Title: PSI 2015/2016 Quantum Gravity - Lecture 7

Date: Mar 01, 2016 10:15 AM

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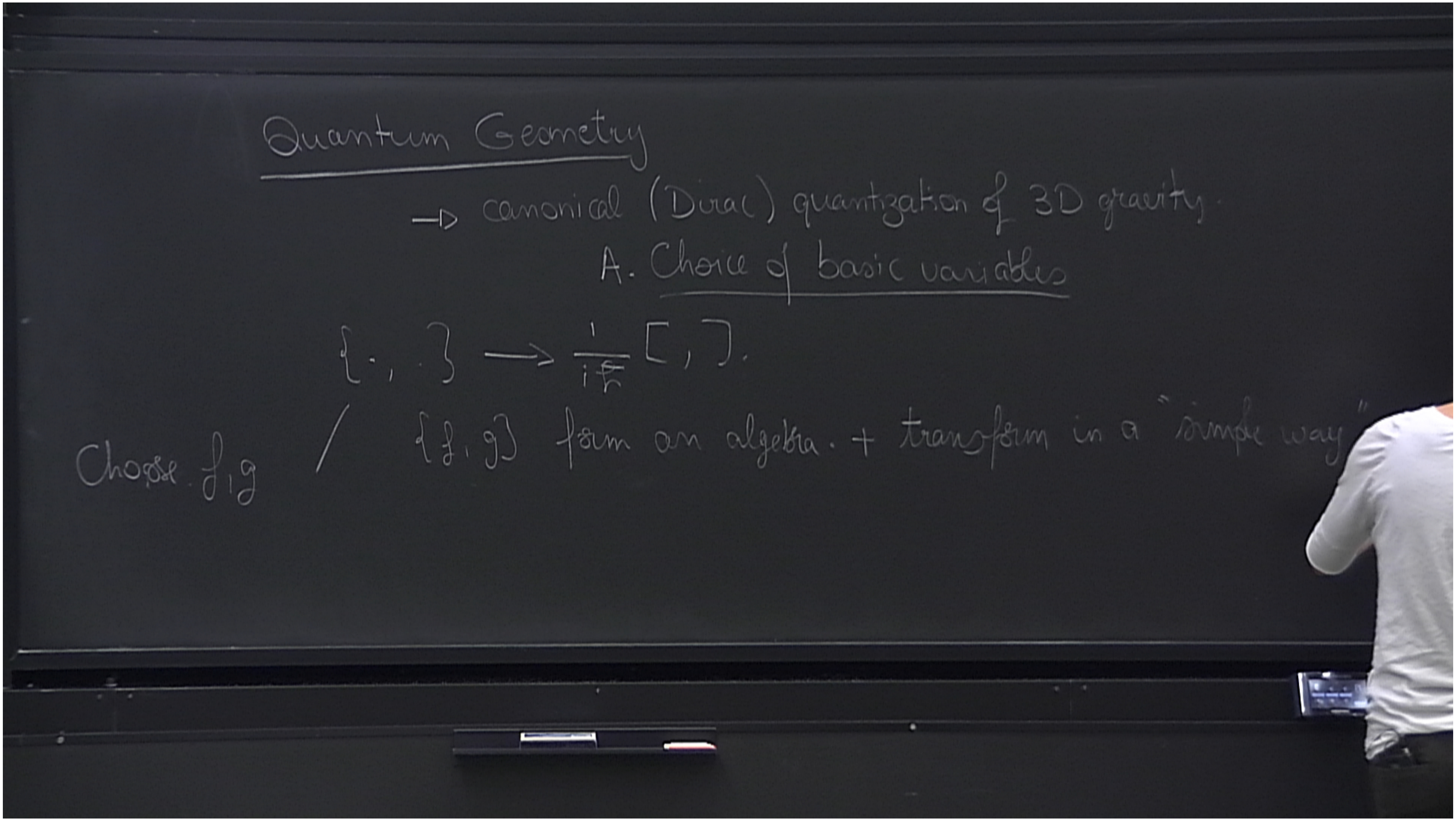
Abstract:

Quantum Geometry

→ canonical (Dirac) quantization of 3D gravity.

A. Choice of basic variables


Quantum Geometry
→ canonical (Dirac) quantization of 3D gravity.
A. Choice of basic variables
$\{\cdot,\cdot\} \longrightarrow \frac{1}{i\hbar}[\,,\,].$
Choose f, g / $\{f,g\}$ form an algebra. + transform in a "simple way"

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Quantum Geometry

→ canonical (Dirac) quantization of 3D gravity.

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$$\{\cdot,\cdot\} \longrightarrow \frac{1}{i\hbar}[\,,\,].$$

- Choose f, g / $\{f,g\}$ form an algebra. + transform in a "simple way" under gauge transfo.
- In field theory, $\{\phi(x), \pi(y)\} = \delta(x,y) \longrightarrow \hat{\phi}(x), \hat{\pi}(y)$. not proper operators. →

y

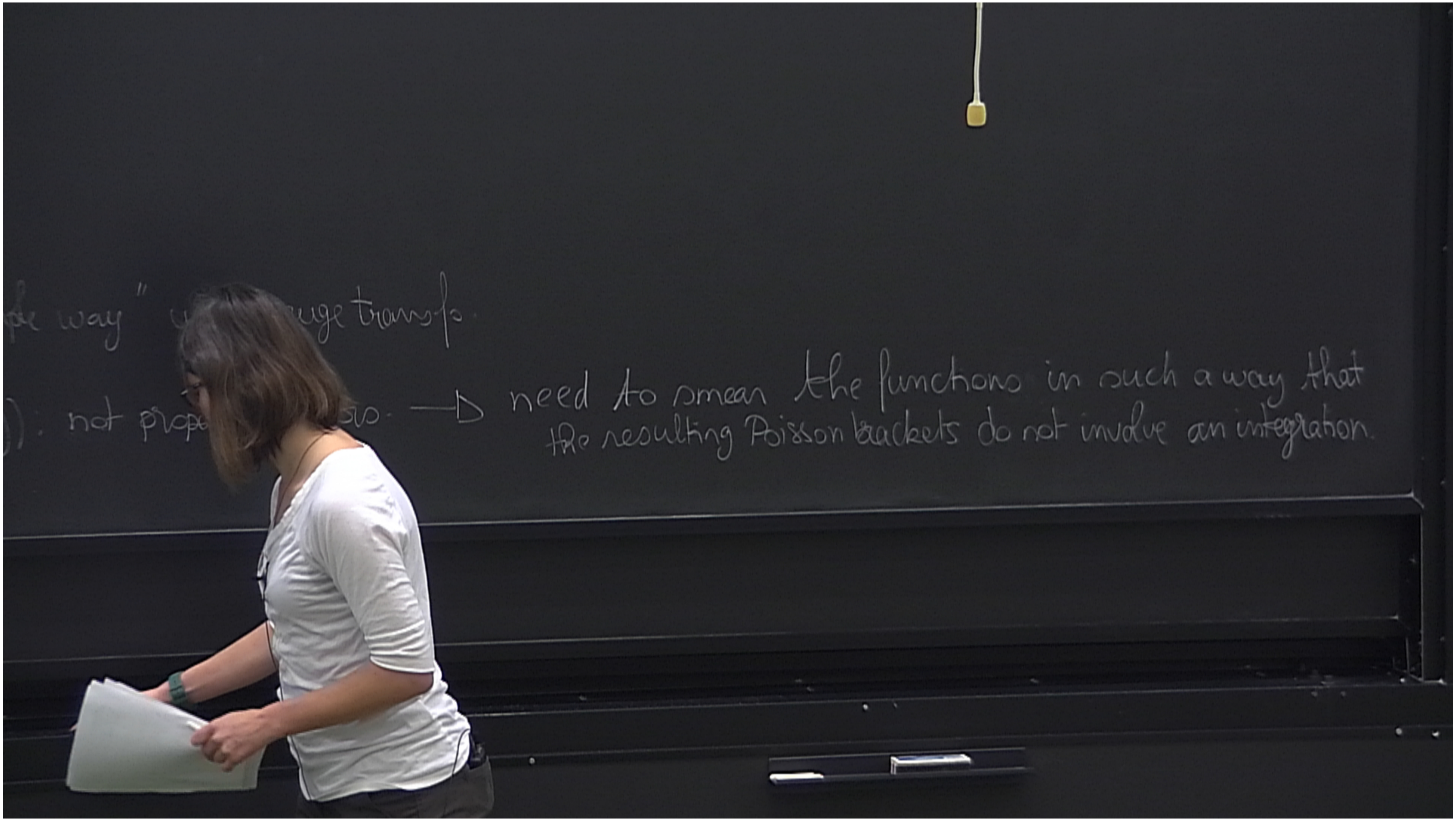
(Dirac) quantization of 3D gravity.

. Choice of basic variables

$= [\,,\,]$.

n an algebra. + transform in a "simple way" under gauge transf.

$(y)\} = \delta(x,y)$ $\longrightarrow$ $\hat{\phi}(x), \hat{\Pi}(y)$: not proper operators. $\longrightarrow$ need to smear the functions.


need to smear the functions in such a way that
the resulting Poisson brackets do not involve an integration.

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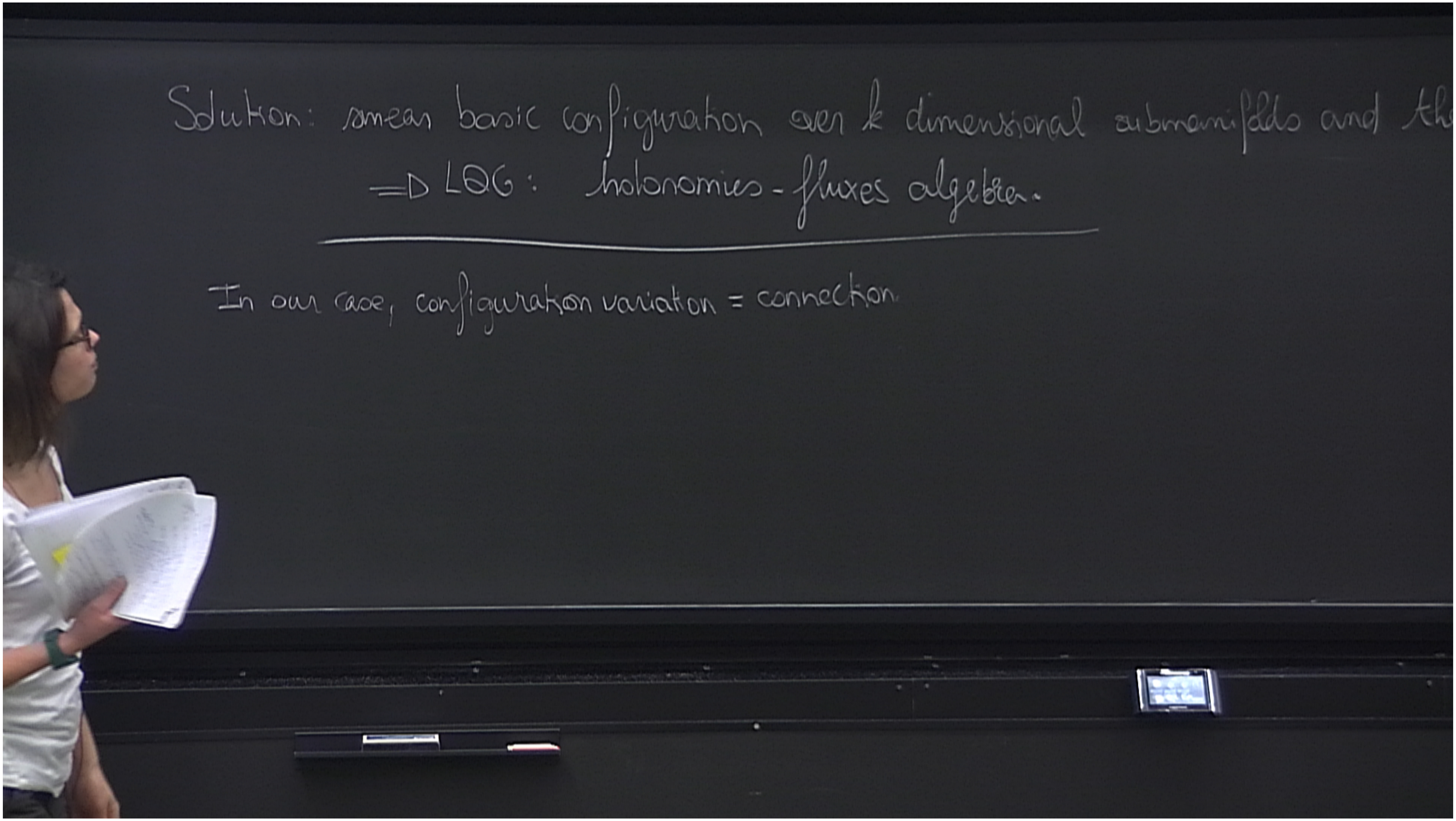
the resulting Poisson brackets do

Solution: smear basic configuration over k dimensional submanifolds and the conjugated variables over $(D-k)$ d ones.



Solution: smear basic configuration over k dimensional submanifolds and th
⇒ LQG: holonomies - fluxes algebra.
In our case, configuration variation = connection

basic configuration over k dimensional submanifolds and the conjugated variables over (D

LQG: holonomies - fluxes algebra.

