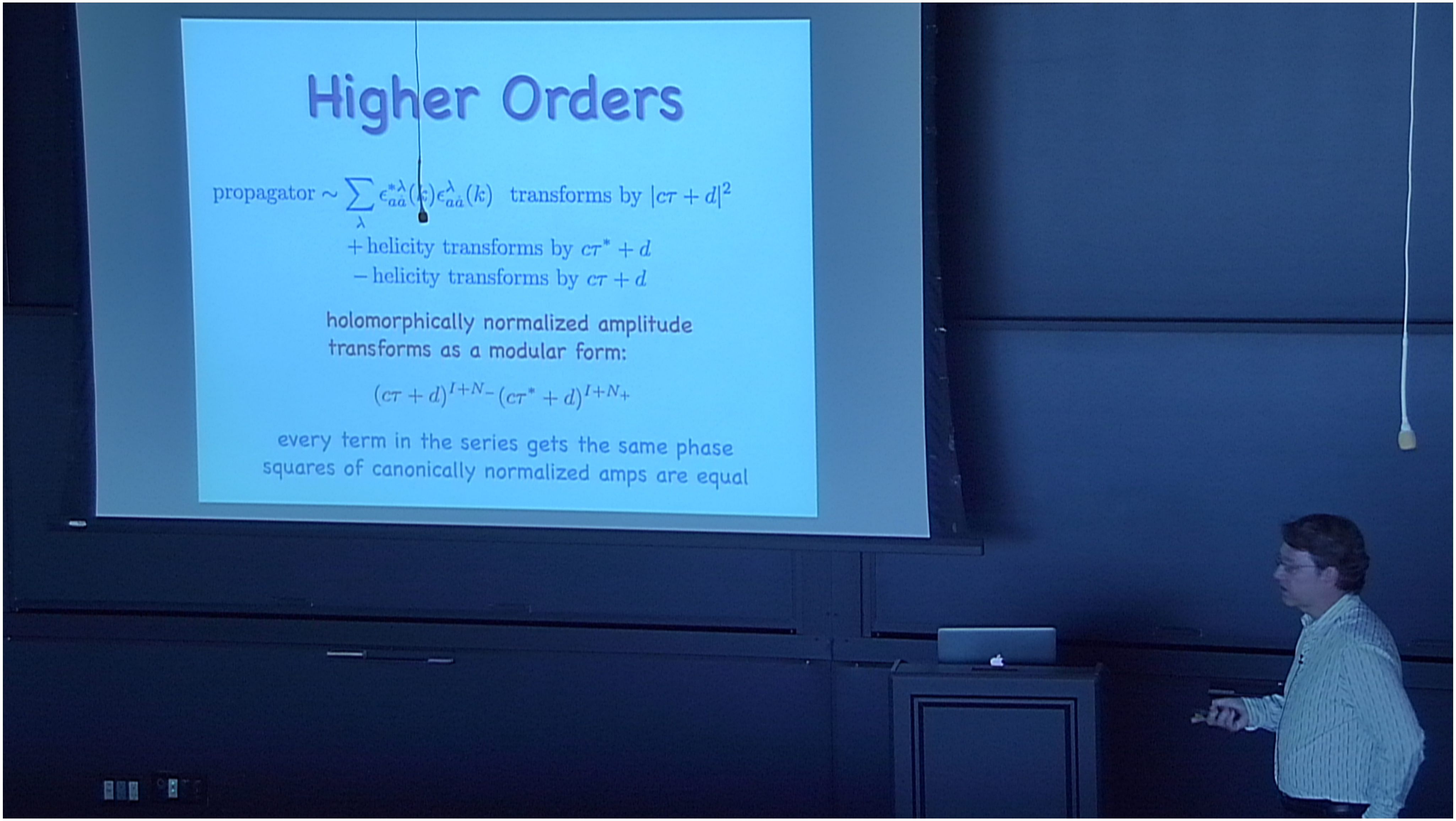

Higher Orders
propagator $\sim \sum_\lambda \epsilon^{*\lambda}_{a\dot{a}}(k)\epsilon^{\lambda}_{a\dot{a}}(k)$ transforms by $|c\tau + d|^2$
$+$ helicity transforms by $c\tau^* + d$
$-$ helicity transforms by $c\tau + d$
holomorphically normalized amplitude
transforms as a modular form:
$(c\tau + d)^{I+N_-}(c\tau^* + d)^{I+N_+}$
every term in the series gets the same phase
