

Title: PHYS 733 - Quantum Many-Body Physics (W2016) - Roger Melko - Lecture 18

Date: Mar 10, 2016 10:00 AM

URL: <http://pirsa.org/16030029>

Abstract:

Thermodynamic Properties in MFT:

As an explicit example $\frac{1}{N} \sum_{\langle ij \rangle} S$

Thermodynamic Properties in MFT:

As an explicit example $H = -J \sum_{\langle ij \rangle} S_i^z S_j^z - h \sum_i S_i^z$

And interpret ϕ as the magnetization m

$m) =$

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$$f(m) = \frac{1}{2} m^2 + \frac{u}{4} m^4 - h m$$

thermodynamics:

igne

And interpret ϕ as the magnetization m

$$f(m) = \frac{r}{2} m^2 + \frac{u}{4} m^4 - hm$$

Some thermodynamics near the critical temperature $T \sim T_c$

1) Magnetization

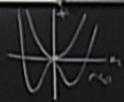
1) Magnetization: $\left. \frac{\partial f}{\partial m} \right|_{m=\bar{m}} = 0 = r\bar{m} + u\bar{m}^3 - h$

set $h=0$ $\bar{m} = \begin{cases} 0 & \text{for } r > 0 \\ \sqrt{\frac{-r}{u}} & \text{for } r < 0 \end{cases}$

ie. if we wrote this $\bar{m} = |T - T_c|^\beta$

This would give an exponent $\beta = \frac{1}{2}$
 Your first "critical exponent" $\sim$ MFT value of β

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Exponents are "universal" numbers describing
 singular power laws in the transition region.
 e.g. $\#$ 2D square $T_c = 2.27$ $T_c \sim 3$ but $\beta = \frac{1}{2}$ (mean-field)



Sound is coming

Singular power laws in the transition region.

e.g.  2D square $T_c = 2.27$ not universal but $\beta = \frac{1}{2}$
 $T_c \sim 3$ (mean-field)

2) Susceptibility: magnetic response.

$$\chi = \frac{\partial \overline{m}}{\partial h}$$

$f(m) = \frac{r}{2} m^2 + \frac{u}{4} m^4 - hm$
 Some thermodynamics near the critical temperature $T \sim T_c$
 Magnetization: $\frac{\partial f}{\partial m} \Big|_{h=0} = r\bar{m} + u\bar{m}^3 = 0$
 at $h=0$ $\bar{m} = \begin{cases} 0 & \text{for } r > 0 \\ \sqrt{\frac{-r}{u}} & \text{for } r < 0 \end{cases}$
 ie if we wrote this $\bar{m} = |T - T_c|^\beta$
 This would give an exponent $\beta = \frac{1}{2}$
 for first "critical exponent" (FT value is β)

Sound is coming

Exponents are "universal" numbers describing
 singular power laws in the transition region.
 e.g. $\frac{1}{2}$ and 3 are $T_c = 2.47$ not universal but $\beta = \frac{1}{2}$ (mean-field)
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 2) Susceptibility: magnetic response
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$$\chi = \left. \frac{\partial \bar{m}}{\partial h} \right|_{h=0}$$

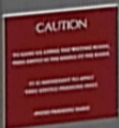
$$\chi^{-1} = \left. \frac{\partial h}{\partial \bar{m}} \right|_{h=0}$$

$$h = r\bar{m} + u\bar{m}^3 \Rightarrow \frac{\partial h}{\partial \bar{m}} = r + 3u\bar{m}^2$$

$$\chi = \frac{1}{r + 3u\bar{m}^2}$$

2 cases, $T < T_c$, $T > T_c$

$$T > T_c, \bar{m} = 0, \text{ and } \chi = \frac{1}{r} \sim \frac{1}{T - T_c}$$



2) Susceptibility: magnetic response.

$$\chi = \left. \frac{\partial \bar{m}}{\partial h} \right|_{h=0}$$

$$\chi' = \left. \frac{\partial h}{\partial \bar{m}} \right|_{h=0}$$

$$h = r \bar{m} + u \bar{m}^3 \Rightarrow \frac{\partial h}{\partial \bar{m}} = r + 3u \bar{m}^2$$

