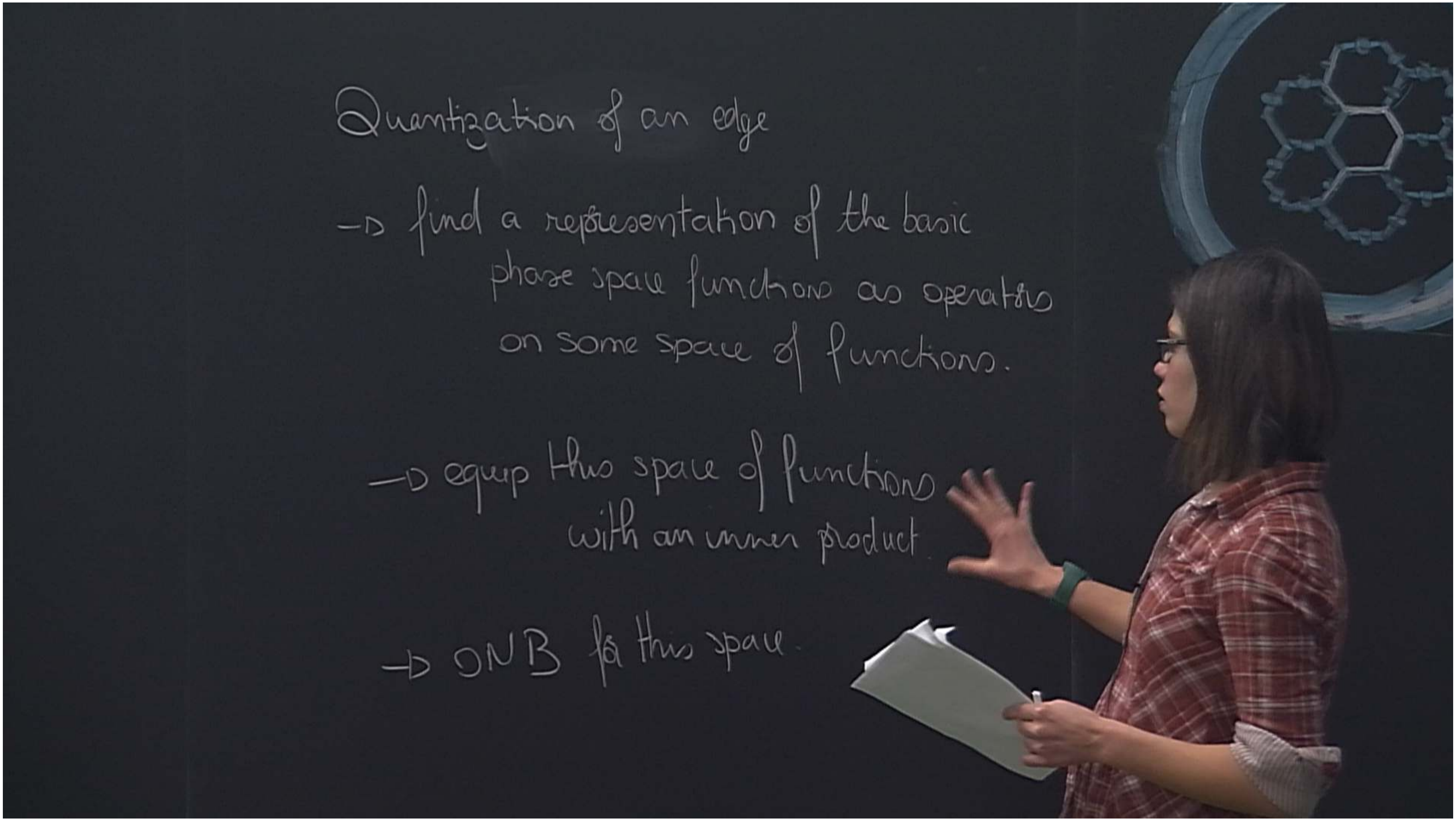
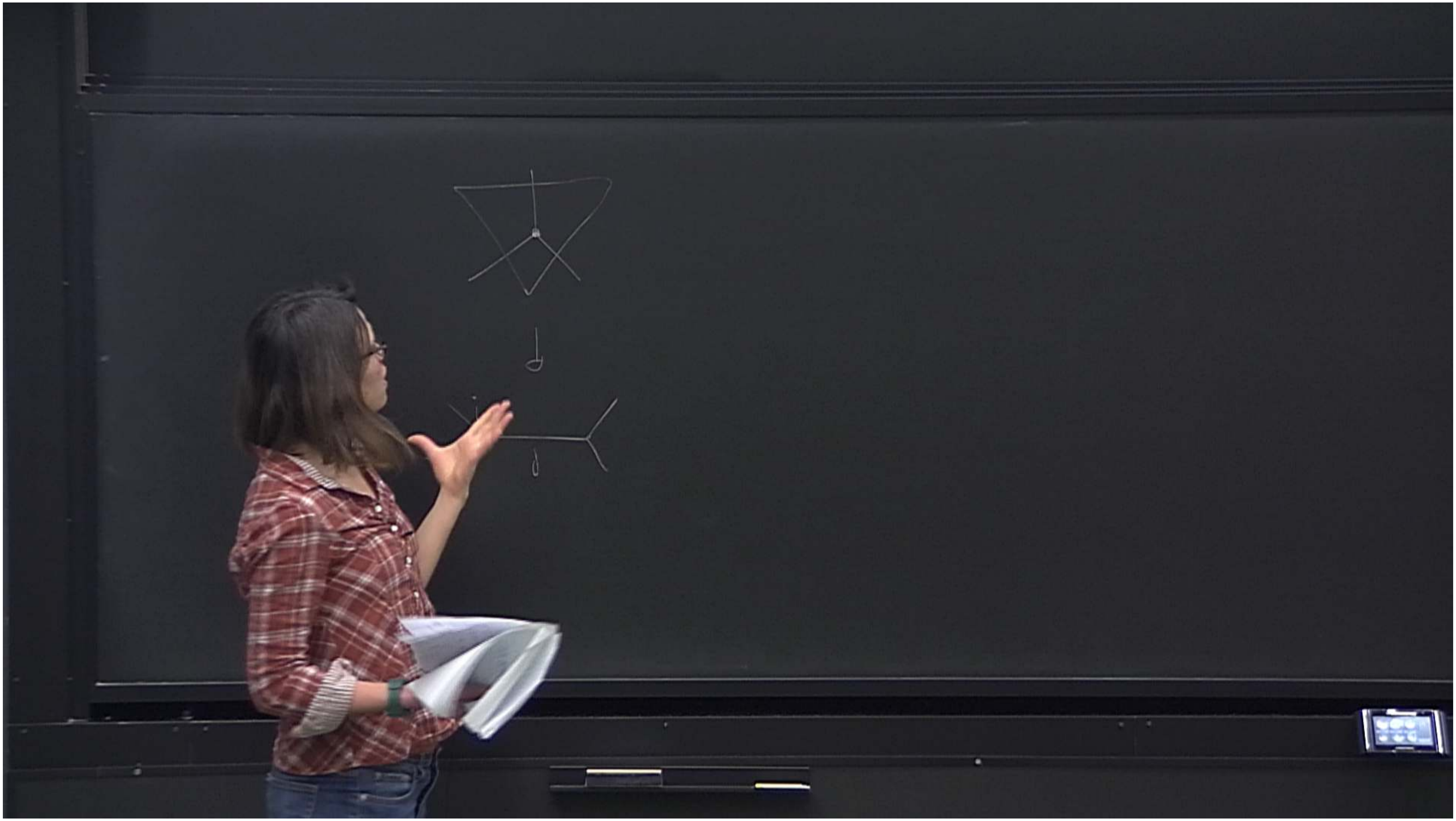
Title: PSI 2015/2016 Quantum Gravity - Lecture 10

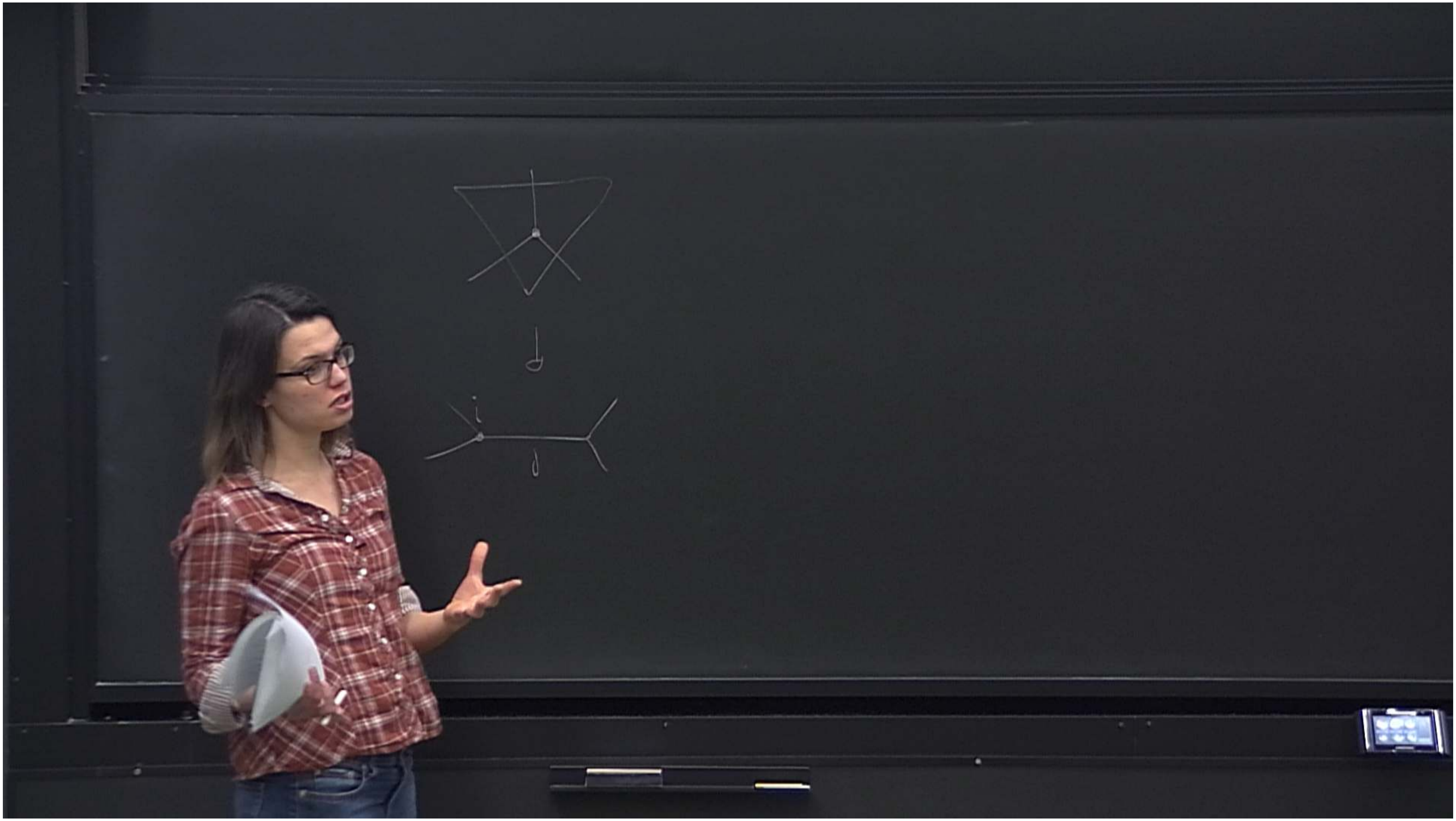
Date: Mar 04, 2016 10:15 AM

URL: http://pirsa.org/16030048

Abstract:


Quantization of an edge
→ find a representation of the basic phase space functions as operators on some space of functions.
→ equip this space of functions with an inner product
→ ONB for this space.