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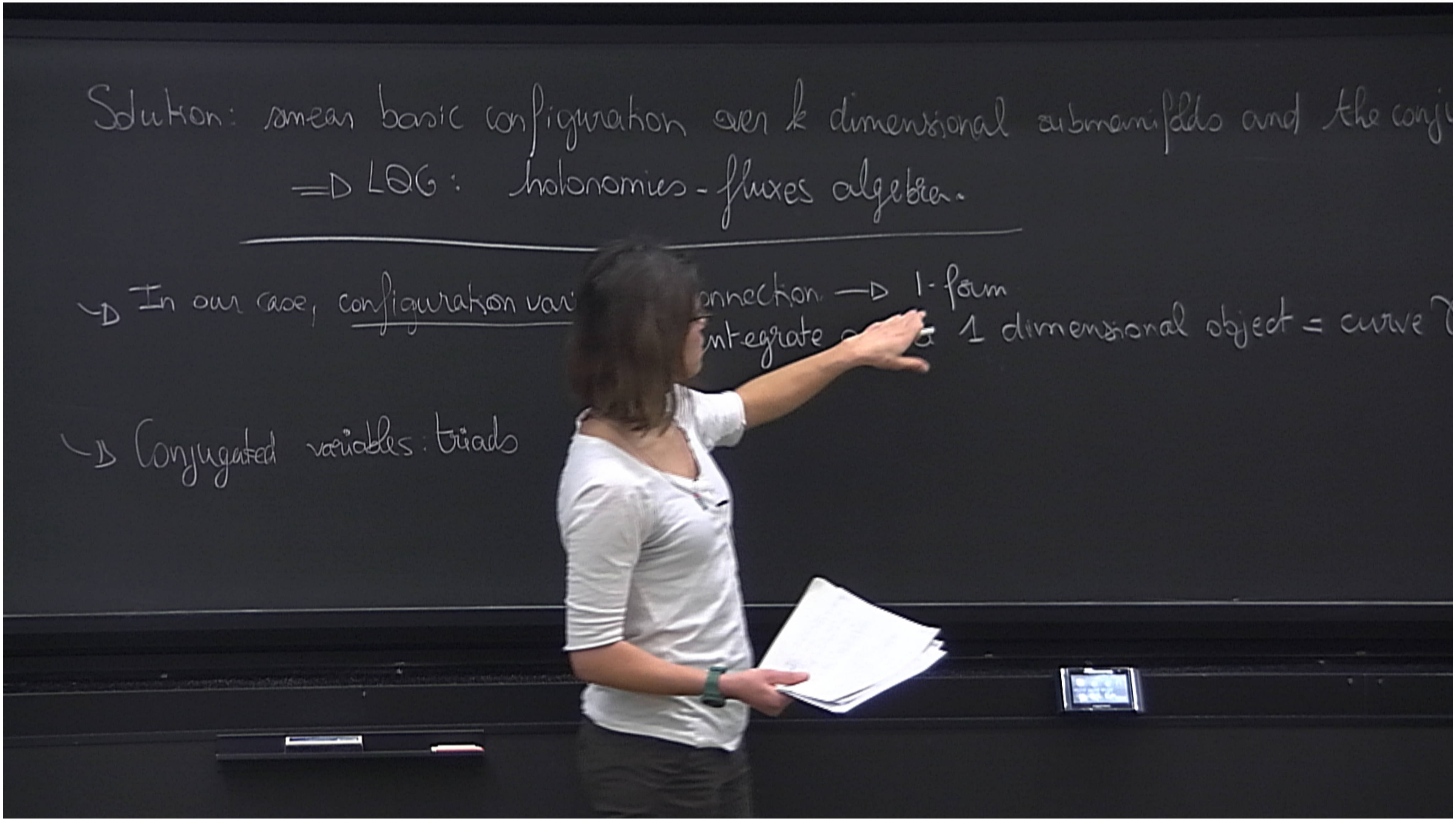
integrate over a 1 dimensional object = curve γ. ⟶ holonomy l submanifolds and the conjugated variables over $(D-k)$ d ones.

dimensional object = curve γ. $\longrightarrow$ holonomy (// transport)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1 dimensional object = curve
Conjugated variables: triads

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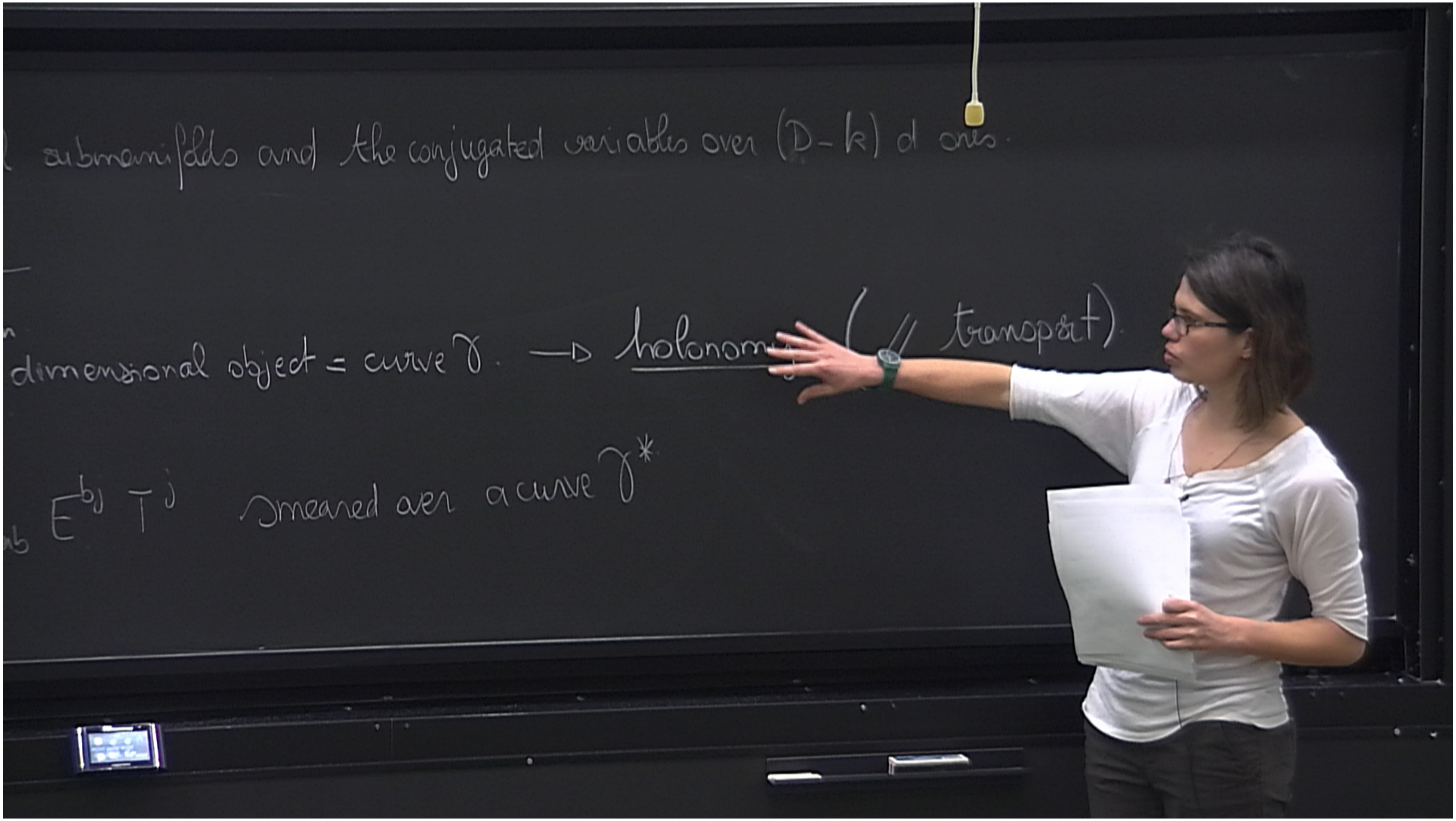
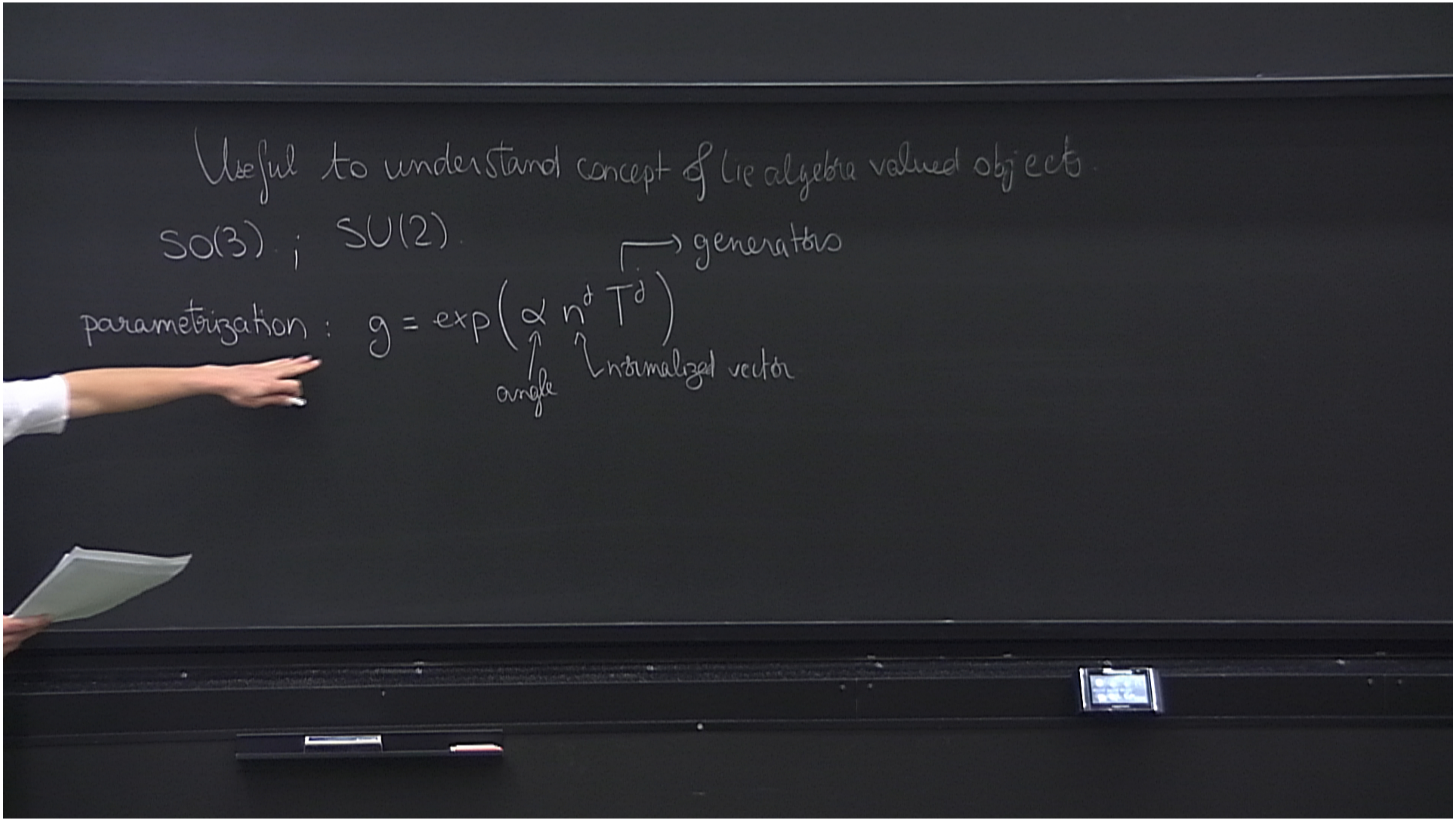
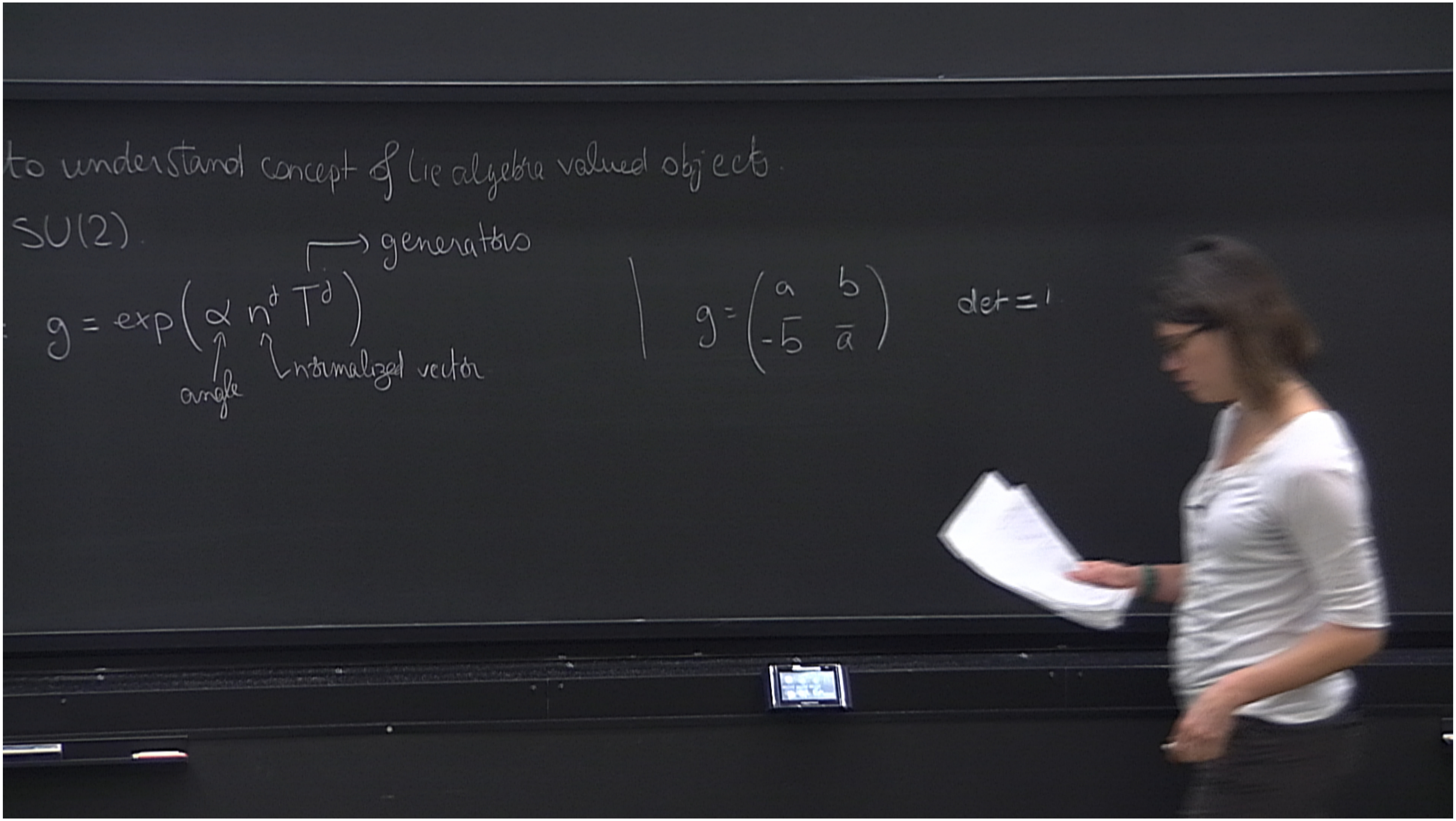
$\Rightarrow$ LQG: holonomies - fluxes algebra.

