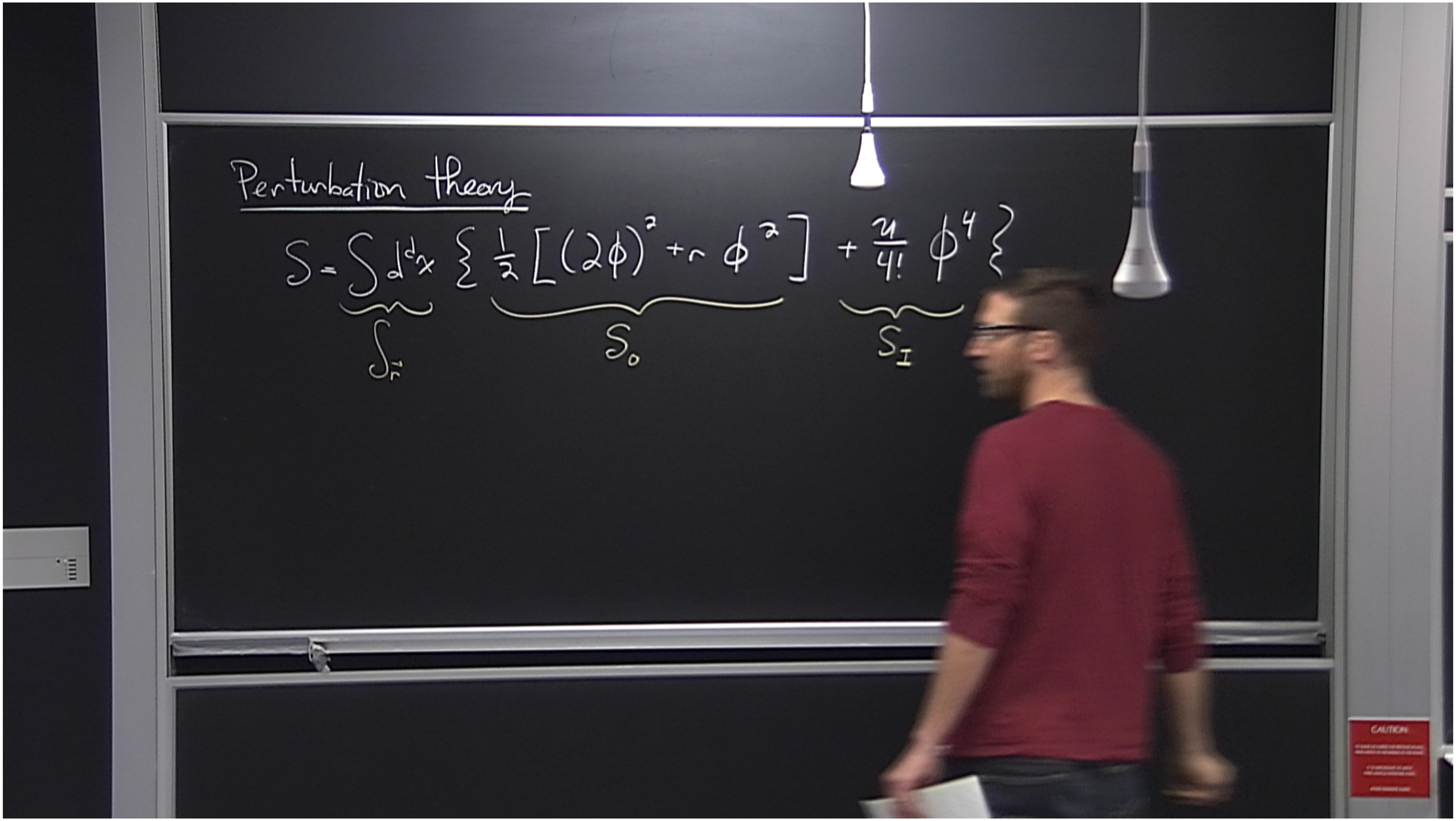
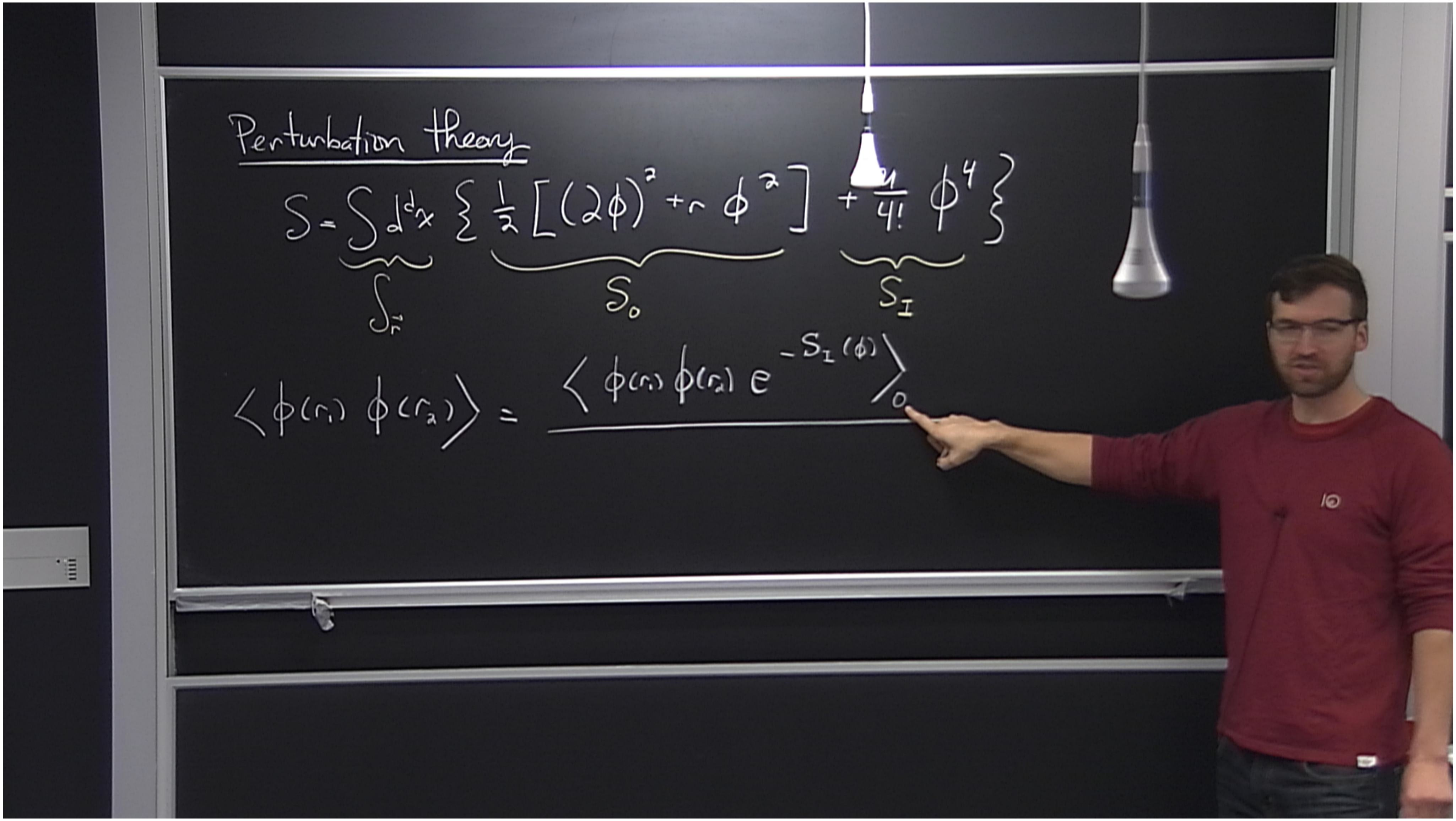
Title: PHYS 733 - Quantum Many-Body Physics (W2016) - Roger Melko - Lecture 15

Date: Mar 01, 2016 10:00 AM

URL: http://pirsa.org/16030026

Abstract:


Perturbation theory
$S = \underbrace{\int d^dx}_{\int_{\vec{r}}} \left\{ \underbrace{\frac{1}{2}\left[(\partial\phi)^2 + r\,\phi^2\right]}_{S_0} + \underbrace{\frac{u}{4!}\,\phi^4}_{S_I} \right\}$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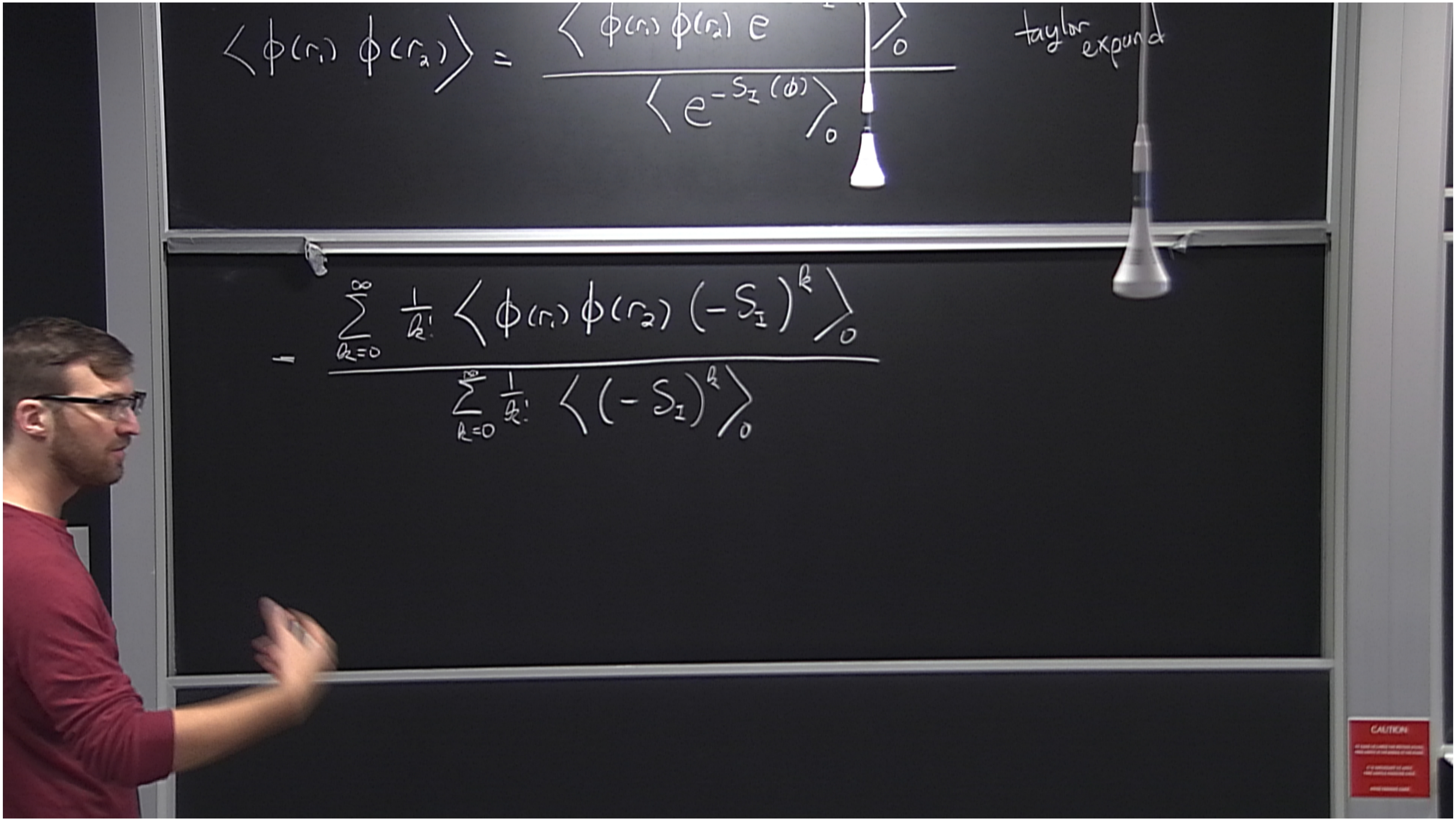
$\langle \phi(r_1)\,\phi(r_2) \rangle = \frac{\langle \phi(r_1)\,\phi(r_2)\,e^{-S_I(\phi)} \rangle_0}{}$


