Title: PHYS 733 - Quantum Many-Body Physics (W2016) - Roger Melko - Lecture 17

Date: Mar 08, 2016 10:00 AM

URL: http://pirsa.org/16030028

Abstract:

$$\int \mathcal{D}[x]\, e \quad = \int d[x]\, e$$

But, to write the partition function, wick rotate

$t \to -i\tau$

$$Z = \int \mathcal{D}[x]\, e^{i\int (-i)\, d\tau \left[\frac{1}{2} m \left(\frac{dx}{d(-i\tau)}\right)^2 - V(x)\right]}$$

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$$= \int \mathcal{D}[x]\, e^{+\int d\tau \left[-\frac{1}{2}m\dot{x}^2 - V(x)\right]}$$

$$= \int \mathcal{D}[x]\, e^{-\int d\tau \left[\frac{1}{2}m\dot{x}^2 + V(x)\right]}$$

Similar for the ϕ^4 theory.

$$U(x',t';x,t) = \int D[\phi]\, e^{i\int d^dx\, \mathcal{L}(\phi)}$$

then $$\mathcal{L}(\phi) = \frac{1}{2}\left[(\partial\phi)^2 - m^2\phi^2\right] = \frac{1}{2}\left[(\partial\phi)^2 - V(\phi)\right]$$

$$(\partial\phi)^2 = \left(\frac{\partial\phi}{\partial t}\right)^2 - \left(\frac{\partial\phi}{\partial x}\right)^2 - \cdots \Rightarrow$$

sign structure

$+ - - -$ Minkowski

Wick rotate $t \to i\tau$

$\mathcal{L}_0 \to$ has $\left(\frac{\partial\phi}{\partial(-i\tau)}\right)^2 - \left(\frac{\partial\phi}{\partial x}\right)^2 - \left(\frac{\partial\phi}{\partial y}\right)^2 - \cdots$

$$\Rightarrow \quad Z = \int D[\phi]\, e^{-\int d^2x\, \mathcal{L}(\phi)}$$

where $\mathcal{L}(\phi) = \frac{1}{2}\left[(\partial\phi)^2 + V(\phi)\right]$

Euclidean $++++$

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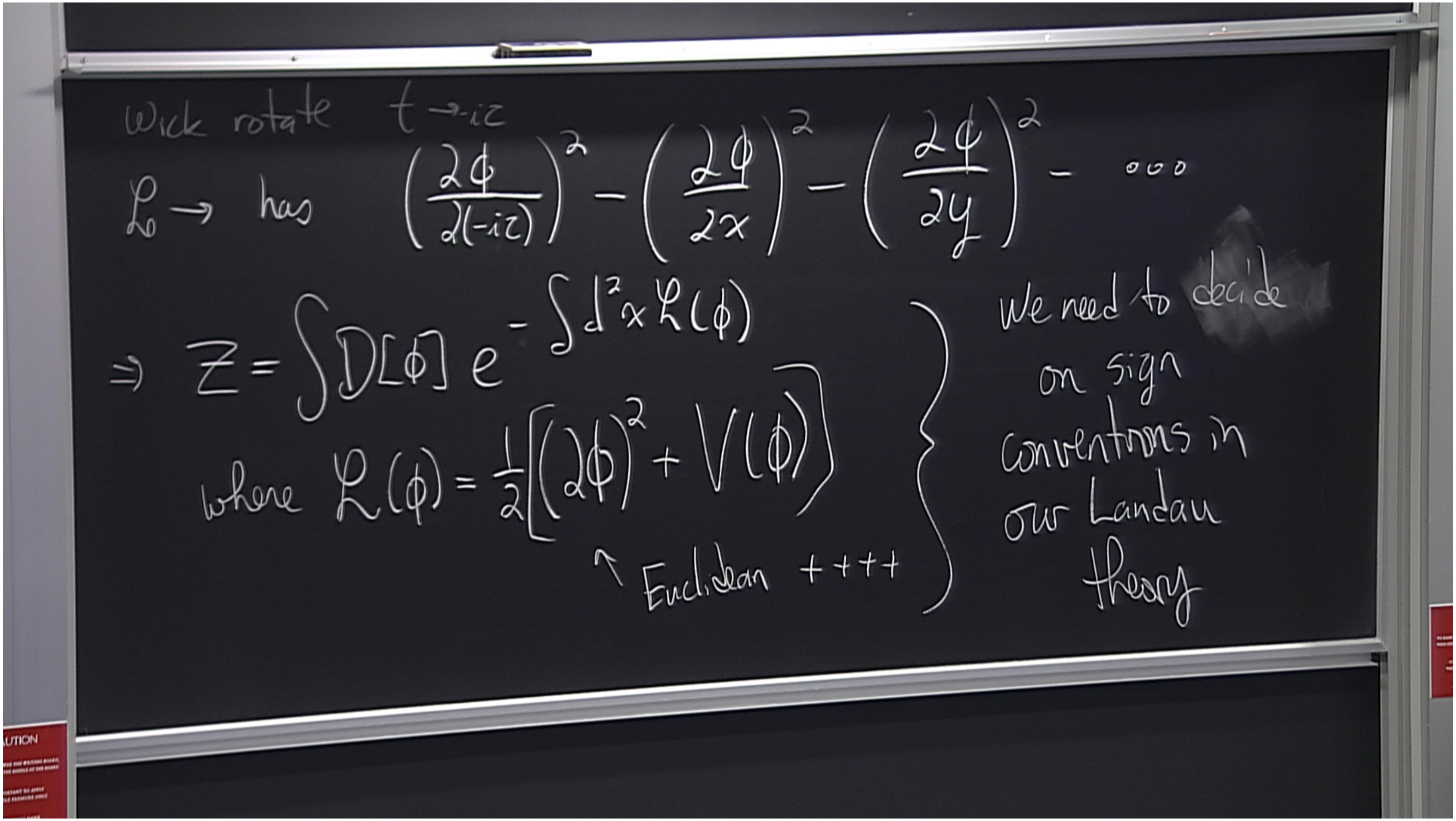
↑ Euclidean $+ + + +$

We need to decide on sign conventions in our Landau theory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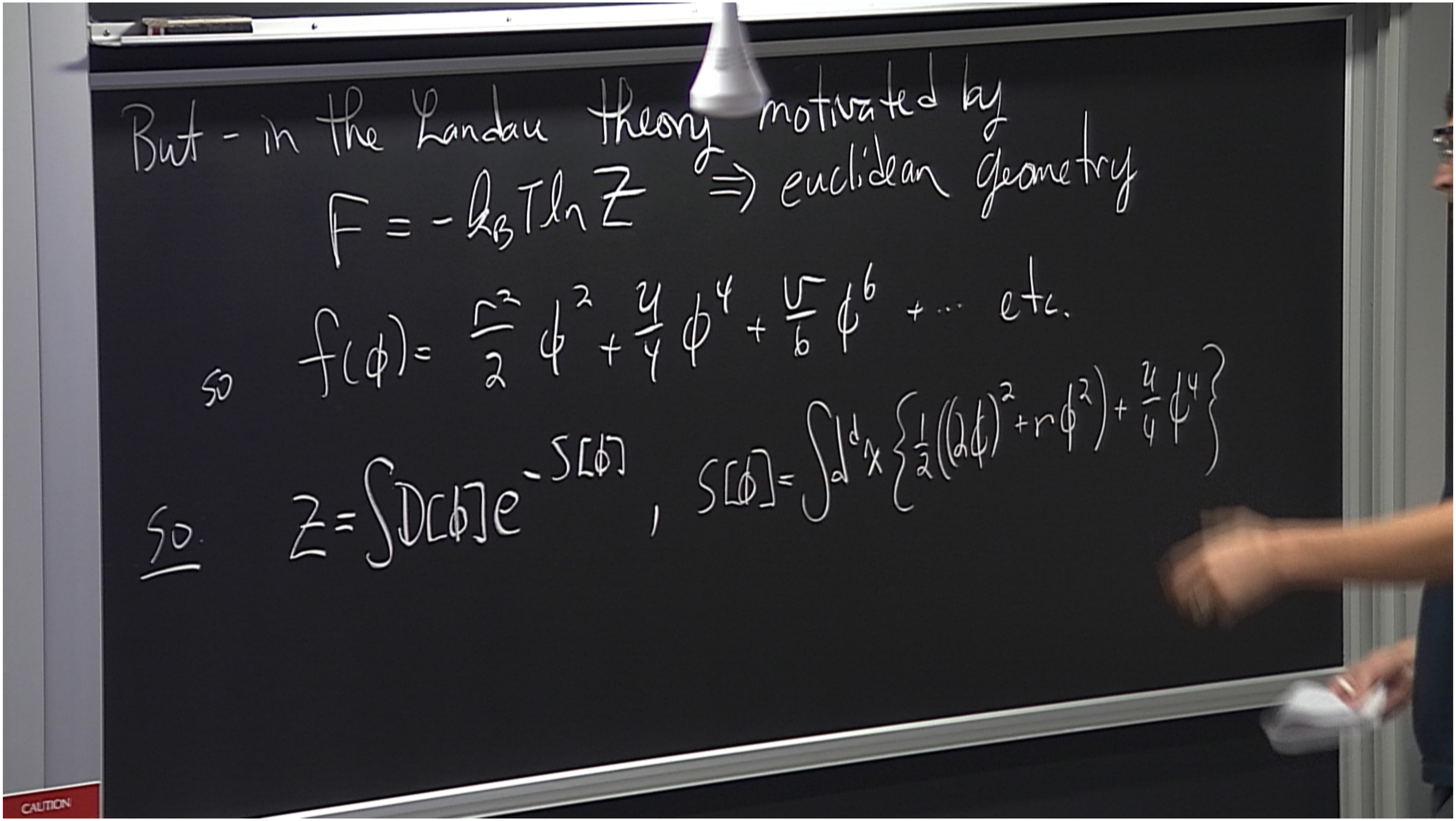
Euclidean $++++$
We need to decide on sign conventions in our Landau theory

e.g.) $f(\phi) = V(\phi) = \frac{r}{2}\phi^2 + \frac{u}{4}\phi^4$

When I derived the K-G equation, c.f. Minkowski, so

$$\mathcal{L}(\phi) = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{r}{2}\phi^2 - \frac{u}{4}\phi^4$$

$$u = 0 \Rightarrow (\partial^2 + r)\phi = 0 \overset{\text{cf}}{\Rightarrow} (\partial^2 + m^2)\phi = 0$$


But - in the Landau theory motivated by
$F = -k_B T \ln Z$ $\Rightarrow$ euclidean geometry
so $f(\phi) = \frac{r}{2}\phi^2 + \frac{u}{4}\phi^4 + \frac{v}{6}\phi^6 + \cdots$ etc.
So. $Z = \int D[\phi]\, e^{-S[\phi]}$, $S[\phi] = \int d^d x \left\{ \frac{1}{2}\left((\partial\phi)^2 + r\phi^2\right) + \frac{u}{4}\phi^4 \right\}$
CAUTION

But - in the Landau theory motivated by $\Rightarrow$ euclidean geometry

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Spontaneous Symmetry breaking: $r \to \tilde{r}(T - T_c)$