$\rightarrow$ In our case, configuration variable = connection $\rightarrow$ 1-form
integrate over a 1 dimensional object = curve γ

$\rightarrow$ Conjugated variables: triads
$\hookrightarrow$ co-triad (one-form): $e_a = e^j_a T_j = \hat{\varepsilon}_{ab} E^{bj} T^j$ smeared o


submanifolds and the conjugated variables over (D-k) d ones.
dimensional object = curve γ. → holonomy (// transport).
E^bj T^j smeared over a curve γ*


Useful to understand concept of Lie algebra valued objects.
SO(3) ; SU(2).
generators
parametrization : $g = \exp(\alpha\, n^a T^a)$
angle
normalized vector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$g = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$
det = 1

to understand concept of Lie algebra valued objects.

SU(2).

$$g = \exp(\underset{\text{angle}}{\alpha}\ \underset{\text{normalized vector}}{n^a}\ \overset{\text{generators}}{T^a})$$

$$g = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \quad \det = 1$$

nderstand concept of Lie algebra valued object.

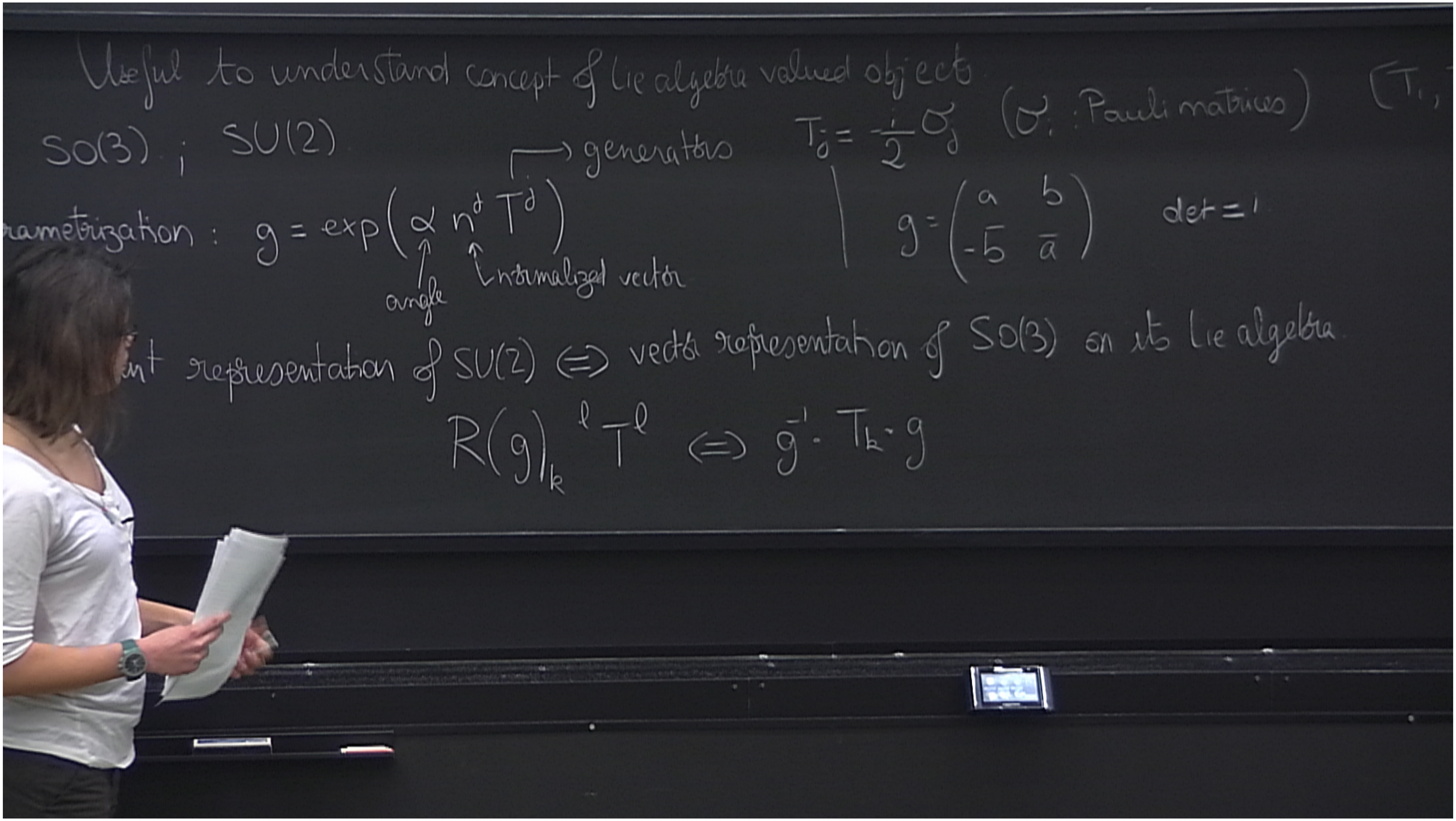
$[T_i, T_j] = \varepsilon_{ijk} T_k$

2).

$T_j = -\frac{i}{2}\sigma_j$ (σ_i : Pauli matrices)

$\exp(\alpha\, n^j\, T^j)$ — T^j: generators; α: angle; n^j: normalized vector

$g = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$ det = 1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nt representation of SU(2) ⟺ vector representation of SO(3) on its Lie algebra
$R(g)_k{}^l\, T^l \iff g^{-1} \cdot T_k \cdot g$

lgebra valued objects.

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$SO(3) = SU$

$g = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$ det = 1

n

nts representation of SO(3) on its Lie algebra.

$\boxed{\Leftrightarrow\ g^{-1} \cdot T_k \cdot g}$ $\longrightarrow$ infinitesimal form $\Rightarrow$ commutation relation of the T^j

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$[T_i, T_j] = \varepsilon_{ijk} T_k$

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$\exp(\alpha\, n^j\, T^j)$ → generators

angle ↑ ↑ normalized vector

$g = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$ det = 1

...ation of SU(2) ⟺ vector representation of SO(3) on its Lie algebra.

$\boxed{R(g)_k{}^l\, T^l \Leftrightarrow g^{-1} \cdot T_k \cdot g}$ ⟶ infinitesimal form ⇒ commutation relation ects.

σ_j (σ_i : Pauli matrices) $\quad [T_i, T_j] = \varepsilon_{ijk} T_k$

$SO(3) = SU(2)/\mathbb{Z}_2$

$g = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$ $\quad$ det $= 1$

of $SO(3)$ on its Lie algebra

$g \longrightarrow$ infinitesimal form $\Rightarrow$ commutation relation of the T^j

to understand concept of Lie algebra valued objects.

$[T_i, T_j] = \varepsilon_{ijk}$

SU(2)

$T_j = -\frac{i}{2}\sigma_j$ (σ_i : Pauli matrices)

$g = \exp(\alpha\, n^j T^j)$ → generators

angle; normalized vector

$g = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$ det = 1

representation of SU(2) ⟺ vector representation of SO(3) on its Lie algebra.

$$\boxed{R(g)_k{}^l\, T^l = g^{-1} \cdot T_k \cdot g}$$

⟶ infinitesimal form ⇒ commutation

Lie Algebra valued object: $B = B^j T_j \rightarrow$ triad, connection, curvature.

We have seen: e^j_a, F^j_{ab} with the internal index j transform in the vector representation of the rotation group.

↳ co-triad (one-form) : $e_a = e_a^j T_j = \hat{\varepsilon}_{ab} E^{bj} T^j$ smeared over a curve

to understand concept of lie algebra valued object

SU(2) $\quad T_j = -\frac{i}{2}\sigma_j$ (σ_i : Pauli matrices) $\quad [T_i, T_j] = \varepsilon_{ijk} T_k$ $\quad$ SO(3

$g = \exp(\alpha\, n^j T^j)$ — α: angle; n^j: normalized vector; T^j: generators $\qquad g = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$ det = 1

presentation of SU(2) $\Leftrightarrow$ vector representation of SO(3) on its lie algebra

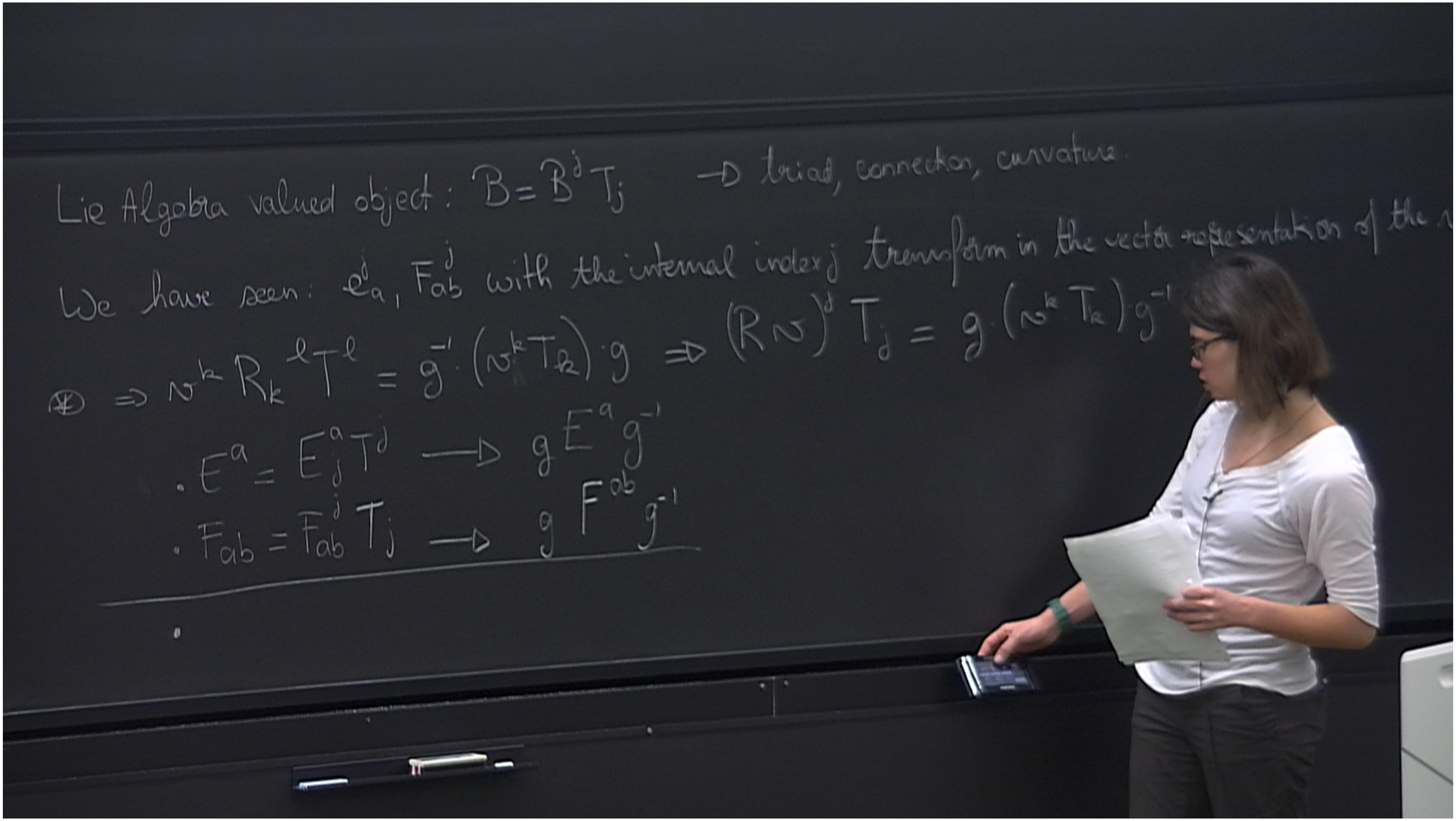
$$\boxed{R(g)_k{}^l\, T^l = g^{-1} \cdot T_k \cdot g} \quad (*)$$

$\longrightarrow$ infinitesimal form $\Rightarrow$ commutation relation of the T^j

Lie Algebra valued object: $B = B^j T_j$ → triad, connection, curvature.