$\langle \phi(r_1)\, \phi(r_2) \rangle = \dfrac{\langle \phi(r_1)\, \phi(r_2)\, e \rangle_0}{\langle e^{-S_I(\phi)} \rangle_0}$
taylor expand
$= \dfrac{\sum_{k=0}^{\infty} \frac{1}{k!} \langle \phi(r_1)\, \phi(r_2)\, (-S_I)^k \rangle_0}{\sum_{k=0}^{\infty} \frac{1}{k!} \langle (-S_I)^k \rangle_0}$
CAUTION

$$\langle \phi(r_1)\,\phi(r_2)\rangle = \frac{\langle \phi(r_1)\,\phi(r_2)\, e^{-S_I(\phi)} \rangle_0}{\langle e^{-S_I(\phi)}\rangle_0}$$

taylor expand

$$= \frac{\sum_{k=0}^{\infty} \frac{1}{k!} \langle \phi(r_1)\,\phi(r_2)\,(-S_I)^k \rangle_0}{\sum_{k=0}^{\infty} \frac{1}{k!} \langle (-S_I)^k \rangle_0}$$

Set up to evaluate nicely with Wick's theorem

$$\langle \phi_{i_1}\,\phi_{i_2} \cdots \phi_{i_n}\rangle = \sum_{\text{pairings}} G_{j_1 j_2}\, G_{j_3 j_4} \cdots G_{j_{n-1} j_n}$$

$$\langle \phi(r_1)\,\phi(r_2)\rangle = \frac{\langle \phi(r_1)\,\phi(r_2)\, e^{-S_I(\phi)} \rangle_0}{\langle e^{-S_I(\phi)}\rangle_0}$$

taylor expand

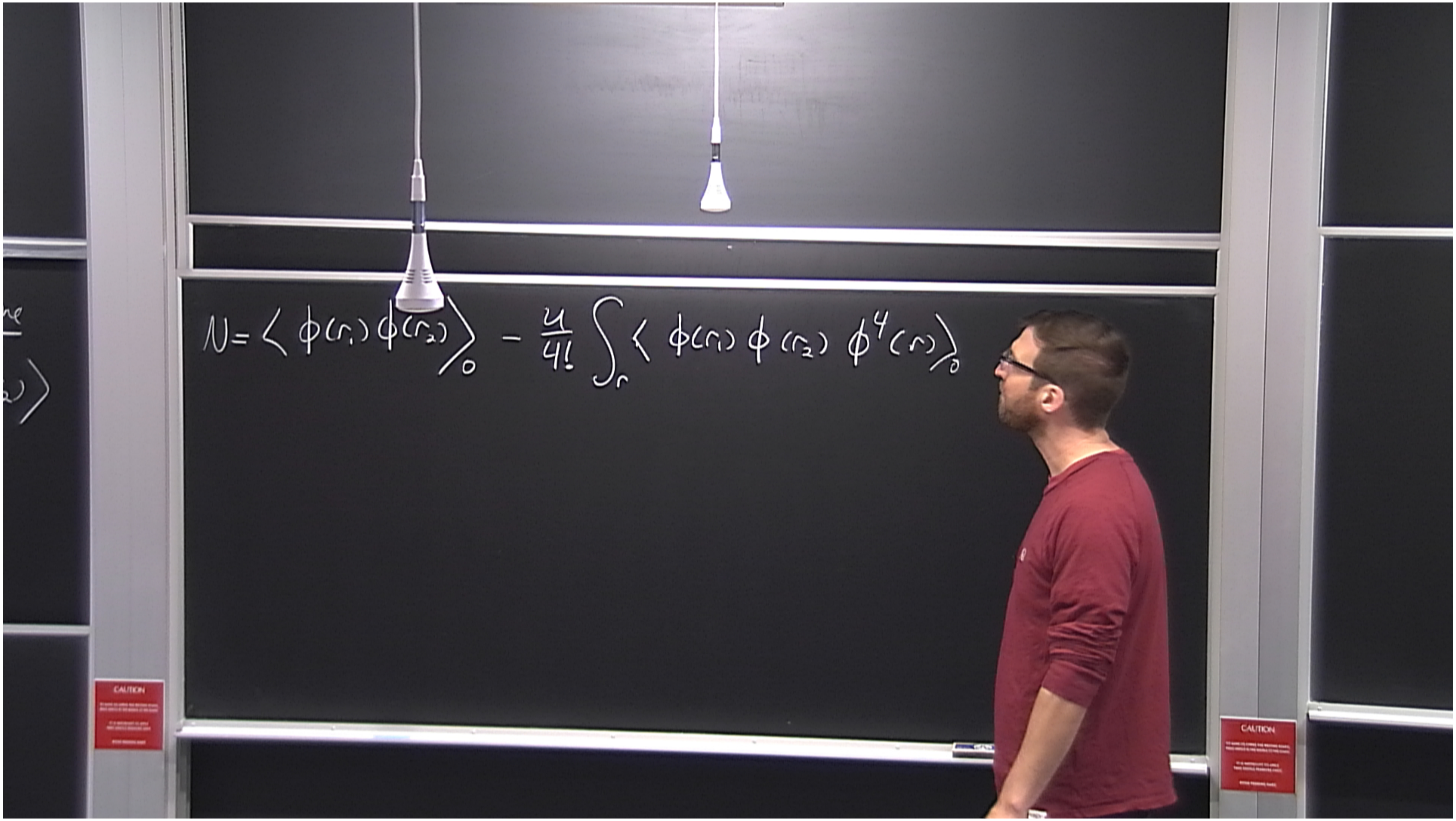
$$= \frac{\sum_{k=0}^{\infty} \frac{1}{k!} \langle \phi(r_1)\,\phi(r_2)\,(-S_I)^k \rangle_0}{\sum_{k=0}^{\infty} \frac{1}{k!} \langle (-S_I)^k \rangle_0}$$

last time

$$\langle \phi(r_1)\,\phi(r_2)\rangle = \frac{N}{D}$$

Set up to evaluate nicely with Wick's theorem

$$\langle \phi_{i_1}\,\phi_{i_2} \cdots \phi_{i_n}\rangle = \sum_{\text{pairings}} G_{j_1 j_2}\, G_{j_3 j_4} \cdots G_{j_{n-1} j_n}$$


$N = \langle \phi(r_1)\phi(r_2)\rangle_0 - \frac{u}{4!}\int_r \langle \phi(r_1)\phi(r_2)\phi^4(r)\rangle_0$