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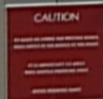
$$T < T_c, \bar{m}^2 = \left(\frac{-r}{u} \right) \Rightarrow \chi = \frac{1}{r + 3u \left(\frac{-r}{u} \right)} \sim \frac{1}{T_c - T}$$

Define another critical exponent

$$\chi = |T - T_c|^{-\gamma}$$

So at the MF level, $\gamma=1$.
3) Specific Heat. $C_v = -T \frac{\partial^2 f}{\partial T^2}$

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= 0

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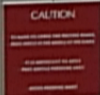
3) Specific Heat. $C_v = -T \frac{\partial^2 f}{\partial T^2}$

$T > T_c$, $f(\bar{m}) = 0$ since $\bar{m} = 0$.

$T < T_c$, $f(\bar{m}) = \frac{1}{2} \left(-\frac{r}{u} \right) + \frac{u}{4} \left(-\frac{r}{u} \right)^2 = -\frac{r}{4u}$

$\Rightarrow f = -\frac{1}{4u}(T-T_c)^2$, $\frac{\partial f}{\partial T} = -\frac{1}{2u}(T-T_c)$, $\frac{\partial^2 f}{\partial T^2} = -\frac{1}{2u}$

Define another
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 $\chi = |T-T_c|^{-\gamma}$

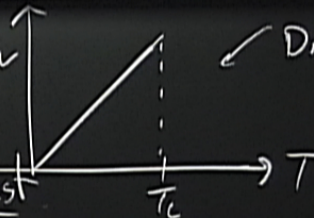


$$\Rightarrow f = -\frac{1}{4u}(T-T_c)^2, \quad \frac{\partial f}{\partial T} = -\frac{2}{4u}(T-T_c), \quad \frac{\partial^2 f}{\partial T^2} = -\frac{2}{4u}$$

$$\text{So } C_v = \frac{I}{2u} (T < T_c) \quad \text{Discontinuity}$$

$$\text{If we write } C_v = |T-T_c|^{-\alpha} + \underline{\text{const}}$$

Then $\alpha=0$ in MFT.

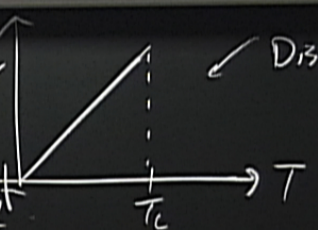


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$$\text{So } C_v = \frac{T}{2u} (T < T_c) \quad C_v \quad \checkmark \text{ Discontinuity}$$