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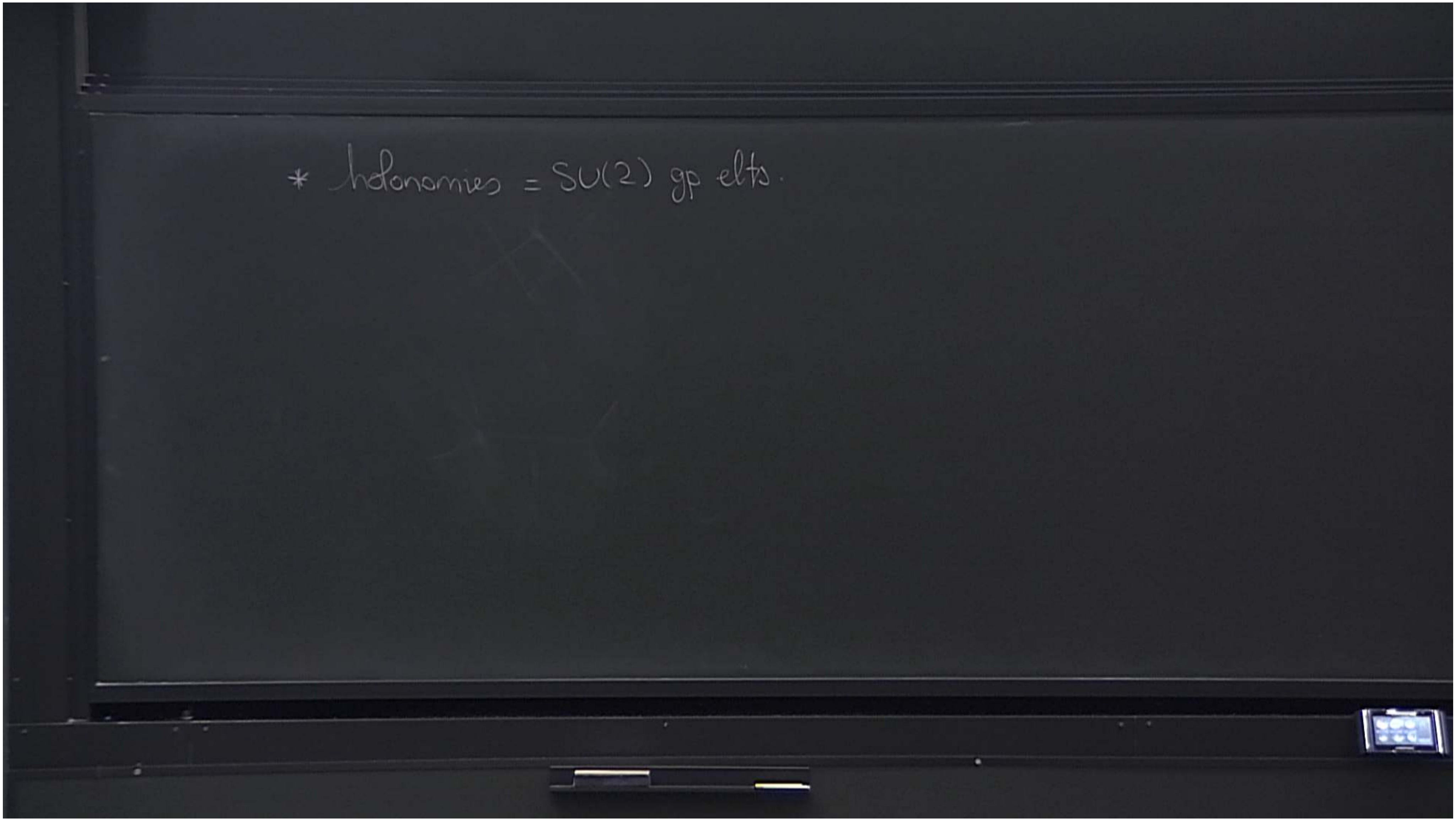

* holonomies = SU(2) gp elts.

$$\{g, g\} = 0$$

$$\{E^i, E^j\} = \epsilon^{ij}{}_k E^k$$

$$\{E^i, g\} = g T^i$$

* holonomies = SU(2) gp elts.

$f : G \longrightarrow \mathbb{C}$

* holonomies = SU(2) gp elts.

$$f : G \to \mathbb{C}$$

→ Quantized as multiplication op.

$$\Psi_e : SU(2) \to \mathbb{C}$$

$$\hat{f}\,\Psi_e(g_e) = f(g_e)\,\Psi_e(g_e)$$

The state $|\Psi_e\rangle$ in the holonomy rep.

$\{g,g\}=0$

$\{E^i, E^j\}=$

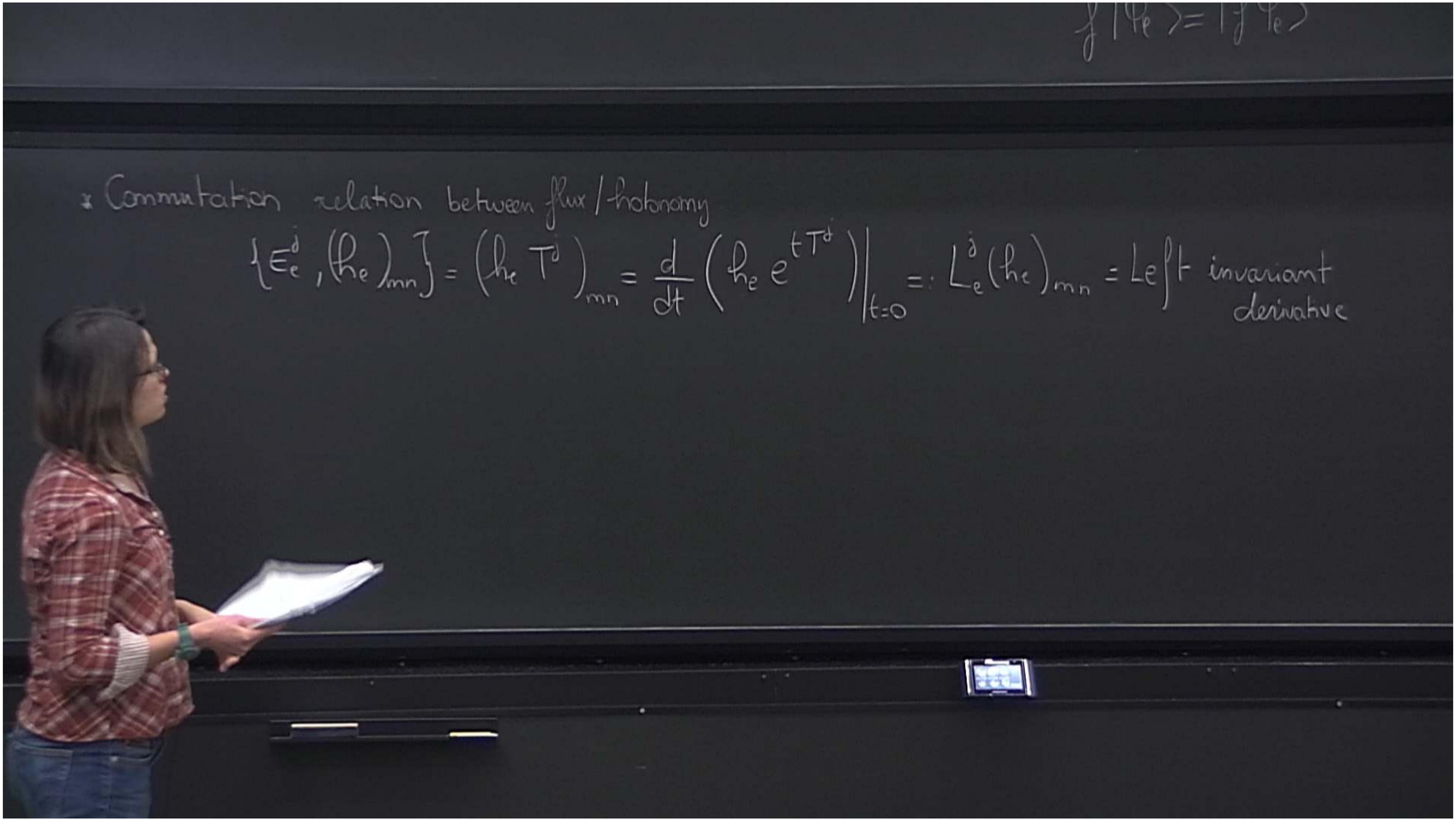
$\{E^i, g\}=$

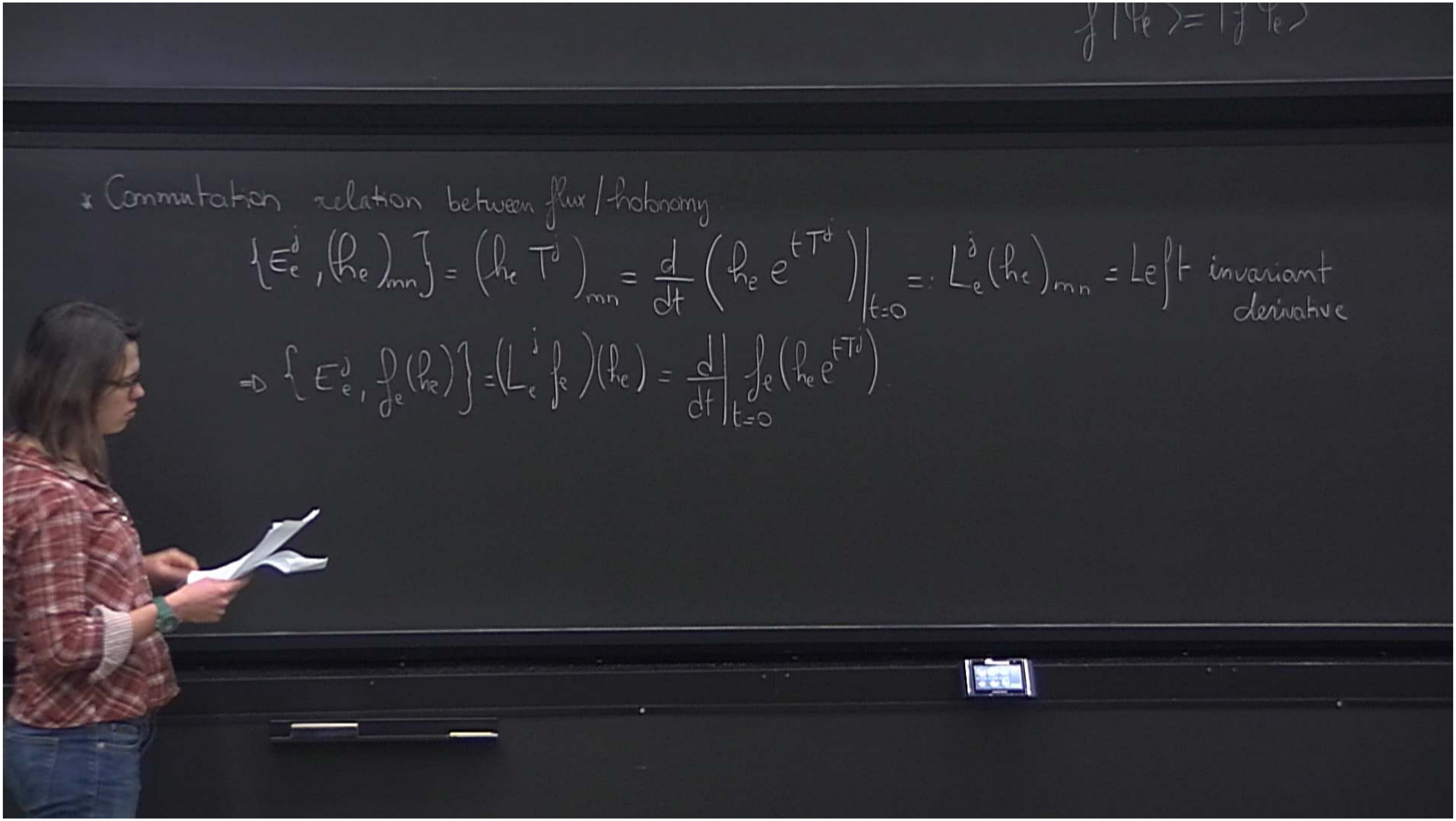
$\Psi_e(g_e)$.