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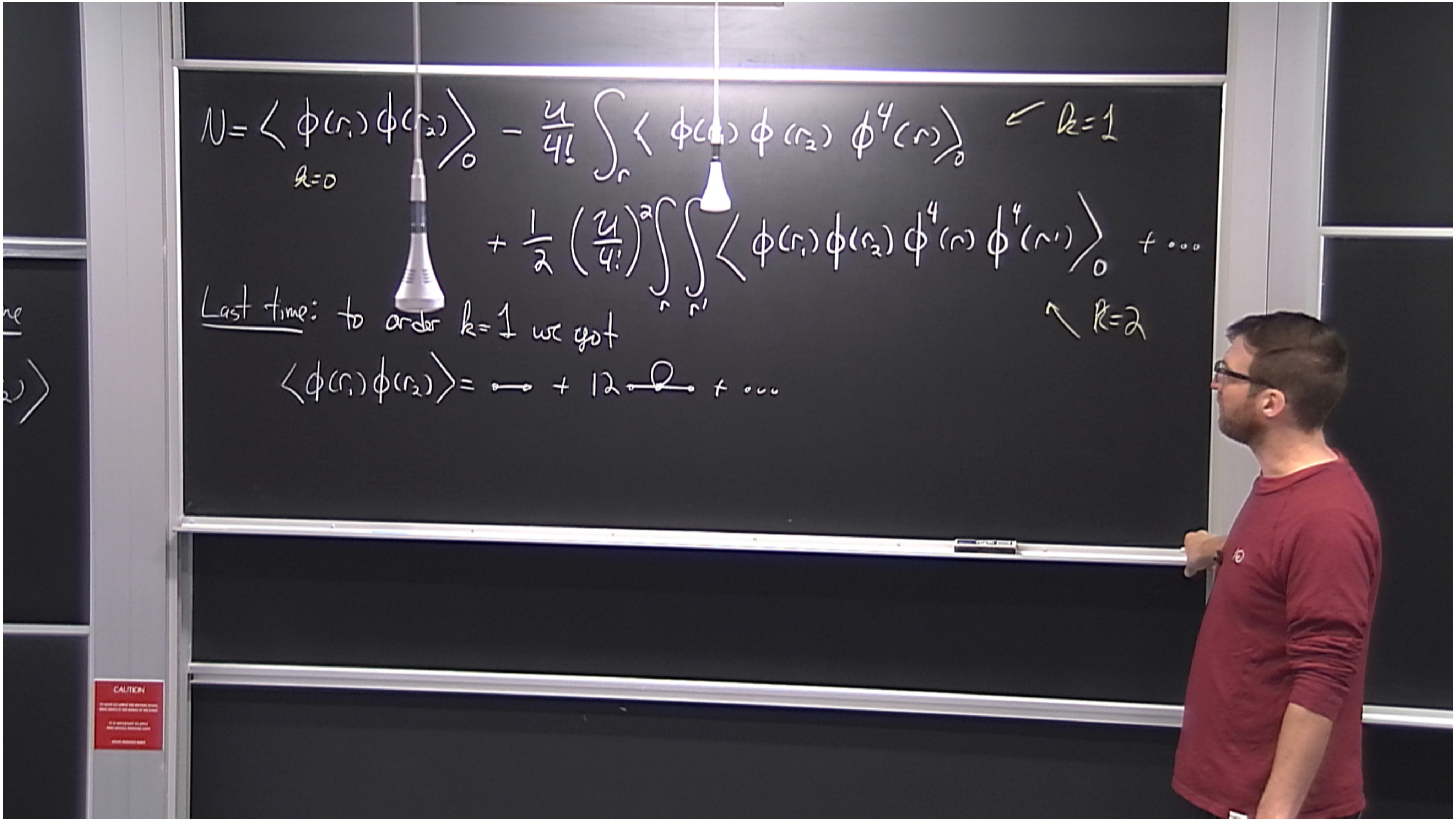
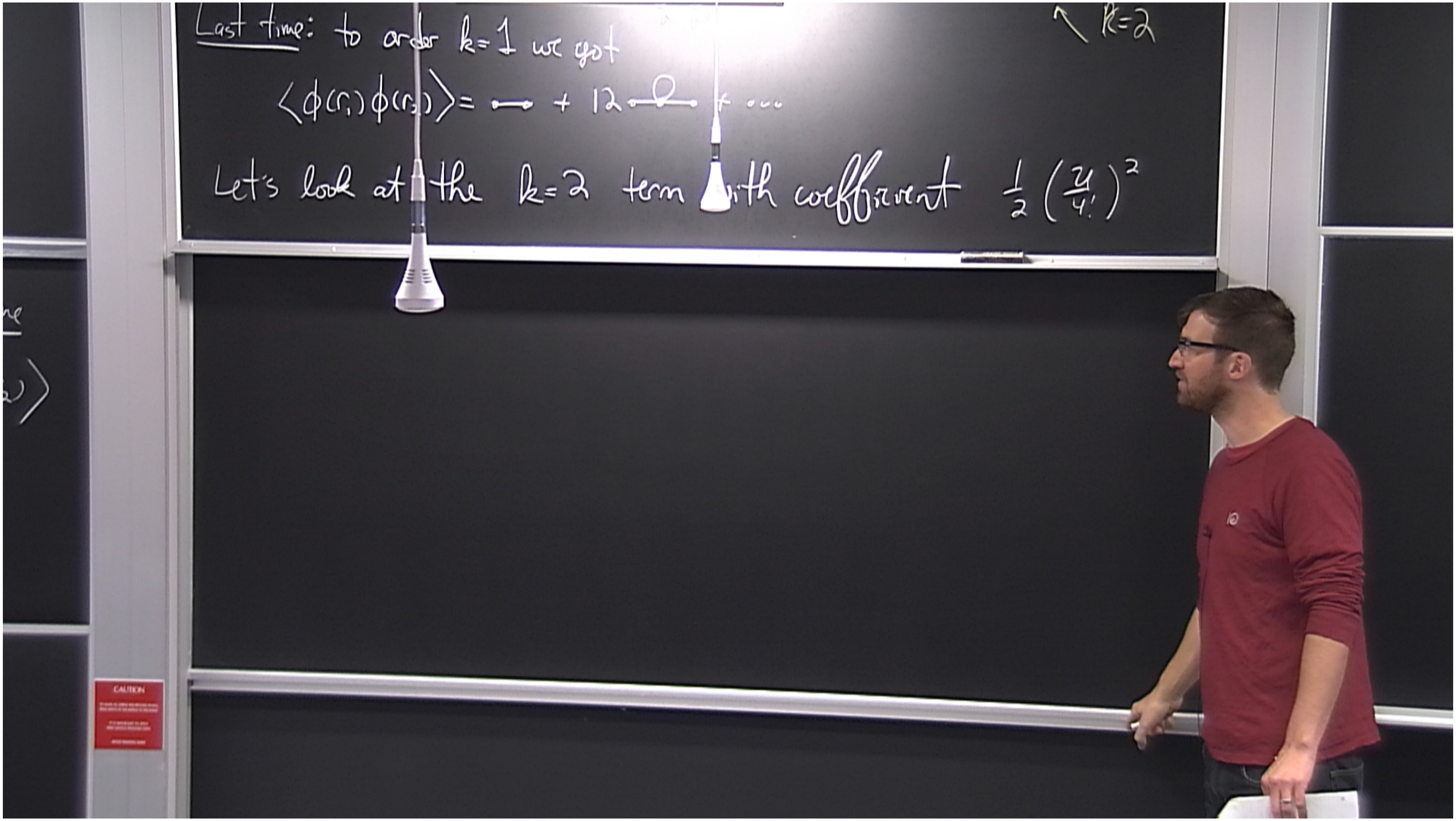
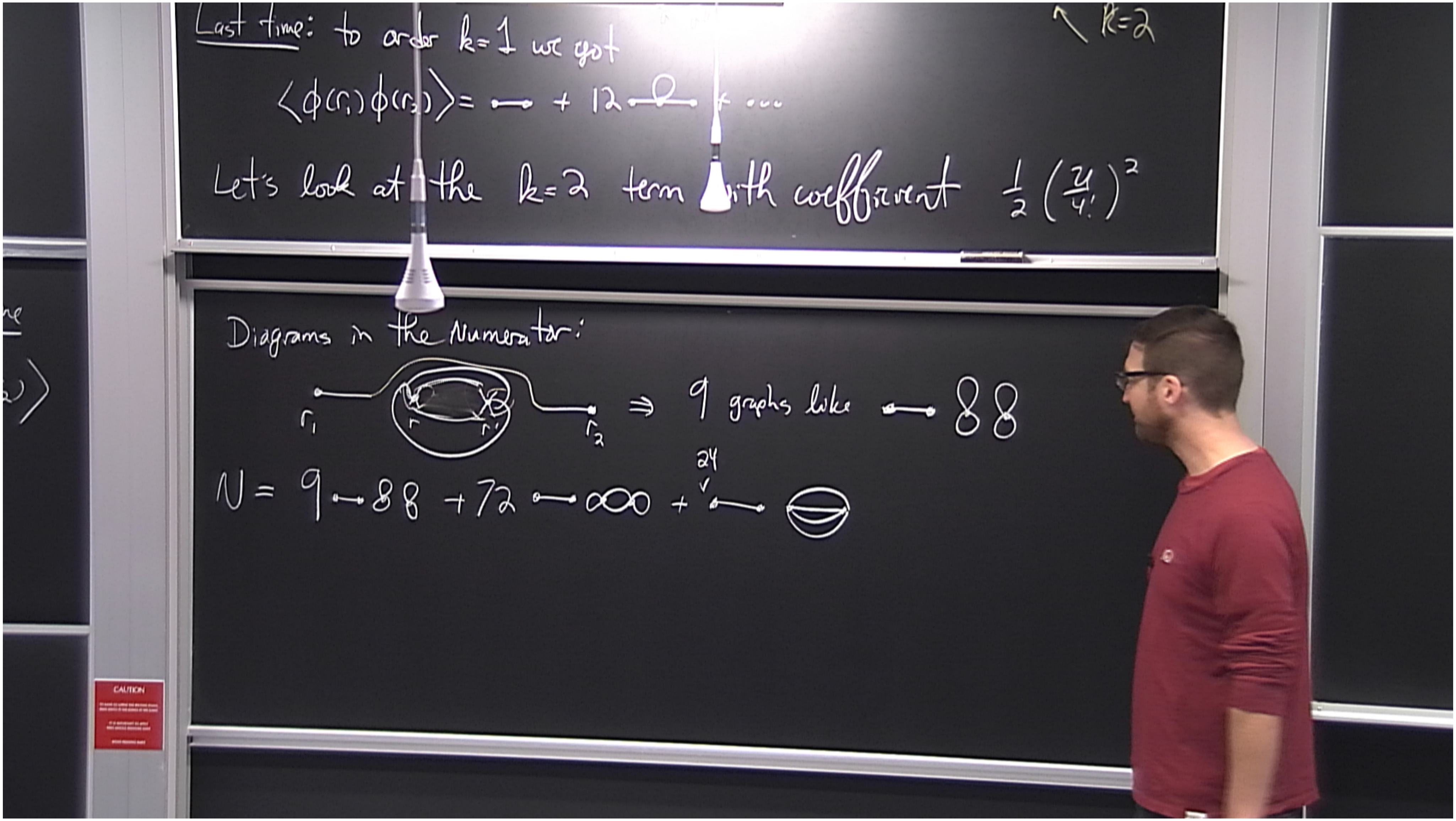
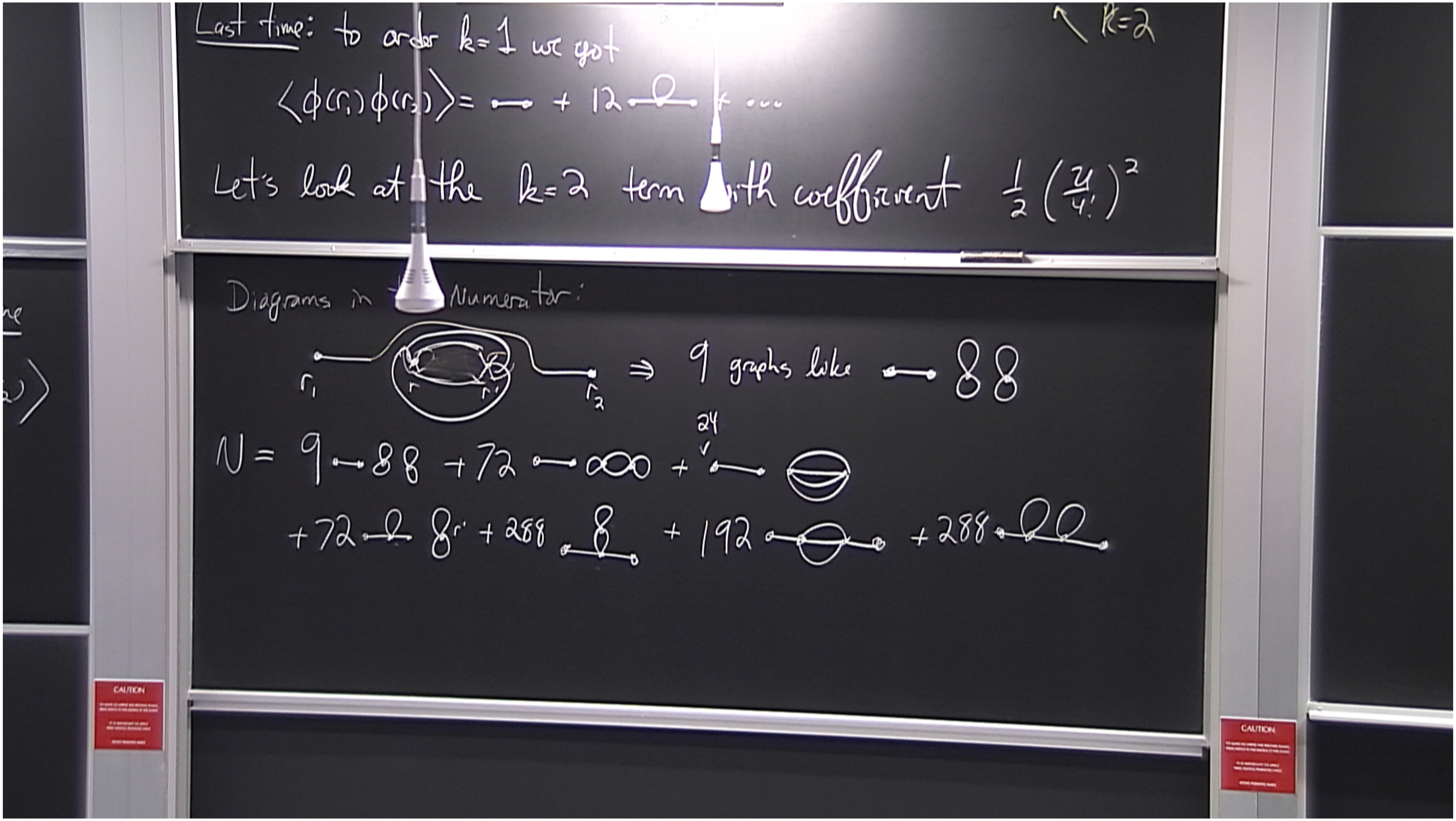
$$+ \frac{1}{2}\left(\frac{u}{4!}\right)^2 \int_r\int_{r'} \langle \phi(r_1)\phi(r_2)\phi^4(r)\phi^4(r')\rangle_0 + \cdots$$

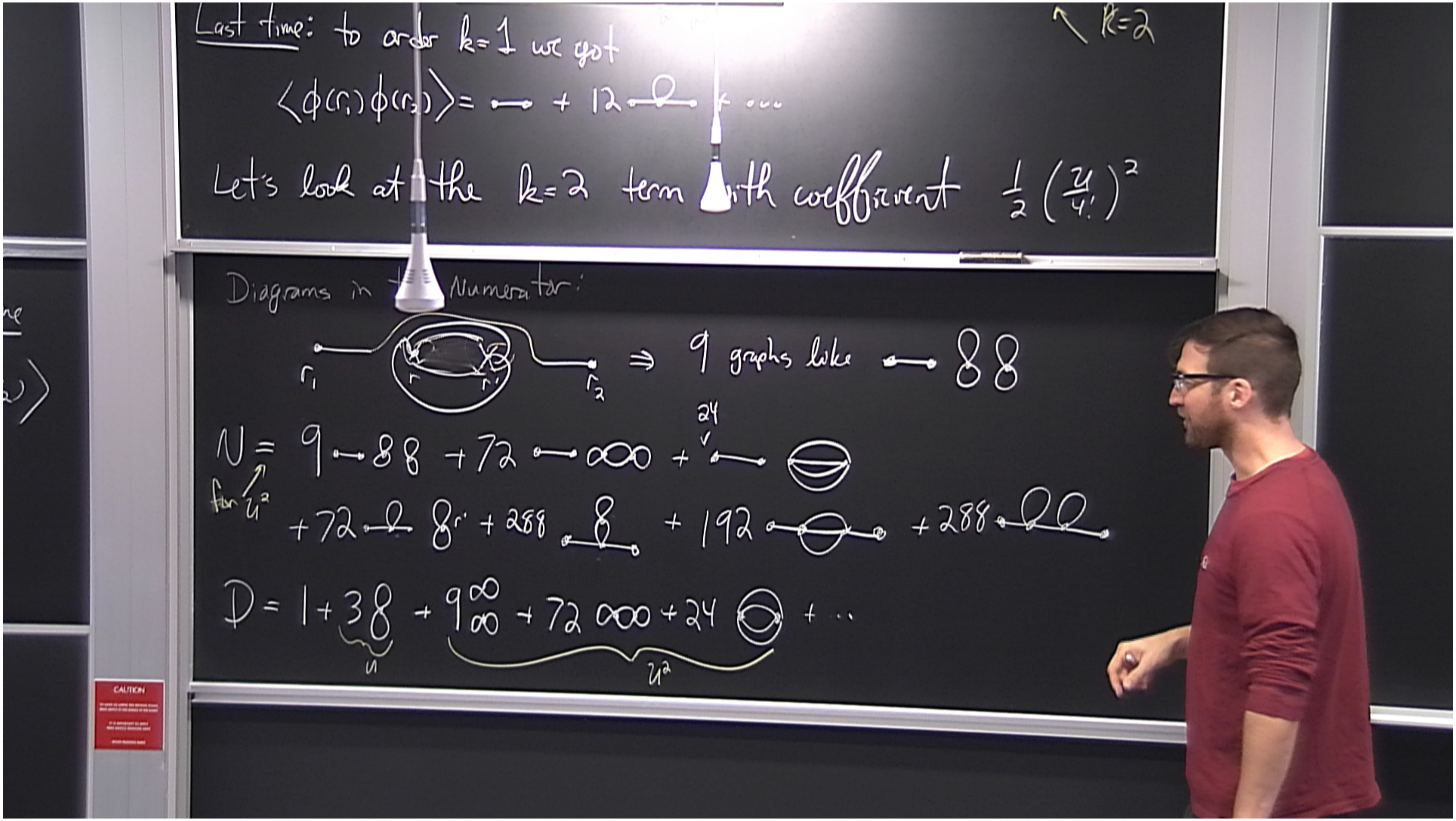


$N = \langle \phi(r_1)\phi(r_2) \rangle_0 - \frac{u}{4!}\int_r \langle \phi(r_1)\phi(r_2)\phi^4(r) \rangle_0$ ← $k=1$
$k=0$
$+ \frac{1}{2}\left(\frac{u}{4!}\right)^2 \int_r \int_{r'} \langle \phi(r_1)\phi(r_2)\phi^4(r)\phi^4(r') \rangle_0 + \dots$
↖ $k=2$
Last time: to order $k=1$ we got
$\langle \phi(r_1)\phi(r_2) \rangle =$ + 12 + ...
CAUTION