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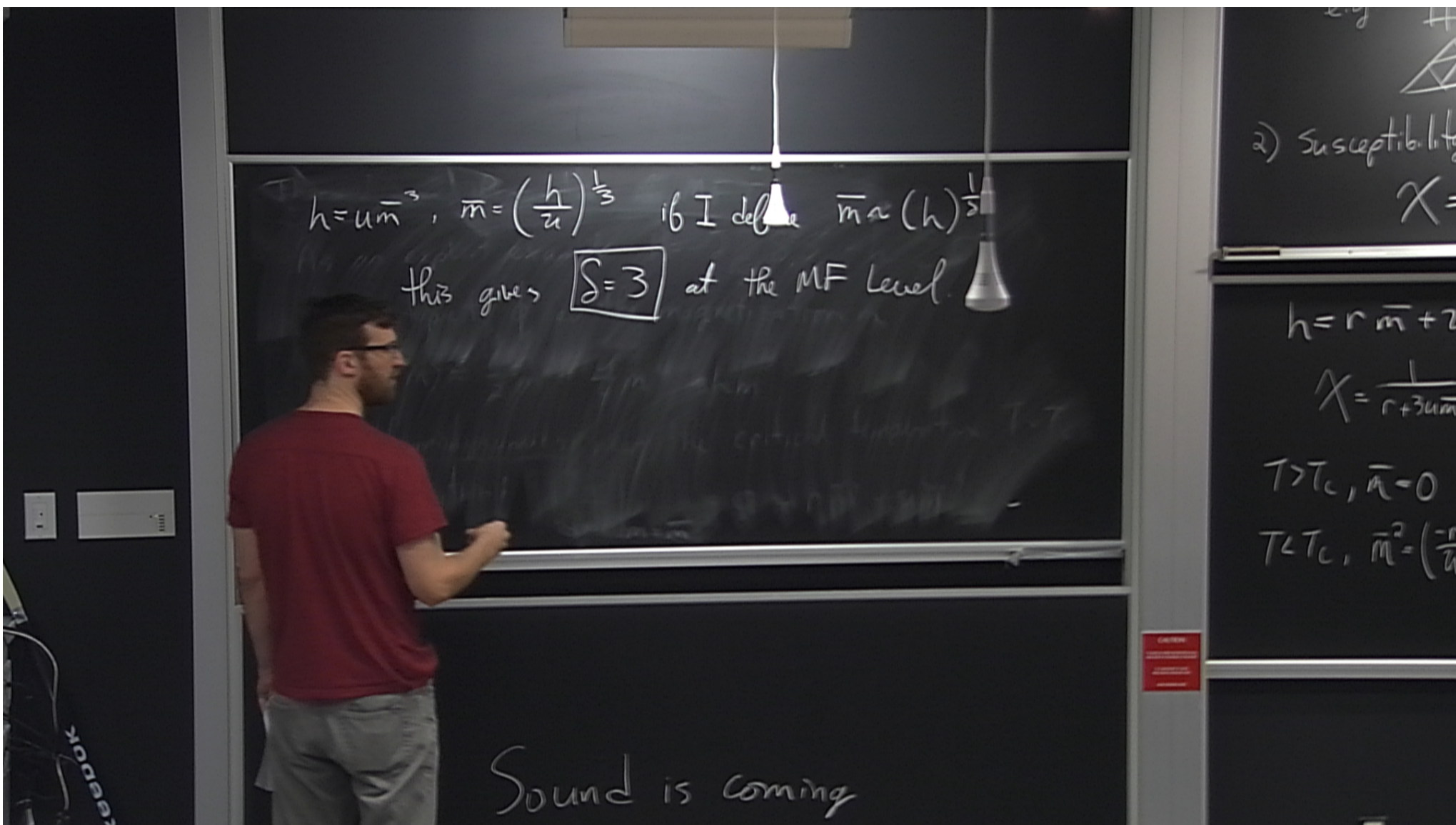


Then $\boxed{\alpha=0}$ in MFT.

4) Critical isotherm: $T=T_c$, finite h

Solve $\left. \frac{\partial f}{\partial m} \right|_{m=\bar{m}} = 0 = r\bar{m} + u\bar{m}^3 - h$ but $r=0$

Define another
critical exponent
 $\chi = |T-T_c|^{-\gamma}$



$$h = u\bar{m}^3, \quad \bar{m} = \left(\frac{h}{u}\right)^{\frac{1}{3}} \quad \text{if I define } \bar{m} \propto (h)^{\frac{1}{d}}$$

this gives $\boxed{d=3}$ at the MF level.