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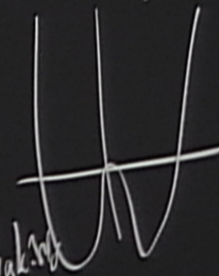
Spontaneous Symmetry breaking: $r \to \tilde{r}(T - T_c)$, $r < 0$

symmetry breaking

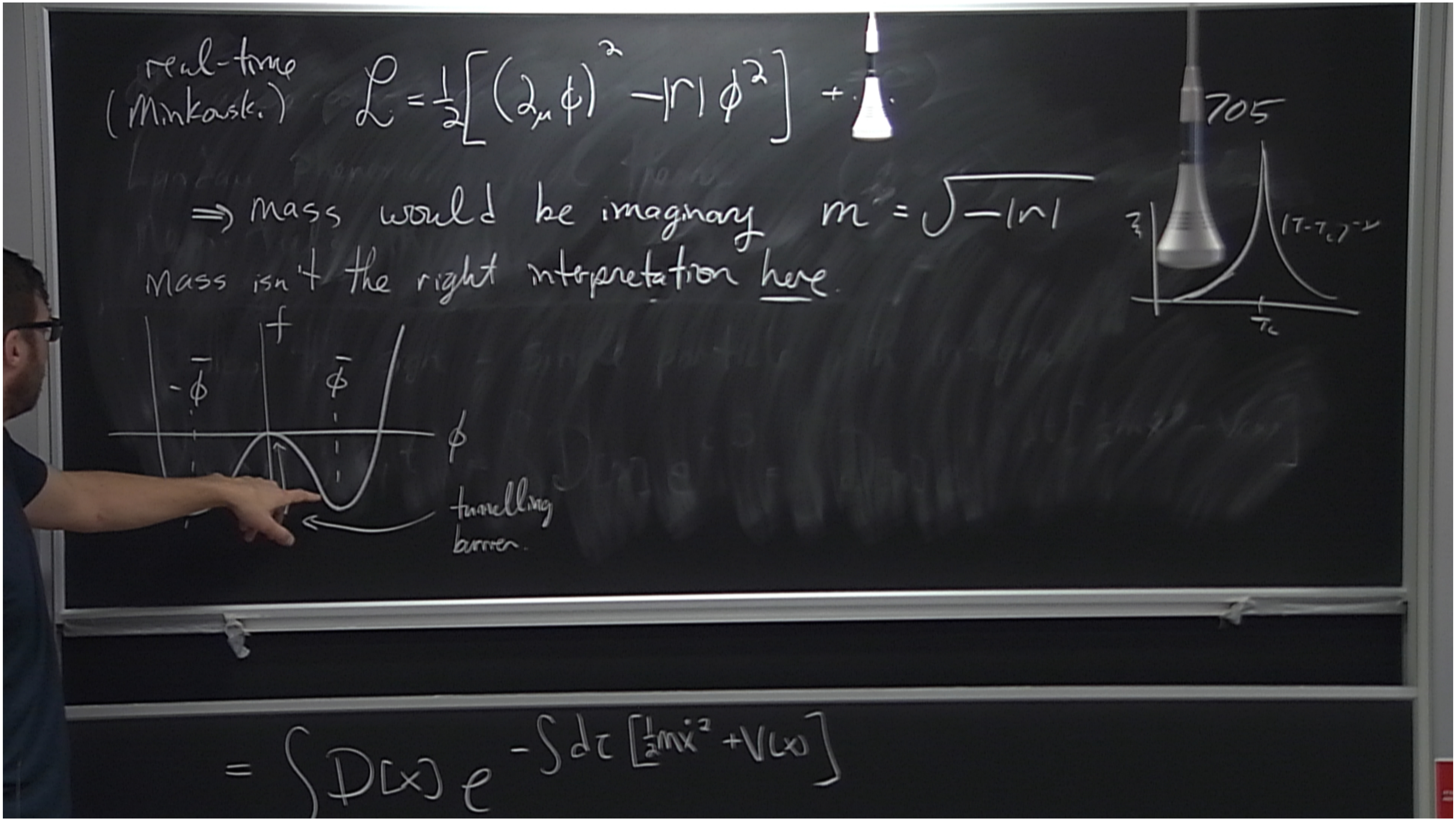


real-time
(Minkowski)

$$\mathcal{L} = \frac{1}{2}\left[(\partial_\mu \phi)^2 - |r|\phi^2\right] + \dots$$

$\Rightarrow$ mass would be imaginary $m = \sqrt{-|r|}$

$$= \int D[x]\, e^{-\int d\tau \left[\frac{1}{2}m\dot{x}^2 + V(x)\right]}$$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Mass isn't the right interpretation here.
705
ξ
$|T-T_c|^{-\nu}$
T_c
f
$-\bar{\phi}$
$\bar{\phi}$
ϕ
tunnelling
barrier.
$= \int D[x]\, e^{-\int d\tau \left[\frac{1}{2} m \dot{x}^2 + V(x)\right]}$

real-time (Minkowski) $\mathcal{L} = \frac{1}{2}\left[(\partial_\mu \phi)^2 - |r|\phi^2\right] + \dots$

705

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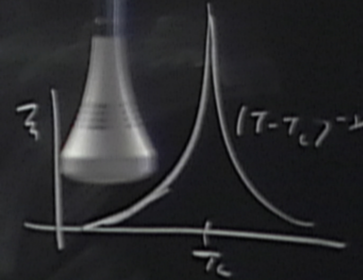
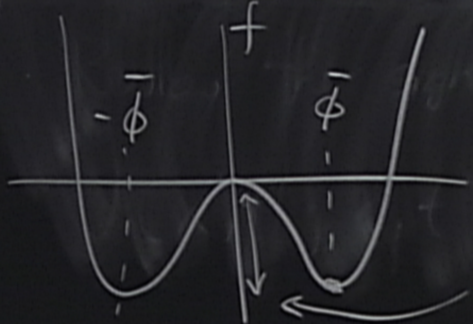
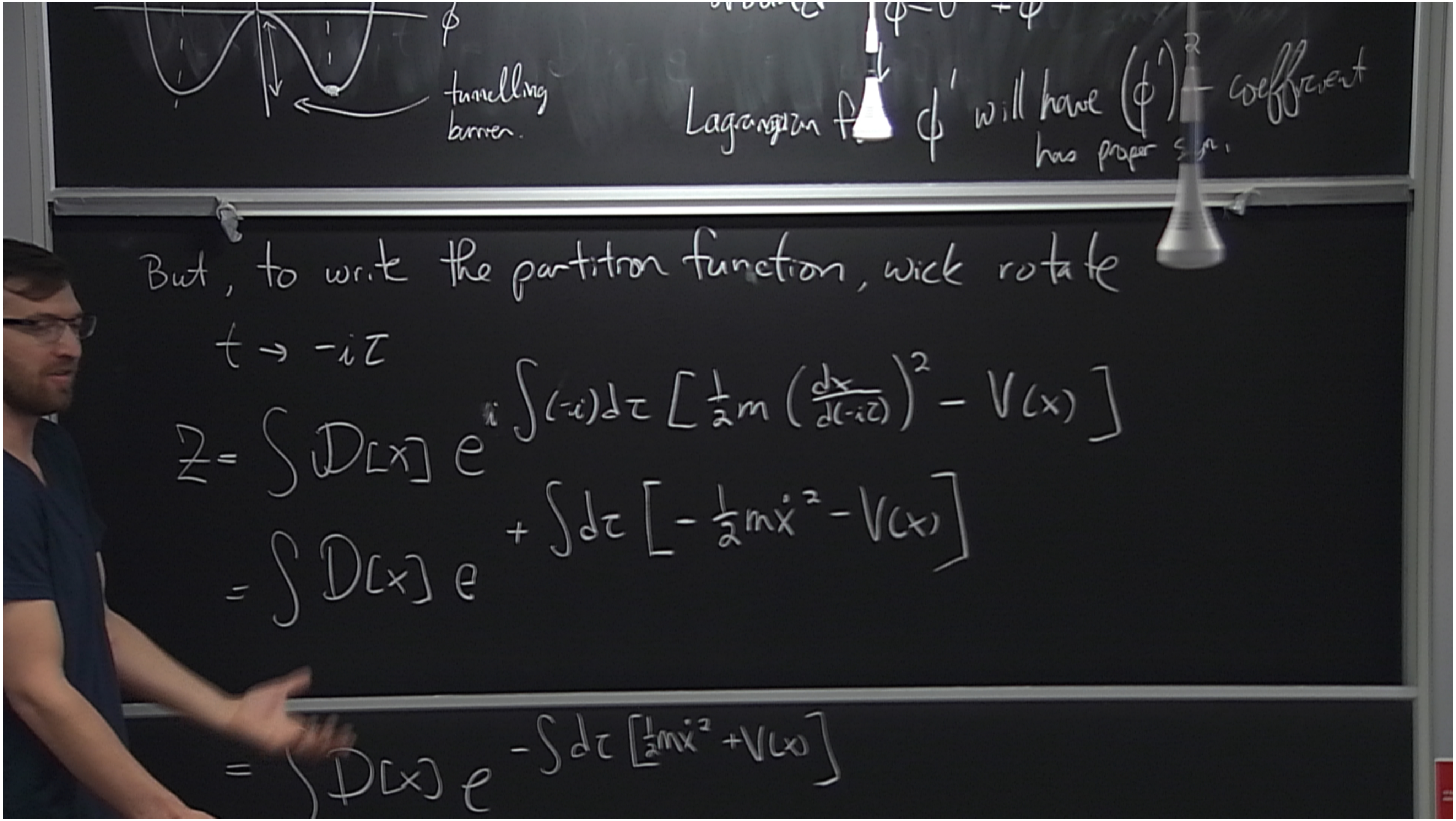


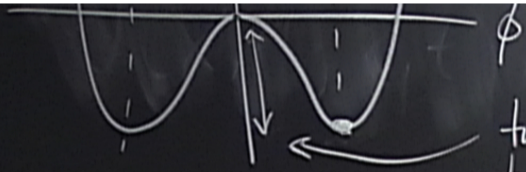


tunnelling barrier.

look at fluctuations around $\bar\phi = v$

$$= \int D[x]\, e^{-\int d\tau \left[\frac{1}{2}m\dot{x}^2 + V(x)\right]}$$


tunnelling barrier
Lagrangian f
ϕ' will have $(\phi')^2$ coefficient
has proper sign.
But, to write the partition function, wick rotate
$t \to -i\tau$
$Z = \int D[x]\, e^{i\int(-i)d\tau\left[\frac{1}{2}m\left(\frac{dx}{d(-i\tau)}\right)^2 - V(x)\right]}$
$= \int D[x]\, e^{+\int d\tau\left[-\frac{1}{2}m\dot{x}^2 - V(x)\right]}$
$= \int D[x]\, e^{-\int d\tau\left[\frac{1}{2}m\dot{x}^2 + V(x)\right]}$



tunnelling barrier.

Lagrangian for ϕ' will have $(\phi')^2$ coefficient has proper sign.

– only (ϕ^2) terms with the correct sign can be interpreted as massive particles.

– If ϕ^2 is hidden in $\mathcal{L}$, expand to see it

e.g. $\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - e^{\alpha\phi}$

$$= \int \mathcal{D}[x]\, e^{-\int d\tau \left[\frac{1}{2} m \dot{x}^2 + V(x)\right]}$$

tunnelling barrier.