Last time: to order k=1 we got
⟨φ(r1)φ(r2)⟩ = + 12 + ...
Let's look at the k=2 term with coefficient 1/2 (u/4!)^2
k=2


Last time: to order k=1 we got
k=2
Let's look at the k=2 term with coefficient
Diagrams in the Numerator:
9 graphs like
N = 9 + 72 +
24


Last time: to order k=1 we got
⟨φ(r₁)φ(r₂)⟩ = + 12 + ...
↖ k=2
Let's look at the k=2 term with coefficient ½(λ/4!)²
Diagrams in the Numerator:
r₁ r r' r₂
⇒ 9 graphs like
N = 9 + 72 + 24
+ 72 + 288 + 192 + 288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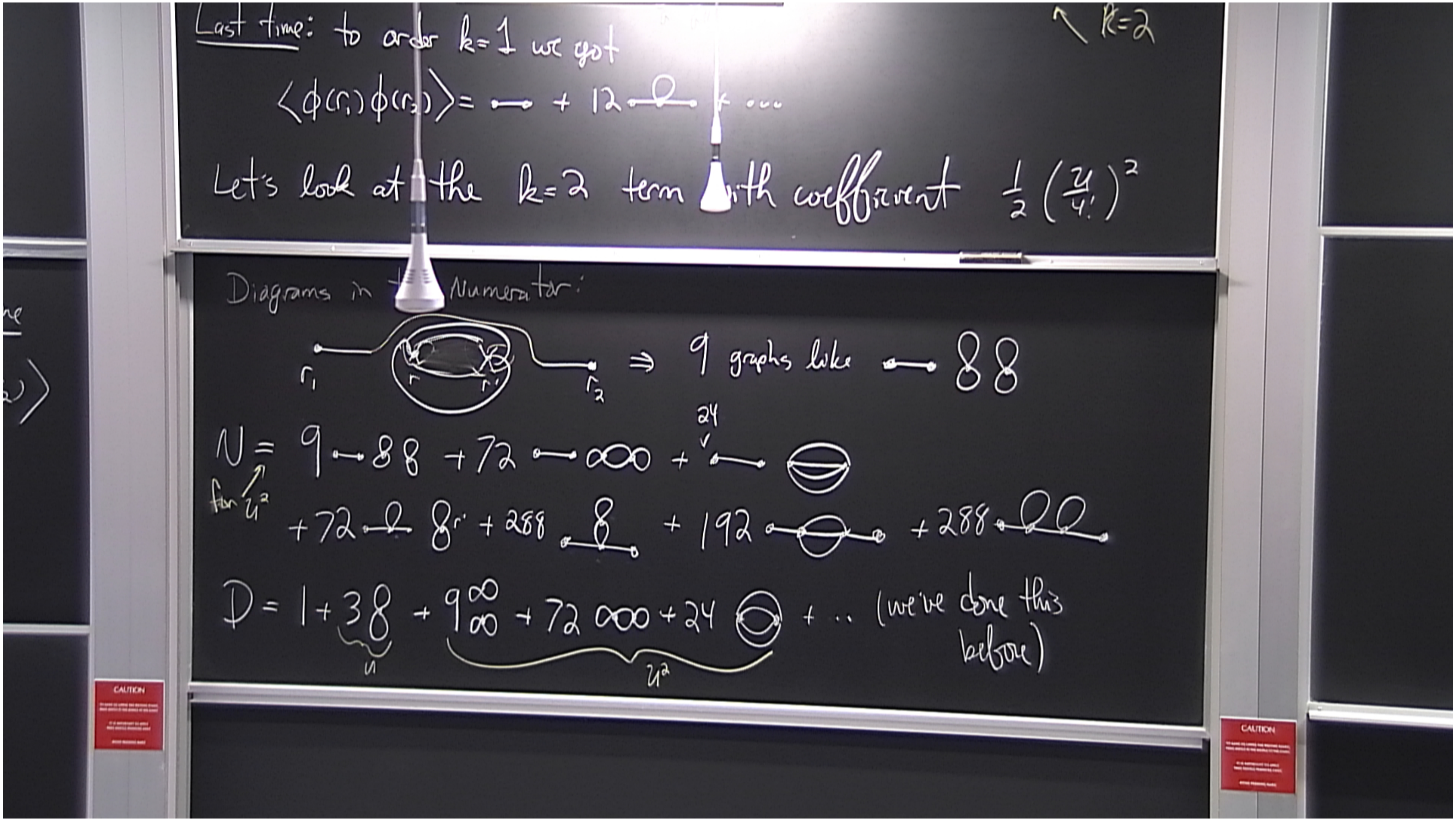
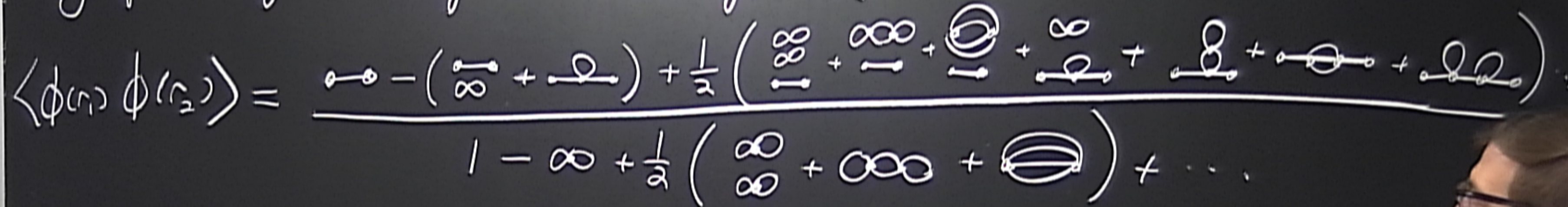
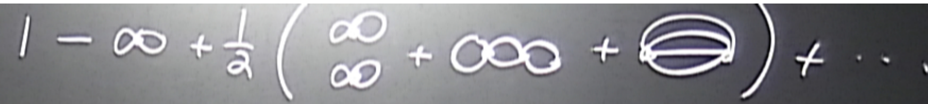
Diagrams in the Numerator:
r₁ r r' r₂
⇒ 9 graphs like
N = 9 + 72 + 24
for u²
+ 72 + 288 + 192 + 288
D = 1 + 3 + 9 + 72 + 24 + ...
u
u²


Last time: to order k=1 we got
$\langle \phi(r_1)\phi(r_2) \rangle =$ + 12 + ...
↖ k=2
Let's look at the k=2 term with coefficient $\frac{1}{2}\left(\frac{u}{4!}\right)^2$
Diagrams in the Numerator:
r_1 r r' r_2
⇒ 9 graphs like
24
N = 9 + 72 +
for u^2
+ 72 + 288 + 192 + 288
D = 1 + 3 + 9 + 72 + 24 + .. (we've done this before)
u
u^2

graphically we get something like



Again, expanding the denominator $\frac{1}{1+\epsilon} \simeq 1 - \epsilon$



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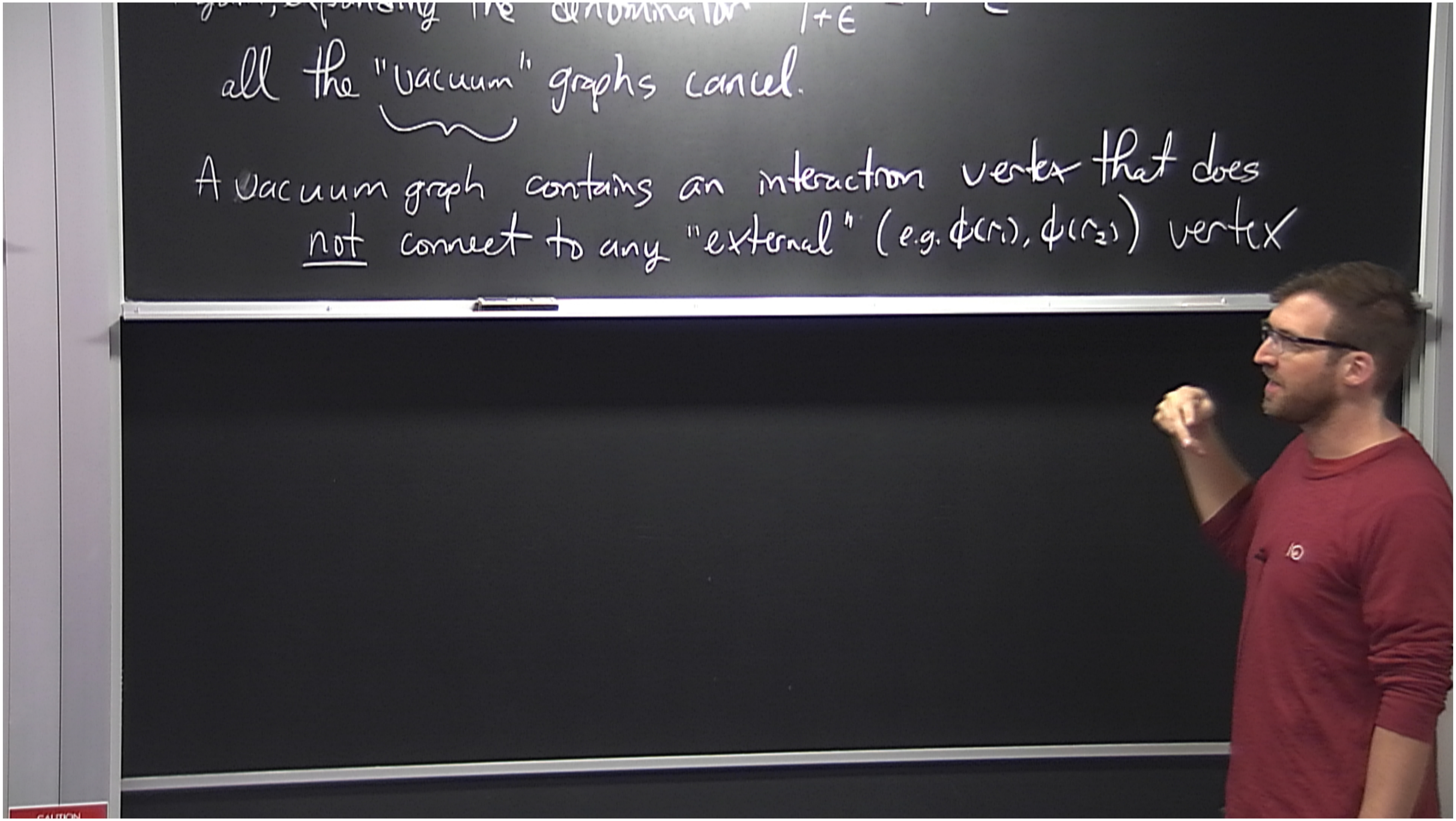
all the $\underbrace{\text{"vacuum"}}$ graphs cancel.