We have seen: e^j_a, F^j_{ab} with the internal index j transform in the vector representation of the

$$(*) \Rightarrow v^k R_k{}^l T^l = g^{-1} \cdot (v^k T_k) \cdot g \Rightarrow (Rv)^j T_j = g \cdot (v^k T_k) g$$


Lie Algebra valued object: B = B^j T_j → triad, connection, curvature.
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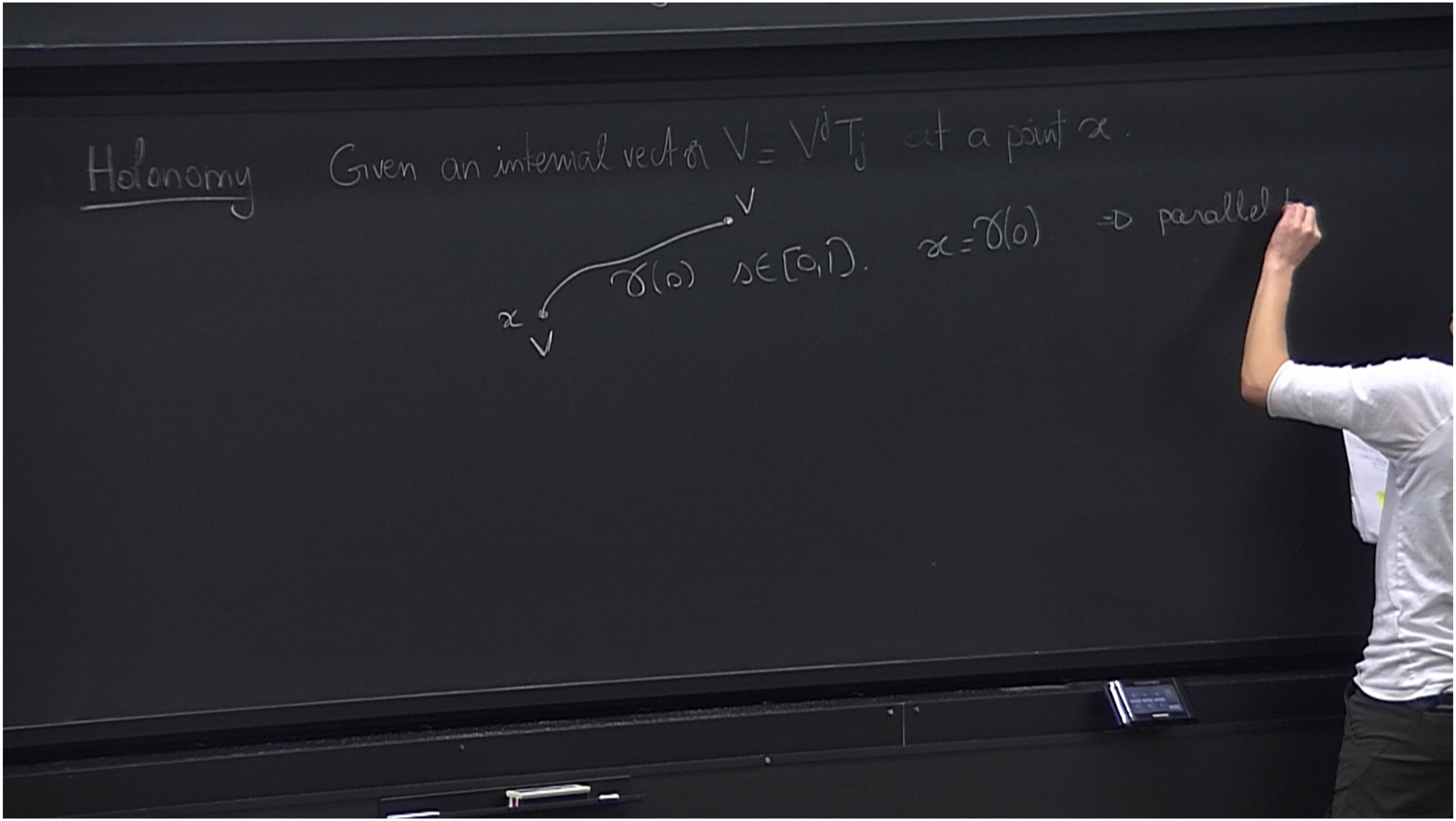
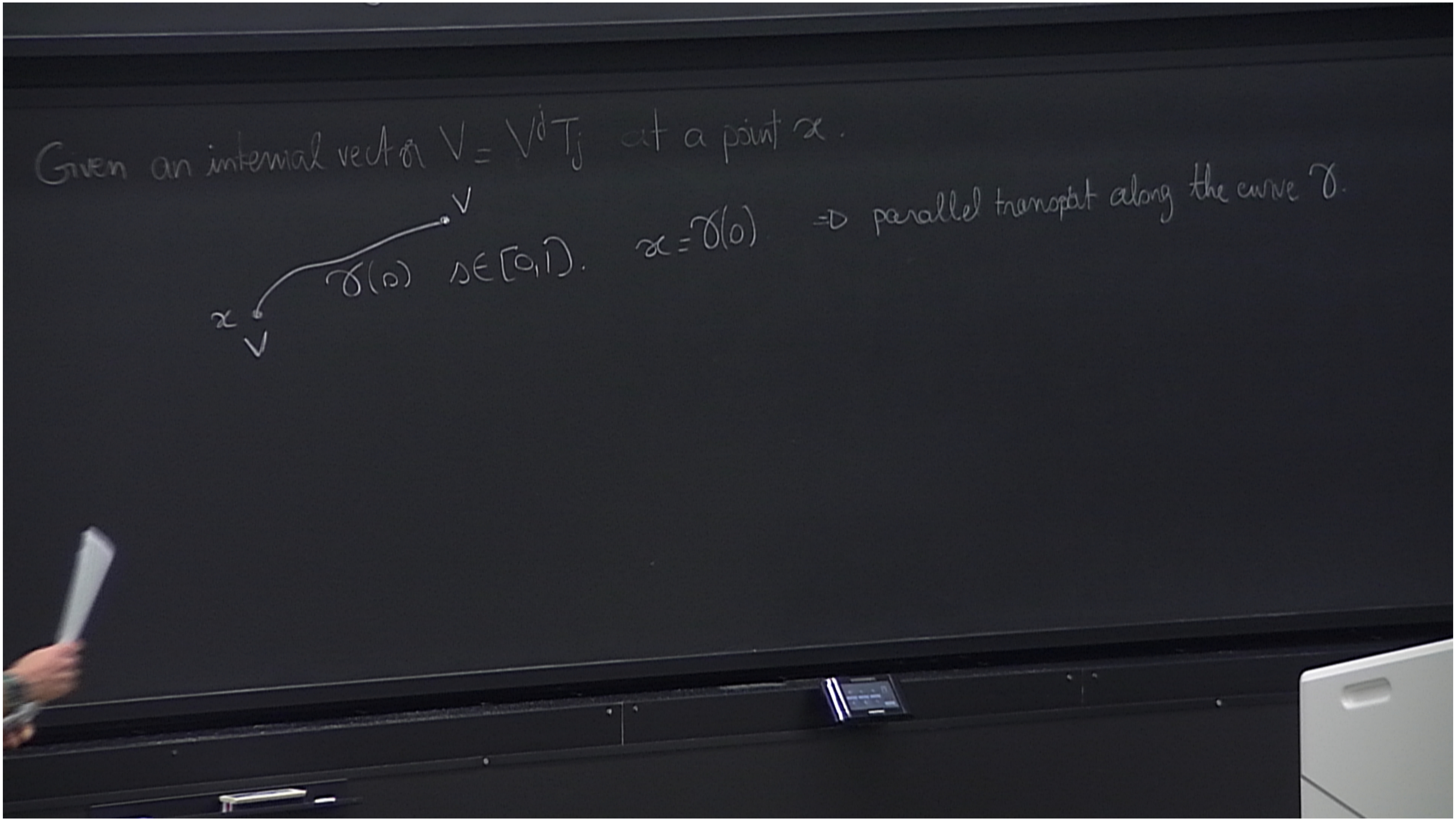
⇒ v^k R_k^l T^l = g^-1 · (v^k T_k) · g ⇒ (Rv)^j T_j = g · (v^k T_k) g^-1
· E^a = E^a_j T^j → g E^a g^-1
· F_ab = F^j_ab T_j → g F^ab g^-1

$$(*) \Rightarrow v^k {R_k}^l T^l = g^{-1} \cdot (v^k T_k) \cdot g \Rightarrow (Rv)^j T_j = g \cdot (v^k T_k) g^{-1}$$

- $E^a = E^a_j T^j \longrightarrow g E^a g^{-1}$
- $F_{ab} = F^j_{ab} T_j \longrightarrow g F^{ab} g^{-1}$

- $A_a = A^j_a T_j \longrightarrow g A_a g^{-1} + g \partial_a g^{-1}$


Holonomy
Given an internal vector V = V^j T_j at a point x.
V
γ(s) s ∈ [0,1].
x = γ(0)
⇒ parallel
x
V


Given an internal vector $V = V^j T_j$ at a point x.
V
$\gamma(s)$ $s \in [0,1]$. $x = \gamma(0)$ $\Rightarrow$ parallel transport along the curve γ.
x
V

• $A_a = A_a^j T_j \longrightarrow g A_a g^{-1} + g \partial_a g^{-1}$

nomy — Given an internal vector $V = V^j T_j$ at a point x.

V

$\gamma(s)$ $\quad s \in [0,1]$. $\quad x = \gamma(0)$ $\Rightarrow$ parallel transport along the curve γ.

x

$V(0)$

V transforms in the adjoint rep. so we define the // transport

$$V(s) = h(s)\, V(0)\, h(s)^{-1}$$

$h(s)$ holonomy from $\gamma(0) = x$ to $\gamma(s)$.

transforms in the adjoint rep. so we define the // transport $h(s)$ holonomy from $\gamma(0) =$

$$V(s) = h(s)\, V(0)\, h(s)^{-1}$$

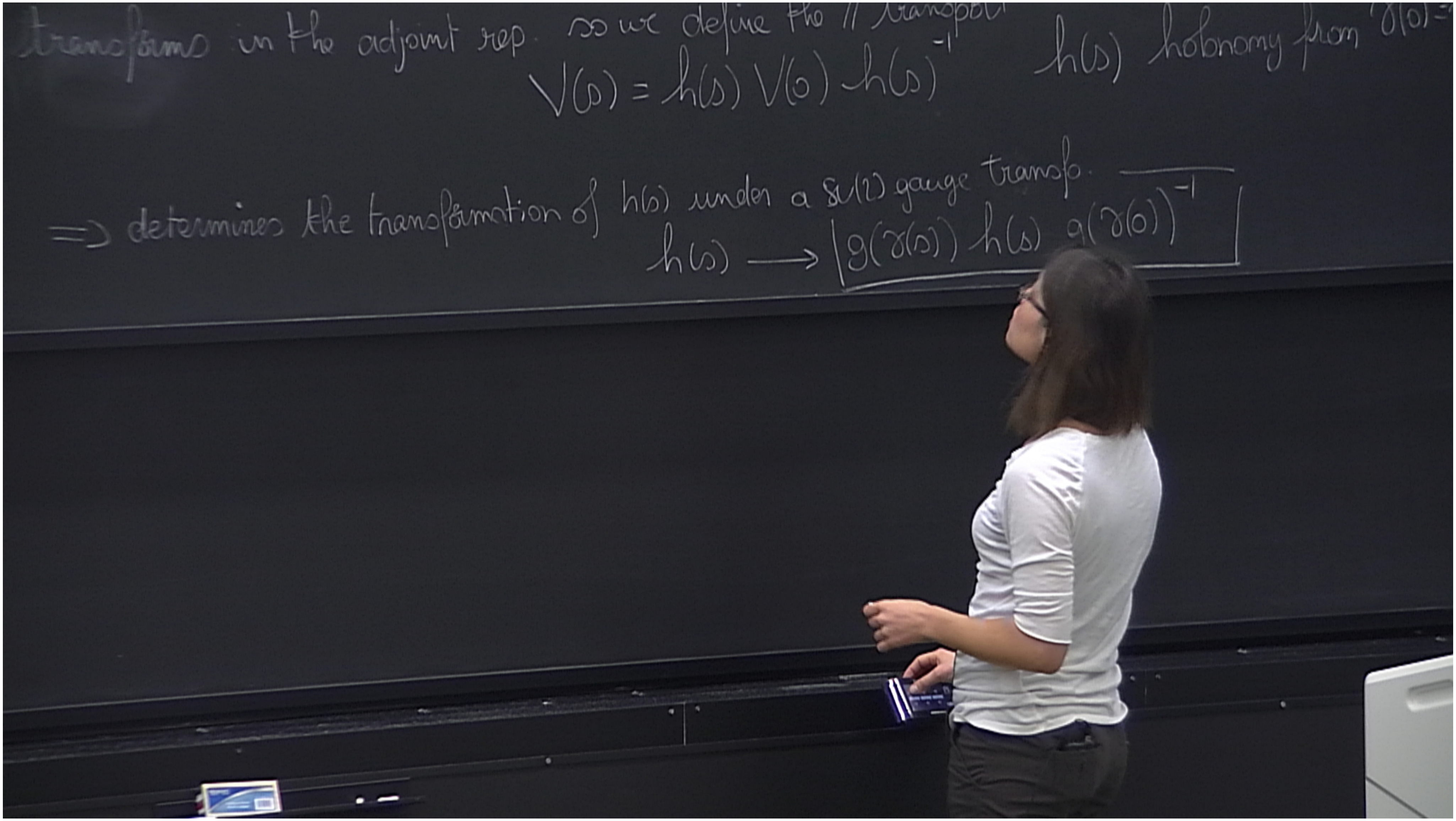
$\Longrightarrow$ determines the transformation of $h(s)$ under a $SU(2)$ gauge transfo.