The state $|\Psi_e\rangle$ in the holonomy rep. $\langle g|\Psi_e\rangle = \Psi_e(g)$

we define the action of a function of the holonomy

$$\hat{f}\,|\Psi_e\rangle = |f\,\Psi_e\rangle$$


$f|\psi_e\rangle = |f\psi_e\rangle$
* Commutation relation between flux / holonomy
$\{E_e^j, (h_e)_{mn}\} = (h_e T^j)_{mn} = \frac{d}{dt}\left(h_e e^{tT^j}\right)\Big|_{t=0} =: L_e^j(h_e)_{mn}$ = Left invariant derivative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$\Rightarrow \{E_e^j, f_e(h_e)\} = (L_e^j f_e)(h_e) = \frac{d}{dt}\Big|_{t=0} f_e(h_e e^{tT^j})$

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- Using that Poisson brackets should be mapped as commutators $[\hat{f}, \hat{g}] = i\hbar \,\widehat{\{f, g\}}$

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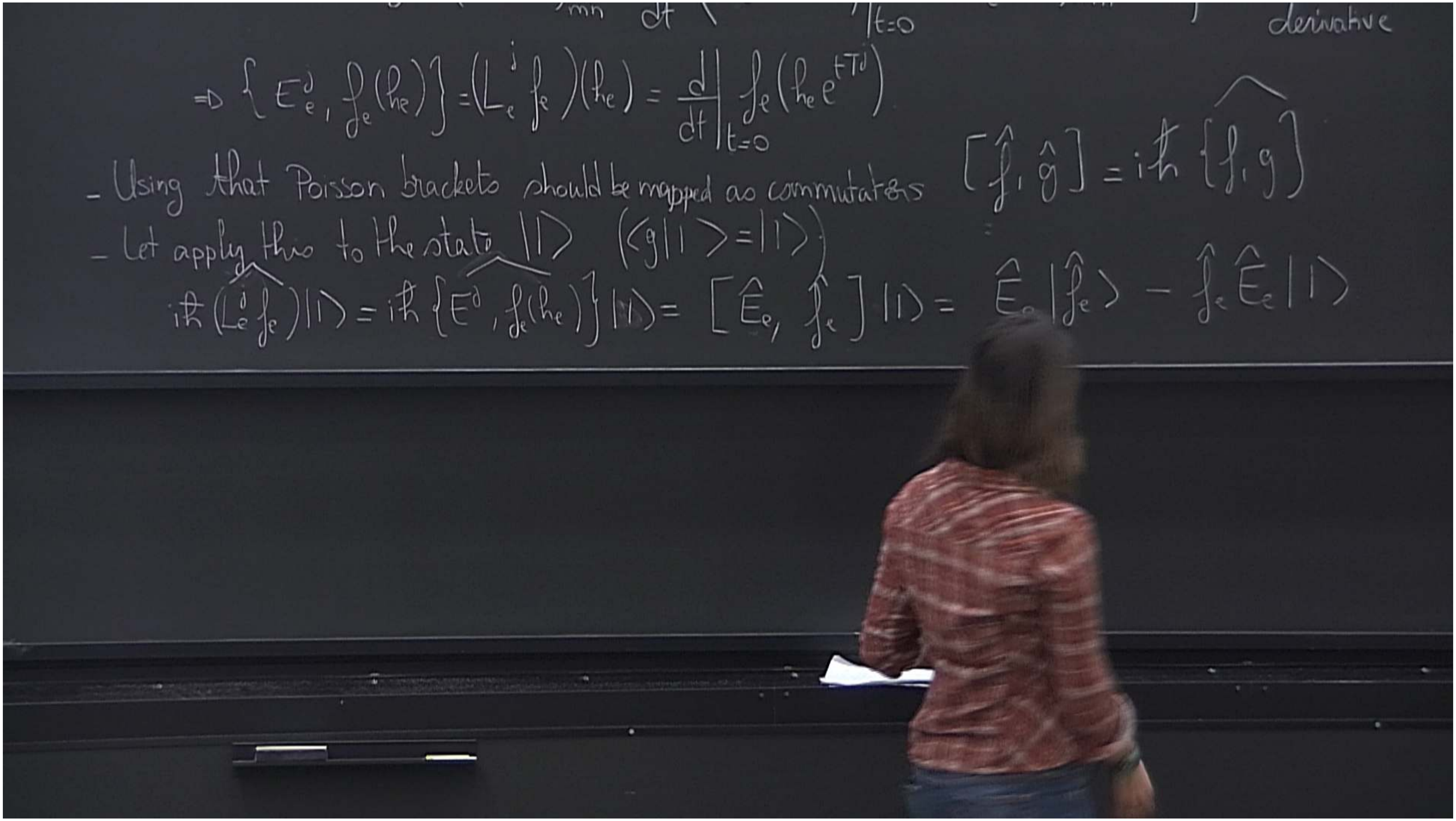
derivative
$\Rightarrow \{E^j_e, f_e(h_e)\} = (L^j_e f_e)(h_e) = \frac{d}{dt}\Big|_{t=0} f_e(h_e e^{tT^j})$
- Using that Poisson brackets should be mapped as commutators $[\hat{f}, \hat{g}] = i\hbar \widehat{\{f, g\}}$
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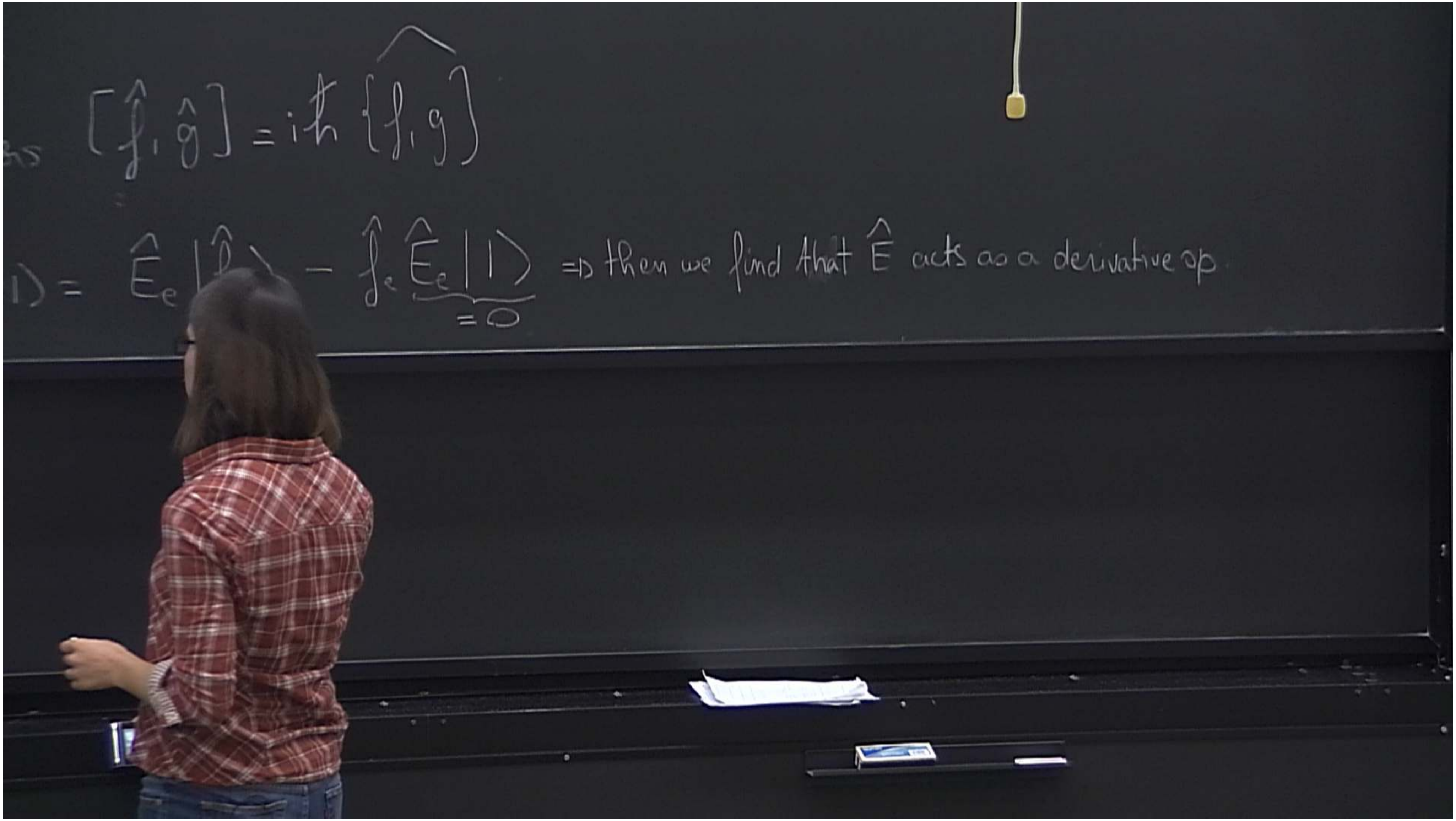
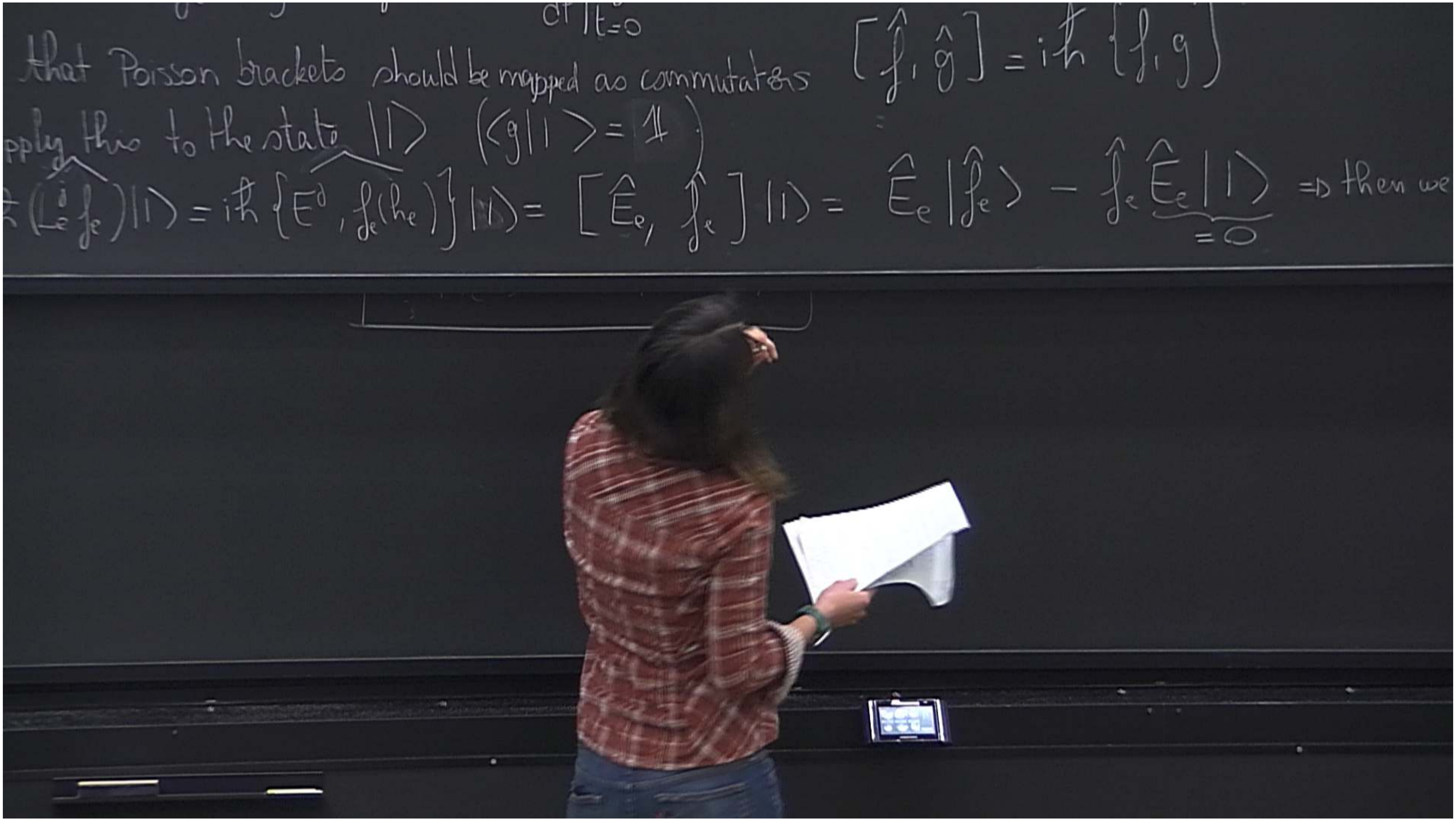
$i\hbar (\widehat{L^j_e f_e})|1\rangle = i\hbar \widehat{\{E^j, f_e(h_e)\}}|1\rangle = [\hat{E}_e, \hat{f}_e]|1\rangle = \hat{E}_e|\hat{f}_e\rangle - \hat{f}_e\hat{E}_e|1\rangle$