A vacuum graph contains


all the "vacuum" graphs cancel.
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Thus the contribution to the correlation function $\langle \phi(r_1) \cdots \phi(r_n) \rangle$ to k^{th} order is the sum of all ... excluding vacuum graphs

- no need to calculate both N and D

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Thus the contribution to the correlation function $\langle \phi(r_1) \cdots \phi(r_n) \rangle$ to k^{th} order is the sum of all graphs excluding vacuum graphs

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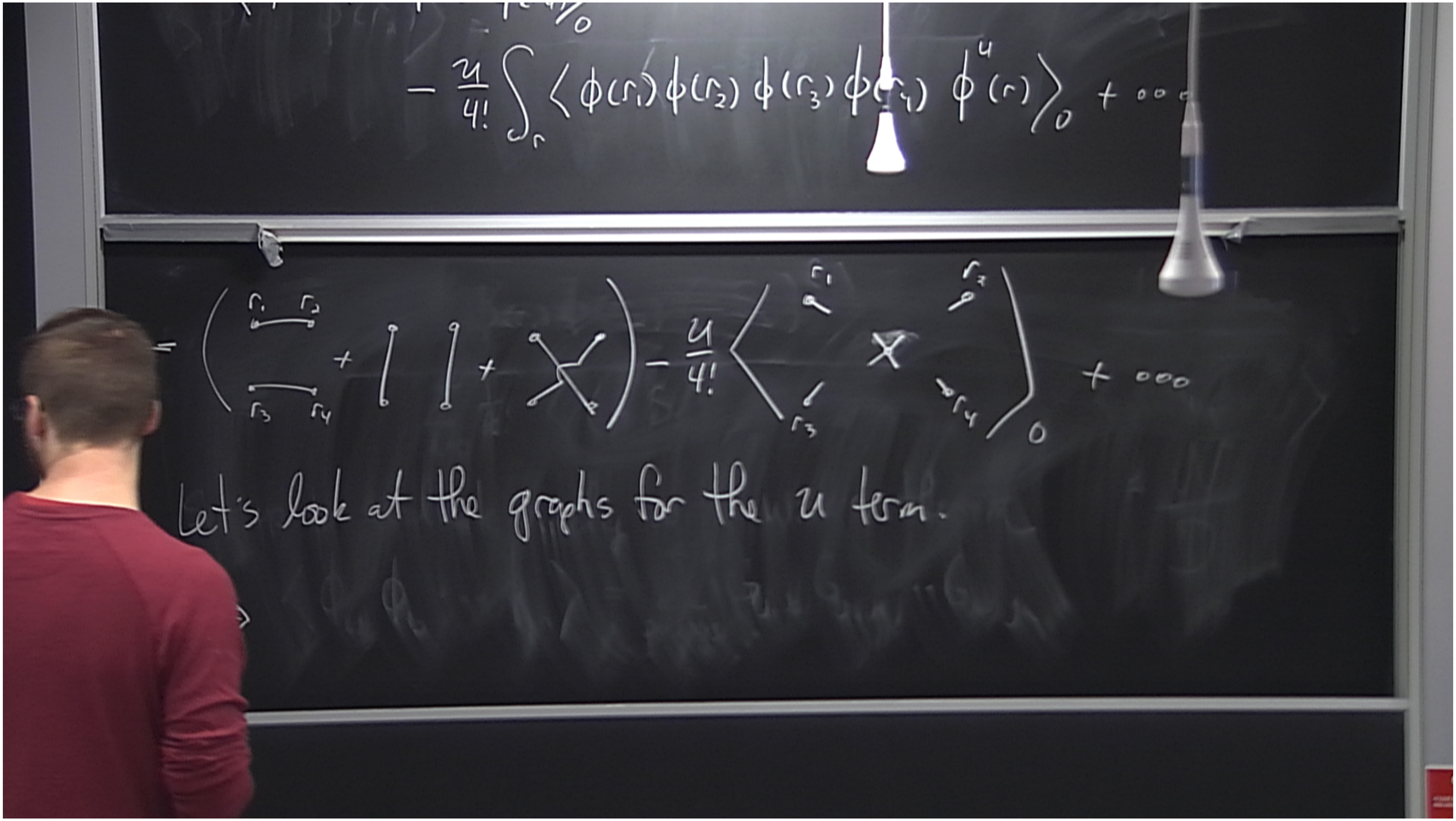
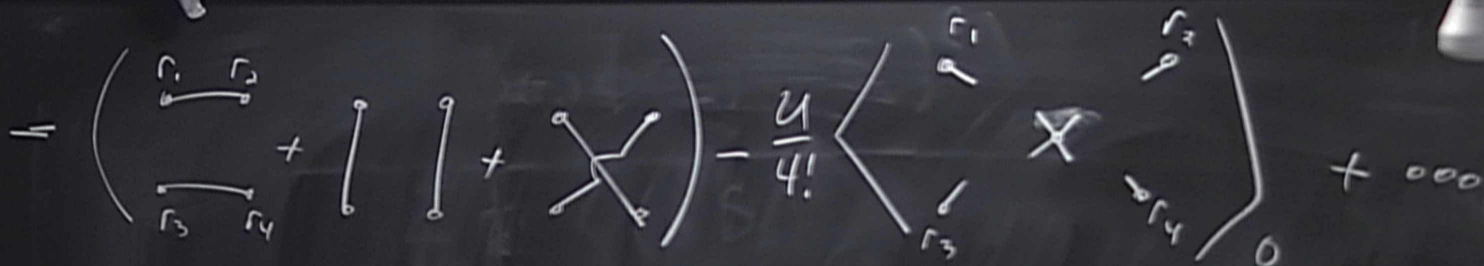


$- \frac{u}{4!} \int_r \langle \phi(r_1)\phi(r_2)\phi(r_3)\phi(r_4)\,\phi^4(r) \rangle_0 + \dots$
r_1 r_2 r_3 r_4
$- \frac{u}{4!}$
$+ \dots$
Let's look at the graphs for the u term.

$$- \frac{u}{4!} \int_r \langle \phi(r_1)\phi(r_2)\phi(r_3)\phi(r_4)\, \phi^4(r) \rangle_0 + \cdots$$



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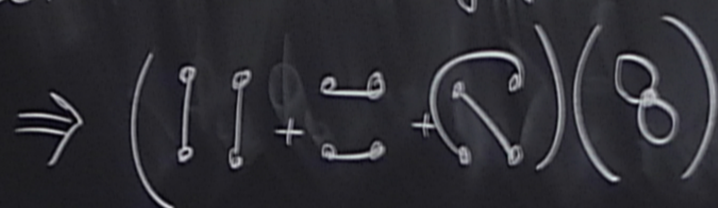

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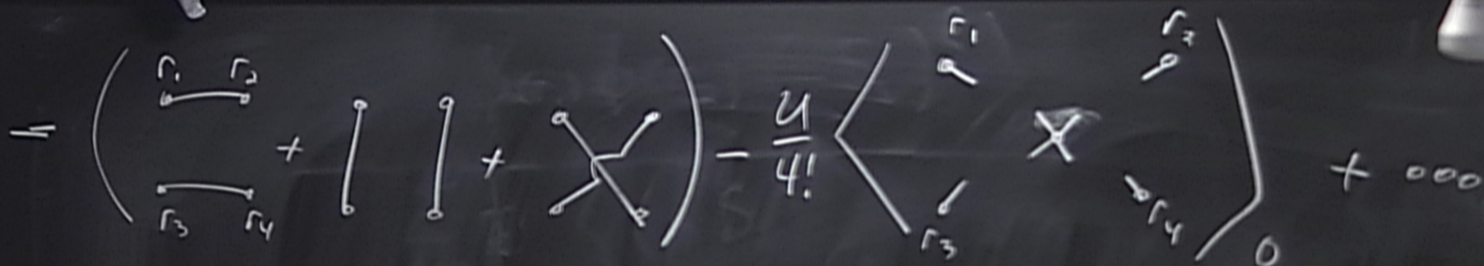