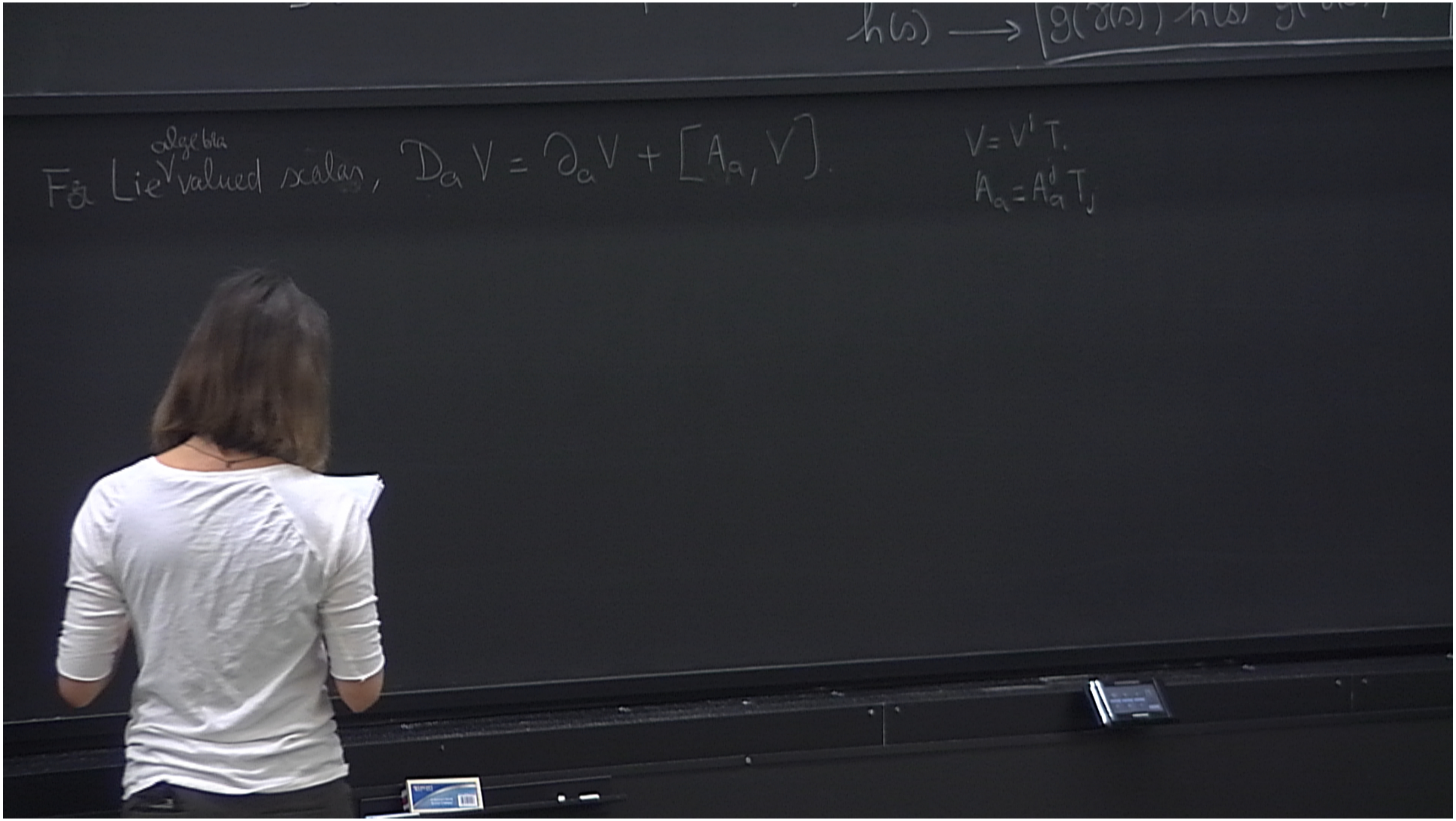
transforms in the adjoint rep. so we define the // transport

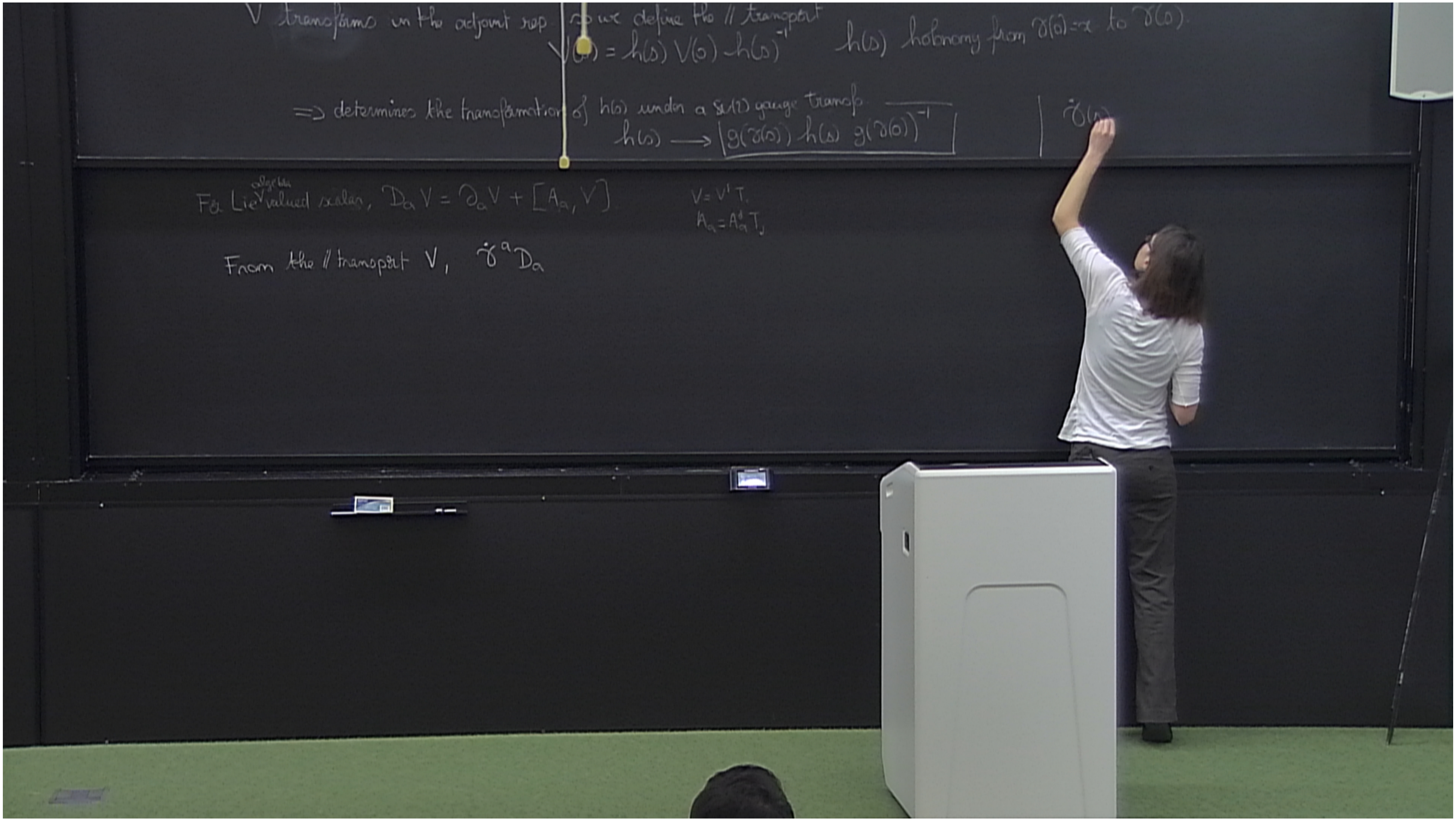
$$V(s) = h(s)\, V(0)\, h(s)^{-1} \qquad h(s) \text{ holonomy from } \gamma(0) =$$

$\Longrightarrow$ determines the transformation of $h(s)$ under a $SU(2)$ gauge transfo.

$$h(s) \longrightarrow g(\gamma(s))\, h(s)\, g(\gamma(0))^{-1}$$


transforms in the adjoint rep. so we define the // transport
V(s) = h(s) V(0) h(s)^-1
h(s) holonomy from γ(0) =
⇒ determines the transformation of h(s) under a SU(2) gauge transfo.
h(s) → g(γ(s)) h(s) g(γ(0))^-1


For Lie algebra valued scalars, $D_a V = \partial_a V + [A_a, V]$.
$V = V^i T_i$
$A_a = A_a^j T_j$


V transforms in the adjoint rep so we define the // transport
V(s) = h(s) V(0) h(s)^{-1}
h(s) holonomy from γ(0)=x to γ(s).
⇒ determines the transformation of h(s) under a SU(2) gauge transf.
h(s) ⟶ g(γ(s)) h(s) g(γ(0))^{-1}
γ̇(s)
For Lie algebra valued scalar, D_a V = ∂_a V + [A_a, V]
V = V^i T_i
A_a = A_a^i T_i
From the // transport V, γ̇^a D_a

$h(s) \longrightarrow \boxed{g(\gamma(s))\, h(s)\, g^{-1}(\dots)}$

For Lie algebra valued scalars, $D_a V = \partial_a V + [A_a, V]$.

$V = V^i T_i$

$A_a = A_a^j T_j$

From the // transport V, $\dot{\gamma}^a D_a$

$h(s)$ $h(s)$ holonomy ...

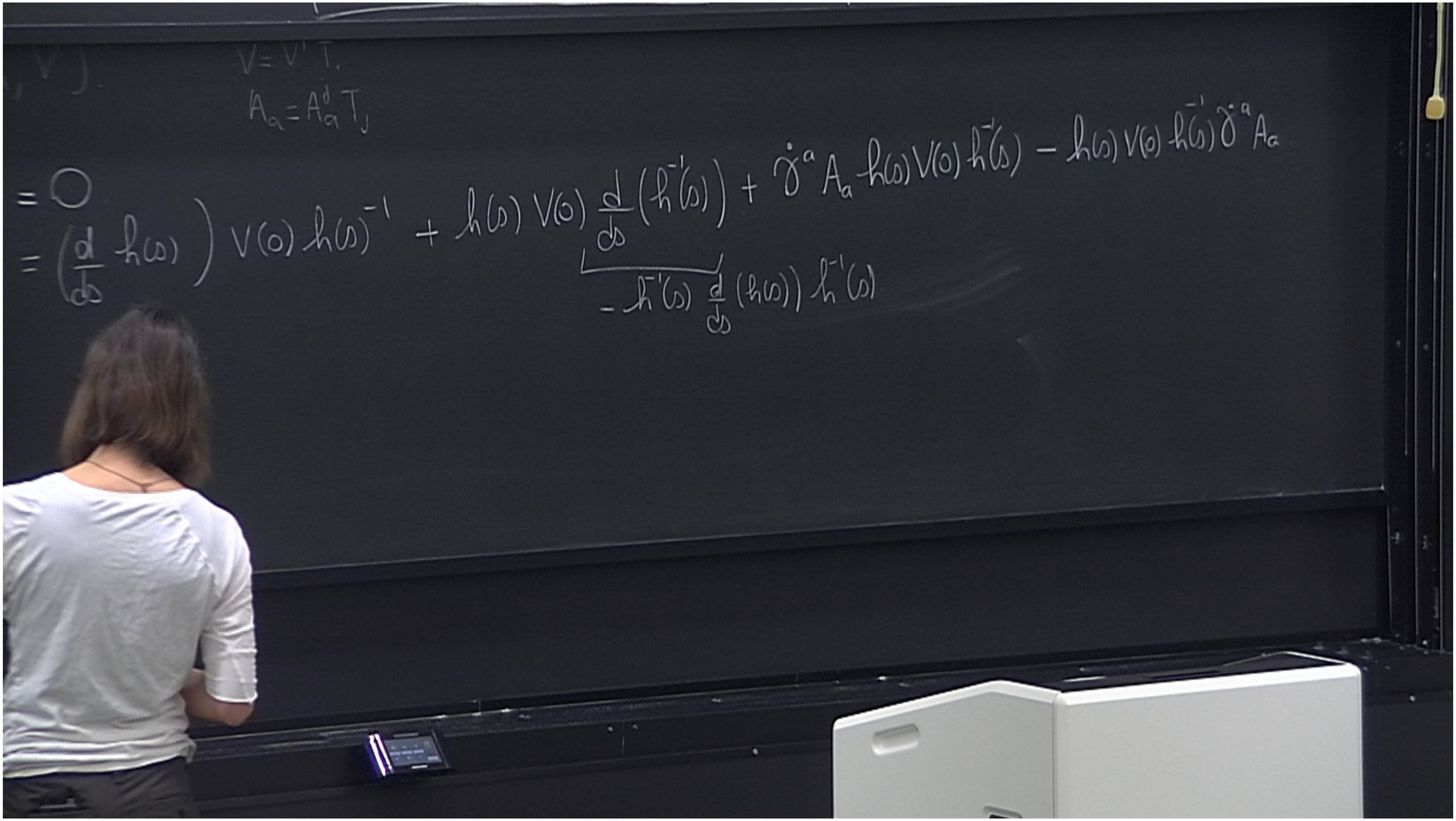
$U(2)$ gauge transfo.