[f̂, ĝ] = iħ {f, g}^
|1⟩ = Ê_e |f̂ ... − f̂_e Ê_e |1⟩ = 0
⇒ then we find that Ê acts as a derivative op.

$$\hbar(\hat{L}^j_e f_e)|1\rangle = i\hbar\{E^j, f_e(h_e)\}|1\rangle = [\hat{E}_e, \hat{f}_e]|1\rangle = \hat{E}_e|f_e\rangle - f_e \underbrace{\hat{E}_e|1\rangle}_{=0} \Rightarrow \text{then w}$$

$\Rightarrow$ Action of the fluxes $\hat{E}^j$ on a state $|\psi_e\rangle = \hat{\psi}_e|1\rangle$

$$\boxed{\hat{E}^j|\psi_e\rangle = i\hbar\,|L^j_e\psi_e\rangle}.$$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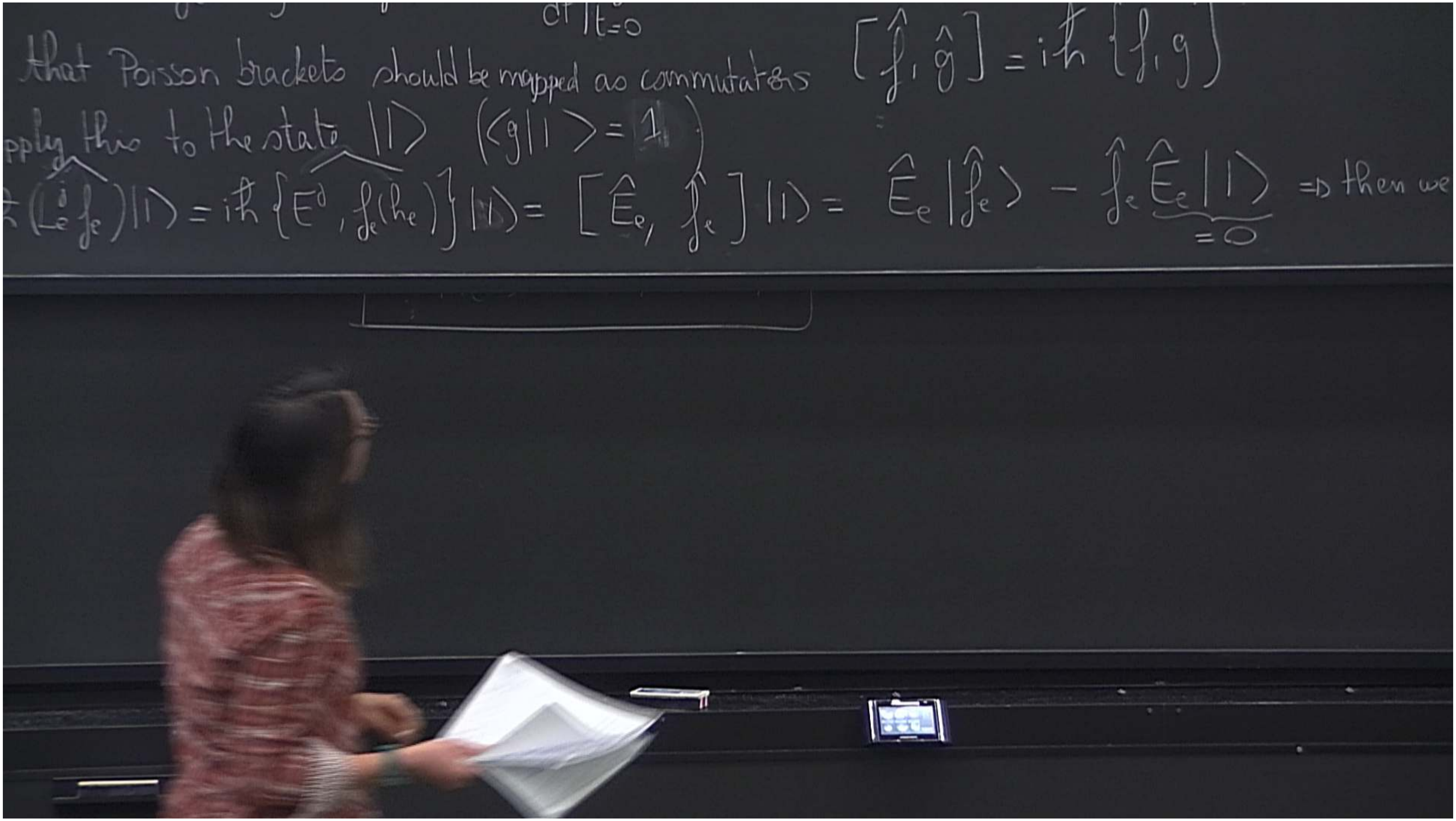
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= iħ {E^j, f_e(h_e)} |1⟩ = [Ê_e, f̂_e] |1⟩ = Ê_e |f̂_e⟩ − f̂_e Ê_e |1⟩
= 0
⇒ then we

$$i\hbar (L_e^j f_e)|1\rangle = i\hbar \{E^j, f_e(h_e)\}|1\rangle = [\hat{E}_e, f_e]|1\rangle = E_e|f_e\rangle - f_e \underbrace{E_e|1\rangle}_{=0}$$

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$$\boxed{\hat{E}^j|\Psi_e\rangle = i\hbar\, |L_e^j \Psi_e\rangle .}$$

Consistency of the commutation relation between the fluxes?

$$[\hat{E}^j, \hat{E}^k]|\Psi_e\rangle = (\hat{E}^j\hat{E}^k - \hat{E}^k\hat{E}^j)|\Psi_e\rangle = (i\hbar)^2 |(L^j \circ L^k - L^k \circ L^j)\Psi_e\rangle .$$

$$\overset{?}{=}$$

$i\hbar\,(L_e^j f_e)|1\rangle = i\hbar\,\{E^j, f_e(h_e)\}|1\rangle = [\hat{E}_e, f_e]\,|1\rangle = E_e|f_e\rangle - f_e\underbrace{E_e|1\rangle}_{=0}$

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$$\overset{?}{=} i\hbar\,\widehat{\{E_e^j\, E_e^k\}}|\Psi_e\rangle$$

$\rightarrow$ on a matrix el^t $(h_e)_{mn}$

$$\left(L^j_e \circ L^k_e - L^k_e \circ L^j_e\right)(h_e)_{mn} = \frac{d}{ds}\Big|_{s=0} \frac{d}{dt}\Big|_{t=0} \left(\left(h_e\, e^{sT^j} e^{tT^k}\right)_{mn} - \left(h_e\, e^{tT^k} e^{sT^j}\right)_{mn} \right)$$

→ on a matrix el^t $(h_e)_{mn}$

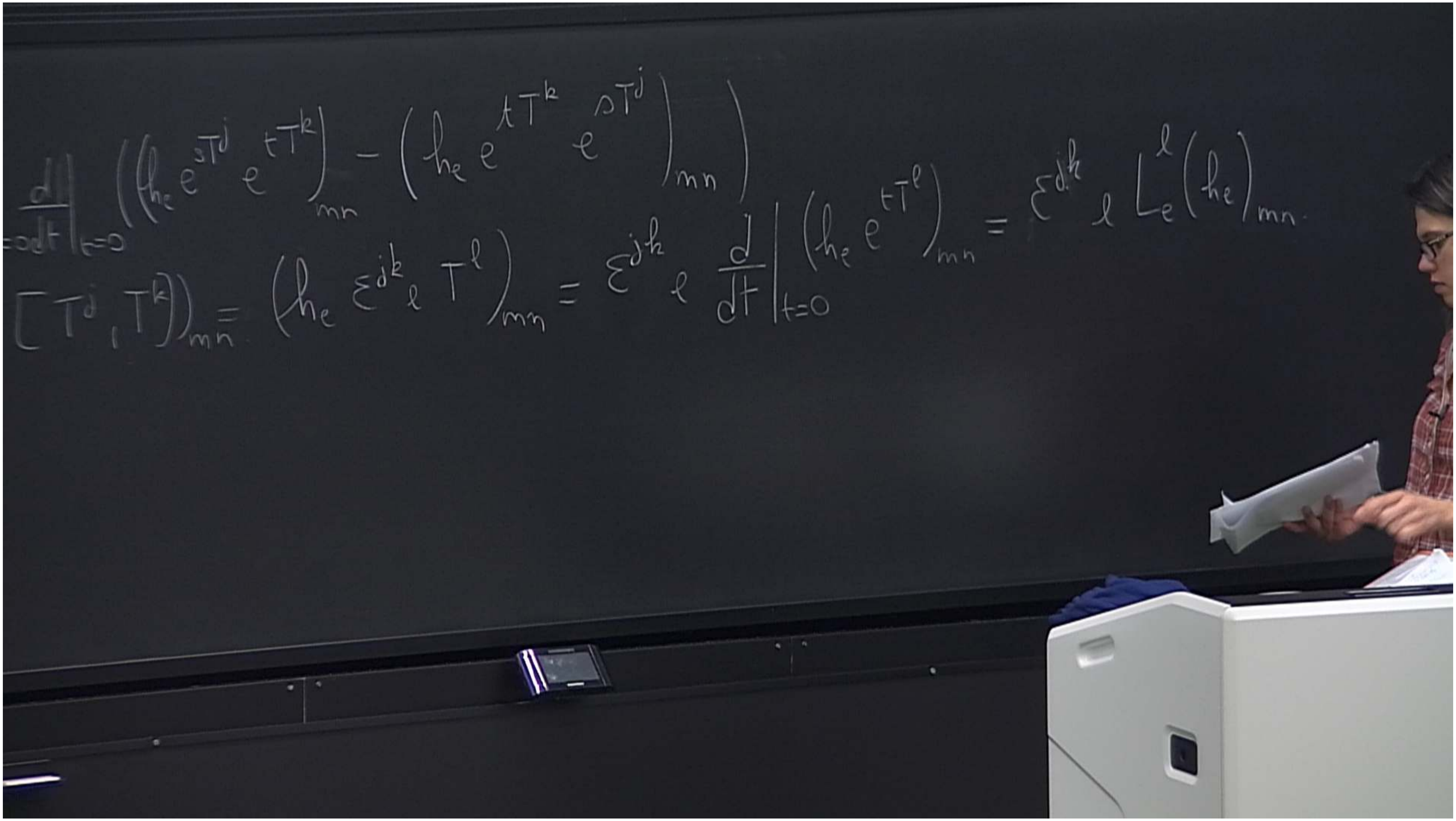
$$\left(L^j \circ L^k - L^k_e \circ L^j_e\right)(h_e)_{mn} = \frac{d}{ds}\Big|_{s=0}\frac{d}{dt}\Big|_{t=0}\left(\left(h_e\, e^{sT^j} e^{tT^k}\right)_{mn} - \left(h_e\, e^{tT^k} e^{sT^j}\right)_{mn}\right)$$

