Title: PHYS 733 - Quantum Many-Body Physics (W2016) - Roger Melko - Lecture 16

Date: Mar 03, 2016 10:00 AM

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Abstract:

Recall the single-particle path integral

$$U(x',t';x,t) = \int \mathcal{D}[x]\, e^{\frac{i}{\hbar} S(\{x\})}$$

with $S = \int \left[\frac{m\dot{x}^2}{2} - V(x)\right] dt = \int dt\, L(\dot{x}, x)$

This path integral has no approximations but may be hard to solve. Let's develop a semi-classical approach

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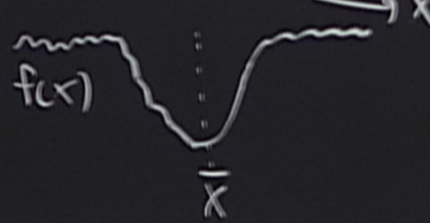
to solve. Let's develop a semi-classical approach

Saddle-point approximation

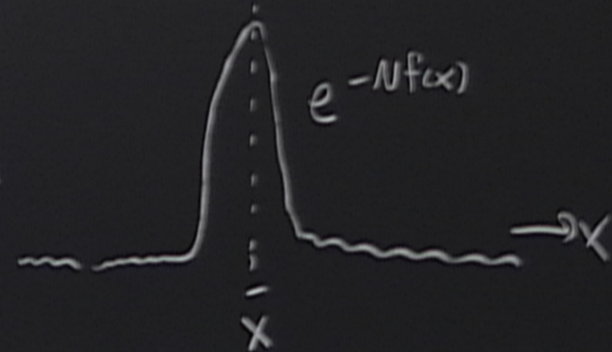
consider first $\quad I = \int_{-\infty}^{\infty} dx\, e^{-Nf(x)}$

consider the limit $N \to$ large

if $f(x)$ looks like



then



If we concentrate near the minimum of $f(x)$

and note $f'(\bar{x}) = 0$ and $f''(\bar{x}) > 0$ then

$$I = \int_{-\infty}^{\infty} dx \, e^{-N\left[f(\bar{x}) + \frac{1}{2} f''(\bar{x})(x-\bar{x})^2 + \cdots\right]}$$

$$\simeq e^{-N f(\bar{x})} \sqrt{\frac{2\pi}{N f''(\bar{x})}}$$

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$$\simeq e^{-Nf(\bar{x})} \sqrt{\frac{2\pi}{N f''(\bar{x})}}$$ ⇐ in most cases the prefactor doesn't matter much.

I is approximated by $e^{-Nf(\bar{x})}$: the maximum value of the integrand.

Back to our Path integral $S = \int \left[\frac{m\dot{x}^2}{2} - V(x) \right] dt$

The minimum is found by taking the variation of S, $\delta S = 0$

this gives $m\ddot{x} = V'(x)$

and setting $\ddot{x} = 0$ gives the equilibrium value $x = \bar{x}$

Now consider ϕ^4 theory - we have been writing

$$S[\phi] = \int d^dx \left\{ \frac{1}{2}\left[(\partial\phi)^2 + r\phi^2 \right] + \frac{u}{4}\phi^4 \right\}$$

Similarly, we can find the minimum of the action in the case where $\phi(x)$ is space-time independent, ϕ

(since $(\partial\phi)^2$ terms increase the action)

$\Rightarrow$ mean-field theory.

This is what you are doing on assignment 4 Q2

The solution to $\delta S = 0$ gives the "vacuum" state of the system $\Rightarrow \phi = \bar{\phi} = v$

Note - not necessarily $\bar{\phi} = 0$: $\bar{\phi} \neq 0$ corresponds to symmetry breaking.

On your assignment. consider fluctuations by considering a small perturbation $\phi \to v + \phi'$

Recall the classical action $\Big\}$ $S[\vec{q}(t)] = \int dt\, \mathcal{L}(\vec{q}(t), \dot{\vec{q}}(t), t)$

The $\delta S = 0$ solution gives the classical equations of motion $\Rightarrow$ $\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_j}\right) - \frac{\partial \mathcal{L}}{\partial q_j} = 0$ Euler-Lagrange equations

Then for our ϕ^4 Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 + \frac{r}{2}\phi^2 + \frac{u}{4}\phi^4$$

Let me label $(\partial\phi^2)$ as $(\partial_\mu\phi)^2$

$$(\partial_\mu\phi)^2 = \left(\frac{\partial\phi}{\partial t}\right)^2 - \left(\frac{\partial\phi}{\partial x}\right)^2 - \left(\frac{\partial\phi}{\partial y}\right)^2 + \ldots$$