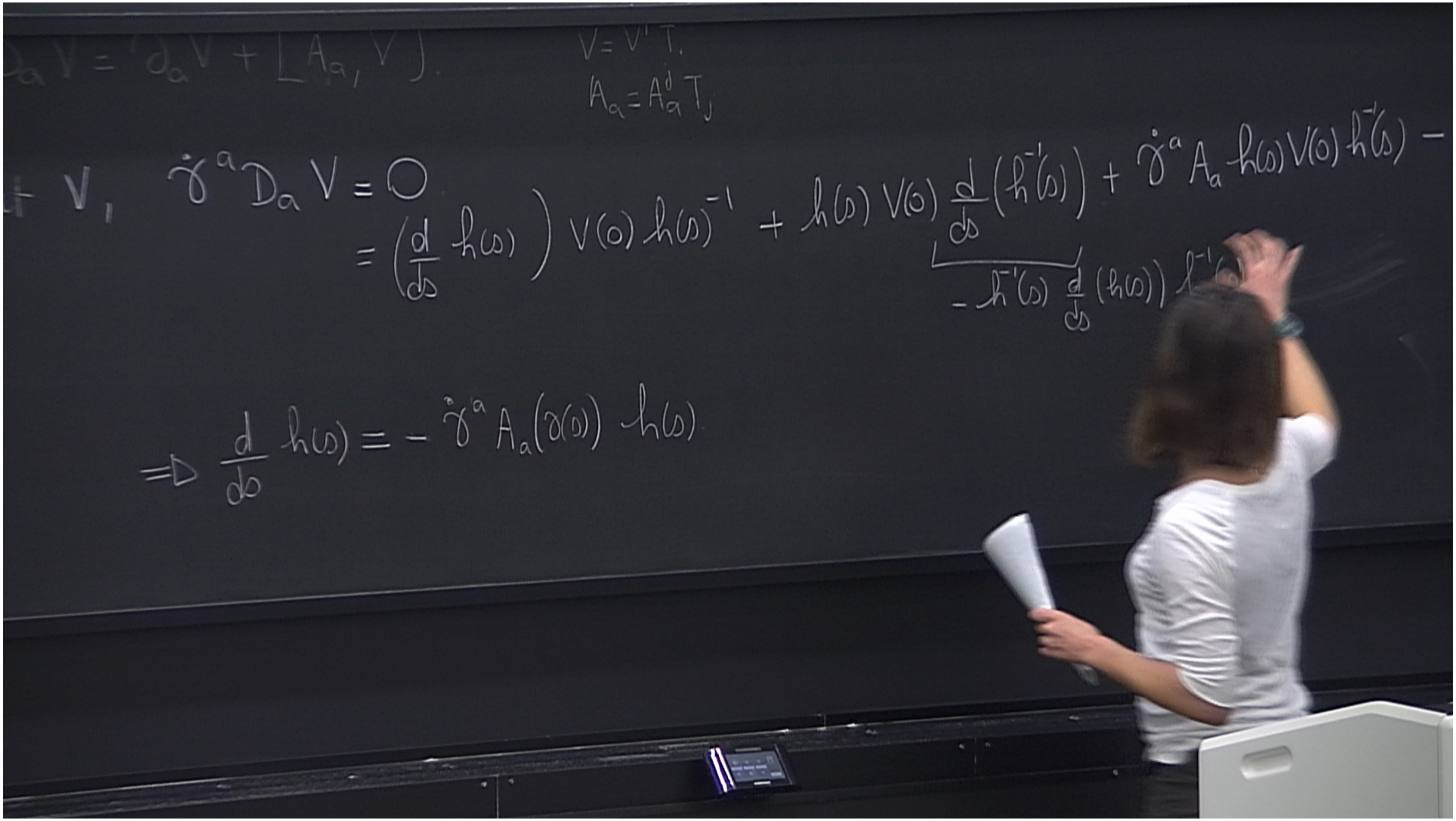
$$\boxed{g(\gamma(s))\, h(s)\, g(\gamma(0))^{-1}}$$

$$\dot{\gamma}(s) = \frac{d}{ds}\gamma$$

T_i

$A_a^i\, T_i$


$V = V^i T_i$
$A_a = A_a^j T_j$
$= 0$
$= \left(\frac{d}{ds} h(s)\right) V(0) h(s)^{-1} + h(s) V(0) \underbrace{\frac{d}{ds}\left(h^{-1}(s)\right)}_{- h^{-1}(s) \frac{d}{ds}(h(s)) h^{-1}(s)} + \dot{\gamma}^a A_a h(s) V(0) h^{-1}(s) - h(s) V(0) h^{-1}(s) \dot{\gamma}^a A_a$



$D_a V = \partial_a V + [A_a, V]$.
$V = V^i T_i$
$A_a = A_a^j T_j$
$V, \quad \dot{\gamma}^a D_a V = 0$
$= \left(\frac{d}{ds} h(s)\right) V(0) h(s)^{-1} + h(s) V(0) \frac{d}{ds}\left(h^{-1}(s)\right) + \dot{\gamma}^a A_a h(s) V(0) h^{-1}(s) -$
$- h^{-1}(s) \frac{d}{ds}(h(s))$
$\Rightarrow \frac{d}{ds} h(s) = -\dot{\gamma}^a A_a(\gamma(s))\, h(s)$

$D_a V = \partial_a V + [A_a, V]$. $\quad V = V^i T_i$

$A_a = A_a^j T_j$

$\forall\, V,\quad \dot{\gamma}^a D_a V = 0$

$$= \left(\frac{d}{ds} h(s)\right) V(0)\, h(s)^{-1} + h(s)\, V(0) \underbrace{\frac{d}{ds}\left(h^{-1}(s)\right)}_{-h^{-1}(s) \frac{d}{ds}(h(s))\, h^{-1}(s)} + \dot{\gamma}^a A_a\, h(s)\, V(0)\, h^{-1}(s) -$$

$$\boxed{\Rightarrow \frac{d}{ds} h(s) = -\dot{\gamma}^a A_a(\gamma(s))\, h(s).}$$

Defining differential equation for the holonomy.

$$\ldots_a T_j$$

$$\ldots h(s)^{-1} + h(s)\,V(0)\,\underbrace{\frac{d}{ds}\left(h^{-1}(s)\right)}_{-h^{-1}(s)\,\frac{d}{ds}(h(s))\,h^{-1}(s)} + \dot{\gamma}^a A_a\, h(s) V(0) h^{-1}(s) - h(s) V(0) h^{-1}(s)\, \dot{\gamma}^a A_a$$

$$\boxed{\ldots h(s).}$$

Defining differential equation for the holonomy.
with initial condition $h(0) = 1$.

step by step: $h(s) = \mathbb{1} - \int_0^s \dot{\gamma}^a A_a(\gamma(s'))\, h(s')\, ds'$

step by step: $h(s) = \mathbb{1} - \int_0^s \dot{\gamma}^a A_a(\gamma(s')) \, h(s') \, ds'$

iterate.

$$\Rightarrow \quad h_\gamma(s) = \mathbb{P} \exp\left(-\int_\gamma A\right) = \sum_{n=0}^{\infty} (-1)^n \int_0^s ds_n \int_0^{s_n} ds_{n-1} \cdots \int_0^{s_2} ds_1 \, A$$

path ordered exponential.

$$s')) \, h(s') \, ds'$$

$$= \sum_{n=0}^{\infty} (-1)^n \int_0^{s} ds_n \int_0^{s_n} ds_{n-1} \cdots \int_0^{s_2} ds_1 \, \underbrace{A(s_n) \, A(s_{n-1}) \cdots A(s_1)}_{\text{ordered from the largest parameters to the smallest.}}$$

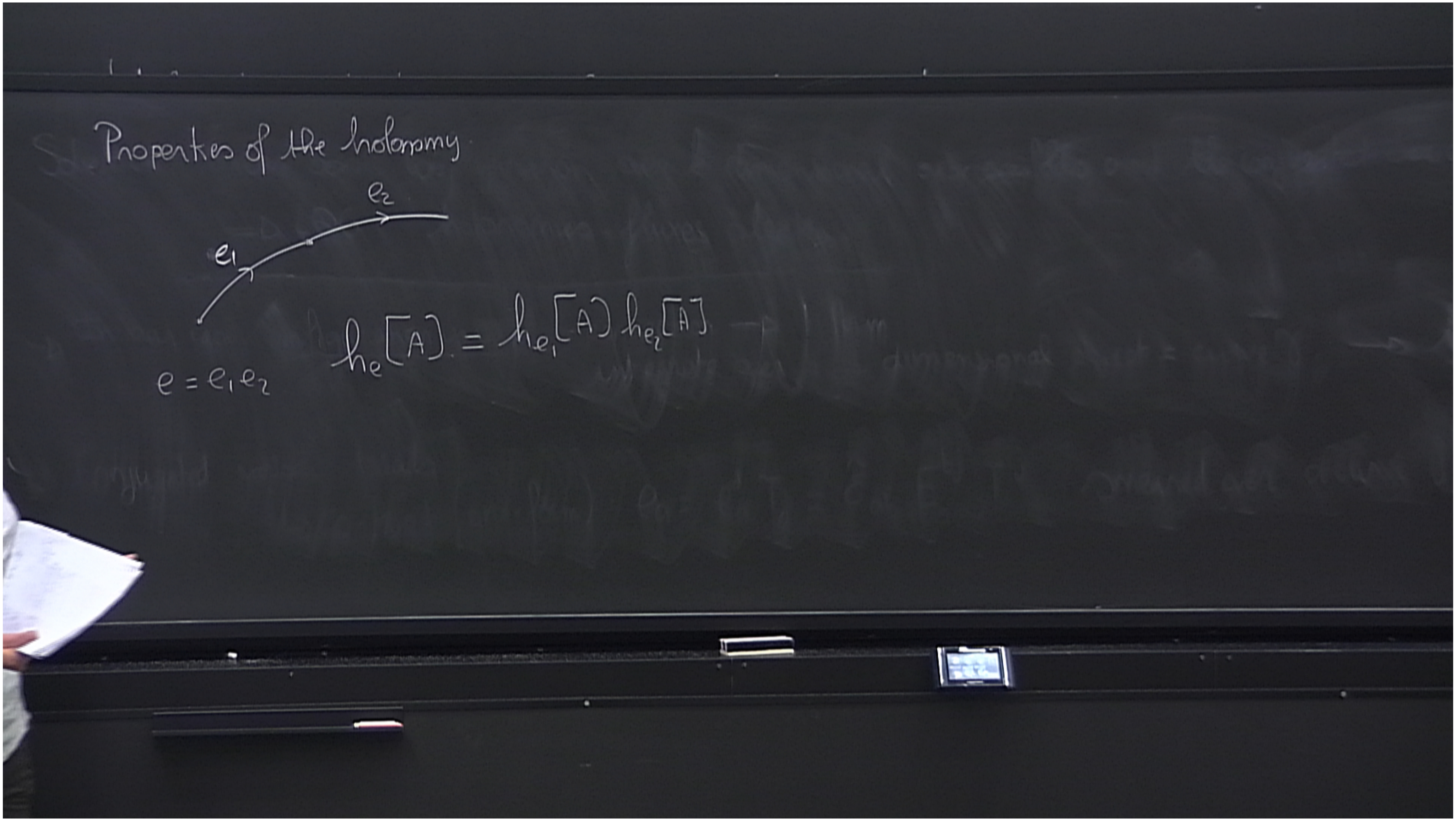
$$A(s) = \dot{\gamma}^a(s) \, A_a(\gamma(s))$$

step by step: $h(s) = \mathbb{1} - \int_0^s \dot{\gamma}^a A_a(\gamma(s'))\, h(s')\, ds'$

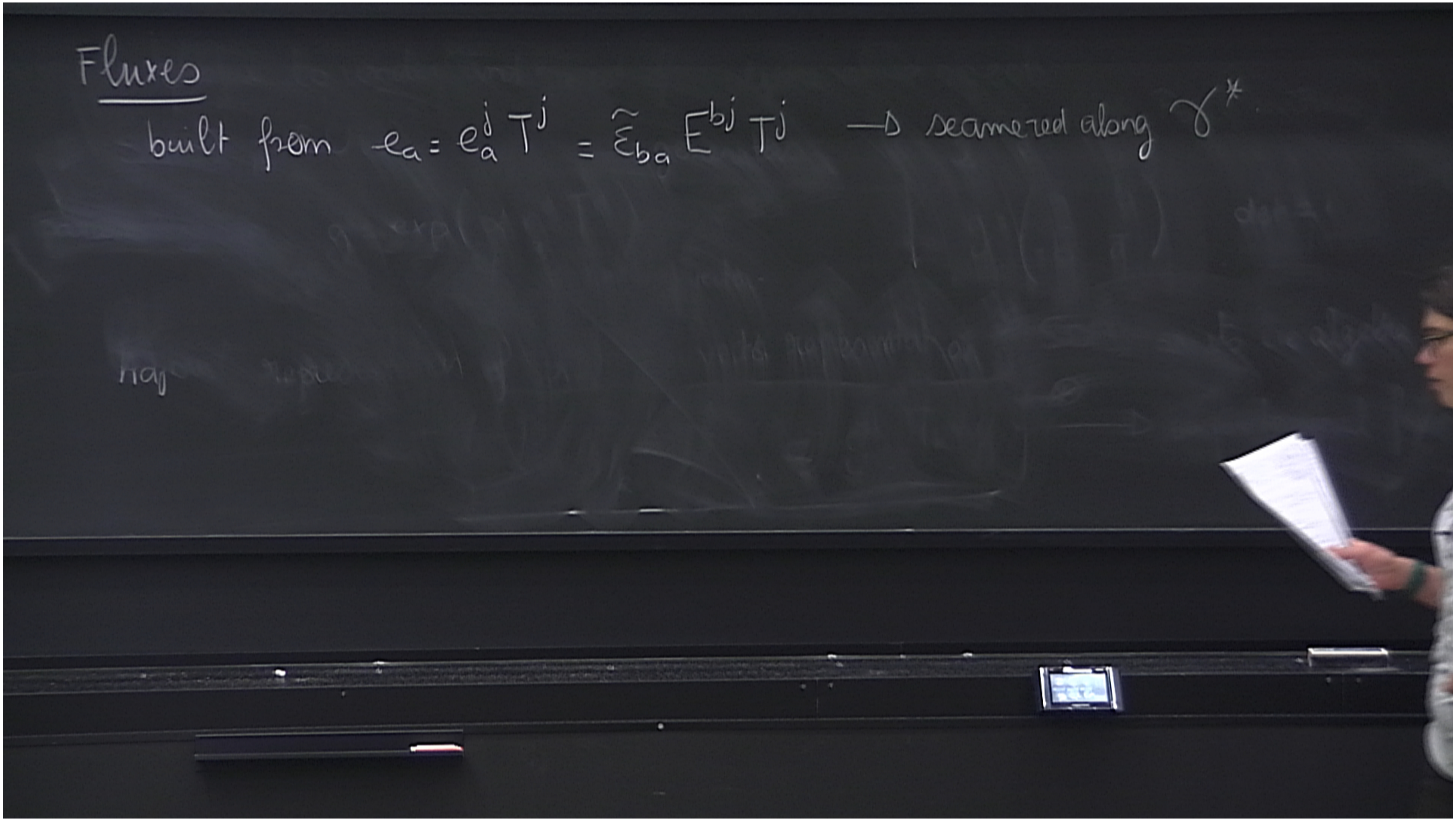
iterate.