squares of canonically normalized amps are equal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propagator $\sim \sum_\lambda \epsilon^{*\lambda}_{a\dot{a}}(k)\epsilon^{\lambda}_{a\dot{a}}(k)$ transforms by $|c\tau + d|^2$
$+$ helicity transforms by $c\tau^* + d$
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holomorphically normalized amplitude
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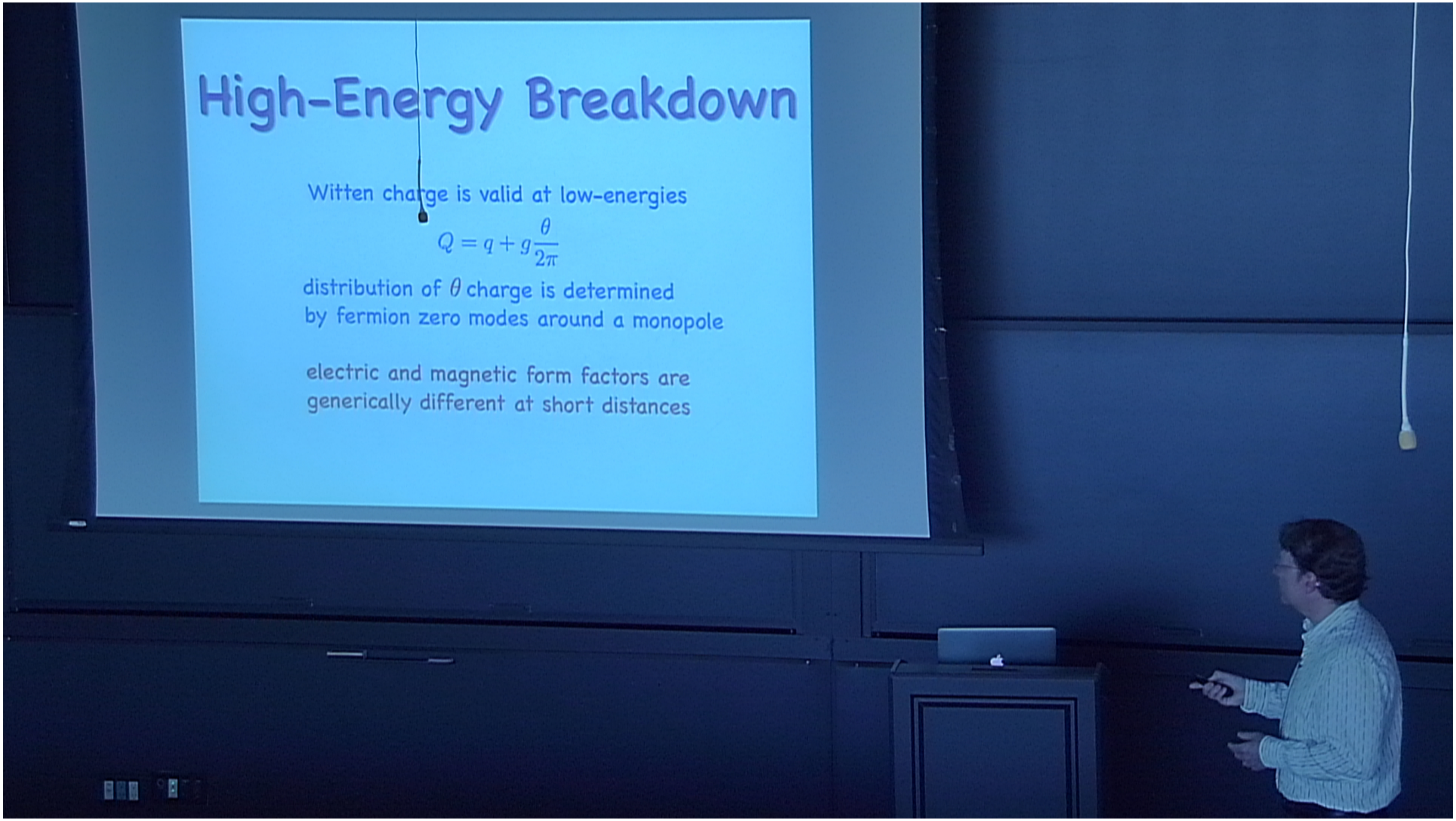