2) Susceptibility

$$\chi =$$

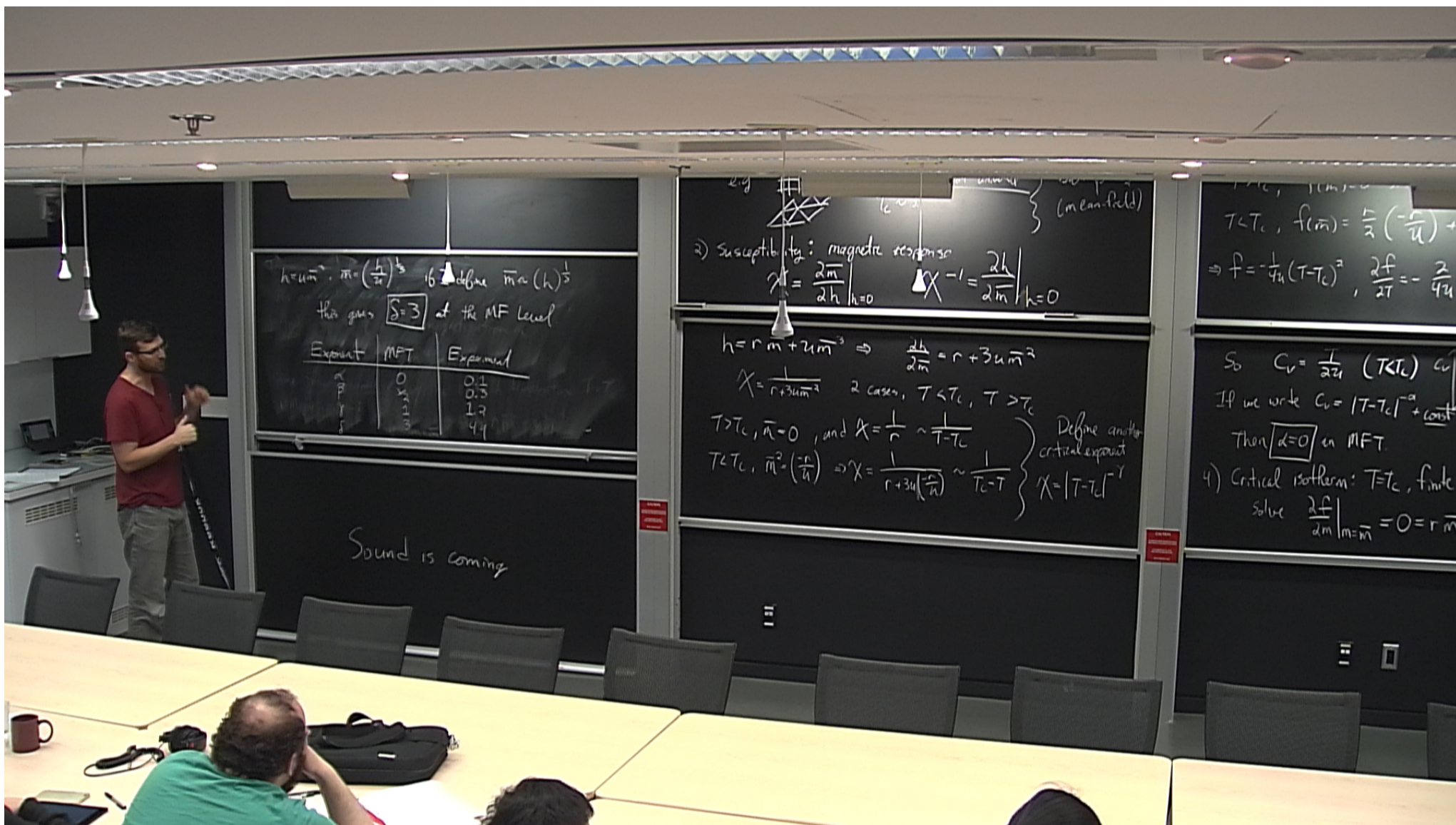
$$h = r\bar{m} + 2$$

$$\chi = \frac{1}{r + 3u\bar{m}}$$

$$T > T_c, \bar{m} = 0$$

$$T < T_c, \bar{m}^2 = \left(\frac{-r}{2u}\right)$$

Sound is coming



$$h = u\bar{m}^3, \bar{m} = \left(\frac{h}{u}\right)^{\frac{1}{3}} \text{ if I define } \bar{m} \propto (h)^{\frac{1}{3}}$$

this gives $\boxed{\delta=3}$ at the MF level.

Exponent	MFT	Experiment
α	0	0.1
β	$\frac{1}{2}$	0.3
γ	1	1.7
δ	3	4.9

Sound is coming

2) Susceptibility
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Exponent	MFT	Experiment
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} dimensionality
matters

- note: for quantum critical points
(e.g. in a D-dimensional TFM)
critical exponents are the same as for the D+1
dimensional classical stat mech model.