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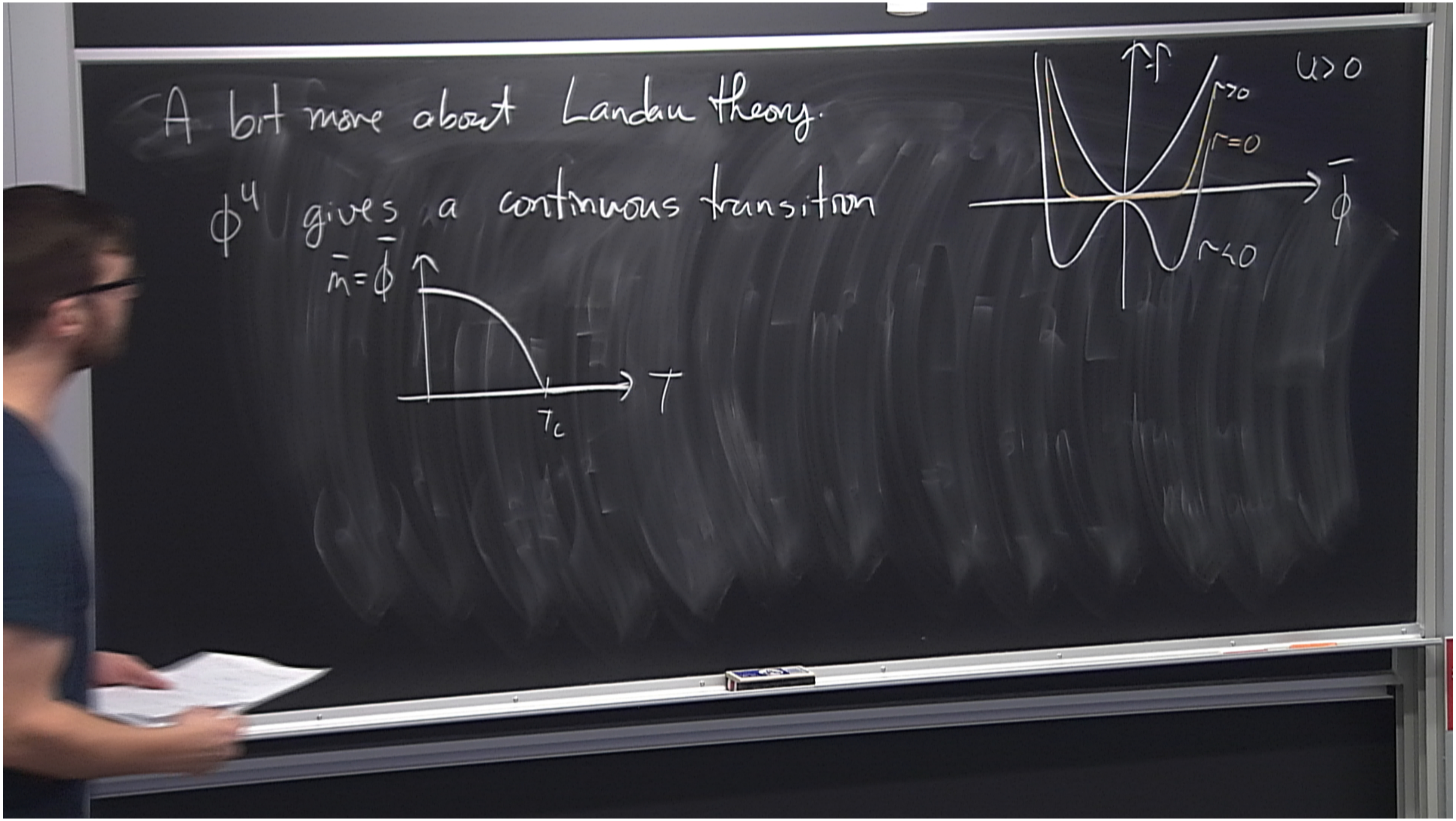
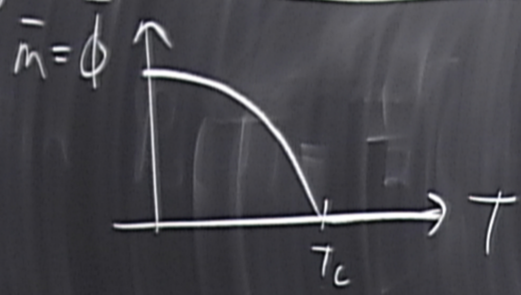
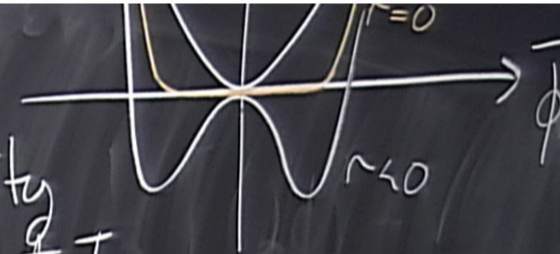

A bit more about Landau theory.
ϕ^4 gives a continuous transition
$\bar{m} = \bar{\phi}$
T
T_c
$-\Gamma$
$u > 0$
$r > 0$
$r = 0$
$r < 0$
$\bar{\phi}$

ϕ^4 gives a continuous transition





note: concavity changes at T_c

$$\Rightarrow \frac{\partial f}{\partial \phi} = 0 \text{ and } \frac{\partial^2 f}{\partial \phi^2} = 0$$

Euclidean + + + +)

theory

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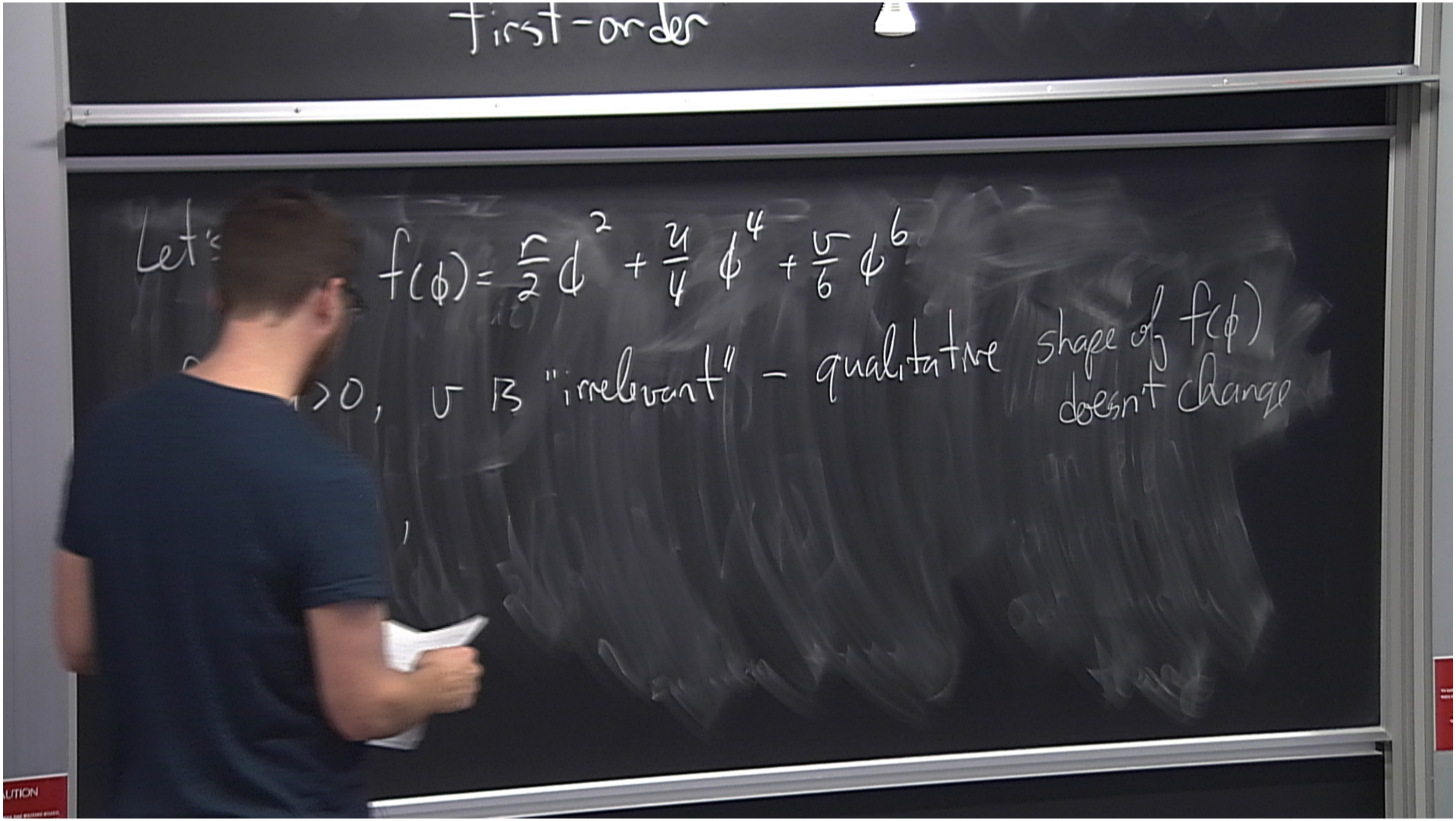
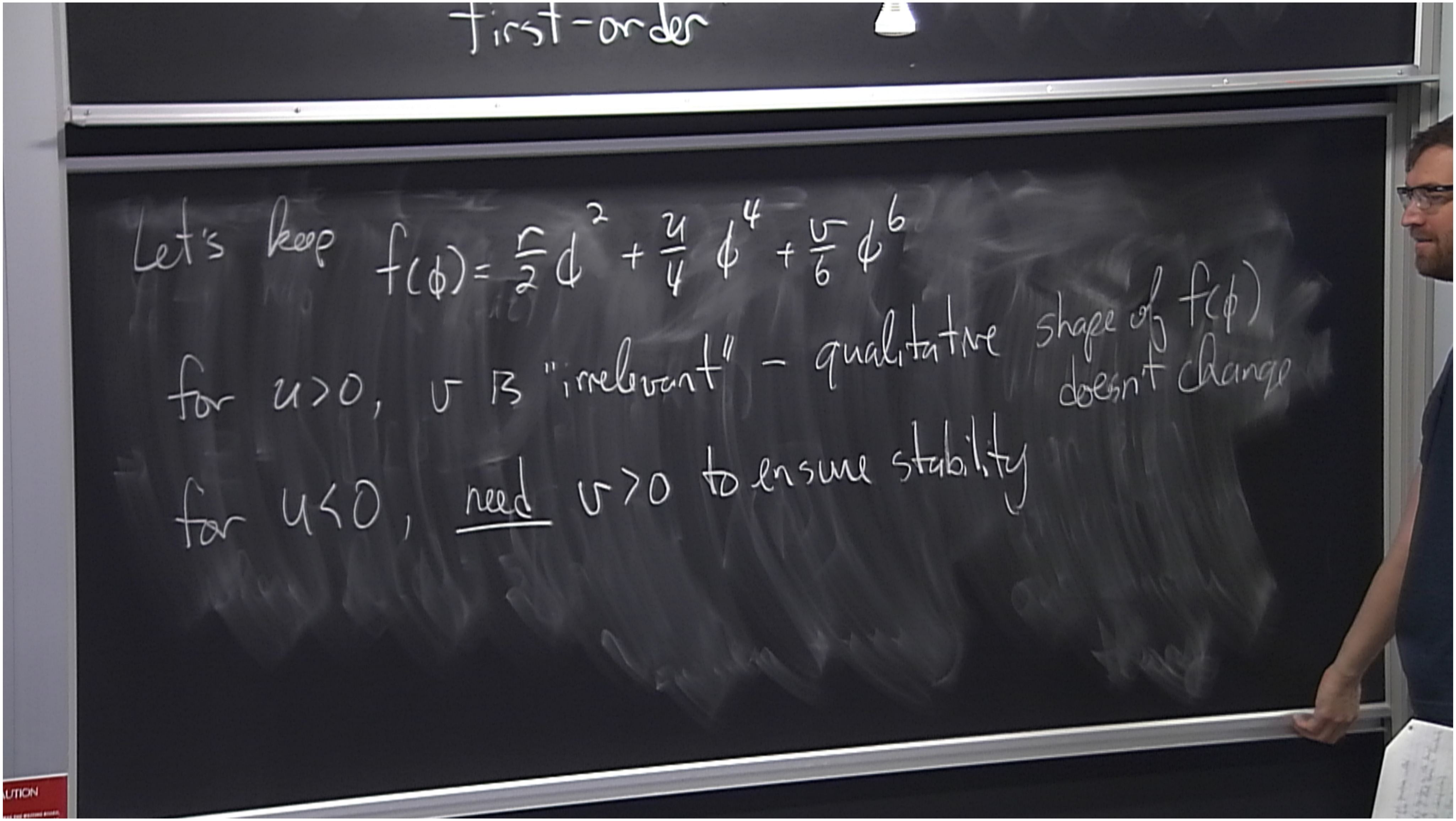
so $f(\phi) = \frac{r}{2}\phi^2 + \frac{u}{4}\phi^4 + \frac{v}{6}\phi^6 + \cdots$ etc.