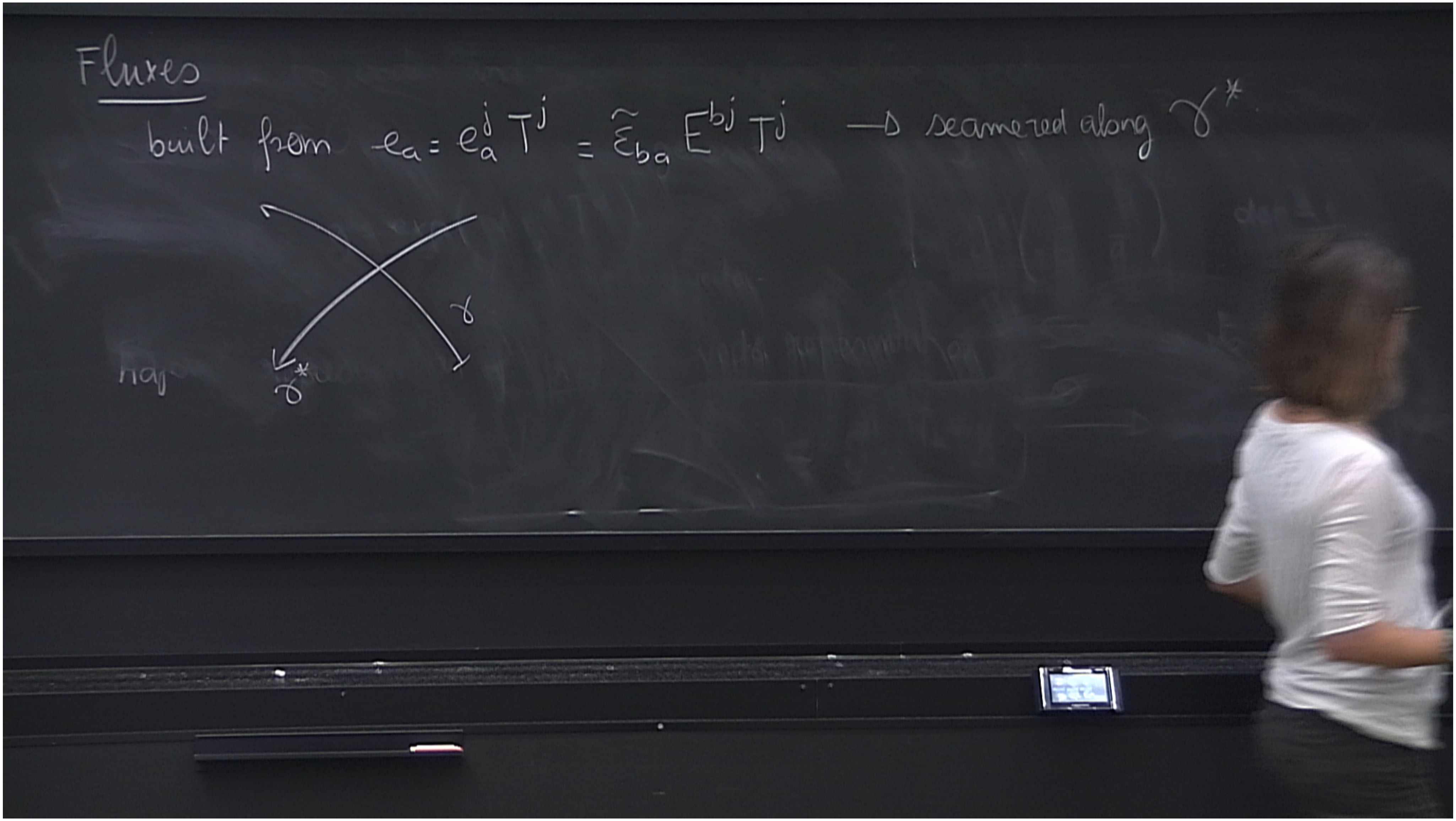
$\Rightarrow \boxed{h_\gamma(s) = \mathbb{P} \exp\left(-\int_\gamma A\right)} = \sum_{n=0}^{\infty} (-1)^n \int_0 d$

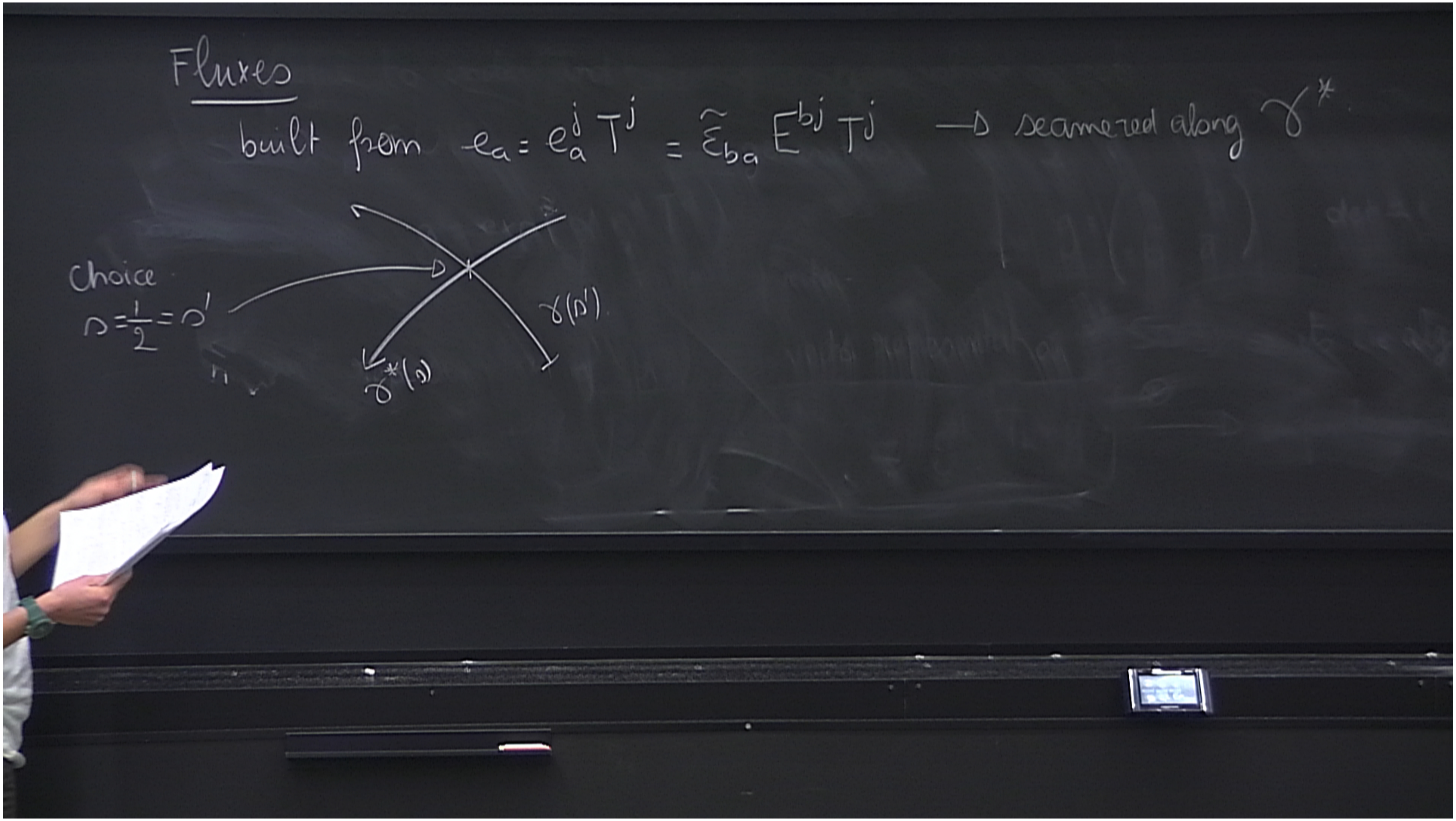
path ordered exponential.

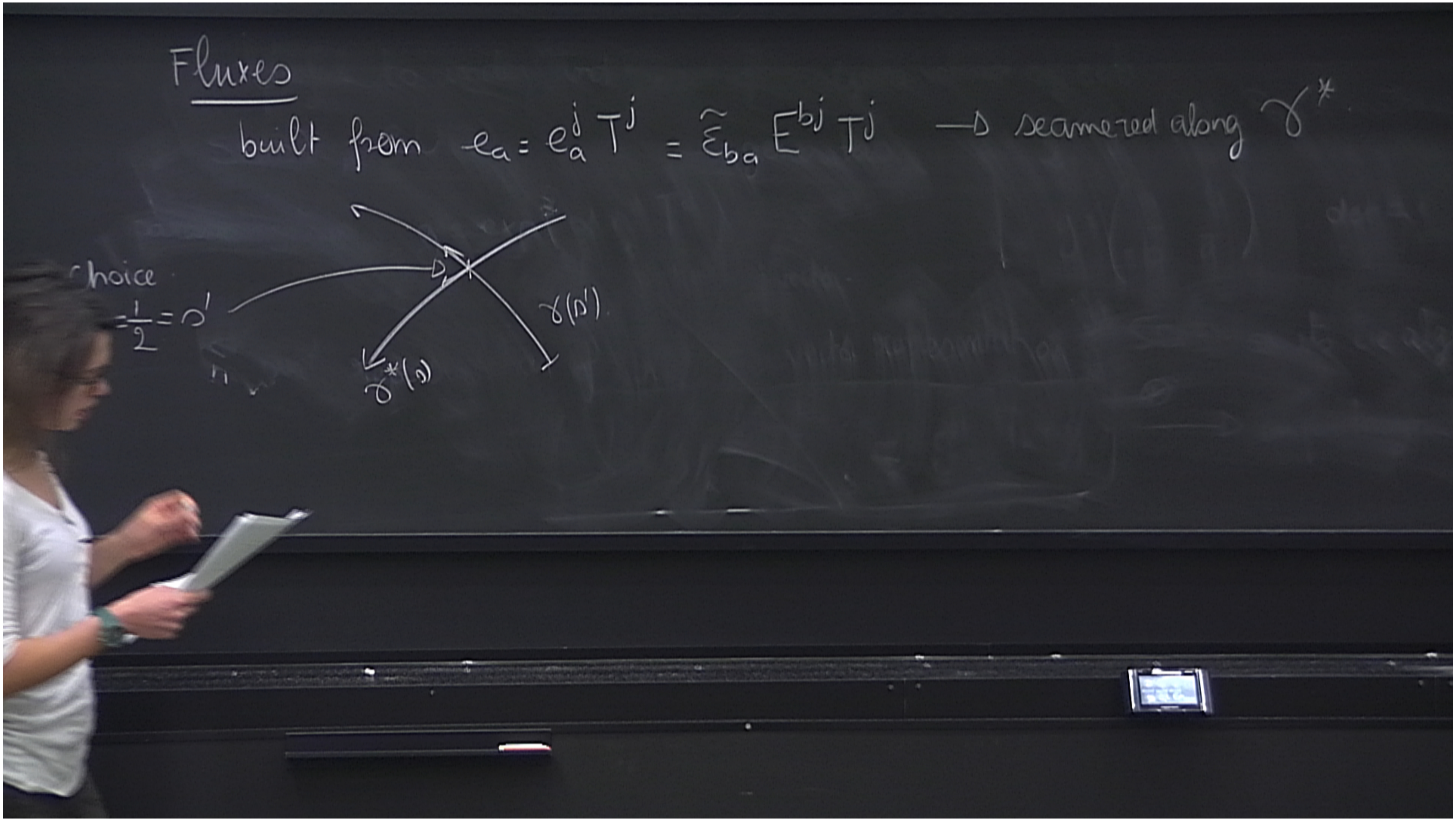

Properties of the holonomy
e_1
e_2
$e = e_1 e_2$
$h_e[A] = h_{e_1}[A]\, h_{e_2}[A]$