The analogous E-L equations are

$$\partial_\mu\left[\frac{\partial\mathcal{L}}{\partial[\partial_\mu\phi]}\right] - \frac{\partial\mathcal{L}}{\partial\phi} = 0$$

Now consider ϕ^4 theory – we have been writing

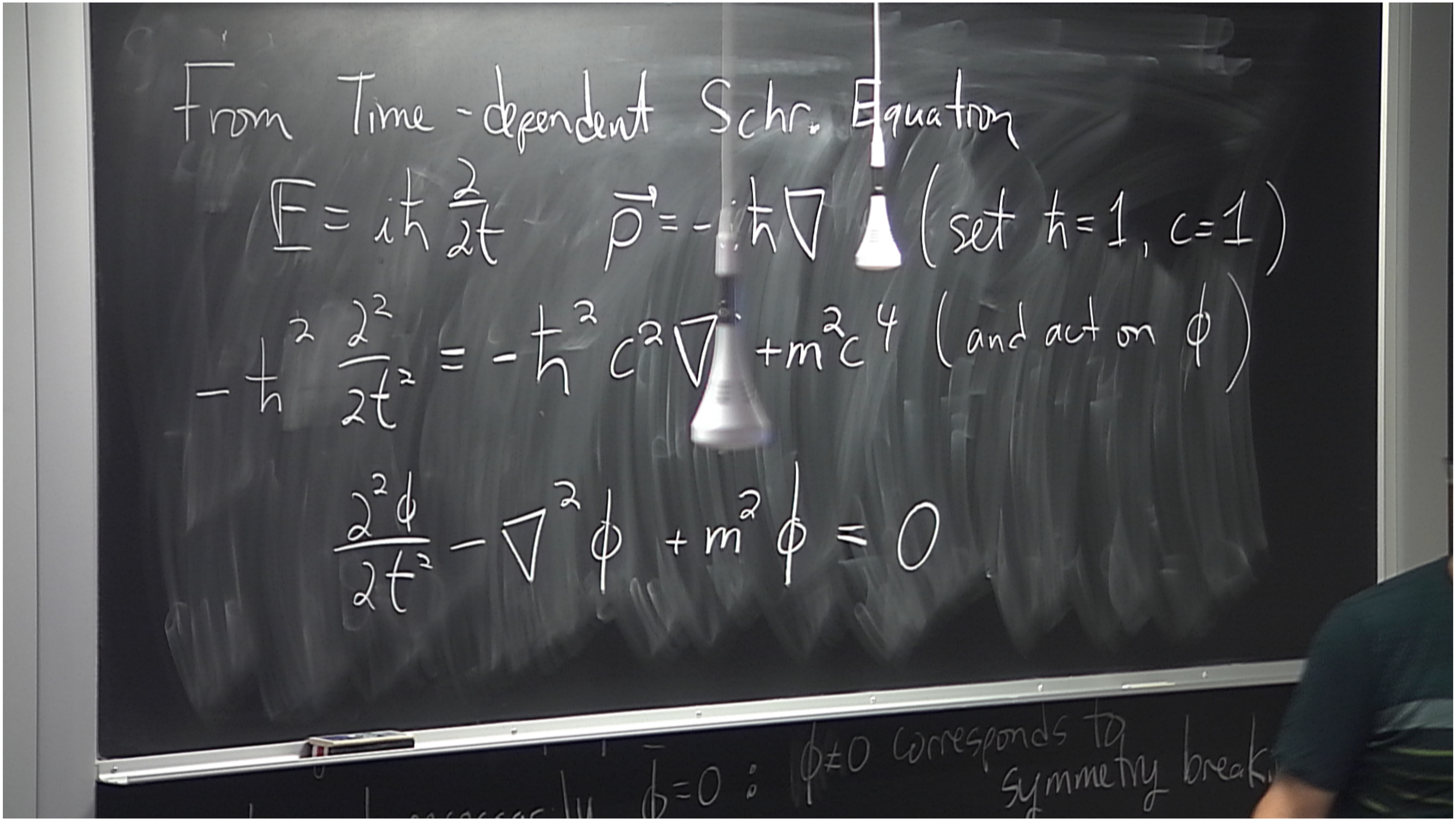
Set $u=0$ here (no interactions) - sub $\mathcal{L}$ into E-L