So. $Z = \int D[\phi]\, e^{-S[\phi]}$, $S[\phi] = \int d^dx \left\{ \frac{1}{2}\left((\partial\phi)^2 + r\phi^2\right) + \frac{u}{4}\phi^4 \right\}$

Spontaneous Symmetry breaking: $r \to \tilde{r}(T - T_c)$, $r < 0$

symmetry breaking


first-order
Let's
$f(\phi) = \frac{r}{2}\phi^2 + \frac{u}{4}\phi^4 + \frac{v}{6}\phi^6$
>0, v is "irrelevant" - qualitative shape of $f(\phi)$ doesn't change


first-order
Let's keep $f(\phi) = \frac{r}{2}\phi^2 + \frac{u}{4}\phi^4 + \frac{v}{6}\phi^6$
for $u>0$, v is "irrelevant" – qualitative shape of $f(\phi)$ doesn't change
for $u<0$, need $v>0$ to ensure stability

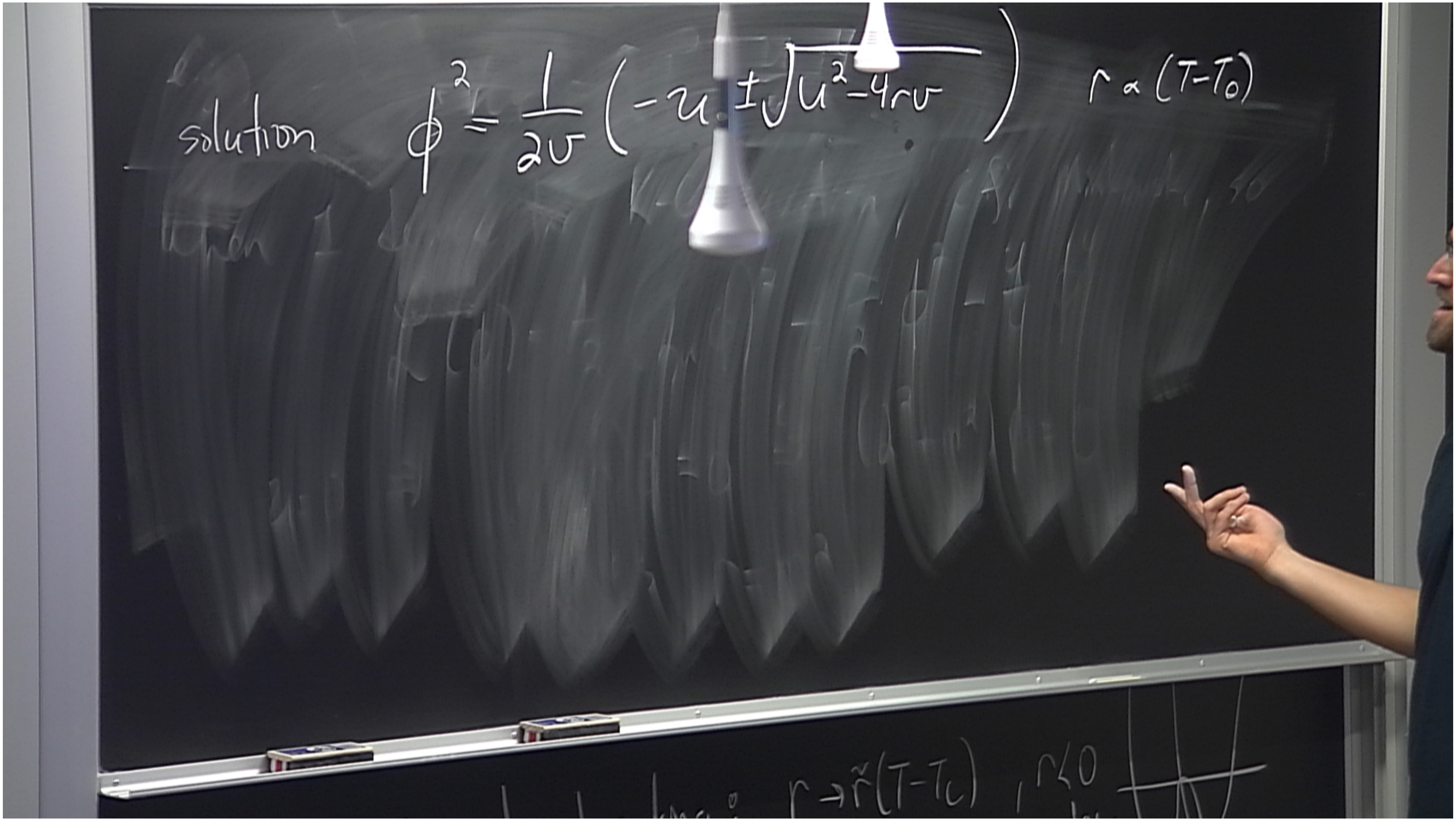
first-order

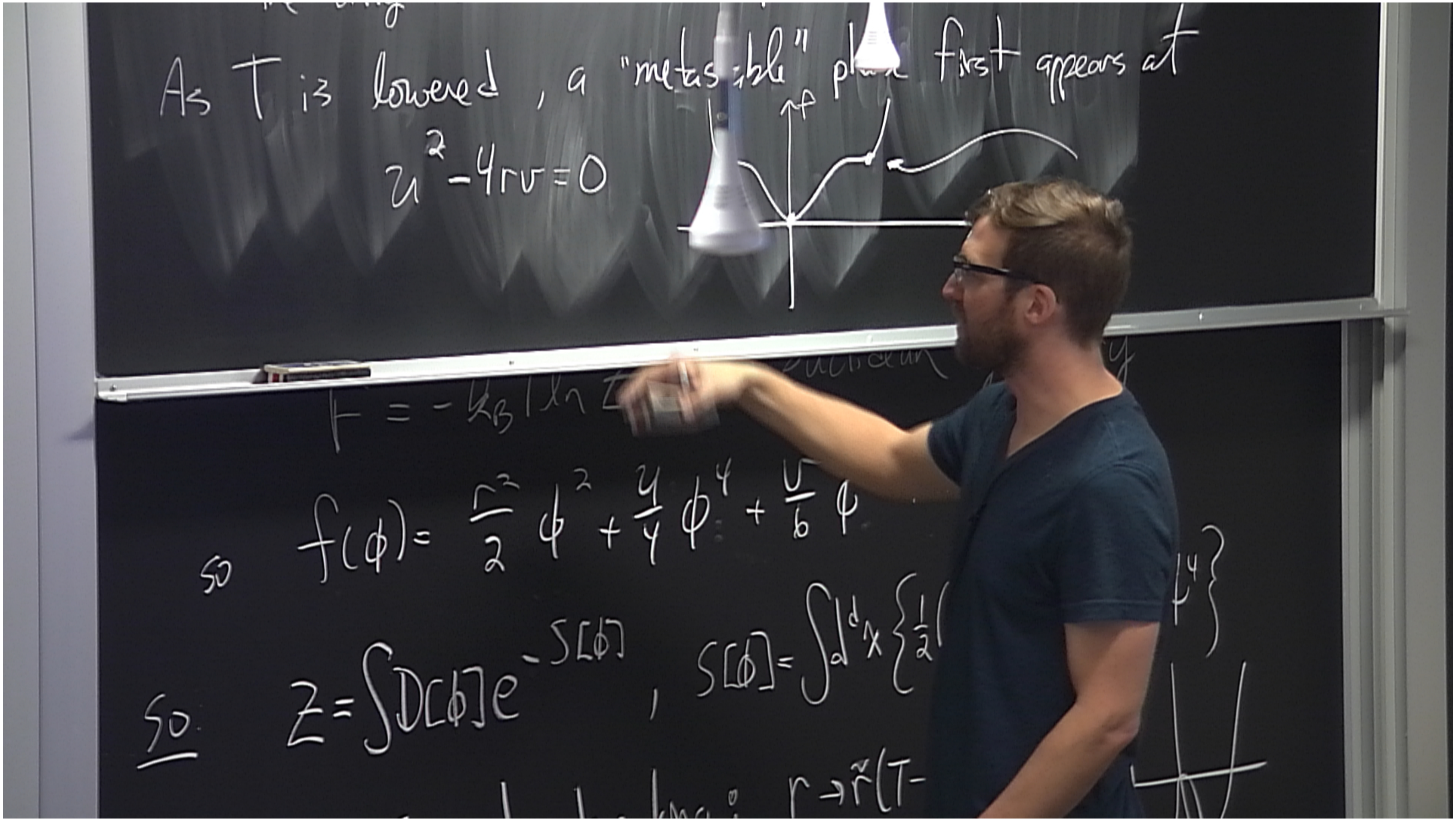
Let's keep $f(\phi) = \frac{r}{2}\phi^2 + \frac{u}{4}\phi^4 + \frac{v}{6}\phi^6$ ↙ Z, R

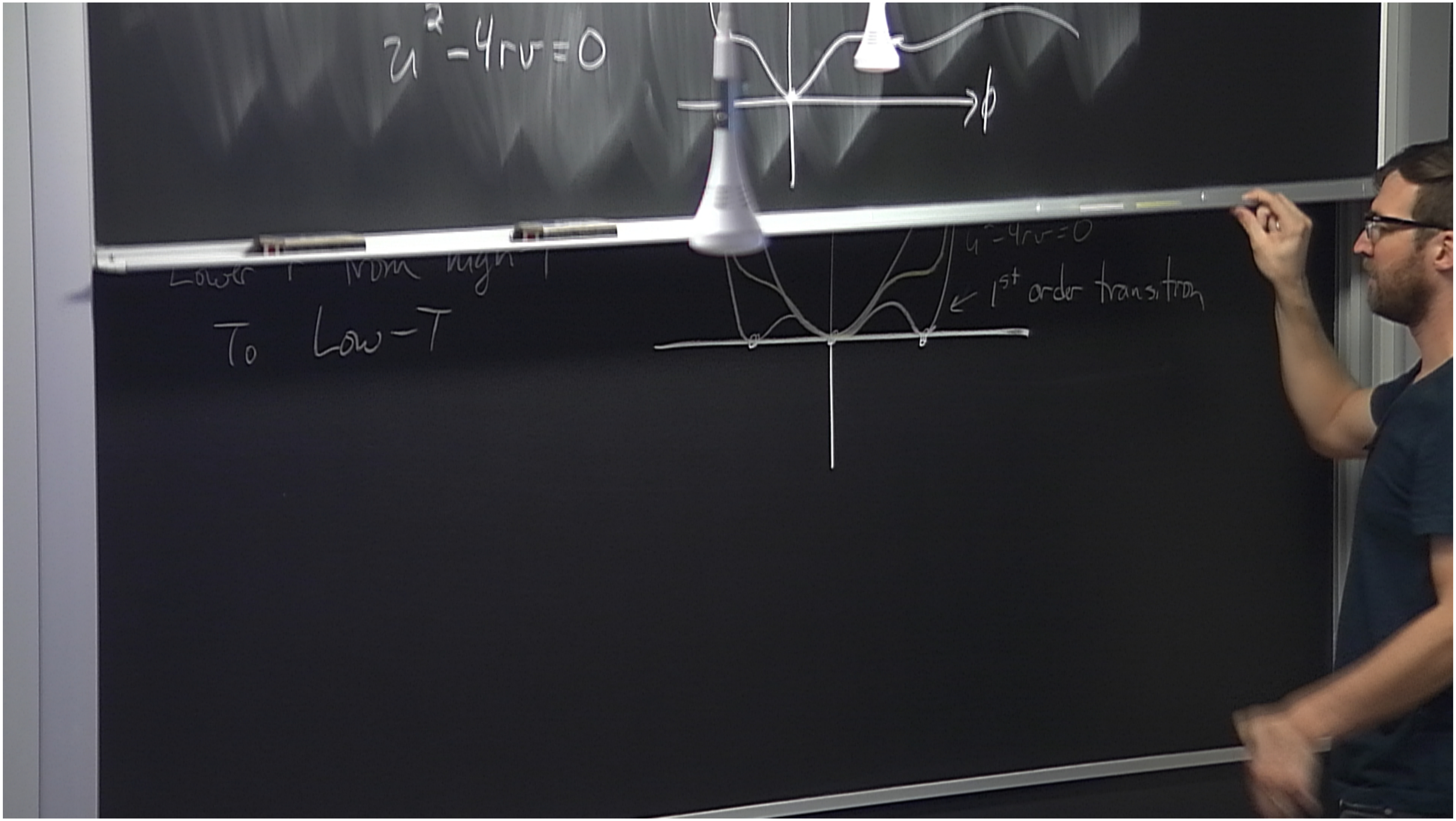
for $u>0$, v is "irrelevant" – qualitative shape of $f(\phi)$ doesn't change