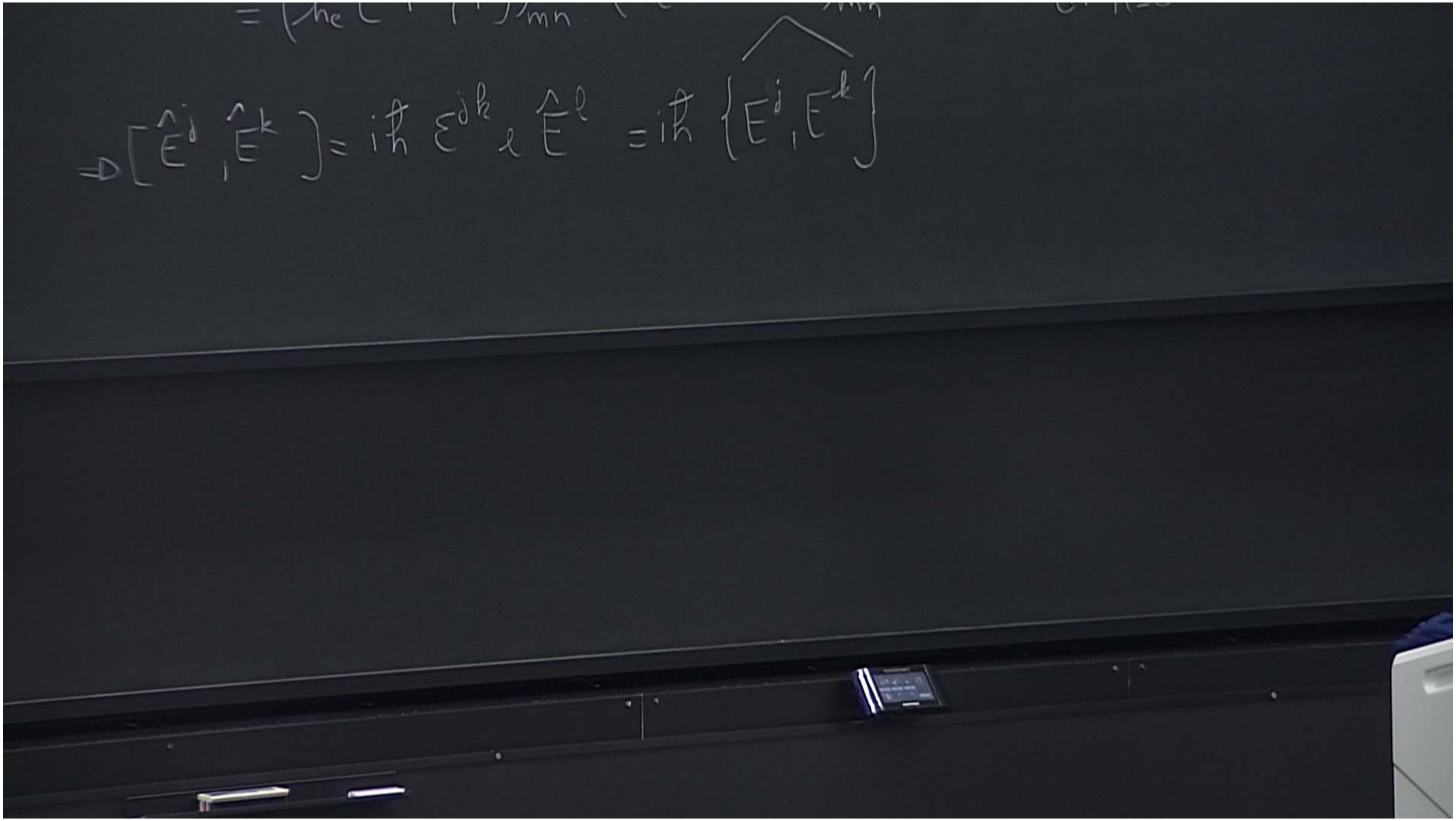
$$= \left(h_e\,[T^j, T^k]\right)_{mn} = \left(h_e\, \varepsilon^{jk}{}_l\, T^l\right)_{mn} = \varepsilon^{jk}{}_l \frac{d}{dt}\Big|$$

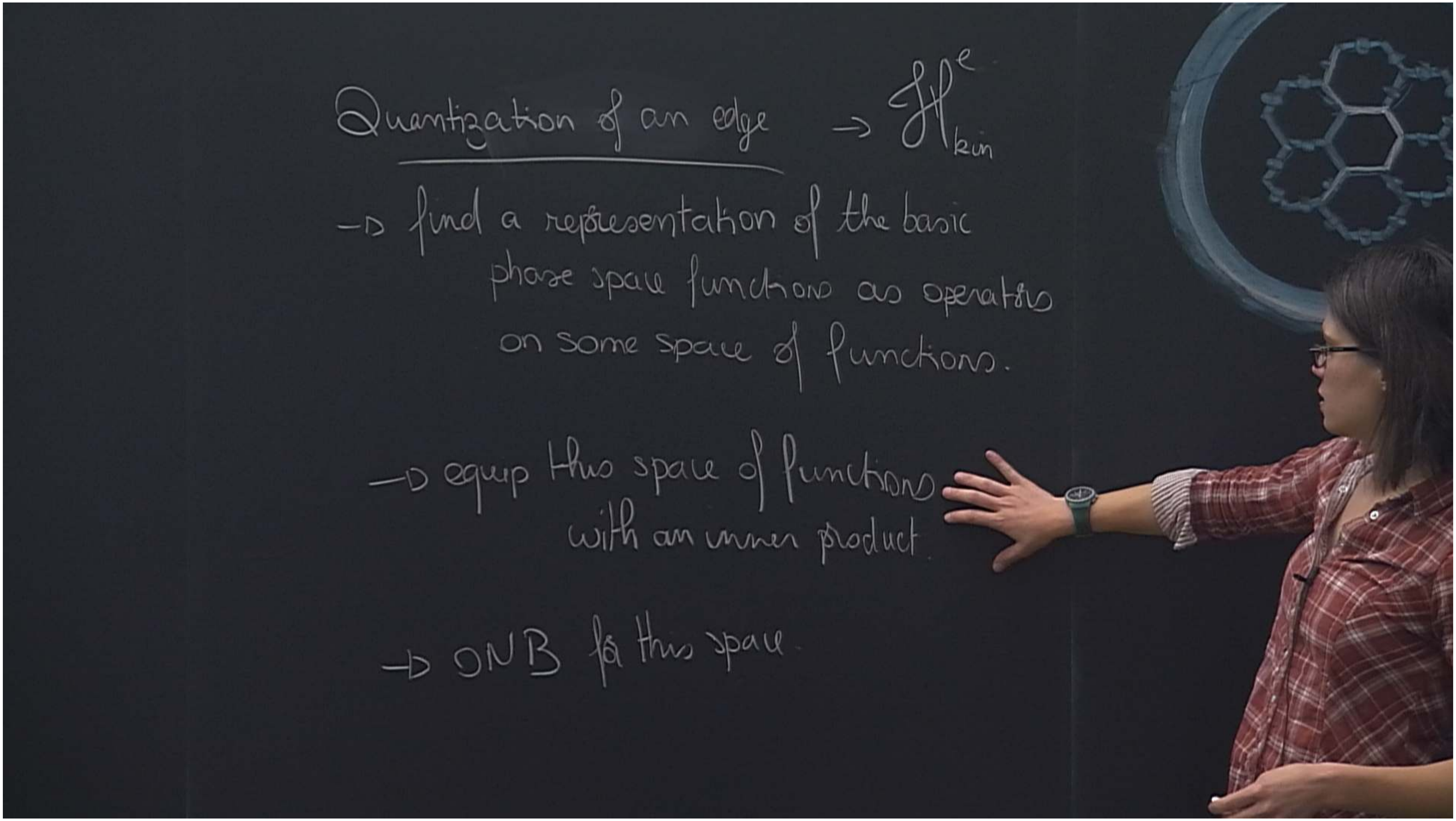
$(h_e)_{mn}$

$$(h_e)_{mn} = \frac{d}{ds}\Big|_{s=0} \frac{d}{dt}\Big|_{t=0} \left(\left(h_e e^{sT^j} e^{tT^k}\right)_{mn} - \left(h_e e^{tT^k} e^{sT^j}\right)_{mn} \right) = \varepsilon^{jk}{}_{\ell} L^{\ell}_{e}($$

$$= \left(h_e [T^j, T^k]\right)_{mn} = \left(h_e \varepsilon^{jk}{}_{\ell} T^{\ell}\right)_{mn} = \varepsilon^{jk}{}_{\ell} \frac{d}{dt}\Big|_{t=0} \left(h_e e^{tT^{\ell}}\right)_{mn} =$$


$\frac{d}{dt}\Big|_{t=0}\left(\left(h_e\, e^{sT^j} e^{tT^k}\right)_{mn} - \left(h_e\, e^{tT^k} e^{sT^j}\right)_{mn}\right)$
$\left(h_e\, e^{tT^l}\right)_{mn} = \varepsilon^{jk}{}_l\, L^l_e(h_e)_{mn}$
$[T^j, T^k])_{mn} = \left(h_e\, \varepsilon^{jk}{}_l\, T^l\right)_{mn} = \varepsilon^{jk}{}_l\, \frac{d}{dt}\Big|_{t=0}$


$\Rightarrow [\hat{E}^j, \hat{E}^k] = i\hbar\, \varepsilon^{jk}{}_l\, \hat{E}^l = i\hbar\, \widehat{\{E^j, E^k\}}$


Quantization of an edge → $\mathcal{H}^e_{kin}$
→ find a representation of the basic phase space functions as operators on some space of functions.
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Fluxes defined as left invariant derivative $\to$ right translation

$$h : f(g) \mapsto f(gh).$$

Fluxes defined as left invariant derivative $\rightarrow$ right translation

$h: f(g) \longmapsto f(gh).$

Left and right action on a group for $h \in G$

Fluxes defined as left invariant derivative $\longrightarrow$ right translation

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Left and right action on a group for $h \in G$.