$$(\partial^2 - r)\,\phi = 0$$

actually $\partial_\mu \partial^\mu = \frac{\partial^2}{\partial t^2} - \nabla^2 = \Box$

c.f. $(\Box + m^2)\,\phi = 0$ Klein-Gordon for a free particle of mass m

comes from $E^2 = p^2c^2 + m^2c^4$


From Time-dependent Schr. Equation
$E = i\hbar \frac{\partial}{\partial t}$ $\vec{p} = -i\hbar\nabla$ (set $\hbar = 1$, $c = 1$)
$-\hbar^2 \frac{\partial^2}{\partial t^2} = -\hbar^2 c^2 \nabla^2 + m^2 c^4$ (and act on ϕ)
$\frac{\partial^2 \phi}{\partial t^2} - \nabla^2 \phi + m^2 \phi = 0$
$\phi = 0$: $\phi \neq 0$ corresponds to
symmetry

Phenomenological Landau Theories

Introduced by Landau to formulate a general theory of phase transitions.

he suggested the free energy should be analytic & obey the symmetries of the Hamiltonian.

e.g., if $q(x) = m$

- if m has little variation at distances compared to the lattice spacing, writing $m(\vec{x})$ is justified
- PHY 705 → why this is justified $(\xi \to \infty)$

The $\delta S = 0$ solution gives the classical equations of motion $\Rightarrow$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_j}\right) - \frac{\partial \mathcal{L}}{\partial q_j} = 0$$

Euler-Lagrange equations

• PHY 705 → why this is justified ($\xi \to \infty$)

A simple free energy functional (consider e.g. an Ising magnet)

$$f(\phi) = \sum_n a_n \phi^n = a_1 \phi + a_2 \phi^2 + a_3 \phi^3 + a_4 \phi^4 \dots$$

Consider e.g. $H = -J \sum_{\langle ij \rangle} S_i^z S_j^z$ without $-h \sum_i S_i^z$

• PHY 705 → why this is justified ($\xi \to \infty$)

A simple free energy functional (consider e.g. an Ising magnet)

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Consider e.g. $H = -J \sum_{\langle ij \rangle} S_i^z S_j^z$ without $-h \sum_i S_i^z$

for $h = 0$, $f(\phi) = + f(\phi)$ because of the Hamiltonian symmetry then $f(\phi) = a_2 \phi^2 + a_4 \phi^4 + \dots = \frac{r}{2}\phi^2 + u\phi^4$

the same as your potential in $\mathcal{L}$

makes sense: $F = -k_B T \log Z$, $Z = \int D[\phi]\, e^{-S(\phi)}$

then $f(\phi) = a_2 \phi^2 + a_4 \phi^4 + \cdots = \frac{r}{2} \phi^2 + u \phi^4$

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So the minimum of $S(\phi)$ should correspond to the minimum in the free energy.

comes from $E^2 = p^2 c^2 + m^2 c^4$

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makes sense: $F = -k_B T \log Z$, $Z = \int D[\phi]\, e^{-S(\phi)}$

So the minimum of $S(\phi)$ should correspond to the minimum in the free energy.

Let's look more closely when $h \neq 0$, add lowest order

$(\partial - r)\,\phi = 0$

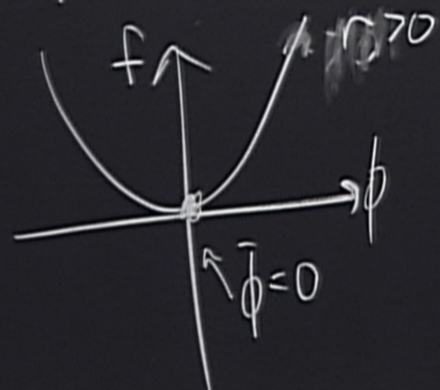
actually $\partial_\mu \partial^\mu = \frac{\partial^2}{\partial t^2} - \nabla^2 = \Box$

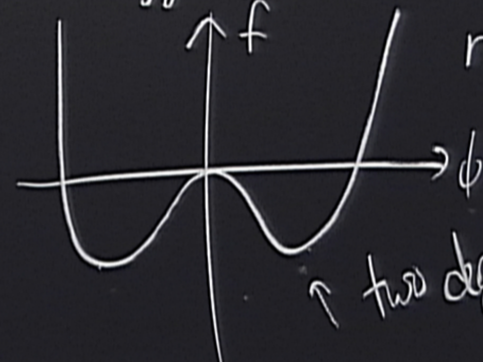
Let's look more closely (when h... case

$$f(x) = \frac{r}{2}\phi^2 + \frac{u}{4}\phi^4 - h\phi$$ (Lowest order terms, add more as necessary)