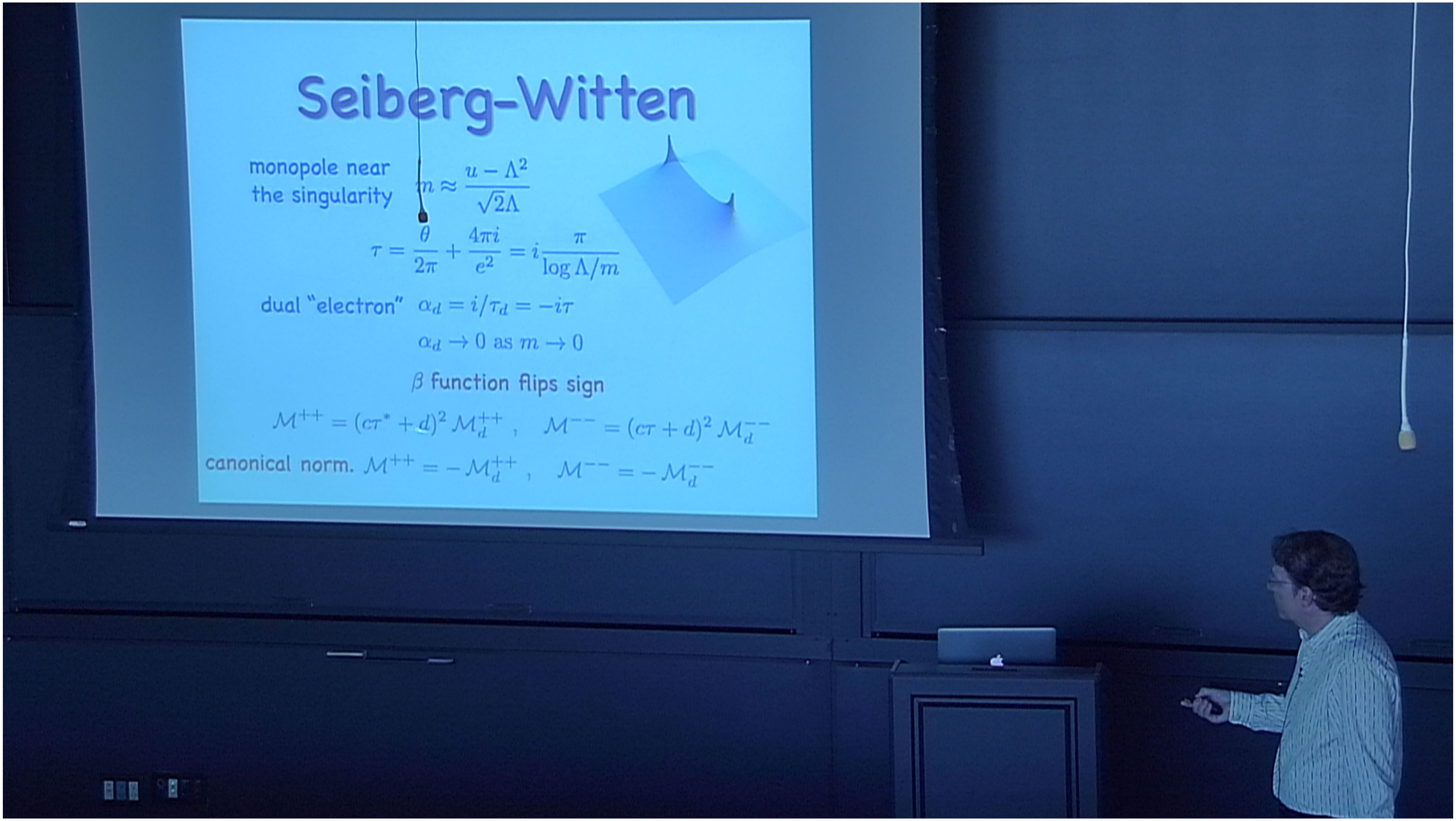

High-Energy Breakdown
Witten charge is valid at low-energies
$Q = q + g\frac{\theta}{2\pi}$
distribution of θ charge is determined
by fermion zero modes around a monopole
electric and magnetic form factors are
generically different at short distances

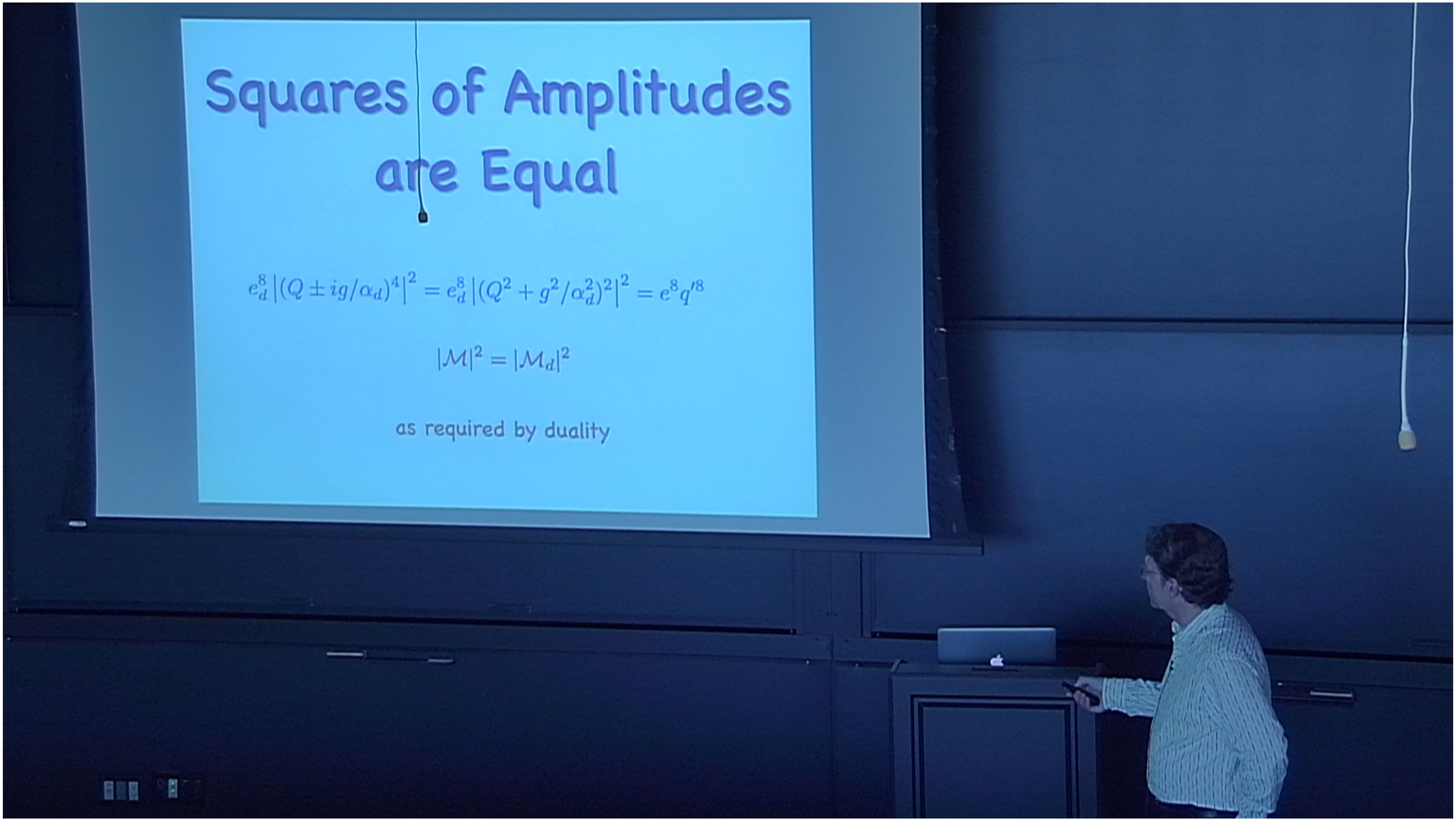

Seiberg-Witten
monopole near the singularity $m \approx \frac{u-\Lambda^2}{\sqrt{2}\Lambda}$