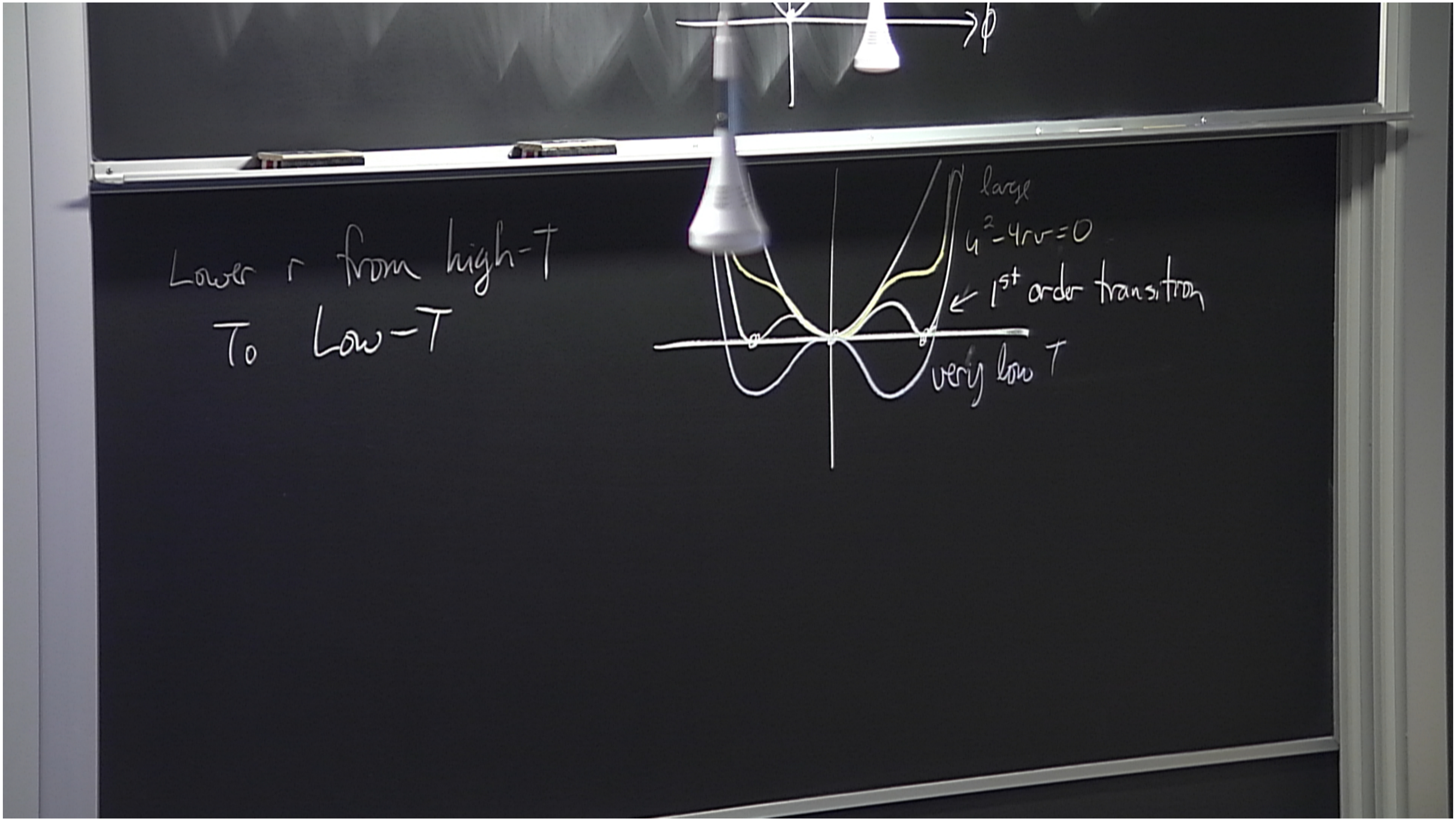
for $u<0$, <u>need</u> $v>0$ to ensure stability

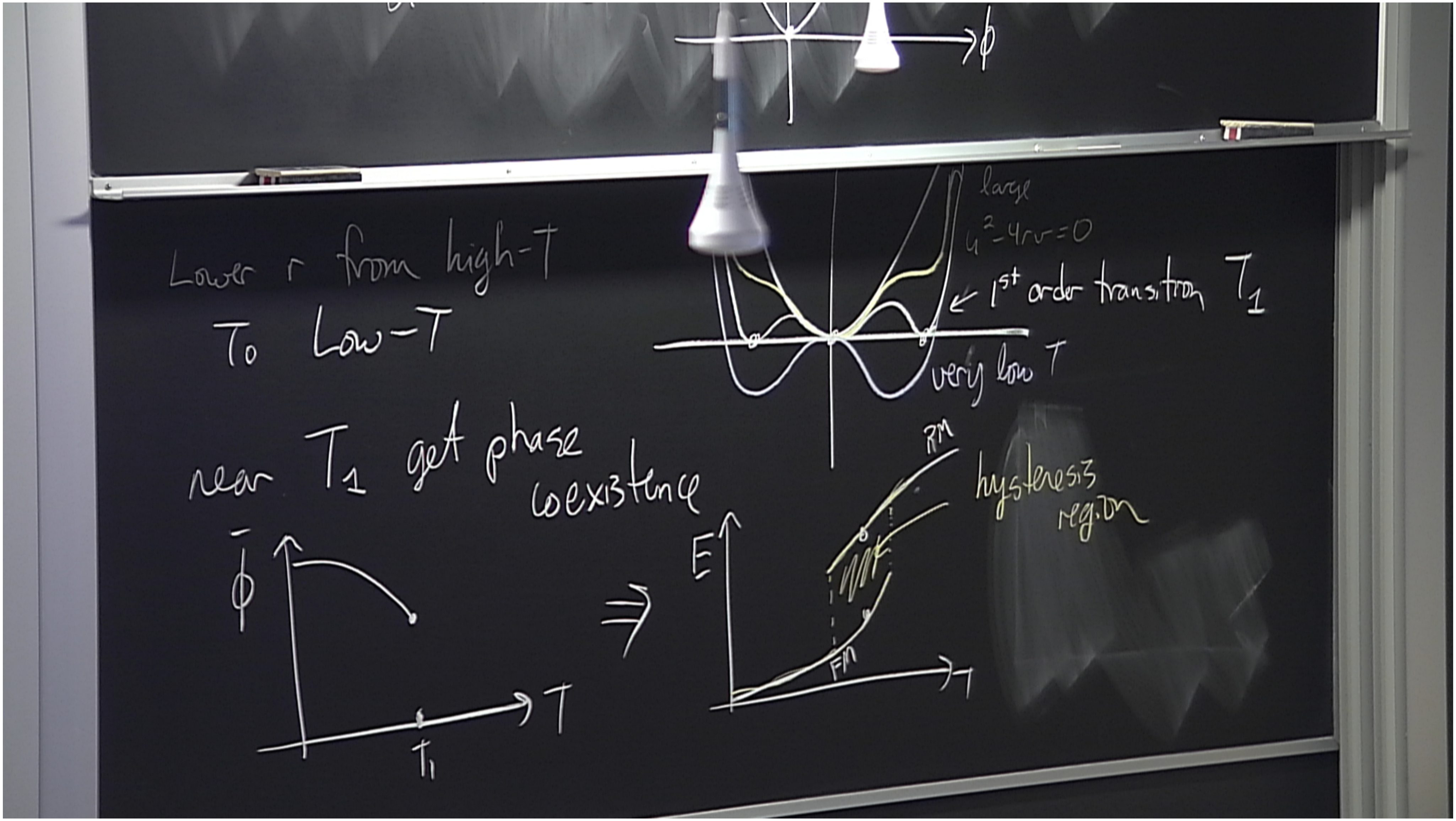
differentiate: $r\phi^2 + u\phi^3 + v\phi^5 = 0$

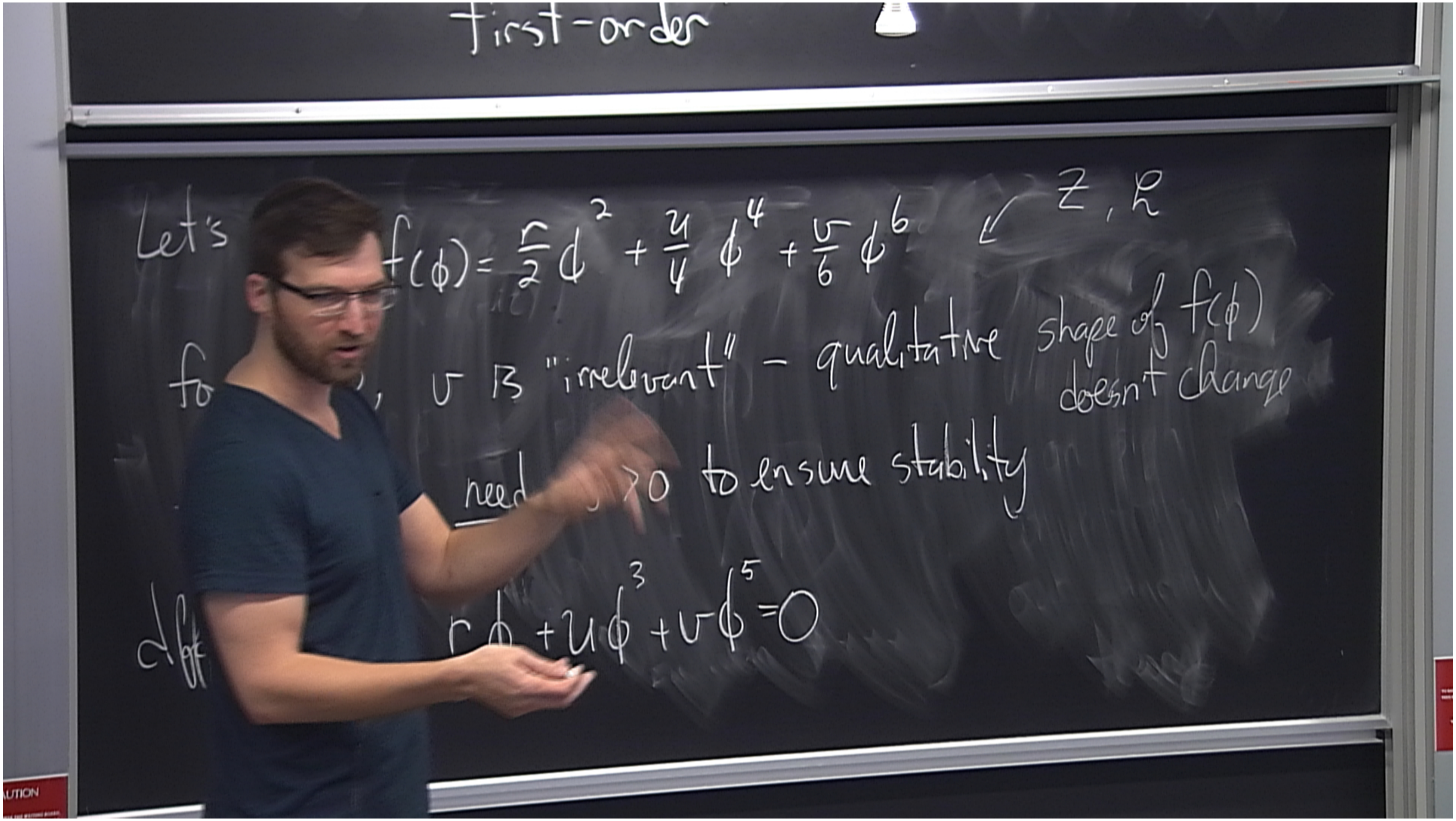

solution
$\phi^2 = \frac{1}{2v}\left(-u \pm \sqrt{u^2-4rv}\right)$
$r \propto (T-T_0)$
$r \to \tilde{r}(T-T_c)$, $r<0$