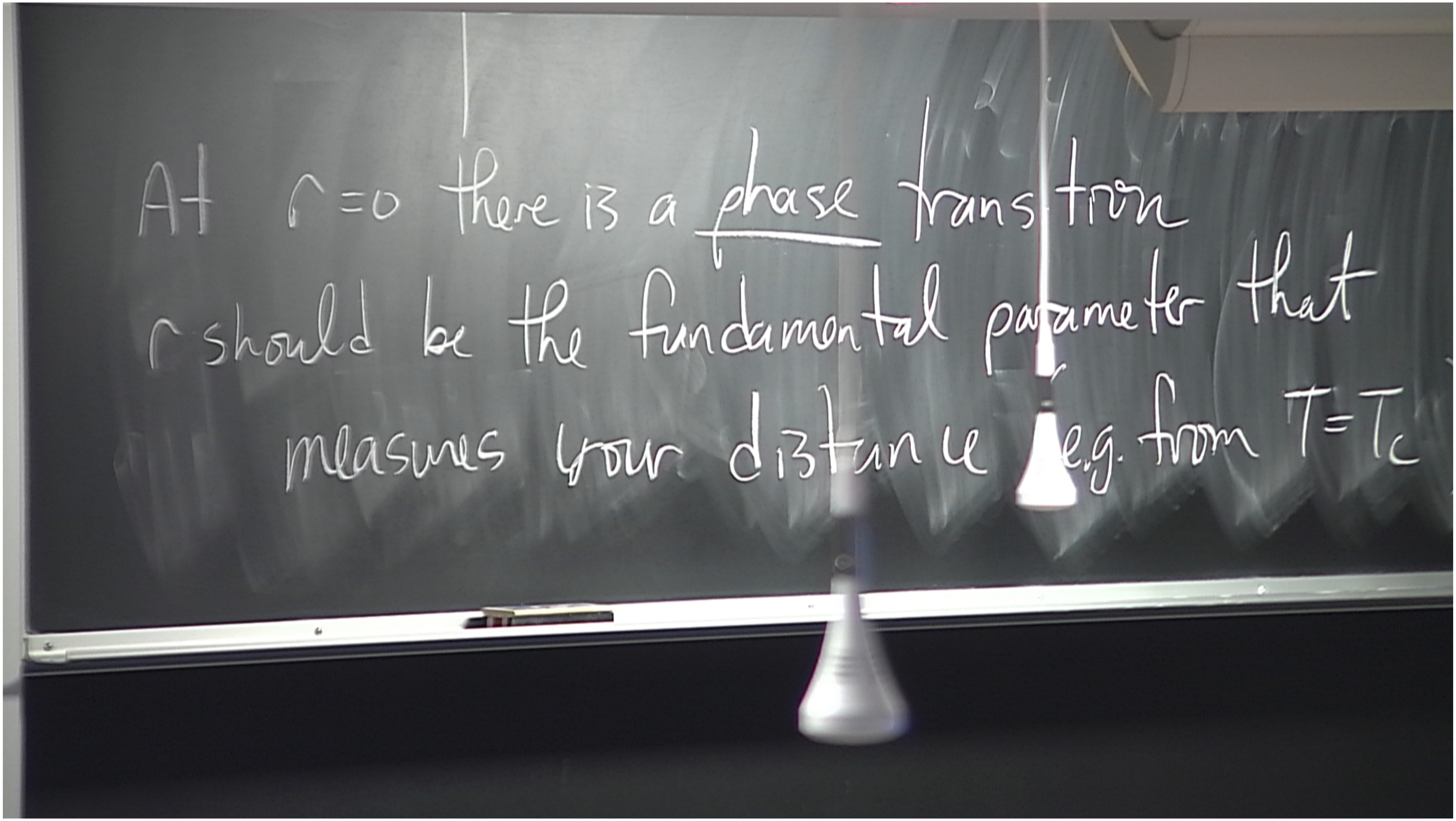
A4Q2: The mean-field free energy possess 2 types of solution.