Sound is coming

2) Susceptibility
 $\chi =$

$$h = r \bar{m} + 2$$

$$\chi = \frac{1}{r + 3u\bar{m}}$$

$$T > T_c, \bar{m} = 0$$

$$T < T_c, \bar{m}^2 = \left(\frac{-r}{2u} \right)$$

A4Q3: Consider the case where you have 2 fields ϕ_1, ϕ_2
 Construct a Lagrangian (assume invariance under $\phi \rightarrow -\phi$)

$$L = \frac{1}{2}(\dot{\phi}_1)^2 - \frac{1}{2}m_1^2 \phi_1^2 - \frac{u}{4}\phi_1^4$$

$$+ \frac{1}{2}(\dot{\phi}_2)^2 - \frac{1}{2}m_2^2 \phi_2^2 - \frac{u}{4}\phi_2^4 - \frac{p}{2}\phi_1^2\phi_2^2$$

$$T < T_c, \quad f(\bar{m}) = \frac{1}{2} \left(\frac{T - T_c}{T_c} \right)^2$$

$$\Rightarrow f = -\frac{1}{4u}(T - T_c)^2, \quad \frac{\partial f}{\partial T}$$

$$S_0 = \frac{I}{2u} (T - T_c)$$

If $m_c = |T - T_c|$

Then $T = T_c$ is a fixed point.

4) Critical

$$L = \frac{1}{2}(\dot{\phi}_1)^2 - \frac{1}{2}m_1^2 \phi_1^2 - \frac{1}{4}\phi_1^4 + \frac{1}{2}(\dot{\phi}_2)^2 - \frac{1}{2}m_2^2 \phi_2^2 - \frac{1}{4}\phi_2^4 - \frac{\rho}{2}\phi_1^2\phi_2^2 \quad \left. \begin{array}{l} U(x,t; x_i, t_i) \\ \text{Minkowski} \end{array} \right\}$$

- with all parameters arbitrary. ϕ_1 & ϕ_2 have no real relationship

However: impose $\phi_1 = \phi_2$
make $\phi_1 \leftrightarrow \phi_2$ interchange

$\Rightarrow$

$$T < T_c, \quad f(\bar{m}) = \frac{1}{2} \left(\frac{T - T_c}{T_c} \right)^2$$

$$\Rightarrow f = -\frac{1}{4u}(T - T_c)^2, \quad \frac{\partial f}{\partial T}$$

$$S_0 \quad C_v = \frac{I}{2u} (T - T_c)$$

If we write $C_v = |T - T_c|$

Then $\boxed{d=0}$ in MFT.

4) Critical isotherm: $T = T_c$
Solve $\frac{\partial f}{\partial m} \Big|_{m=\bar{m}} = 0$

$$L = \frac{1}{2}(\dot{\phi}_1)^2 - \frac{1}{2}m_1^2 \phi_1^2 - \frac{u_1}{4}\phi_1^4 + \frac{1}{2}(\dot{\phi}_2)^2 - \frac{1}{2}m_2^2 \phi_2^2 - \frac{u_2}{4}\phi_2^4 - \frac{p}{2}\phi_1^2\phi_2^2 \quad \left. \begin{array}{l} U(x,t; x',t') \\ \text{Minkowski} \end{array} \right\}$$

- with all parameters arbitrary, ϕ_1 & ϕ_2 have no real relationship

However: impose constraints:

make L invariant under interchange $\phi_1 \leftrightarrow \phi_2$

$$\Rightarrow m_1 = m_2 \text{ and } u_1 = u_2$$

Also let me choose $p = u_1 = u_2$

$$-\frac{1}{4}(\phi_1 + \phi_2)$$

This Lagrangian is invariant under 2D rotation

ie. $\phi_1(x) \rightarrow \cos \theta \phi_1(x) + \sin \theta \phi_2(x)$

$$\phi_2(x) \rightarrow -\sin \theta \phi_1(x) + \cos \theta \phi_2(x)$$

where θ is an arbitrary angle.

$O(2)$ rotation $\rightarrow$ a continuous symmetry.