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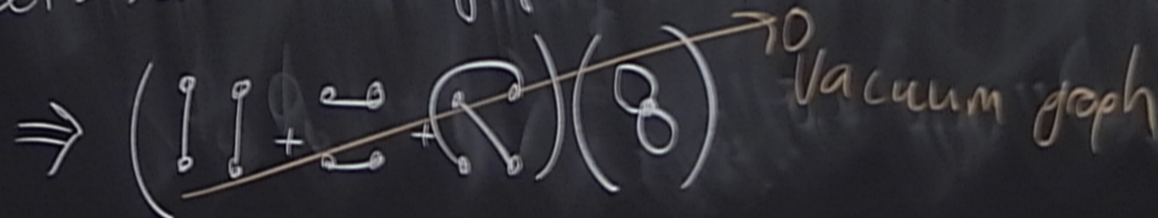
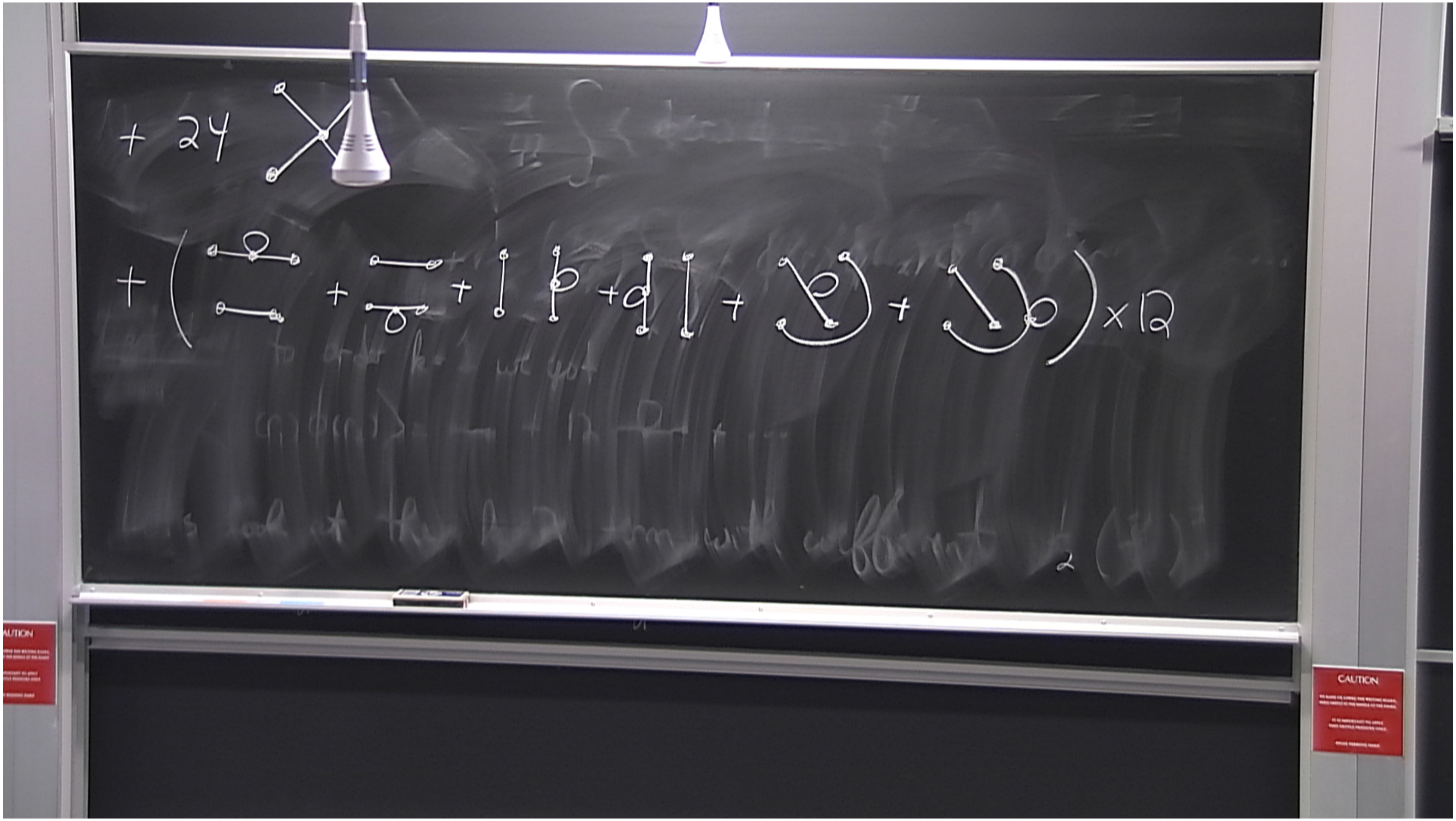
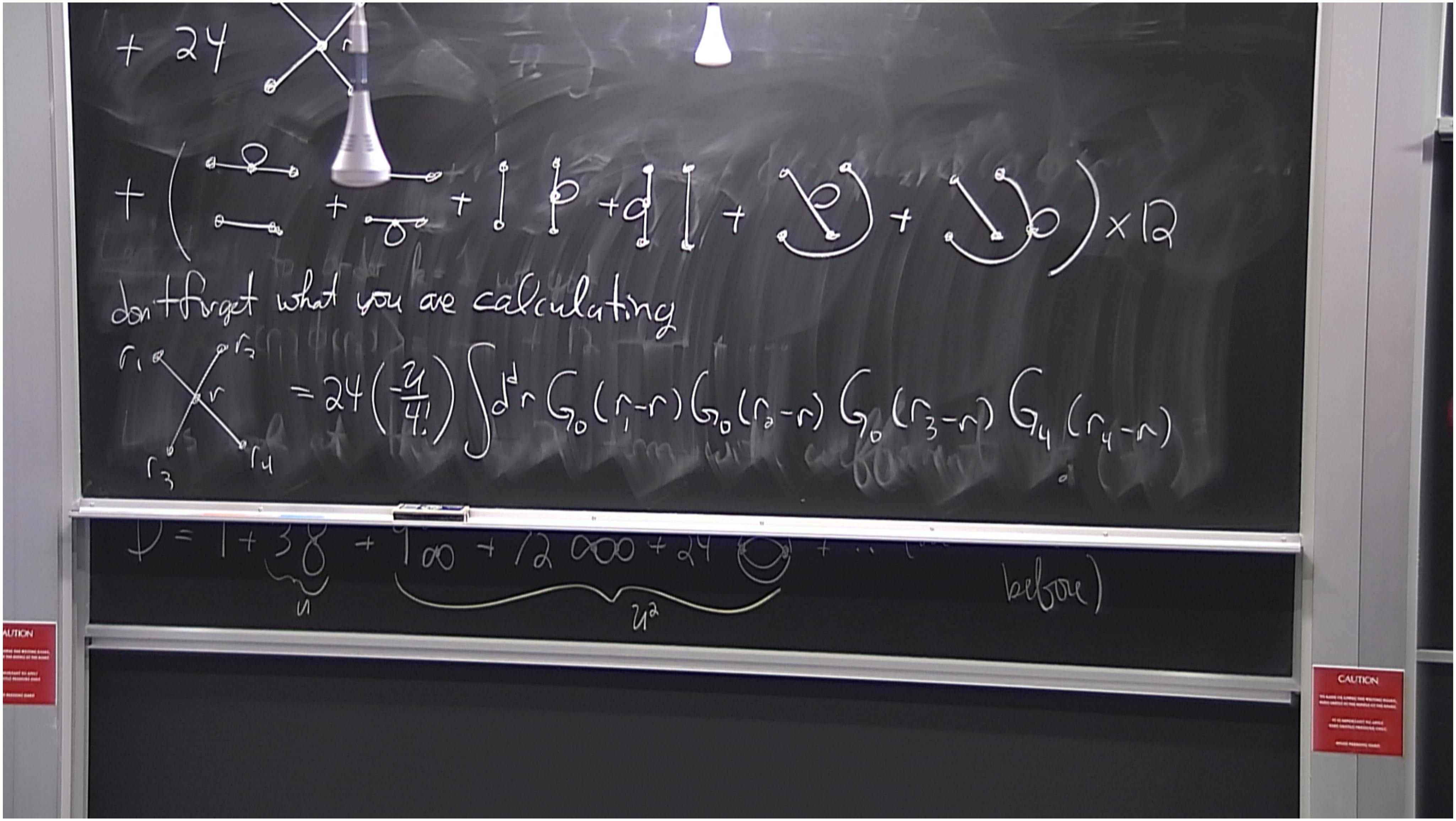
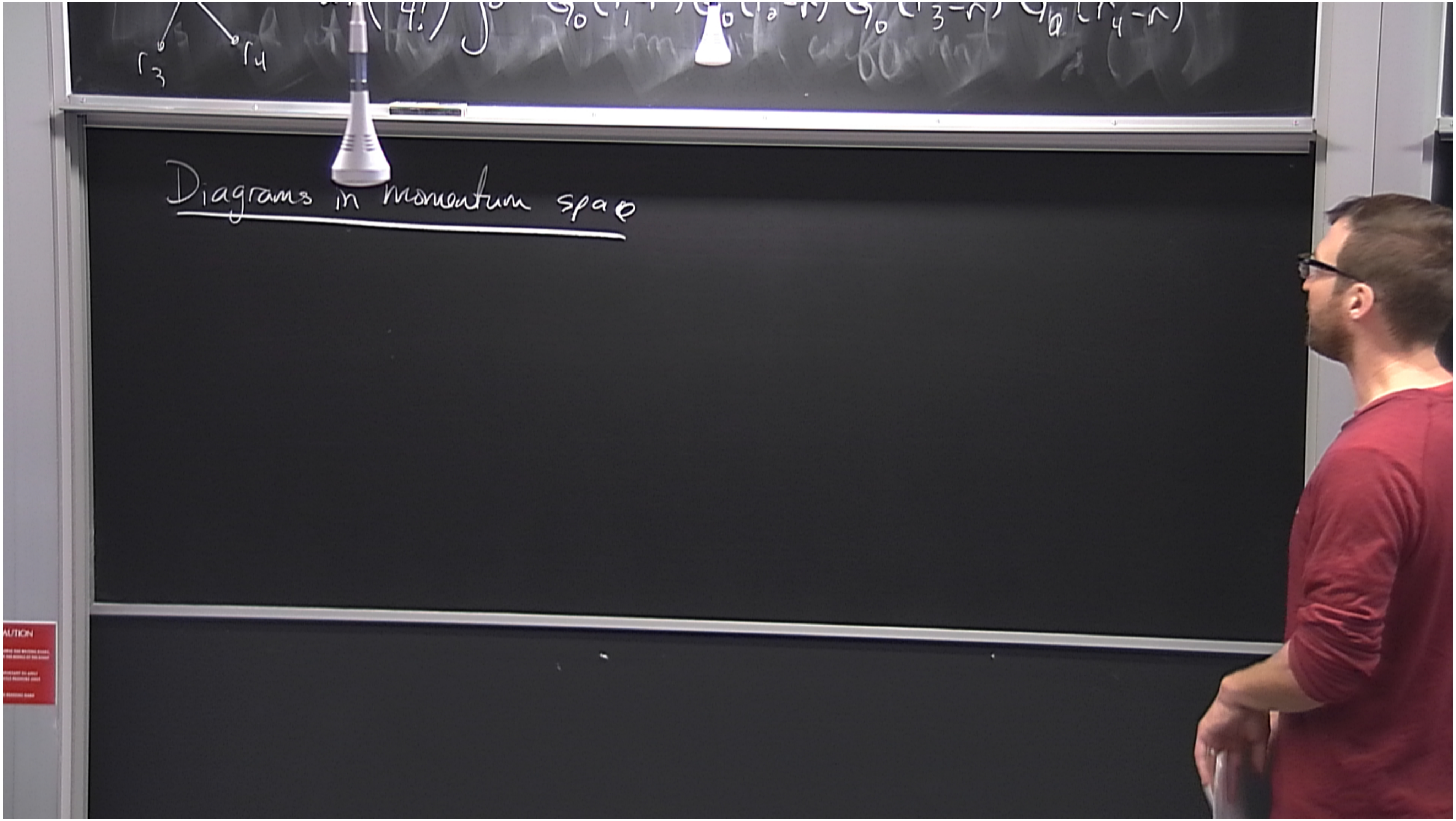



+ 24
× 12


+ 24
+ (+ + + + + +) × 12
don't forget what you are calculating
r_1 r_2 r r_3 r_4
$= 24\left(-\frac{u}{4!}\right)\int d^d r\, G_0(r_1-r)\, G_0(r_2-r)\, G_0(r_3-r)\, G_0(r_4-r)$
before)


Diagrams in momentum space

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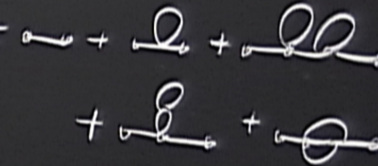
(Back to our two-point c.f.) $\langle \phi(r_1)\, \phi(r_2) \rangle =$



- momentum space: fix some repetition
- simplify the calculation (e.g. we know G_0)

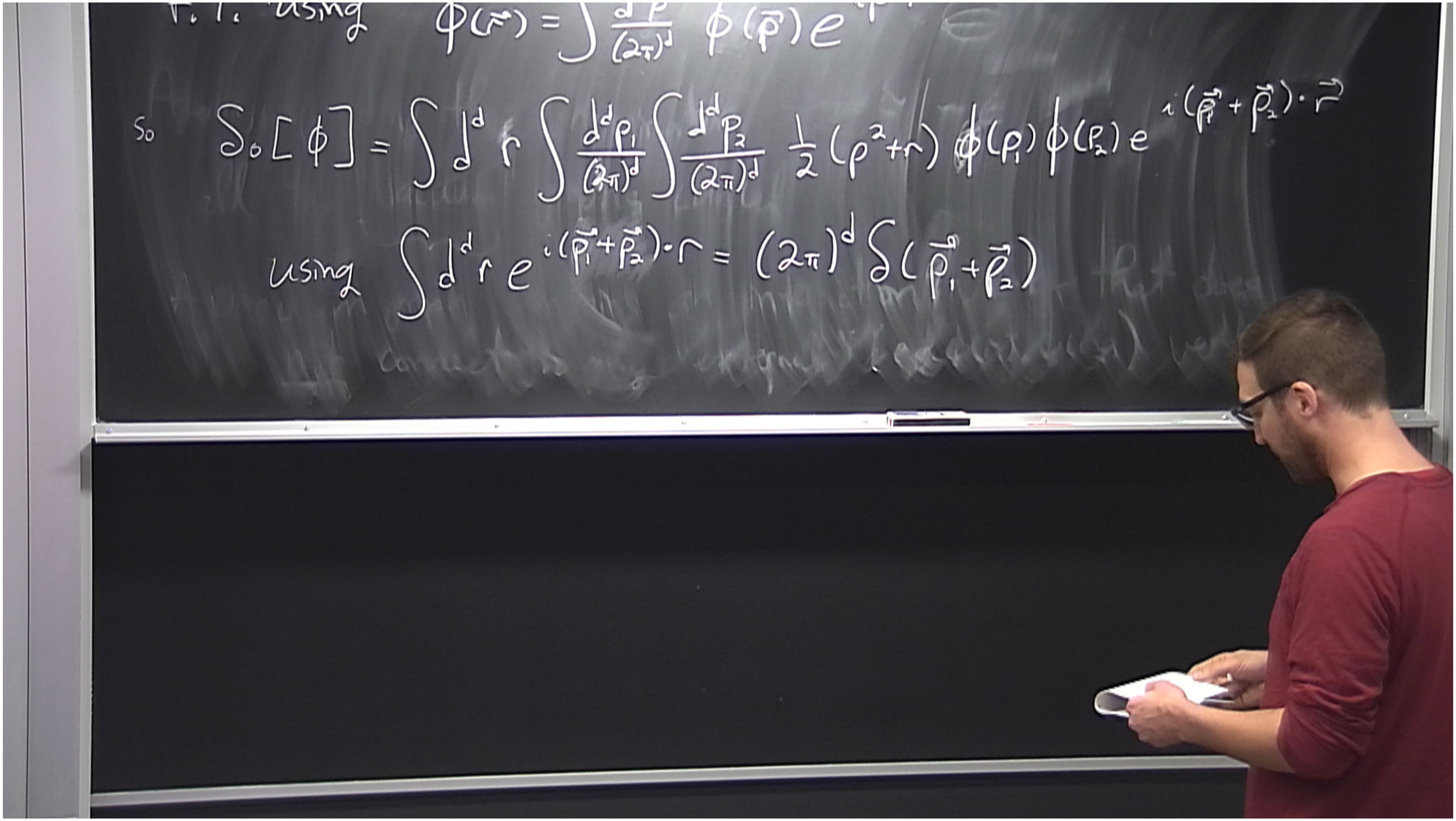
r_3 r_4 $\left(\frac{\lambda}{4!}\right)\int d^dr\, G_0(r_1-r)\, G_0(r_2-r)\, G_0(r_3-r)\, G_0(r_4-r)$