As T is lowered , a
first appears at
u² − 4rv = 0
so f(φ) = r/2 φ² + u/4 φ⁴ + v/6 φ
So Z = ∫D[φ]e^{−S[φ]} , S[φ] = ∫dᵈx {½(


$u^2-4rv=0$
ϕ
1st order transition
To Low-T


φ
Lower r from high-T
To Low-T
large
u²-4rv=0
1st order transition
very low T


Lower r from high-T
To Low-T
near T1 get phase coexistence
large
u^2-4rv=0
1st order transition T1
very low T
RM
hysteresis region
E
FM
T
T1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Let's
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v is "irrelevant" - qualitative shape of f(φ) doesn't change
to ensure stability
$r\phi + u\phi^3 + v\phi^5 = 0$

like ϕ

$$f(\phi) = \frac{r}{2}\phi^2 + \frac{s}{3}\phi^3 + \frac{u}{4}\phi^4 \qquad \text{kept } u>0$$

then $\Rightarrow$ $r\phi - |s|\phi^2 + u\phi^3 = 0$

$$\Rightarrow \quad \phi = \frac{1}{2u}\left(|s| \pm \sqrt{s^2 - 4ru}\right)$$

similar theory of 1st order transitions

e.g. $\mathcal{L} = \frac{1}{2}(\partial\phi) - e^{\alpha\phi}$

$$= \int D[x]\, e^{-\int d\tau \left[\frac{1}{2}m\dot{x}^2 + V(x)\right]}$$

like ϕ

$$f(\phi) = \frac{r}{2}\phi^2 + \frac{s}{3}\phi^3 + \frac{u}{4}\phi^4$$ kept $u > 0$

then $\Rightarrow$ $r\phi - |s|\phi^2 + u\phi^3 = 0$

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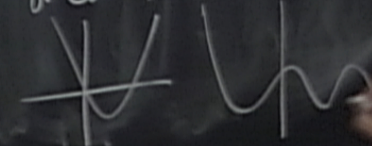
similar theory of 1st order transitions



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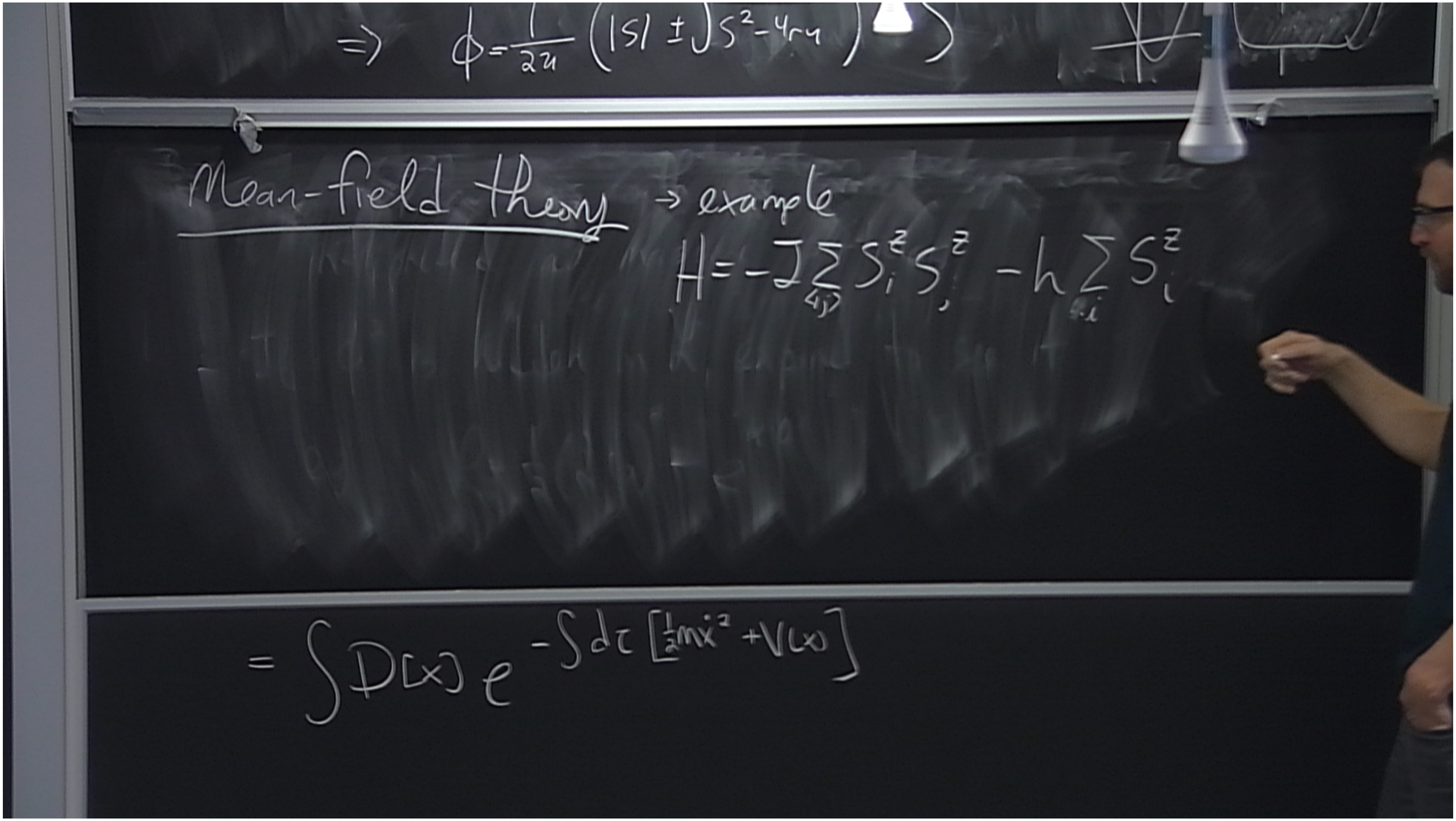

$\Rightarrow \quad \phi = \frac{1}{2u}\left(|s| \pm \sqrt{s^2 - 4ru}\right)$
Mean-field theory → example
$H = -J \sum_{\langle ij \rangle} S_i^z S_j^z - h \sum_i S_i^z$
$= \int D[x]\, e^{-\int d\tau \left[\frac{1}{2} m \dot{x}^2 + V(x)\right]}$

$$\Rightarrow \quad \phi = \frac{1}{2u}\left(|s| \pm \sqrt{s^2 - 4ru}\right)$$

Mean-field theory → example

$$\phi = m = \langle S_i^z \rangle \qquad H = -J\sum_{\langle ij \rangle} S_i^z S_j^z - h\sum_i S_i^z$$

Let's divide S_i^z into an average magnetization m + deviations

$$= \int D[x]\, e^{-\int d\tau \left[\frac{1}{2}m\dot{x}^2 + V(x)\right]}$$

$$\Rightarrow \quad \phi = \frac{1}{2u}\left(|s| \pm \sqrt{s^2 - 4ru} \right)$$

Mean-field theory → example

$$H = -J \sum_{\langle i,j \rangle} S_i^z S_j^z - h \sum_i S_i^z$$

$$\phi = m = \langle S_i^z \rangle$$

Let's divide S_i^z into an average magnetization m + deviations

<u>deviations</u> $\delta S_i^z = S_i^z - \langle S_i^z \rangle$

$$= \int D[x]\, e^{-\int d\tau \left[\frac{1}{2} m \dot{x}^2 + V(x) \right]}$$

$$H = -J\sum_{\langle ij\rangle}\underbrace{(m+\delta S_i^z)(m+\delta S_j^z)}_{m^2 + m\delta S_i^z + m\delta S_j^z + O(\delta S_i^z)^2} - h\sum_i S_i^z$$

N_B
\# bonds

$$= -Jm^2 N_B - Jm\sum_{\langle ij\rangle}(\delta S_i^z + \delta S_j^z) - h\sum_i S_i^z$$

for $u<0$, need $v>0$ to ensure stability

differentiate: $r\phi + u\phi^3 + v\phi^5 = 0$

N_B
bonds

$$m^2 + m\,\delta S_i^z + m\,\delta S_j^z + \mathcal{O}(\delta S_i^z)^2$$

$$= -J m^2 N_B - J m \sum_{\langle ij \rangle} \left(\delta S_i^z + \delta S_j^z\right) - h \sum_i S_i^z$$