$O(2)$ model of your assignment

$$\text{i.e. } L = \frac{1}{2} [(\partial_\mu \phi)^2 - m^2 \phi^2] - \frac{\lambda}{4} (\phi^2)^2$$

Can relate this case to the Lagrangian of a complex field

$$\phi \equiv$$

$$\text{i.e. } L = \frac{1}{2} [(\partial_\mu \vec{\phi})^2 - m^2 \vec{\phi}^2] - \frac{\lambda}{4} (\vec{\phi}^2)^2$$

Can relate in this case to the Lagrangian of a complex field

define $\phi \equiv (\phi_1 + i\phi_2) \frac{1}{\sqrt{2}}$ (a complex field)

$$\therefore \mathcal{L} = \partial_\mu \phi^\dagger \partial^\mu \phi - m^2 \phi^\dagger \phi - \frac{\lambda}{4} (\phi^\dagger \phi)^2$$

Consider: $\mathcal{L} = 2\phi^\dagger 2\phi - m^2 \phi^\dagger \phi + u(\phi^\dagger \phi)^2$
 $= 2(\phi_1 - i\phi_2)/\sqrt{2} \cdot 2(\phi_1 + i\phi_2)/\sqrt{2}$

$$- m^2 (\phi_1 - i\phi_2)/\sqrt{2} \cdot (\phi_1 + i\phi_2)/\sqrt{2} - u \left[\frac{1}{2} (\phi_1 - i\phi_2)(\phi_1 + i\phi_2) \right]^2$$

$$= \frac{1}{2} \left[(2\phi_1)^2 + (2\phi_2)^2 - \frac{1}{2} m^2 (\phi_1^2 + \phi_2^2) - \frac{u}{4} (\phi_1^2 + \phi_2^2)^2 \right]$$

Note: the Lagrangian for the complex field is invariant
under a U(1) transformation
 $\phi \rightarrow e^{i\theta} \phi, \quad \phi^\dagger \rightarrow e^{-i\theta} \phi^\dagger$

This Lagrangian is invariant

ie. $\phi \rightarrow \cos \theta \phi + \sin \theta \phi^\dagger$
 $\phi^\dagger \rightarrow \cos \theta \phi^\dagger - \sin \theta \phi$
any a
ntiv
of you

This is a global gauge symmetry - the freedom of a QM system under transformation by a phase.

Let's parameterize the theory by $\phi(x) = \rho(x) e^{i\theta(x)}$
 then $\partial\phi = (\partial\rho + i\rho\partial\theta) e^{i\theta}$
 $\Rightarrow \mathcal{L} = (\partial\rho - i\rho\partial\theta) e^{-i\theta} (\partial\rho + i\rho\partial\theta) e^{i\theta}$
 $- m^2 \rho^2 e^{-i\theta} e^{i\theta} - u (\rho^2 e^{-i\theta} e^{i\theta})^2$