Diagrams in momentum space

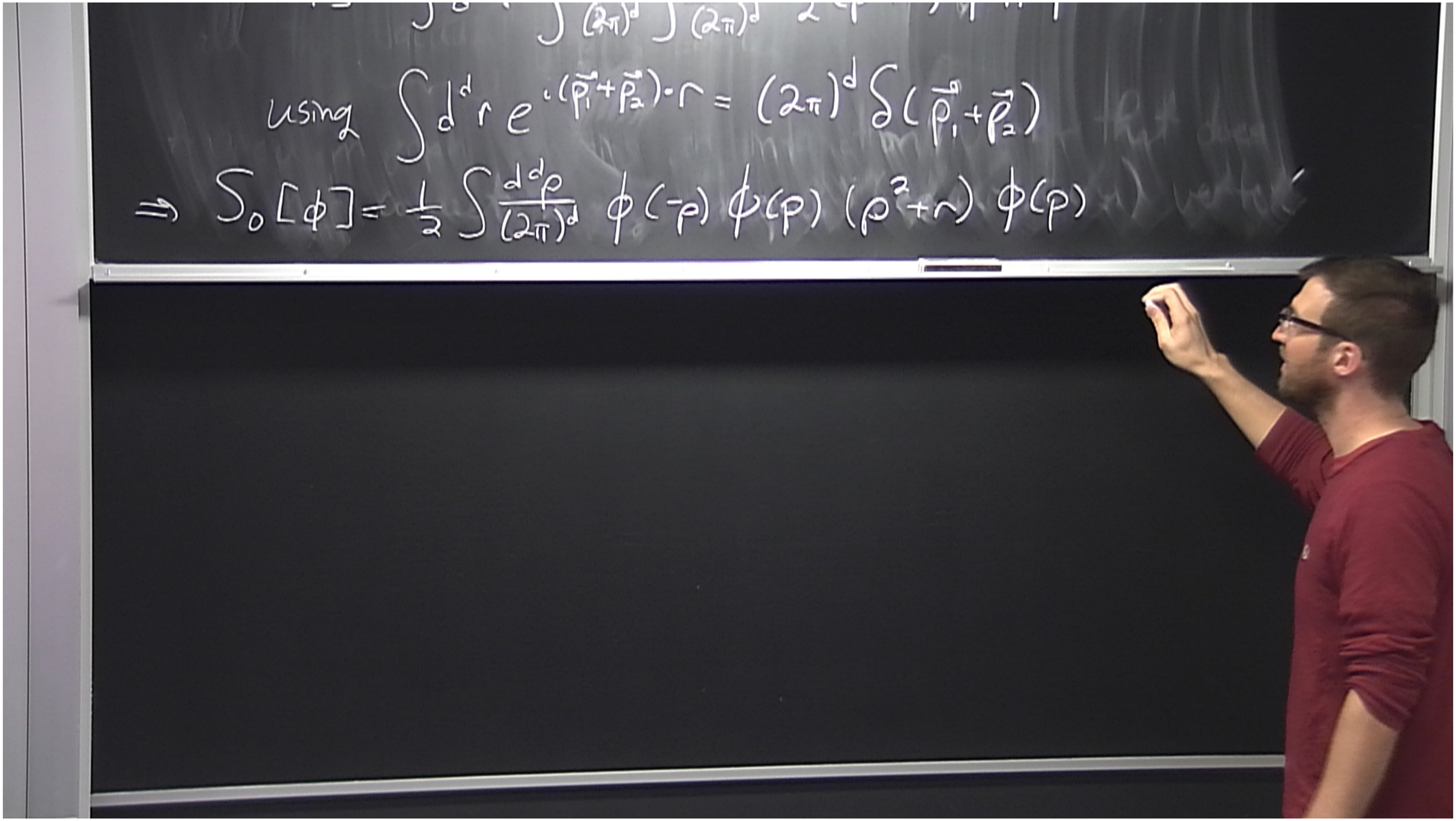
(Back to our two-point c.f.) $\langle \phi(r_1)\, \phi(r_2) \rangle =$

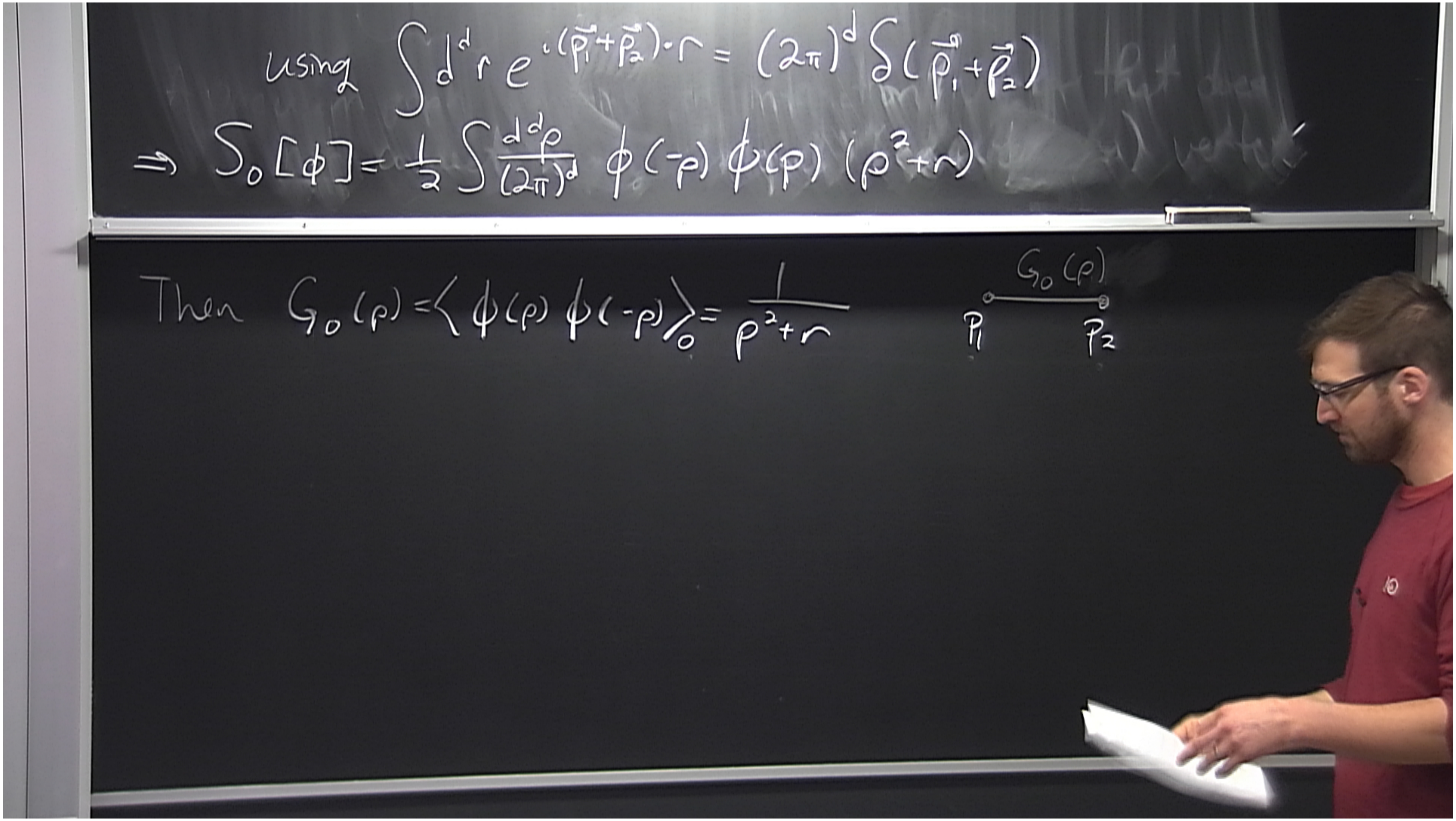
- momentum space: fix some repetition
- simplify the calculation (e.g. we know G_0)

Recall: $G_0(r_1 - r_2) = \langle \phi(r_1)\phi(r_2) \rangle_0$ labelled by r_1 —— r_2



$\phi(\vec{r}) = \int \frac{d^d p}{(2\pi)^d} \phi(\vec{p}) e$
So $S_0[\phi] = \int d^d r \int \frac{d^d p_1}{(2\pi)^d} \int \frac{d^d p_2}{(2\pi)^d} \frac{1}{2}(p^2+r)\, \phi(p_1)\phi(p_2)\, e^{i(\vec{p}_1+\vec{p}_2)\cdot\vec{r}}$
using $\int d^d r\, e^{i(\vec{p}_1+\vec{p}_2)\cdot r} = (2\pi)^d \delta(\vec{p}_1+\vec{p}_2)$


using $\int d^d r \, e^{i(\vec{p}_1+\vec{p}_2)\cdot r} = (2\pi)^d \delta(\vec{p}_1+\vec{p}_2)$
$\Rightarrow S_0[\phi] = \frac{1}{2}\int \frac{d^d p}{(2\pi)^d} \phi(-p)\,\phi(p)\,(p^2+r)\,\phi(p)$



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$\Rightarrow S_0[\phi] = \frac{1}{2}\int \frac{d^d p}{(2\pi)^d} \phi(-p)\phi(p)(p^2+r)$
Then $G_0(p) = \langle \phi(p)\phi(-p) \rangle_0 = \frac{1}{p^2+r}$
$G_0(p)$
p_1
p_2

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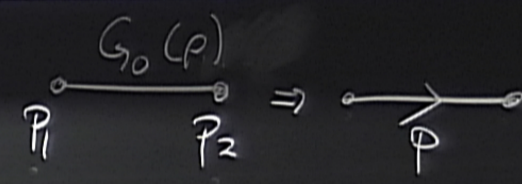

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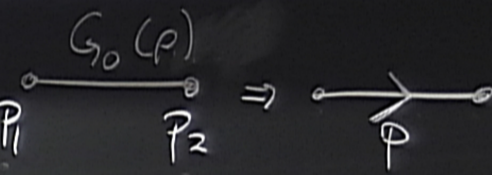


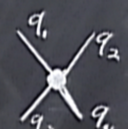
conservation of momentum comes from $\delta(p_1+p_2) \Rightarrow p_1 = -p_2$

Similarly, the ϕ^4 part of the action can be written in momentum space.

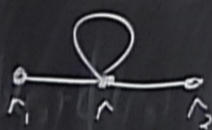
$$S_I = \frac{u}{4!}\int d^d r \, \phi^4(r) = \frac{u}{4!}\int d^d r \int \frac{d^d q_1}{(2\pi)^d} \cdots \frac{d^d q_4}{(2\pi)^d} \phi(q_1)\phi(q_2)\phi(q_3)\phi(q_4) e^{i(\vec{q}_1+\vec{q}_2+\vec{q}_3+\vec{q}_4)\cdot\vec{r}}$$

$p = q$

and $$\int d^d r \, e^{i(\vec{q}_1+\vec{q}_2+\vec{q}_3+\vec{q}_4)\cdot\vec{r}} = (2\pi)^d \, \delta(\vec{q}_1+\vec{q}_2+\vec{q}_3+\vec{q}_4)$$



From this, momentum will be conserved at "internal" (interaction) vertices

e.g. $12 \times$ 

$$= 12\left(\frac{-u}{4!}\right)\int d^d r \, G_0(r-r)\, G_0(r_1-r)\, G_0(r-r_2)$$

$= (2\pi)^d \delta(q_1 + q_2 + q_3 + q_4)$

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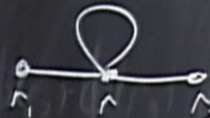
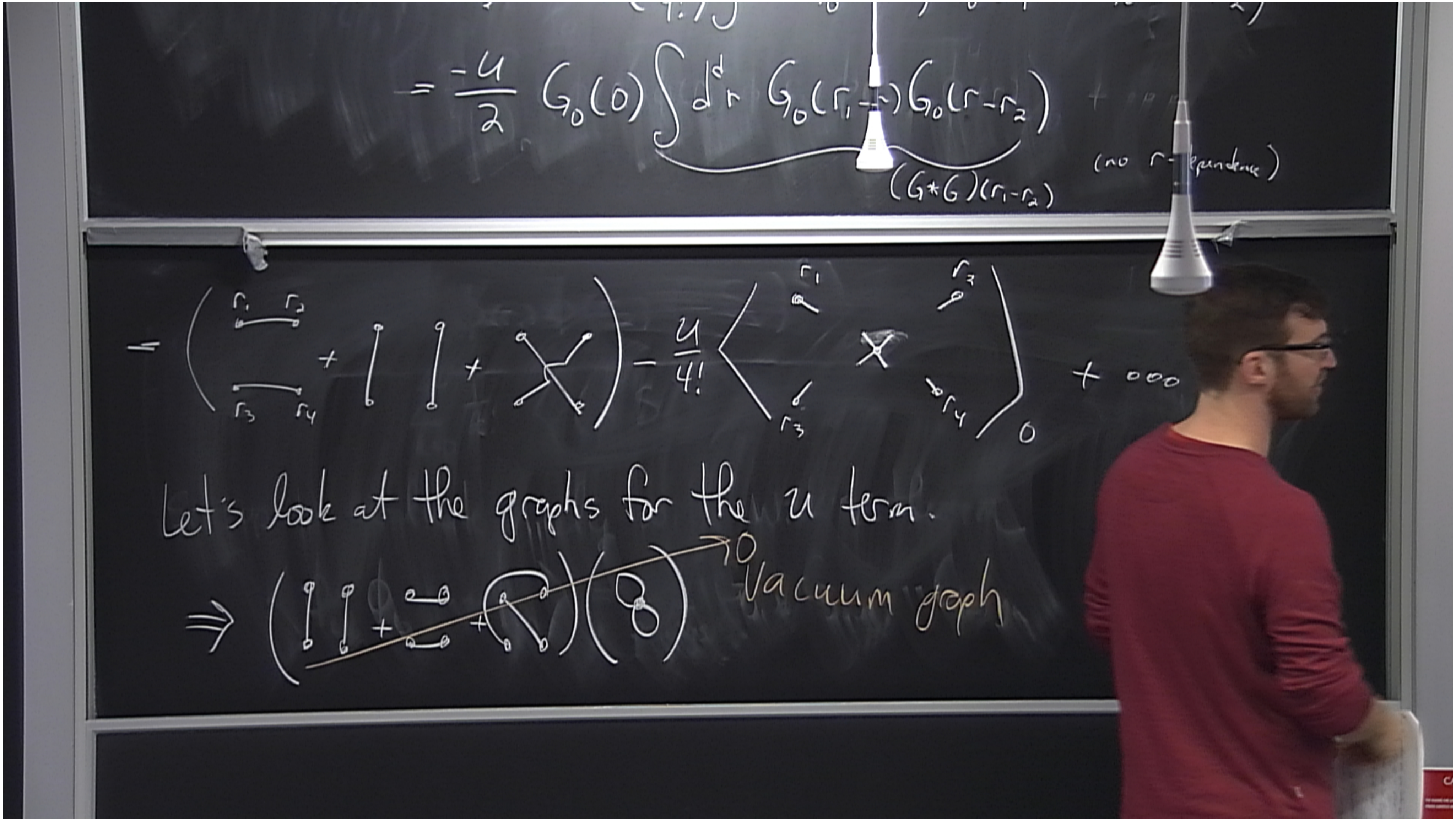
e.g. $12 \times$ 

$= 12\left(\frac{-u}{4!}\right)\int d^d r\, G_0(r-r)\, G_0(r_1-r)\, G_0(r-r_2)$

$= \frac{-u}{2}\, G_0(0)\int d^d r\, G_0(r_1-r)\, G_0(r-r_2)$

... at ... the graphs for the u term.