hypercube

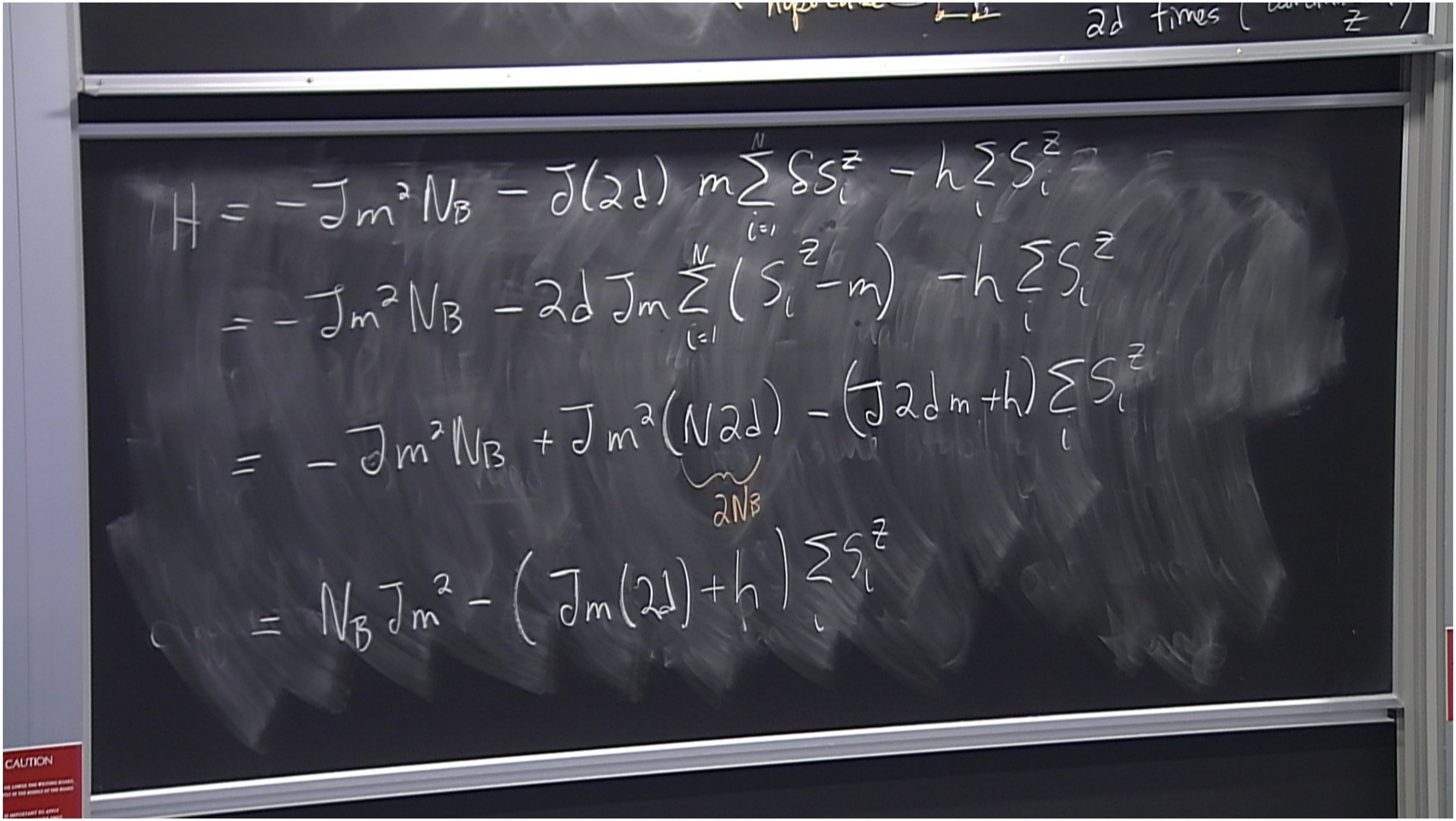
each δS^z appears $2d$ times (coordination z)

2d times (z)

$$H = -Jm^2 N_B - J(2d)\, m \sum_{i=1}^{N} \delta S_i^z - h \sum_i S_i^z$$

$$= -Jm^2 N_B - 2d\, J m \sum_{i=1}^{N} (S_i^z - m) - h \sum_i S_i^z$$

$$= -Jm^2 N_B + J m^2 (N 2d) - (J 2d m + h) \sum_i S_i^z$$



$H = -J m^2 N_B - J(2d)\, m \sum_{i=1}^{N} \delta S_i^z - h \sum_i S_i^z$
$= -J m^2 N_B - 2d\, J m \sum_{i=1}^{N} (S_i^z - m) - h \sum_i S_i^z$
$= -J m^2 N_B + J m^2 (\underbrace{N 2d}_{2N_B}) - (J 2d m + h) \sum_i S_i^z$
$= N_B J m^2 - (J m (2d) + h) \sum_i S_i^z$

2d times

$$H = -Jm^2 N_B - J(2d)\, m \sum_{i=1}^{N} \delta S_i^z - h \sum_i S_i^z$$

$$= -Jm^2 N_B - 2d\, J m \sum_{i=1}^{N} (S_i^z - m) - h \sum_i S_i^z$$

$$= -Jm^2 N_B + J m^2 \underbrace{(N 2d)}_{2N_B} - (J 2d m + h) \sum_i S_i^z$$

$$= N_B J m^2 - (J m (2d) + h) \sum_i S_i^z$$

2d times

$$H = -J m^2 N_B - J(2d)\, m \sum_{i=1}^{N} \delta S_i^z - h \sum_i S_i^z$$

$$= -J m^2 N_B - 2d\, J m \sum_{i=1}^{N} \left(S_i^z - m \right) - h \sum_i S_i^z$$

$$= -J m^2 N_B + J m^2 \underbrace{(N 2d)}_{2N_B} - (J 2d m + h) \sum_i S_i^z$$

$$= N_B J m^2 - \left(J m (2d) + h \right) \sum_i S_i^z \leftarrow$$

calculate free energy density

$2d$ times

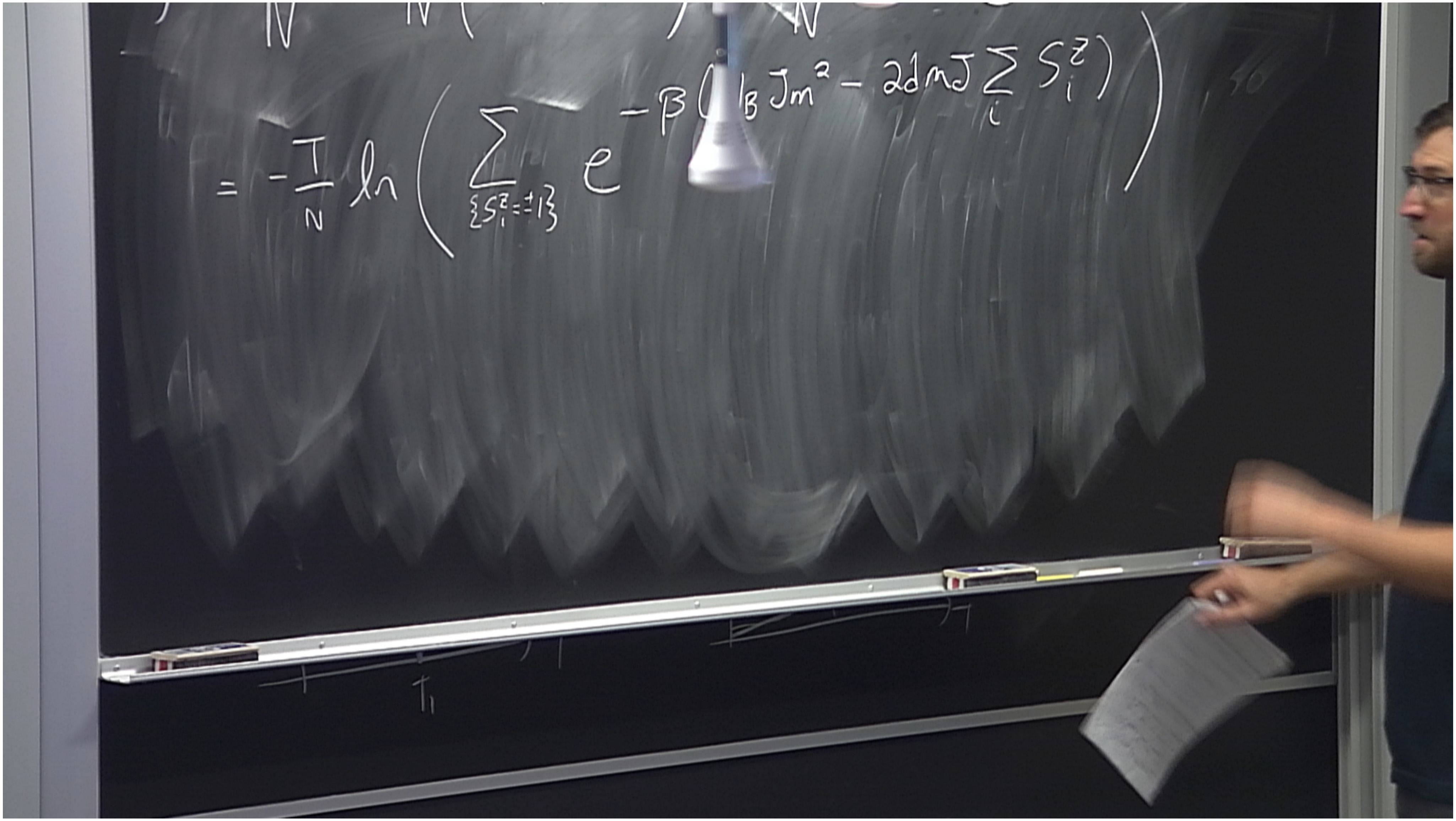
$$H = -Jm^2 N_B - J(2d)\, m \sum_{i=1}^{N} \delta S_i^z - h \sum_i S_i^z$$

$$= -Jm^2 N_B - 2d\, J m \sum_{i=1}^{N} \left(S_i^z - m \right) - h \sum_i S_i^z$$

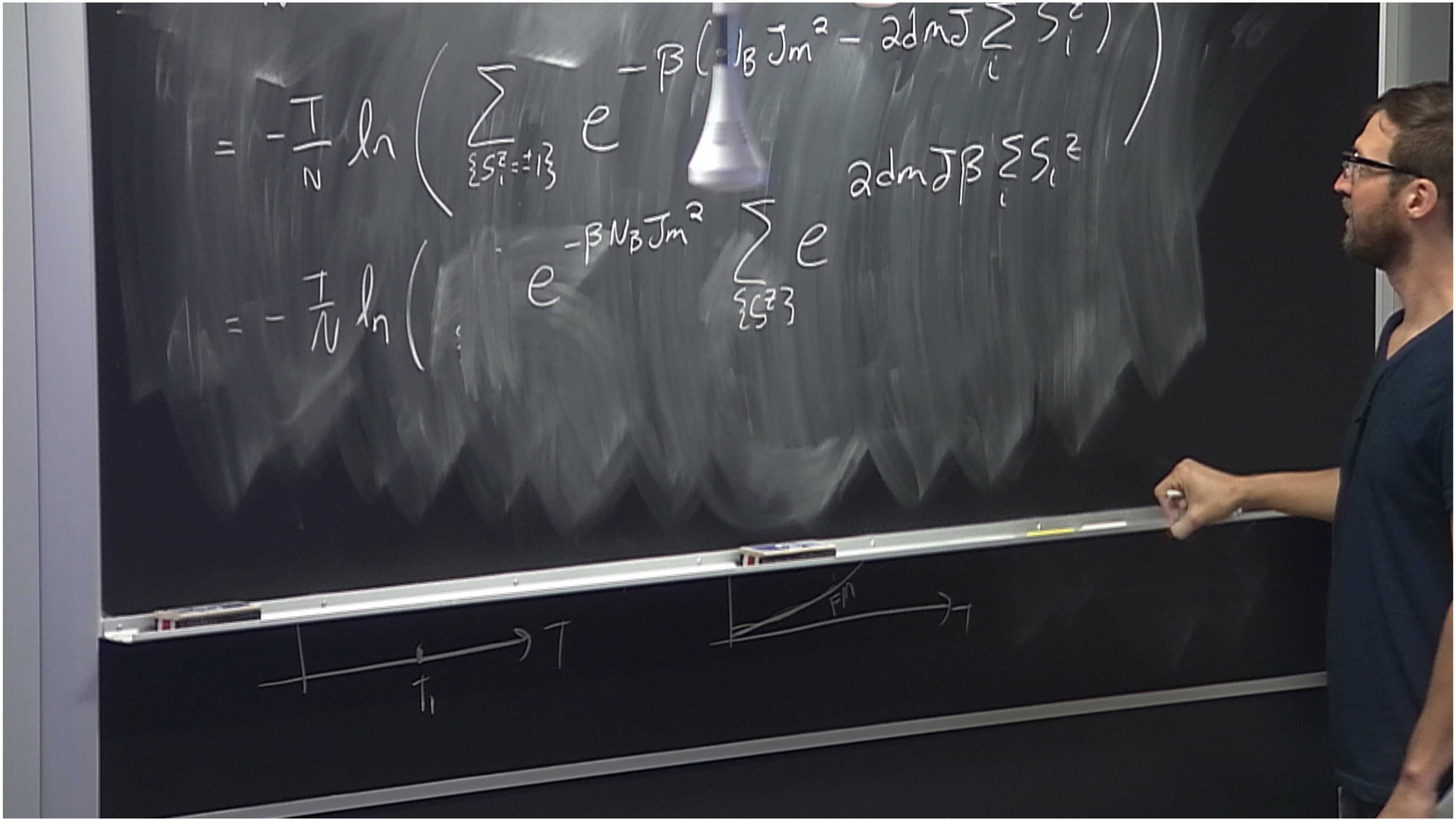
$$= -Jm^2 N_B + J m^2 \underbrace{(N 2d)}_{2N_B} - (J 2d m + h) \sum_i S_i^z$$