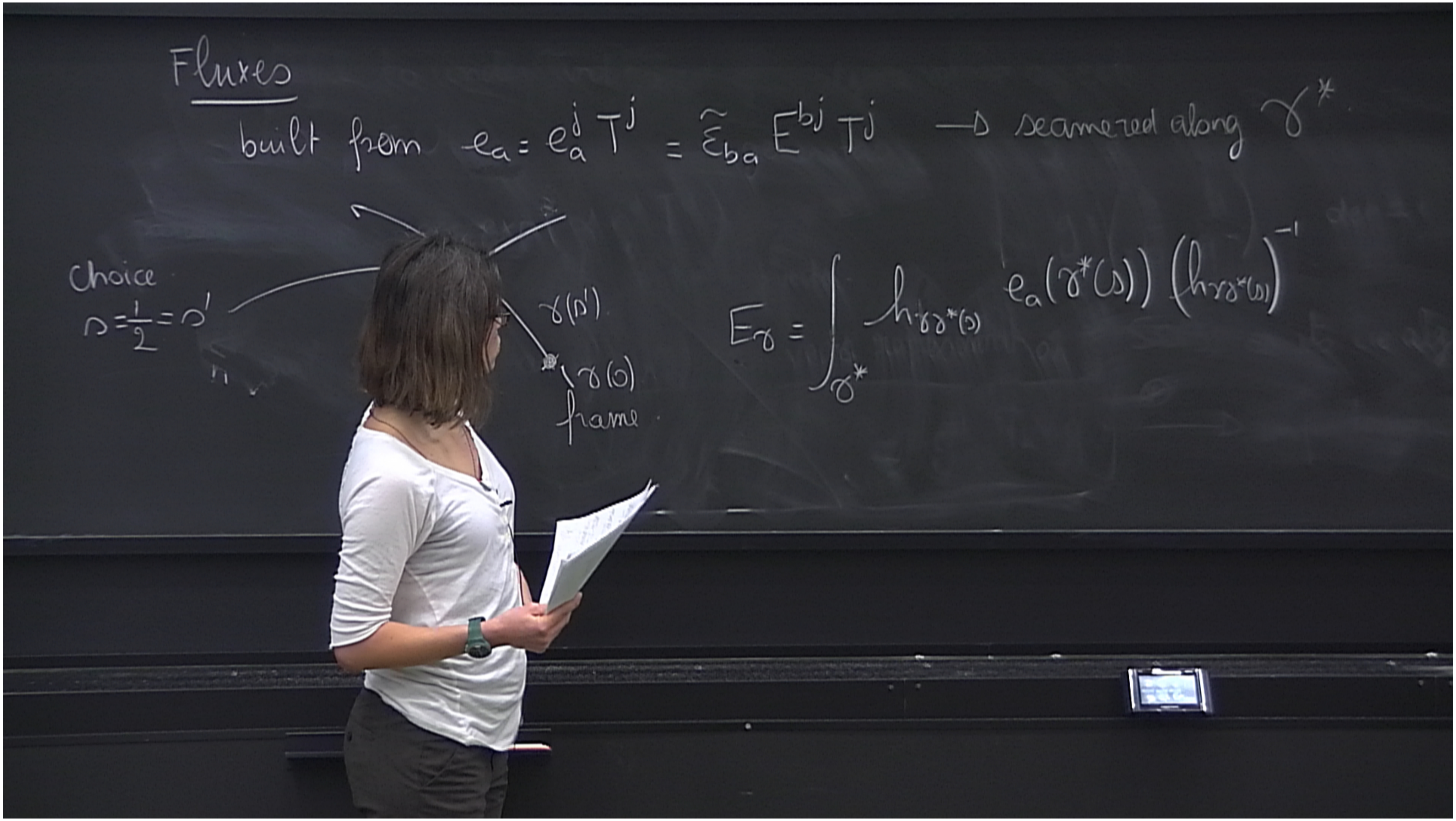
$h_e(A)$ is indpt of the parametrization of e


Fluxes
built from $e_a = e^j_a T^j = \tilde{\varepsilon}_{ba} E^{bj} T^j$ → seamered along γ^*


Fluxes
built from $e_a = e^j_a T^j = \tilde{\varepsilon}_{ba} E^{bj} T^j$ → smeared along γ^*.
γ
γ^*


Fluxes
built from $e_a = e_a^j T^j = \tilde{\varepsilon}_{ba} E^{bj} T^j$ → smeared along γ^*
Choice $s = \frac{1}{2} = s'$
$\gamma(s')$
$\gamma^*(s)$

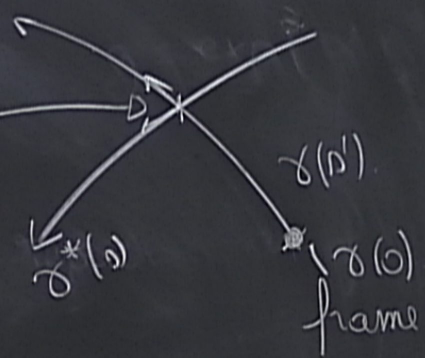

Fluxes
built from $e_a = e^j_a T^j = \tilde{\varepsilon}_{ba} E^{bj} T^j$ —▷ smeared along γ^*
choice
$= \frac{1}{2} = s'$
$\gamma(s')$
$\gamma^*(s)$


Fluxes
built from $e_a = e^j_a T^j = \tilde{\varepsilon}_{ba} E^{bj} T^j$ → smeared along γ^*
Choice
$s = \frac{1}{2} = s'$
$\gamma(s')$
$\gamma(0)$
frame
$E_\gamma = \int_{\gamma^*} h_{\gamma\gamma^*(s)} \, e_a(\gamma^*(s)) \left(h_{\gamma\gamma^*(s)}\right)^{-1}$

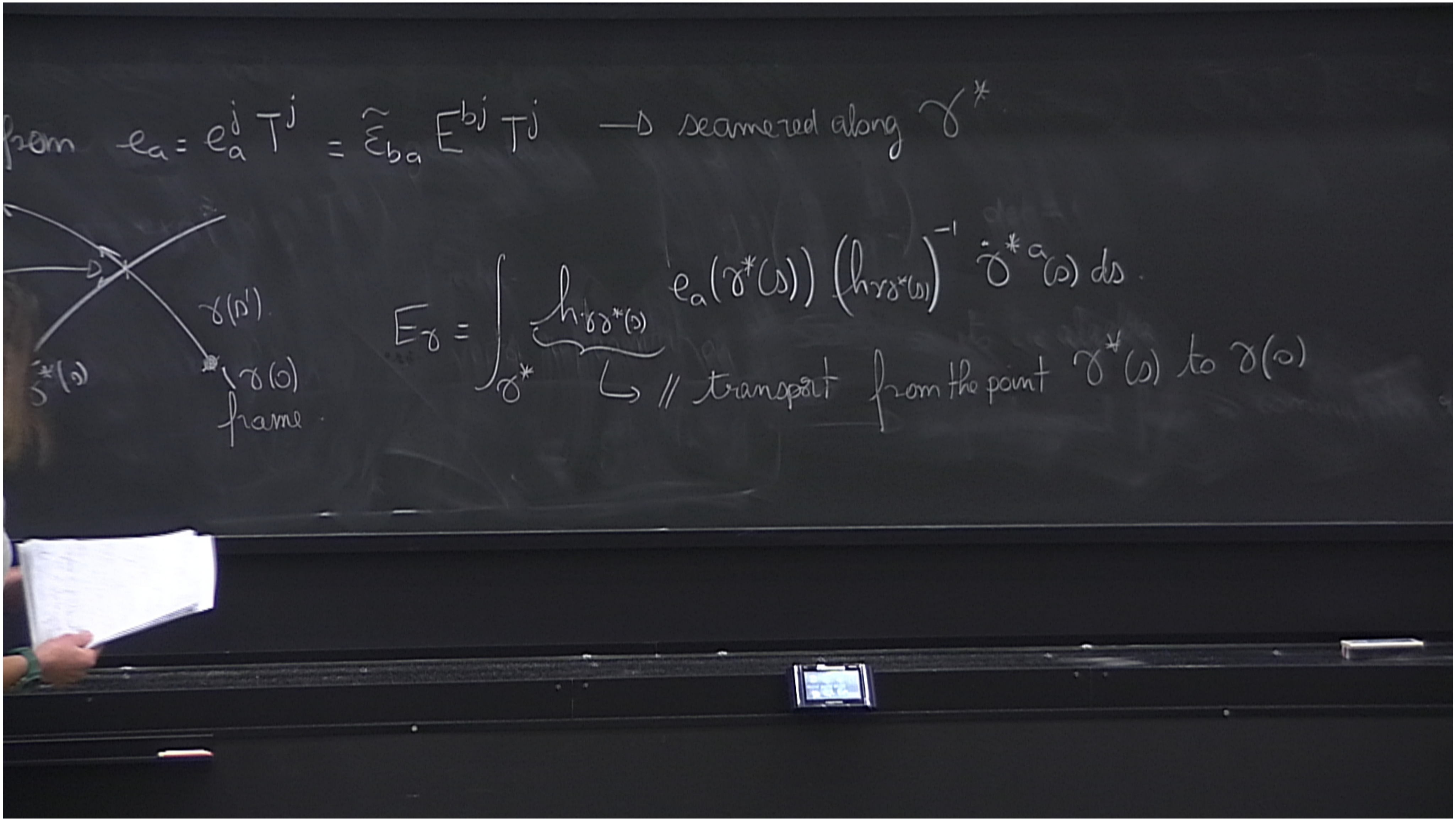
Fluxes

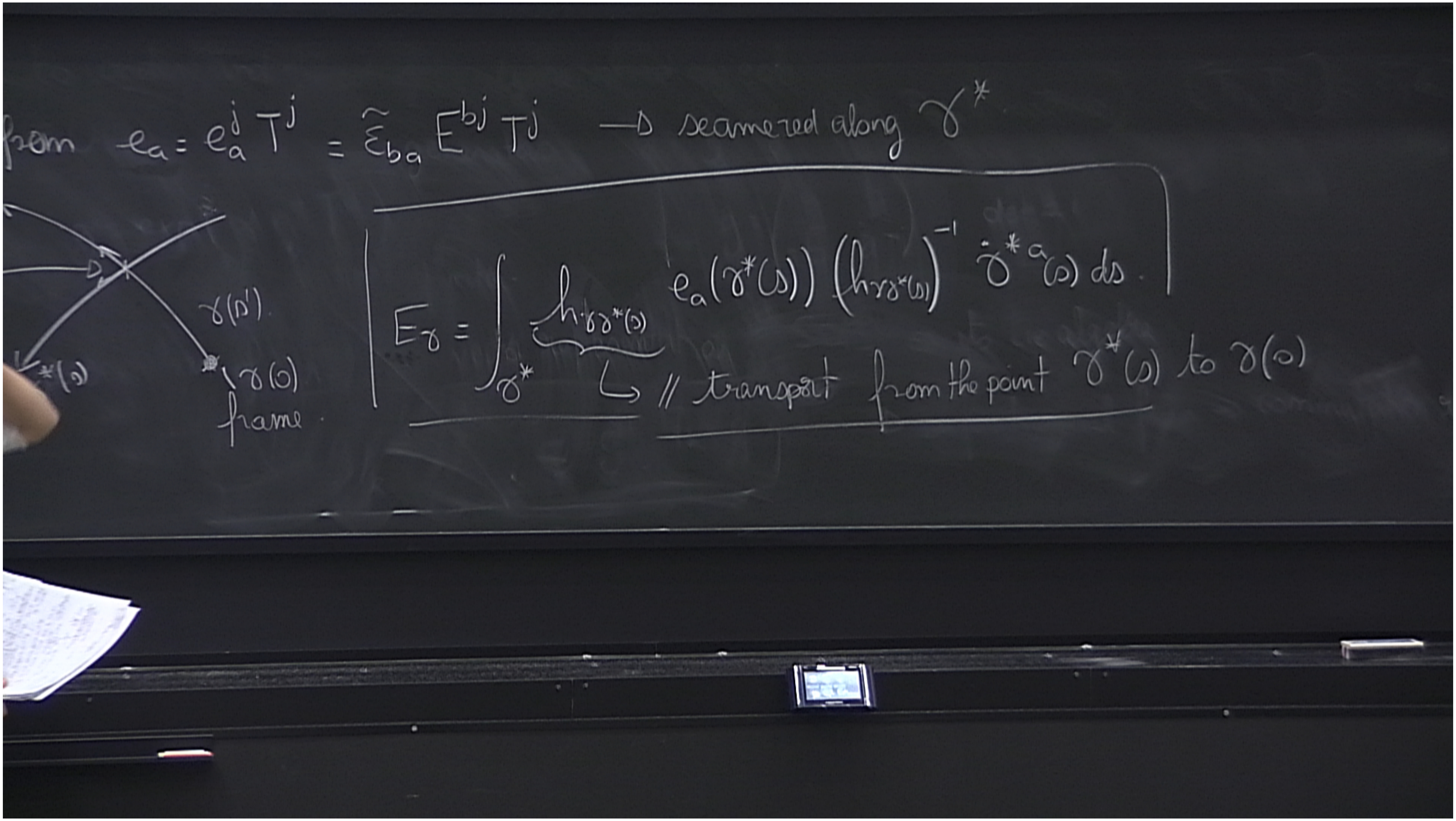
built from $e_a = e_a^j T^j = \tilde{\varepsilon}_{ba} E^{bj} T^j \longrightarrow$ smeared along γ^*

Choice
$s = \frac{1}{2} = s'$



$$E_\gamma = \int_{\gamma^*} h_{\gamma\gamma^*(s)} \, e_a(\gamma^*(s)) \left(h_{\gamma\gamma^*(s)}\right)^{-1}$$


from $e_a = e^j_a T^j = \tilde{\mathcal{E}}_{ba} E^{bj} T^j$ → smeared along γ^*
$E_\gamma = \int_{\gamma^*} \underbrace{h_{\gamma\gamma^*(s)}} e_a(\gamma^*(s)) \left(h_{\gamma\gamma^*(s)}\right)^{-1} \dot{\gamma}^{*a}(s)\, ds$.
↳ // transport from the point $\gamma^*(s)$ to $\gamma(0)$
$\gamma(s')$
$\gamma^*(s)$
$\gamma(0)$
frame


from $e_a = e_a^j T^j = \tilde{\mathcal{E}}_{ba} E^{bj} T^j$ → smeared along γ^*
$E_\gamma = \int_{\gamma^*} h_{\gamma\gamma^*(s)} \, e_a(\gamma^*(s)) \left(h_{\gamma\gamma^*(s)}\right)^{-1} \dot{\gamma}^{*a}(s) \, ds$
↳ // transport from the point $\gamma^*(s)$ to $\gamma(s)$
$\gamma(s')$
$\gamma(s)$
frame