$$\mathcal{L} = \frac{1}{2} [(\partial\phi)^2 + (\partial\phi)^2]$$

is invariant

(2) model of you

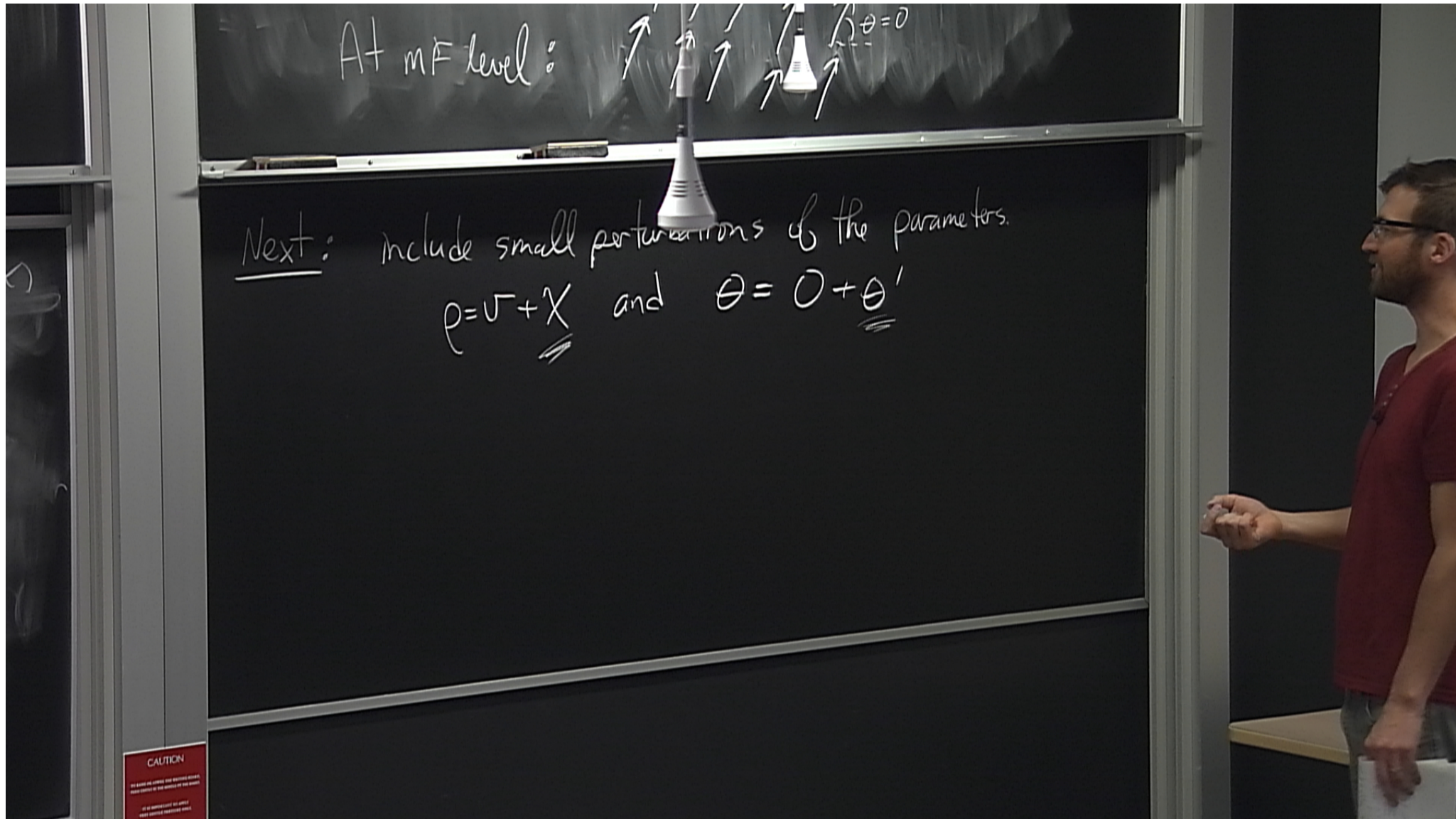
$$\mathcal{L} = \rho^2 (\partial_\theta)^2 + (\partial_\rho)^2 - m^2 \rho^2 - u \rho^4$$

CAUTION

$$\mathcal{L} = \rho^2 (\partial_\theta)^2 + (\partial_\rho)^2 - m^2 \rho^2 - \lambda \rho^4$$

Similar to before, a mean field (vacuum) expectation value exists: $\rho = v$ (find the extrema of the potential)

CAUTION



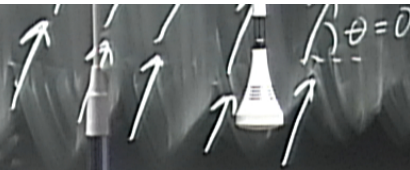
At mF level: $\uparrow \uparrow \uparrow \uparrow \uparrow$ $\hbar\theta=0$

Next: include small perturbations of the parameters.

$$\rho = v + \chi \quad \text{and} \quad \theta = 0 + \theta' \leftarrow \text{drop } \hbar$$

$$\begin{aligned} \mathcal{L} &= (v + \chi)^2 (2\theta')^2 + (2\overset{0}{v} + 2\chi)^2 - m^2(v + \chi)^2 - u(v + \chi)^4 \\ &= (v^2 + 2v\chi + \chi^2)(2\theta'^2) + (2\overset{0}{v})^2 - m^2v^2 - 2m^2v\chi - m^2\chi^2 \end{aligned}$$

At mF level:



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$$\begin{aligned} \mathcal{L} &= (v + \chi)^2 (2\theta')^2 + (2\overset{0}{v} + 2\chi)^2 - m^2(v + \chi)^2 - u(v + \chi)^4 \\ &= (v^2 + 2v\chi + \chi^2)(2\theta'^2) + (2\chi)^2 - \underset{\text{const}}{m^2 v^2} - 2m^2 v\chi - m^2 \chi^2 \\ &\quad - 4uv^3\chi - 6uv^2\chi^2 - 4uv\chi^3 - u\chi^4 + \text{constants} \end{aligned}$$