$$L_h(f)(g) = f(h^{-1} g)$$

$$R_h(f)(g) = f(gh).$$

Fluxes defined as left invariant derivative $\rightarrow$ right translation

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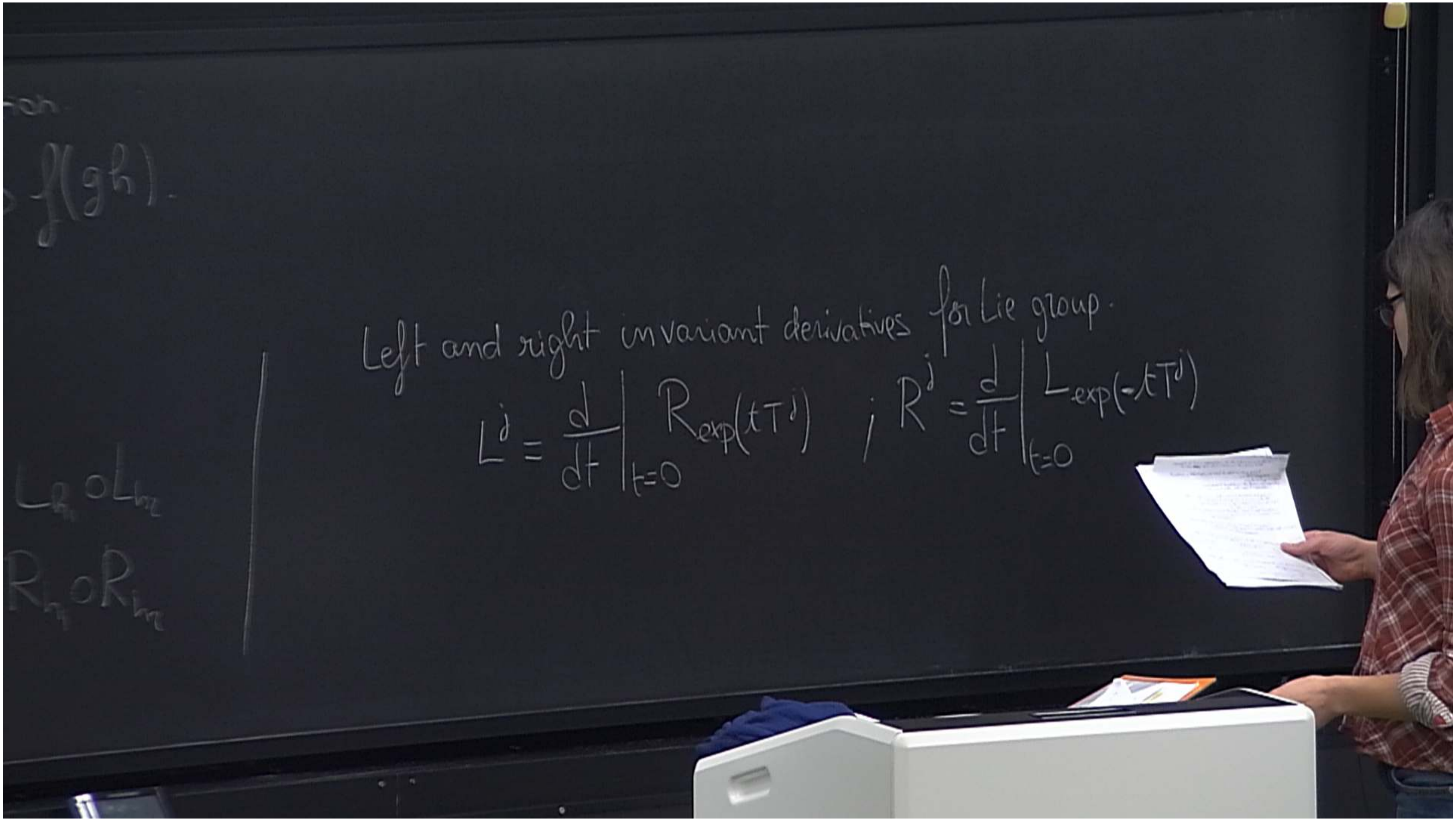
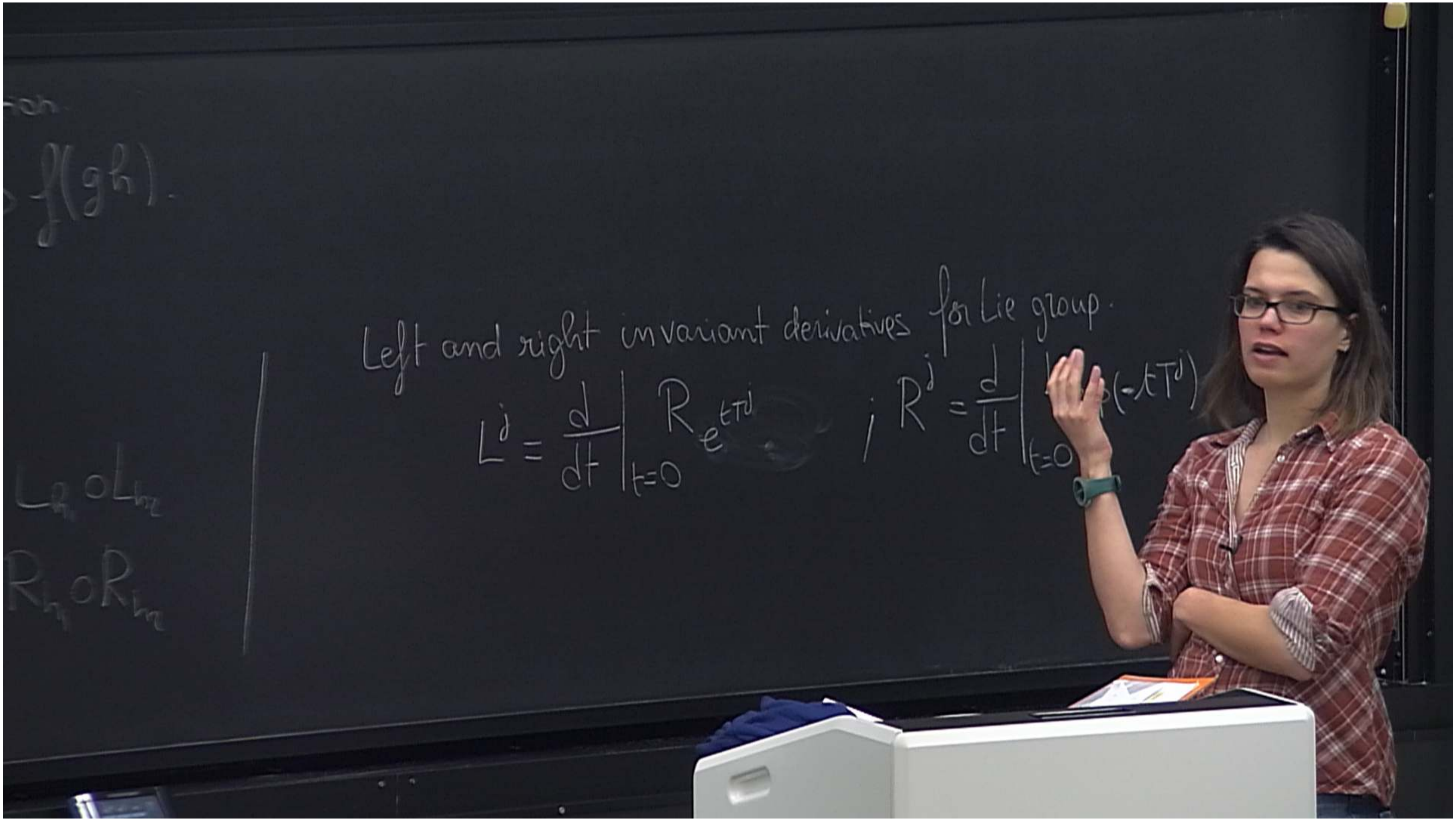
Left and right action on a group for $h \in G$

$$\left.\begin{aligned} L_h(f)(g) &= f(h^{-1} g) \\ R_h(f)(g) &= f(gh). \end{aligned}\right] \longrightarrow \begin{aligned} L_{h_1 h_2} &= L_{h_1} \circ L_{h_2} \\ R_{h_1 h_2} &= R_{h_1} \circ R_{h_2} \end{aligned}$$


f(gh).
L_{h_1} ∘ L_{h_2}
R_{h_1} ∘ R_{h_2}
Left and right invariant derivatives for Lie group.
$L^j = \frac{d}{dt}\Big|_{t=0} R_{\exp(tT^j)}$; $R^j = \frac{d}{dt}\Big|_{t=0} L_{\exp(-tT^j)}$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L^j = d/dt |t=0 R e^tT^j
; R^j = d/dt |t=0
(-tT^j)
L_h1 o L_h2
R_h1 o R_h2

$R_h(f)(g) = f(g$

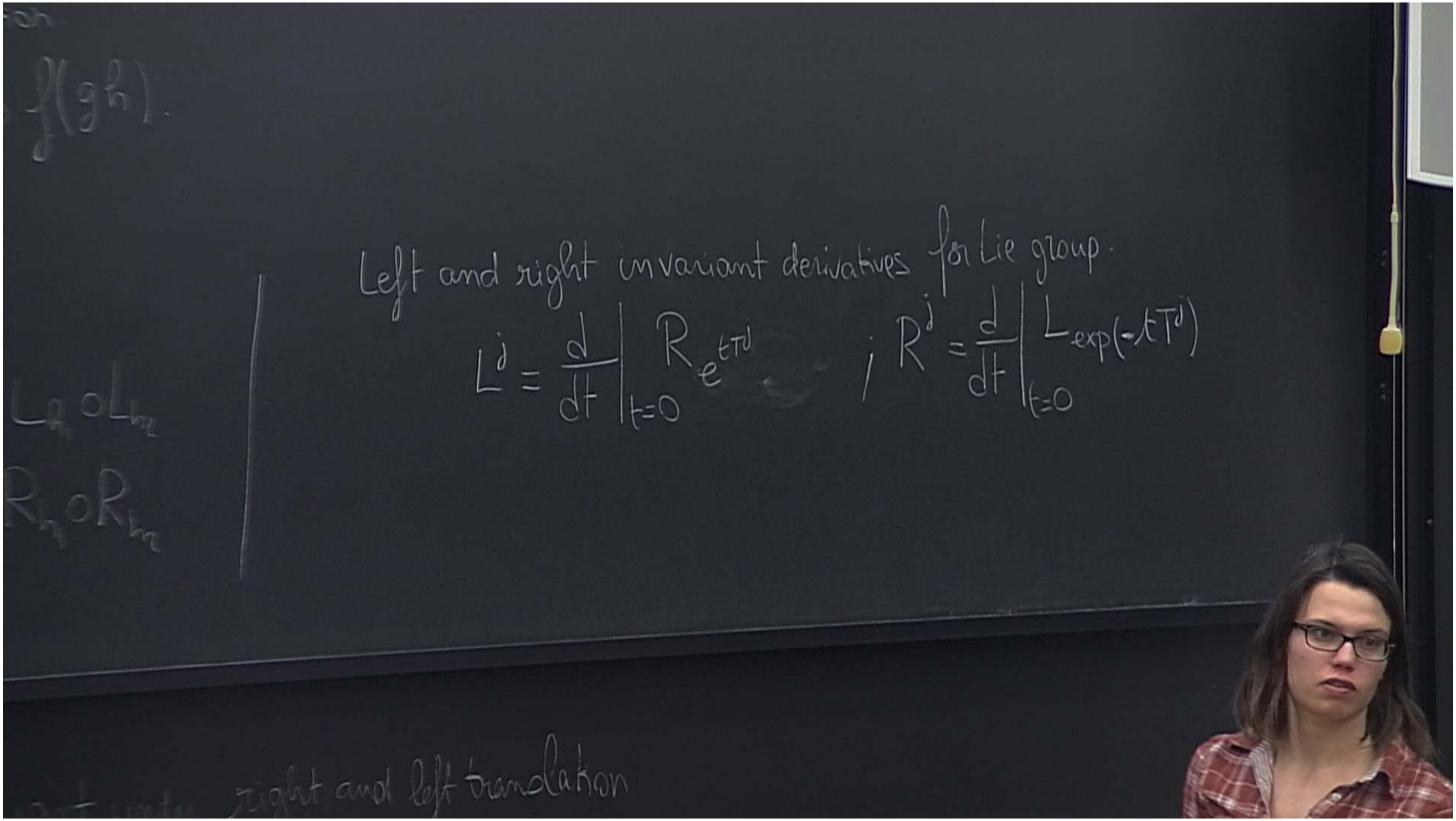
Fluxe → self adjoint op. / exponentiated fluxes (Right translation): unitary op.

⇒ inner product invariant under right translations.