$$= N_B J m^2 - \left(J m (2d) + h \right) \sum_i S_i^z \quad \leftarrow$$

calculate the free energy density

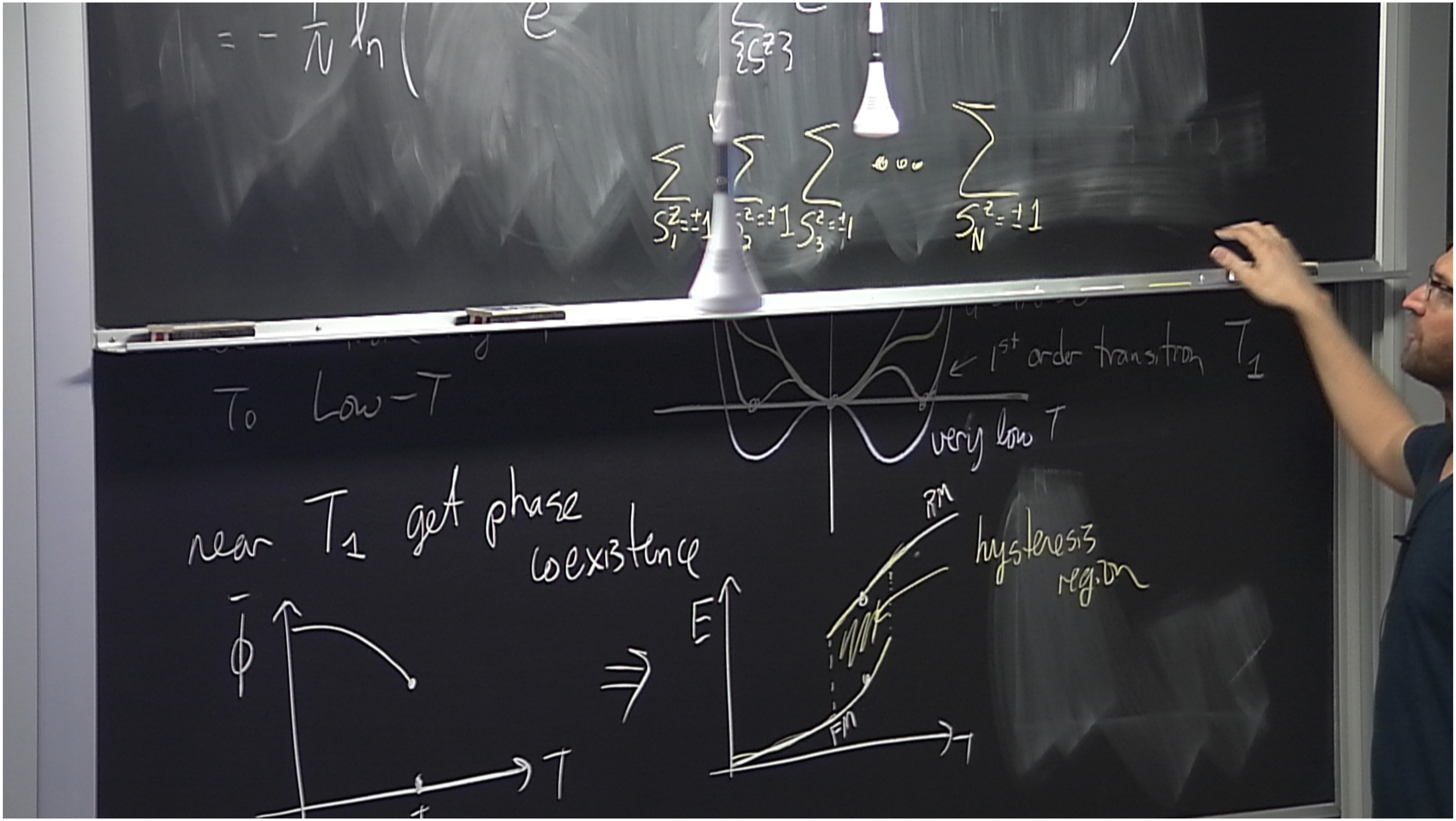


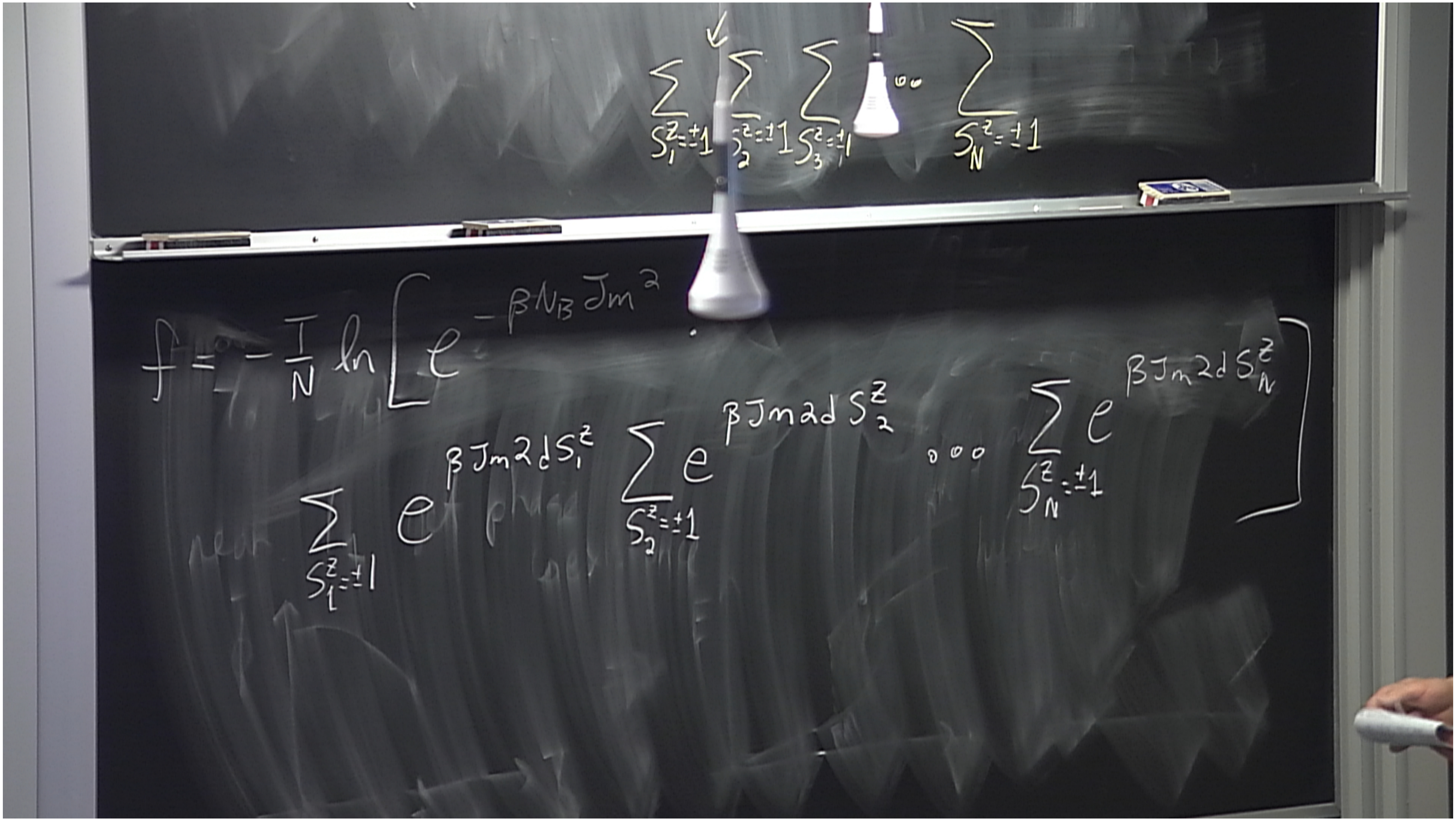
$= -\frac{T}{N} \ln \left(\sum_{\{S_i^z = \pm 1\}} e^{-\beta(_B J m^2 - 2dmJ \sum_i S_i^z)} \right)$
T_1


= - T/N ln (Σ_{S_i^z = ±1} e^{-β(N_B J m^2 - 2dmJ Σ_i S_i^z)})
= - T/N ln (e^{-β N_B J m^2} Σ_{S^z} e^{2dmJβ Σ_i S_i^z}
T_1
T
FM
T

$$= -\frac{T}{N}\ln\left(\sum_{\{S_i^z=\pm1\}} e^{-\beta\left(N_B J m^2 - 2dmJ\sum_i S_i^z\right)}\right)$$

$$= -\frac{T}{N}\ln\left(e^{-\beta N_B J m^2} \sum_{\{S^z\}} e^{2dmJ\beta\sum_i S_i^z} \right)$$


1st order transition T1
To Low-T
very low T
near T1 get phase coexistence
RM
hysteresis region
FM
E
T


$\sum_{S_1^z=\pm1}\sum_{S_2^z=\pm1}\sum_{S_3^z=\pm1}\cdots\sum_{S_N^z=\pm1}$
$f = -\frac{T}{N}\ln\Big[e^{-\beta N_B J m^2} \sum_{S_1^z=\pm1} e^{\beta J m 2d S_1^z}\sum_{S_2^z=\pm1} e^{\beta J m 2d S_2^z}\cdots\sum_{S_N^z=\pm1} e^{\beta J m 2d S_N^z}\Big]$

$$f = -\frac{T}{N}\ln\left[e^{-\beta N_B J m^2} \sum_{S_1^z=\pm1} e^{\beta J m 2d S_1^z} \sum_{S_2^z=\pm1} e^{\beta J m 2d S_2^z} \cdots \sum_{S_N^z=\pm1} e^{\beta J m 2d S_N^z} \right]$$

$$= -\frac{T}{N}\ln\left[e^{-\beta N_B J m^2} \left(e^{\beta J m 2d} + e^{-\beta J m 2d} \right)^N \right]$$

$$f = -\frac{T}{N} \ln\Big[e^{-\beta N_B J m^2} \cdot \sum_{S_1^z=\pm 1} e^{\beta J m 2 d S_1^z} \sum_{S_2^z=\pm 1} e^{\beta J m 2 d S_2^z} \cdots \sum_{S_N^z=\pm 1} e^{\beta J m 2 d S_N^z} \Big]$$

$$= -\frac{T}{N} \ln\Big[e^{-\beta N_B J m^2} \left(e^{\beta J m 2 d} + e^{-\beta J m 2 d} \right)^N \Big]$$

and only keeps terms up to m

$$f = -T\log 2 - \frac{1}{2}J(2d)\left[J(2d)\beta - 1\right]m^2$$

$$+\underbrace{\frac{1}{12}(J2d)^4\beta^4}_{\text{positive.}}m^4 + \ldots$$

changes sign for some $T = T_c$

MFT = Landau theory = ϕ^4 saddle-point approximation.

and only keeps terms up to m

$$f = -T \log 2 - \frac{1}{2} J(2d)\left[J(2d)\beta - 1\right] m^2$$

changes sign for some $T = T_c$

$$+ \underbrace{\frac{1}{12} (J2d)^4}_{\text{positive}} \beta^4 m^4 + \ldots$$

MFT = Landau theory = ϕ^4 saddle-point approximation.