I.e. there is no Θ^2 term

So Θ is called a massless field:

Generally - if we spontaneously break a continuous symmetry, the amplitude (ρ) mode results in massive particles and the phase (Θ) results in a massless $\rho \Rightarrow$ Goldstone mode

$$= \frac{1}{2} \left[(\phi_1^2 + \phi_2^2)^2 - \frac{1}{2} m^2 (\phi_1^2 + \phi_2^2) - \frac{\mu}{4} (\phi_1^2 + \phi_2^2)^2 \right]$$

Generally - if we spontaneously break a continuous symmetry, the amplitude (ρ) mode results in a massive particle, and the phase (θ) results in a massless particle $\Rightarrow$ Goldstone mode

Goldstone mode

Let's
then - a
 $\Rightarrow \mathcal{L} = (2)$
 $-m^2 \rho$

We can examine the effect of the massless mode
on the stability of the phase.

Consider: $\phi_1 = v + \phi_1'$, $\phi_2 = \phi_2'$ $\swarrow$ this gives a massless mode

Consider $\angle 4$.

$$-m^2 p^2 e^{-i\theta} e^{i\theta} - u (p^2 e^{-i\theta} e^{i\theta})^2$$

Value

At

Next: m

$$L = (v+x)$$

$$= (v^2 + 2v$$

-1

Consider $\langle \phi_2'^2(0) \rangle = \frac{1}{Z} \int D[\phi] e^{iS[\phi]} \phi_2'(0) \phi_2'(0)$
 $= \lim_{x \rightarrow 0} \langle \phi_2'(x) \phi_2'(0) \rangle$

$$\simeq \lim_{x \rightarrow 0} G_0(x, 0) = \lim_{x \rightarrow 0} \int \frac{d^d p}{(2\pi)^d} \frac{e^{ipx}}{p^2 + m^2}$$

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At

Next: m

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-1

CAUTION

DO NOT TOUCH THE BOARD WHEN IT IS HOT
 IT IS DANGEROUS TO TOUCH THE BOARD WHEN IT IS HOT

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$\simeq \lim_{x \rightarrow 0} G_0(x, 0) = \lim_{x \rightarrow 0} \int \frac{d^d p}{(2\pi)^d} \frac{e^{ipx}}{p^2 + m^2} \quad m=0$
 write $\int_0^1 \frac{d^d p}{(2\pi)^d} \frac{1}{p^2} = \frac{S_d}{(2\pi)^d} \int_0^1 \frac{p^{d-1} dp}{p^2} = \frac{S_d}{(2\pi)^d} \int_0^1 dp p^{d-3}$

Consider $\langle \phi_2'^2(0) \rangle = \frac{1}{Z} \int D[\phi] e^{iS[\phi]} \phi_2'(0) \phi_2'(0)$
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$\approx \lim_{x \rightarrow 0} G_0(x, 0) = \lim_{x \rightarrow 0} \int \frac{d^d p}{(2\pi)^d} \frac{e^{ipx}}{p^2 + m^2} \quad \underline{m=0}$ lattice
lengthscale

write $\int_0^1 \frac{d^d p}{(2\pi)^d} \frac{1}{p^2} = \frac{S_d}{(2\pi)^d} \int_0^1 \frac{p^{d-1} dp}{p^2} = \frac{S_d}{(2\pi)^d} \int_0^1 dp p^{d-3}$

$d > 2$ integral converges, $d \leq 2$ diverges

$$\phi'_2(0) \phi'_1(0)$$

$$(0)$$

$m=0$

lattice
lengthscale.

$$\frac{S_d}{(2\pi)^d} \int_0^{\Lambda} dp p^{d-3}$$

where θ is an arbitrary angle.
 $O(2)$ rotation $\rightarrow$ a continuous symmetry.

Divergence $\Rightarrow$ transverse fluctuations become "infinite"
 killing the long-range order
 $\Rightarrow$ Mermin-Wagner theorem
 No long-range order in systems with a
 spontaneously broken continuous symmetry in $d \leq 2$

$$-4uv^2x - 6uv^2x - 4uv^2x - 4x + \text{constants}$$