$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{e^2} = i\frac{\pi}{\log \Lambda/m}$
dual "electron" $\alpha_d = i/\tau_d = -i\tau$
$\alpha_d \to 0$ as $m \to 0$
β function flips sign
$\mathcal{M}^{++} = (c\tau^* + d)^2\, \mathcal{M}_d^{++}\,, \quad \mathcal{M}^{--} = (c\tau + d)^2\, \mathcal{M}_d^{--}$
canonical norm. $\mathcal{M}^{++} = -\mathcal{M}_d^{++}\,, \quad \mathcal{M}^{--} = -\mathcal{M}_d^{--}$

Conclusions

Anomalies for monopoles and dyons can be easily calculated

Low-energy effective U(1) theories enjoy an SL(2,Z) duality

SL(2,Z) duality does not interchange strong & weak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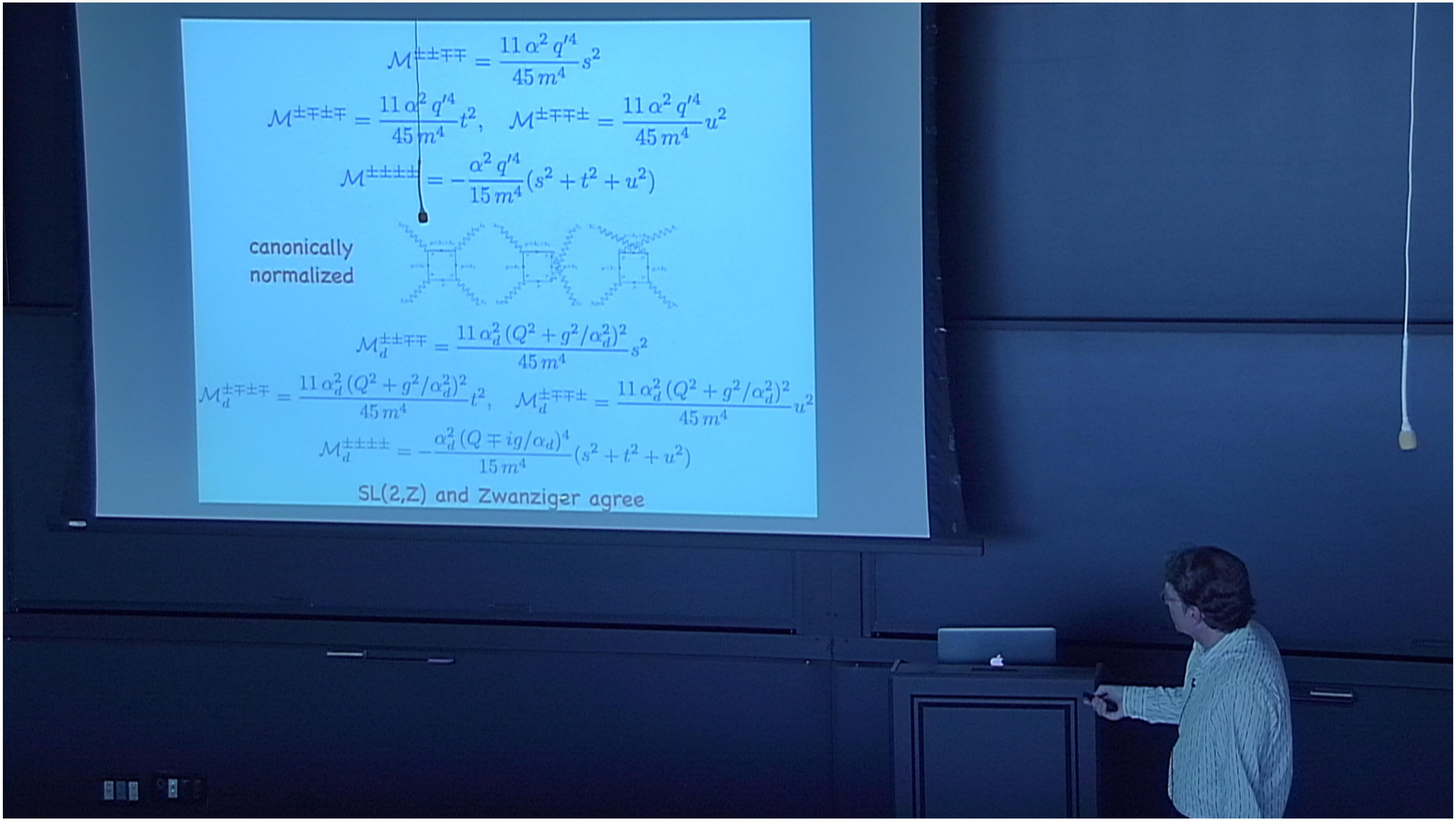

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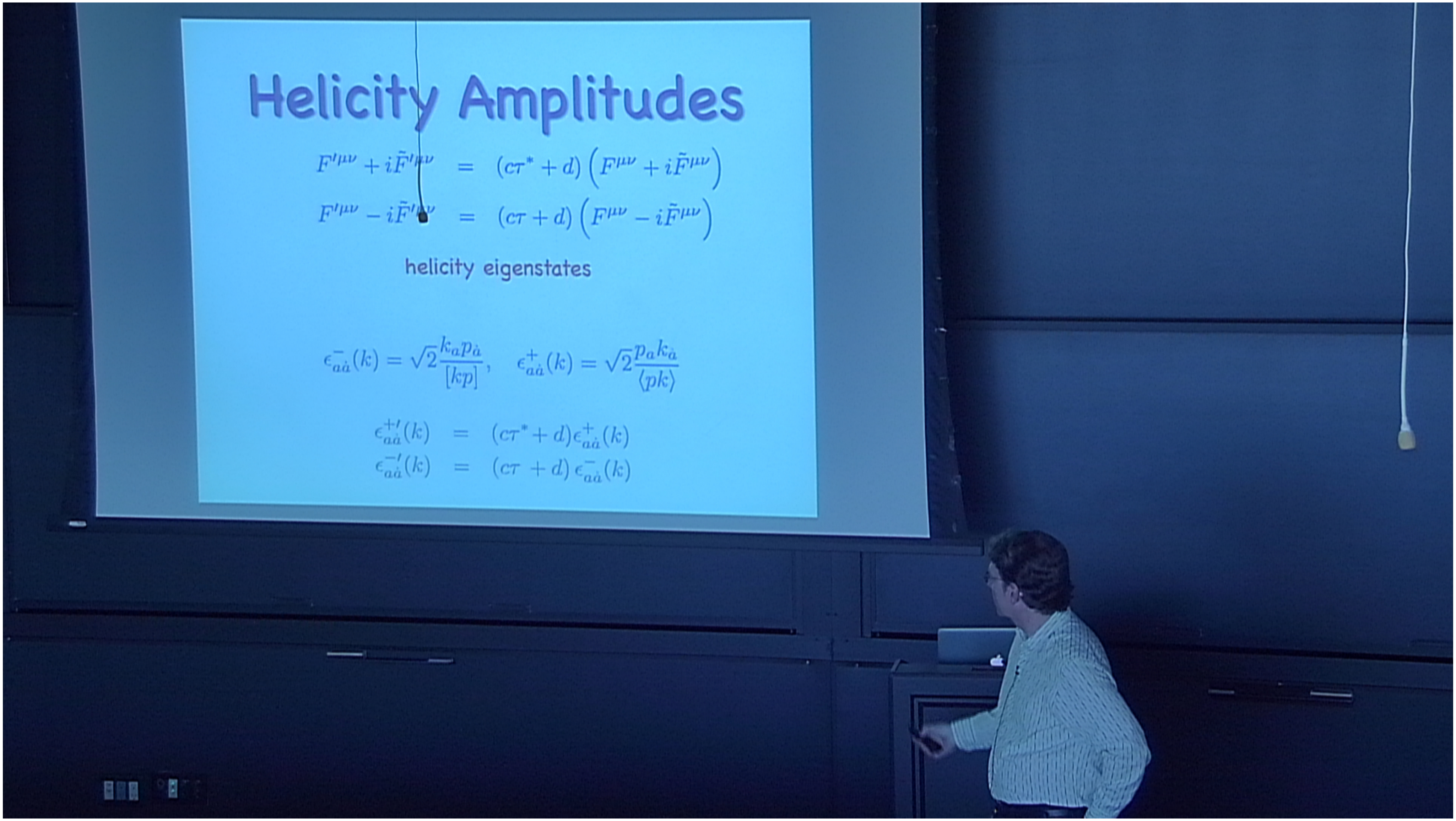
$\mathcal{M}^{\pm\pm\pm\pm} = -\frac{\alpha^2\,q'^4}{15\,m^4}(s^2+t^2+u^2)$
canonically normalized