Can do better for SU(2):

we can find an inner product invariant under right and left translation

⇒ Haar measure.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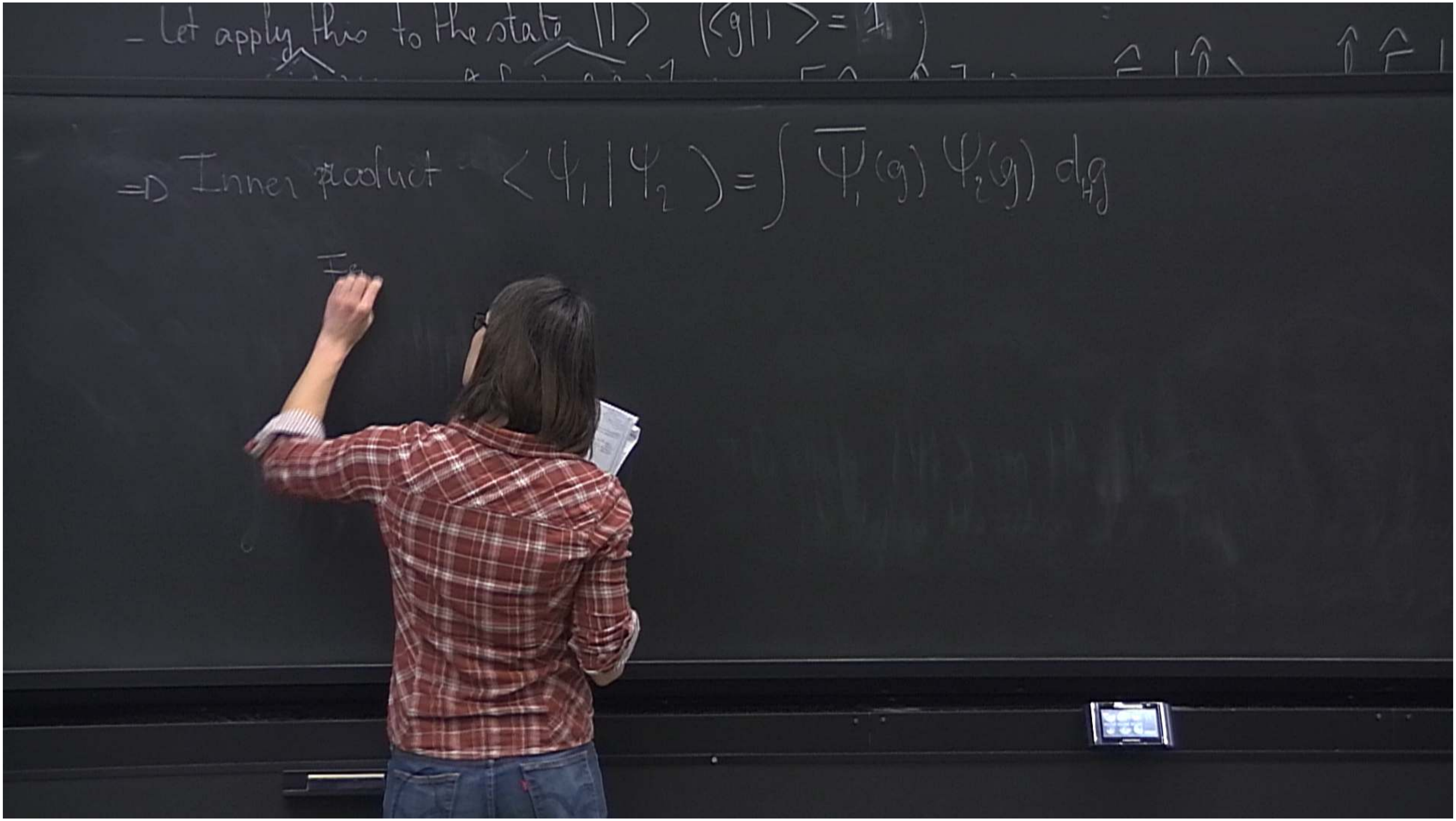
$L_{h_1} \circ L_{h_2}$
$R_{h_1} \circ R_{h_2}$
right and left translation

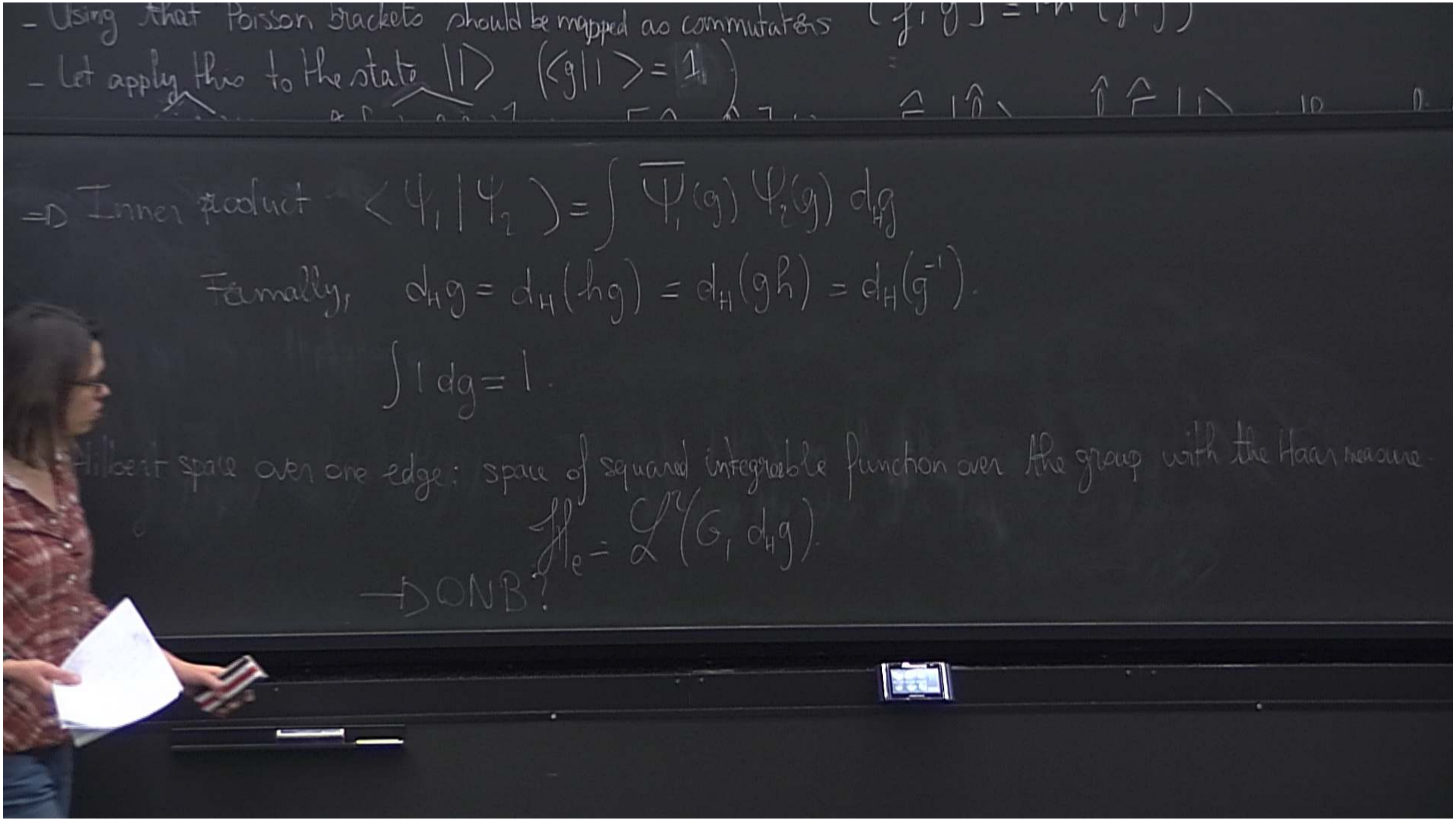

- Let apply this to the state |1> (<g|1> = 1)
⇒ Inner product <Ψ1|Ψ2> = ∫ Ψ̄1(g) Ψ2(g) dHg

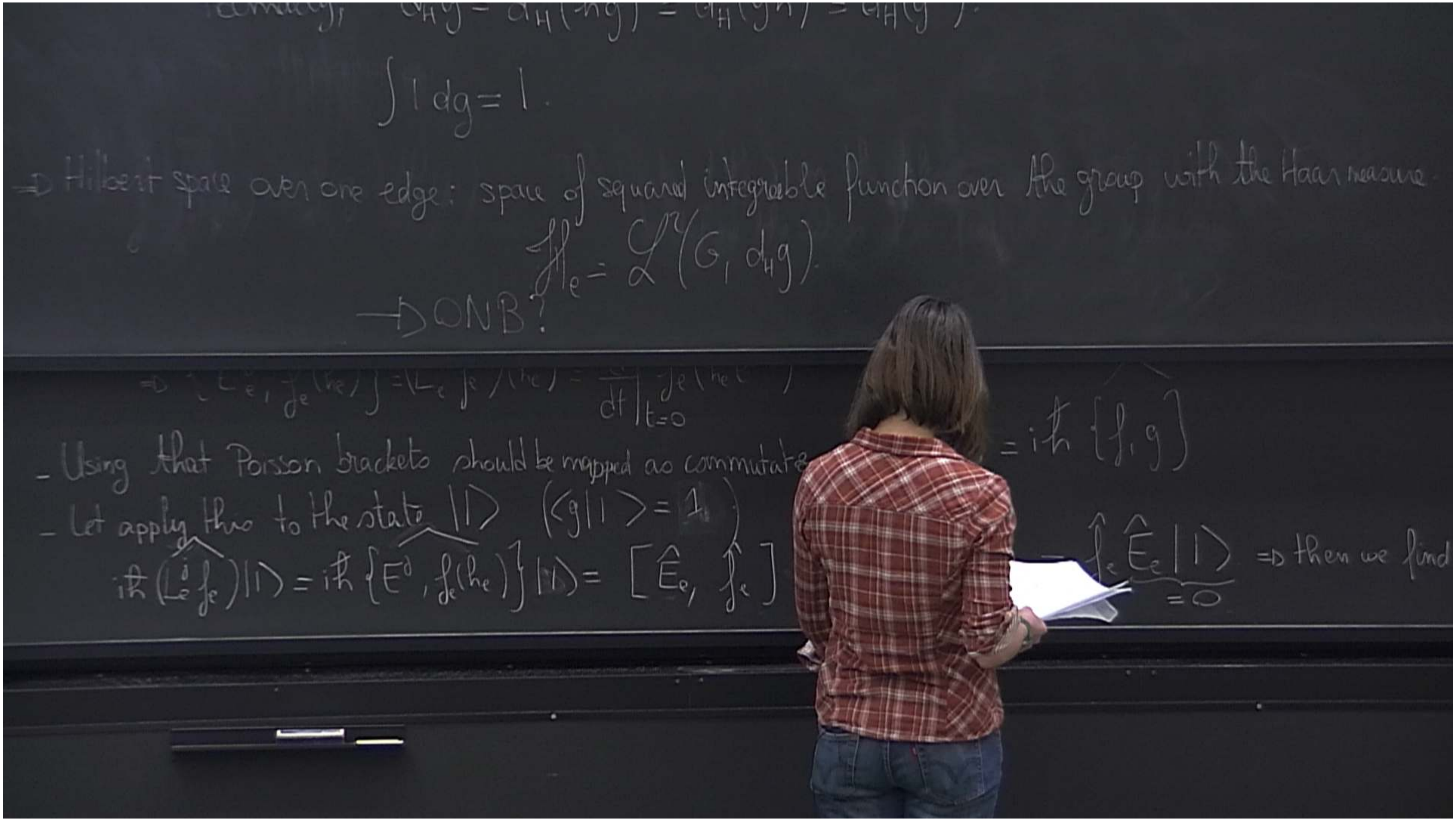
...g this to the state $|1\rangle$ $(\langle g|1\rangle = 1)$

...er product $\langle \Psi_1 | \Psi_2 \rangle = \int \overline{\Psi_1}(g)\, \Psi_2(g)\, d_H g$

Formally, $d_H g = d_H(hg) = d_H(gh) = d_H(g^{-1})$.

$$\int 1 \, dg = 1 .$$


- Using that Poisson brackets should be mapped as commutators
- Let apply this to the state $|1\rangle$ $(\langle g|1\rangle = 1)$
$\Rightarrow$ Inner product $\langle \Psi_1 | \Psi_2 \rangle = \int \overline{\Psi_1}(g)\, \Psi_2(g)\, d_H g$
Formally, $d_H g = d_H(hg) = d_H(gh) = d_H(g^{-1})$.
$\int 1\, dg = 1$.
Hilbert space over one edge: space of squared integrable function over the group with the Haar measure.
$\mathcal{H}_e = \mathcal{L}^2(G, d_H g)$.
$\rightarrow$ ONB?


∫ 1 dg = 1.
⇒ Hilbert space over one edge: space of squared integrable function over the group with the Haar measure
ℋe = L²(G, dHg).
→ DONB?
- Using that Poisson brackets should be mapped as commutat
= iħ {f, g}
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iħ (Le fe)|1⟩ = iħ {E, fe(he)}|1⟩ = [Êe, f̂e]
f̂e Êe|1⟩ ⇒ then we find
= 0

$\int 1\, dg = 1$.