if $h=0$





$r<0$ ↑↑↑↑ or ↓↓↓↓↓

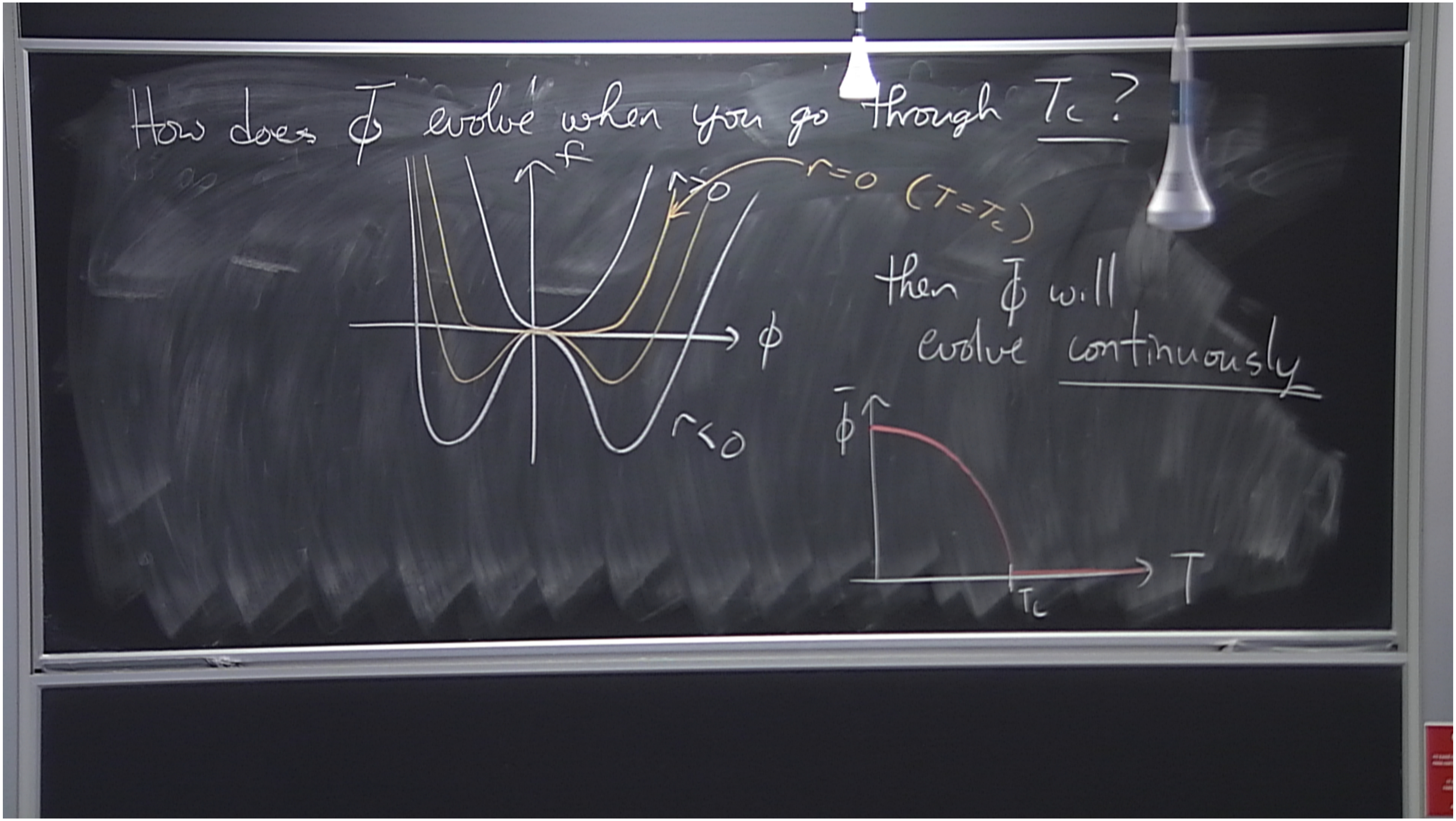

At $r=0$ there is a phase transition
r should be the fundamental parameter that
measures your distance e.g. from $T=T_c$

The parameters a_n (e.g. [illegible] and u) are analytic functions of the temperature (and n) $\Rightarrow$ they can be Taylor expanded about $T=T_c$

e.g. $$r(T) = r_1 + r_2(T-T_c) + O(T-T_c)^2$$

$$u(T) = u_1 + u_2(T-T_c) + O(T-T_c)^2 \Rightarrow u_1 > 0$$

Require r to be monotonic function of $T-T_c$ $\Rightarrow r_1 = 0\,,\ r_2 > 0$


How does $\bar{\phi}$ evolve when you go through T_c?
f
ϕ
$r>0$
$r=0$ $(T=T_c)$
$r<0$
then $\bar{\phi}$ will evolve continuously
$\bar{\phi}$
T
T_c