$\mathcal{M}_d^{\pm\pm\mp\mp} = \frac{11\,\alpha_d^2\,(Q^2+g^2/\alpha_d^2)^2}{45\,m^4}s^2$
$\mathcal{M}_d^{\pm\mp\pm\mp} = \frac{11\,\alpha_d^2\,(Q^2+g^2/\alpha_d^2)^2}{45\,m^4}t^2, \quad \mathcal{M}_d^{\pm\mp\mp\pm} = \frac{11\,\alpha_d^2\,(Q^2+g^2/\alpha_d^2)^2}{45\,m^4}u^2$
$\mathcal{M}_d^{\pm\pm\pm\pm} = -\frac{\alpha_d^2\,(Q\mp ig/\alpha_d)^4}{15\,m^4}(s^2+t^2+u^2)$
SL(2,Z) and Zwanziger agree


Helicity Amplitudes
$F'^{\mu\nu} + i\tilde{F}'^{\mu\nu} = (c\tau^* + d)\Big(F^{\mu\nu} + i\tilde{F}^{\mu\nu}\Big)$
$F'^{\mu\nu} - i\tilde{F}'^{\mu\nu} = (c\tau + d)\Big(F^{\mu\nu} - i\tilde{F}^{\mu\nu}\Big)$
helicity eigenstates
$\epsilon^-_{a\dot{a}}(k) = \sqrt{2}\frac{k_a p_{\dot{a}}}{[kp]}, \quad \epsilon^+_{a\dot{a}}(k) = \sqrt{2}\frac{p_a k_{\dot{a}}}{\langle pk\rangle}$
$\epsilon^{+\prime}_{a\dot{a}}(k) = (c\tau^* + d)\epsilon^+_{a\dot{a}}(k)$
$\epsilon^{-\prime}_{a\dot{a}}(k) = (c\tau + d)\,\epsilon^-_{a\dot{a}}(k)$