$\Rightarrow$ Hilbert space over one edge: space of squared integrable function over the group with the Haar measure.

$$\mathcal{H}_e = \mathcal{L}^2(G, d_H g).$$

$\rightarrow$ ONB?

$\Rightarrow$

$\frac{d}{d}$

- Using that Poisson [br]ackets should be mapped as commutators $[\hat{f}, \hat{g}]$
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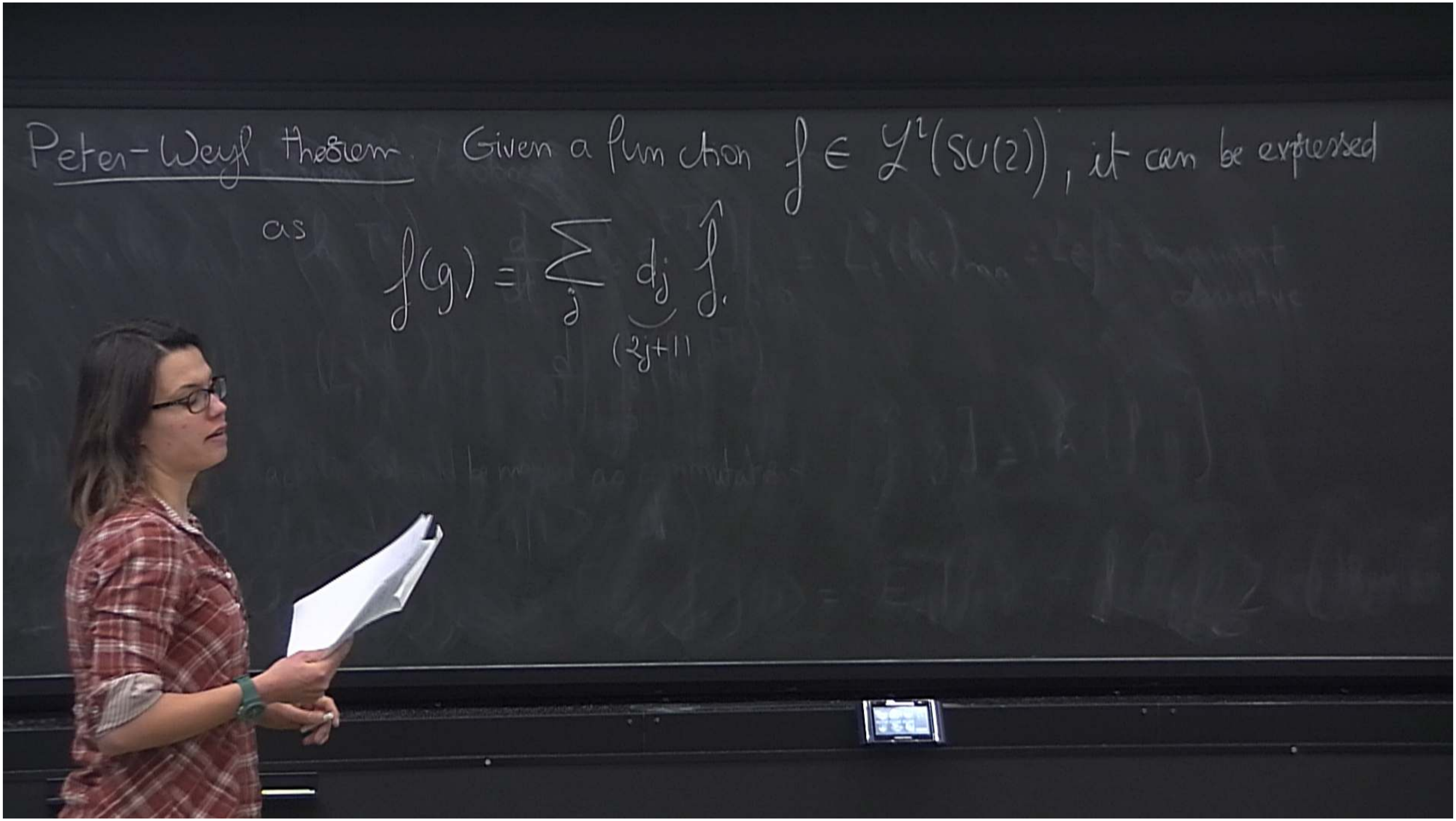
$\hat{E}$ ... $\Rightarrow$ then we find

ONB $\to$ Peter-Weyl theorem Given a function $f \in \mathcal{L}^2(SU(2))$


Peter-Weyl theorem: Given a function $f \in \mathcal{L}^2(SU(2))$, it can be expressed
as
$f(g) = \sum_j \underbrace{d_j}_{(2j+1)} \hat{f}_j$

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as

$$f(g) = \sum_j \underbrace{d_j}_{(2j+1)} \hat{f}^j_{mn} D^j_{mn}(g) \quad \text{where} \quad \hat{f}^j_{mn} = d_j \int_{SU(2)} dg\, D^j_{nm}(g^{-1})\, f(g)$$

<u>Peter-Weyl theorem</u>. Given a function $f \in \mathcal{L}^2(SU(2))$, it can be expressed as

$$f(g) = \sum_j \underbrace{d_j}_{(2j+1)} \hat{f}^j_{mn} D^j_{mn}(g) \quad \text{where} \quad \hat{f}^j_{mn} = d_j \int_{SU(2)} dg \, D^j_{nm} \ldots$$

$\Rightarrow$ Matrix el^t of irreps form an orthonormal basis in $\mathcal{L}^2(SU(2), dg)$

$$\underbrace{\langle g | j\, m\, n \rangle}_{\text{basis state}} = \sqrt{d_j}\; D^j_{mn}(g)$$

$\in \mathcal{L}^2(SU(2))$, it can be expressed

$_{mn}(g)$ where $\hat{f}^j_{mn} = d_j \int_{SU(2)} dg\, D^j_{nn}(g^{-1})\, f(g)$

in $\mathcal{L}^2(SU(2), dg)$

$$\int \overline{D^j_{mn}(g)}\, D^{j'}_{m'n'}(g)\, dg = \frac{1}{d_j} \delta_{jj'}$$

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in $\mathcal{L}^2(SU(2), d_H g)$

$$\int \overline{D^j_{mn}(g)}\, D^{j'}_{m'n'}(g)\, dg = \frac{1}{d_j}\, \delta_{jj'}\, \delta_{mm'}\, \delta_{nn'}$$

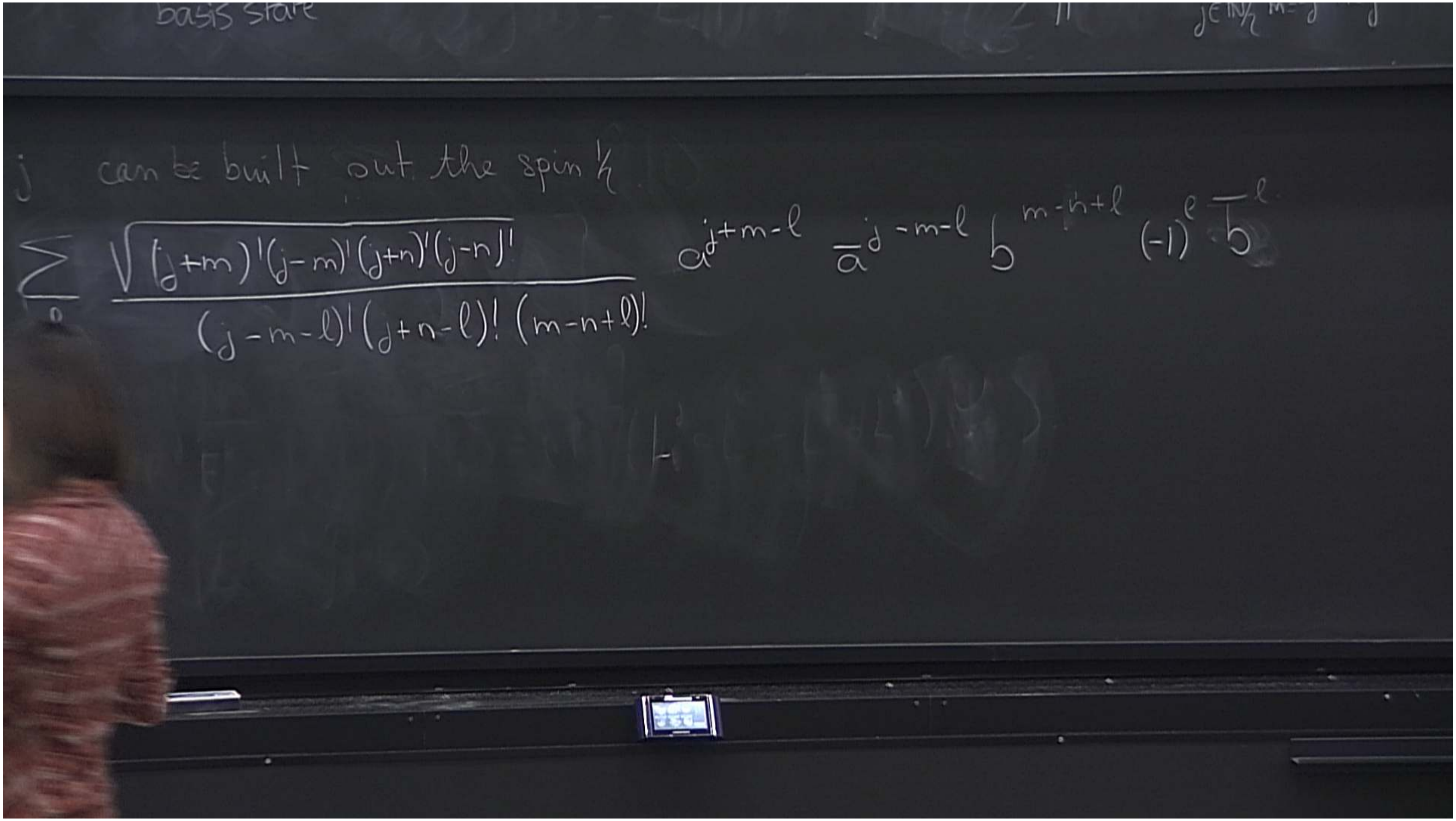
complete basis:

$$\sum_{j \in \mathbb{N}/2} \sum_{m=-j}^{j} \sum_{n=-j}^{j} |j\,m\,n)(j\,m\,n| = \mathbb{1}_{\mathcal{L}^2(SU(2), d_H g)}$$

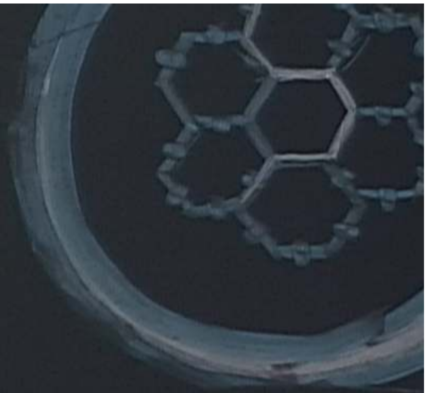
basis state

SU(2)-rep. j can be built out the spin ½

$$\begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} \quad D^{j}_{mn} = \sum_{\ell} \frac{\sqrt{(j+m)!(j-m)!(j+n)!(j-n)!}}{(j-m-\ell)!(j+n-\ell)!(m-n+\ell)!} a^{j+m-\ell} \, \bar{a}^{j-m-\ell} \, b^{m-}$$


basis state
j can be built out the spin 1/2
$\sum_{\ell} \frac{\sqrt{(j+m)!(j-m)!(j+n)!(j-n)!}}{(j-m-\ell)!(j+n-\ell)!(m-n+\ell)!} a^{j+m-\ell} \bar{a}^{j-m-\ell} b^{m-n+\ell} (-1)^{\ell} \bar{b}^{\ell}$

Quantization of an edge $\rightarrow \mathcal{H}_{kin}$



$\rightarrow$ find a representation of the basic phase space functions as operators on some space of functions.

$\rightarrow$ equip this space of functions with an inner product

$\rightarrow$ ONB for this space.

$|0\, m\, n\rangle$ m d n.