$\to 0$ Vacuum graph


$= \frac{-u}{2} G_0(0) \int d^d r \, G_0(r_1 - r) G_0(r - r_2)$
$(G*G)(r_1 - r_2)$
(no r-dependence)
r_1 r_2 r_3 r_4
$- \frac{u}{4!}$
$+ \cdots$
Let's look at the graphs for the u term.
Vacuum graph

$$= \frac{-u}{2}\, G_0(0) \underbrace{\int d^d r\; G_0(r_1 - r)\, G_0(r - r_2)}_{(G*G)(r_1 - r_2)}$$

(no r-dependence)

F.T. of $(G*G)(r_1 - r_2)$ exercise

$$= (2\pi)^d\, \delta(q_1 + q_2) \left[G_0(q_1) \right]^2$$

$$= \frac{-u}{2} G_0(0) \underbrace{\int d^d r \, G_0(r_1 - r) G_0(r - r_2)}_{(G*G)(r_1 - r_2)}$$

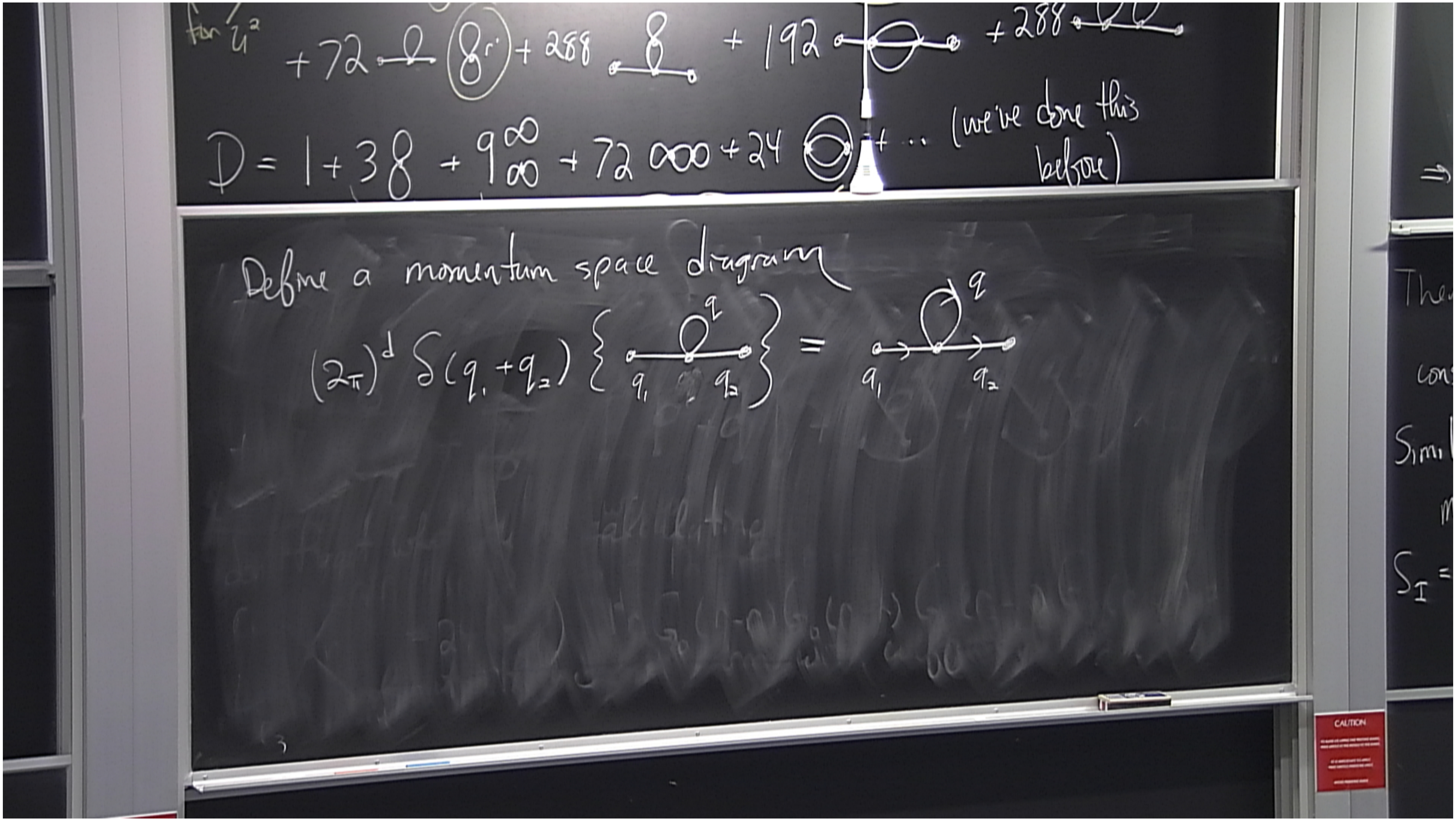
(no r-dependence)

F.T. of $(G*G)(r_1 - r_2)$ exercise

$$= (2\pi)^d \delta(q_1 + q_2) \left[G_0(q_1) \right]^2$$

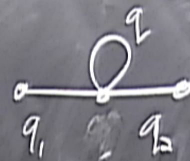
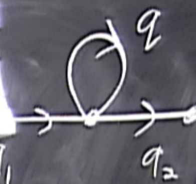
So the F.T. of  in momentum space is given by

$$(2\pi)^d \delta(q_1 + q_2) \left\{ -\frac{u}{2} \left[G_0(q_1) \right]^2 \cdot \int \frac{d^d q}{(2\pi)^d} G_0(q) \right\}$$



for λ^2
+ 72 + 288 + 192 + 288
D = 1 + 3 + 9 + 72 + 24 + ..
(we've done this before)
Define a momentum space diagram
$(2\pi)^d \delta(q_1+q_2)$
q
q_1
q_2
=

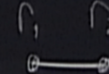
Define a momentum space diagram

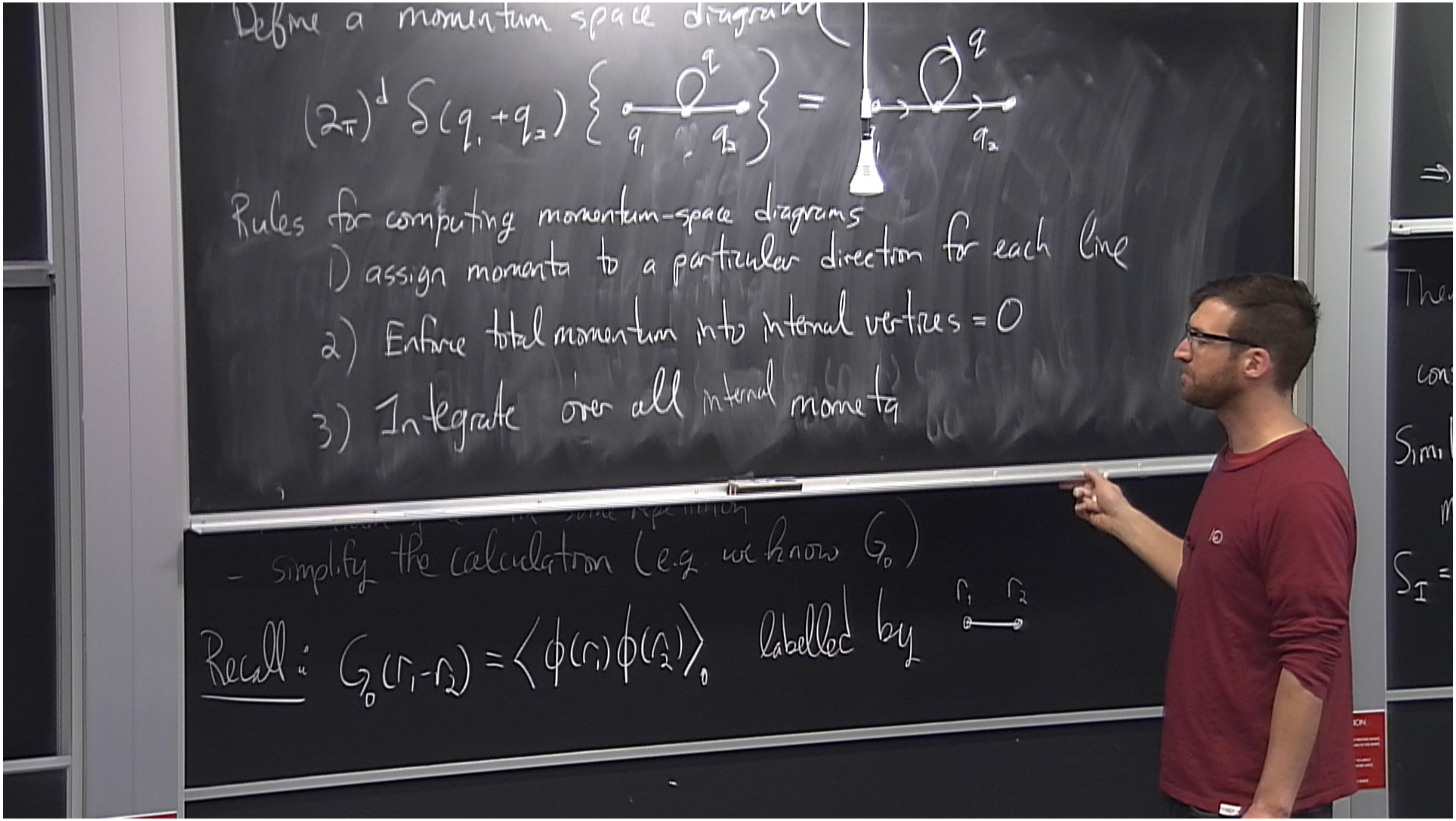
$$(2\pi)^d \delta(q_1+q_2) \left\{ \right.$$  $$\left. \right\} =$$ 

Rules for computing momentum-space diagrams

1) assign momenta to a particular direction for each line

2) Enforce total momentum into internal vertices = 0

Recall: $G_0(r_1 - r_2) = \langle \phi(r_1)\phi(r_2)\rangle_0$ labelled by 

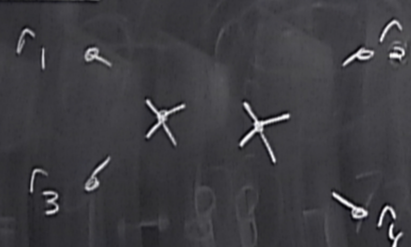

Define a momentum space diagram
(2π)^d δ(q1 + q2) { q1 q2 } =
q1 q2 q
Rules for computing momentum-space diagrams
1) assign momenta to a particular direction for each line
2) Enforce total momentum into internal vertices = 0
3) Integrate over all internal momenta
- simplify the calculation (e.g. we know G0)
Recall: G0(r1 - r2) = ⟨φ(r1)φ(r2)⟩0 labelled by
r1 r2

1) assign momenta to a particular direction for each line

2) Enforce total momentum into internal vertices $= 0$

3) Integrate over all internal momenta

<u>example:</u> a 2^{nd} order contribution to a 4-point correlation function



$\Rightarrow$

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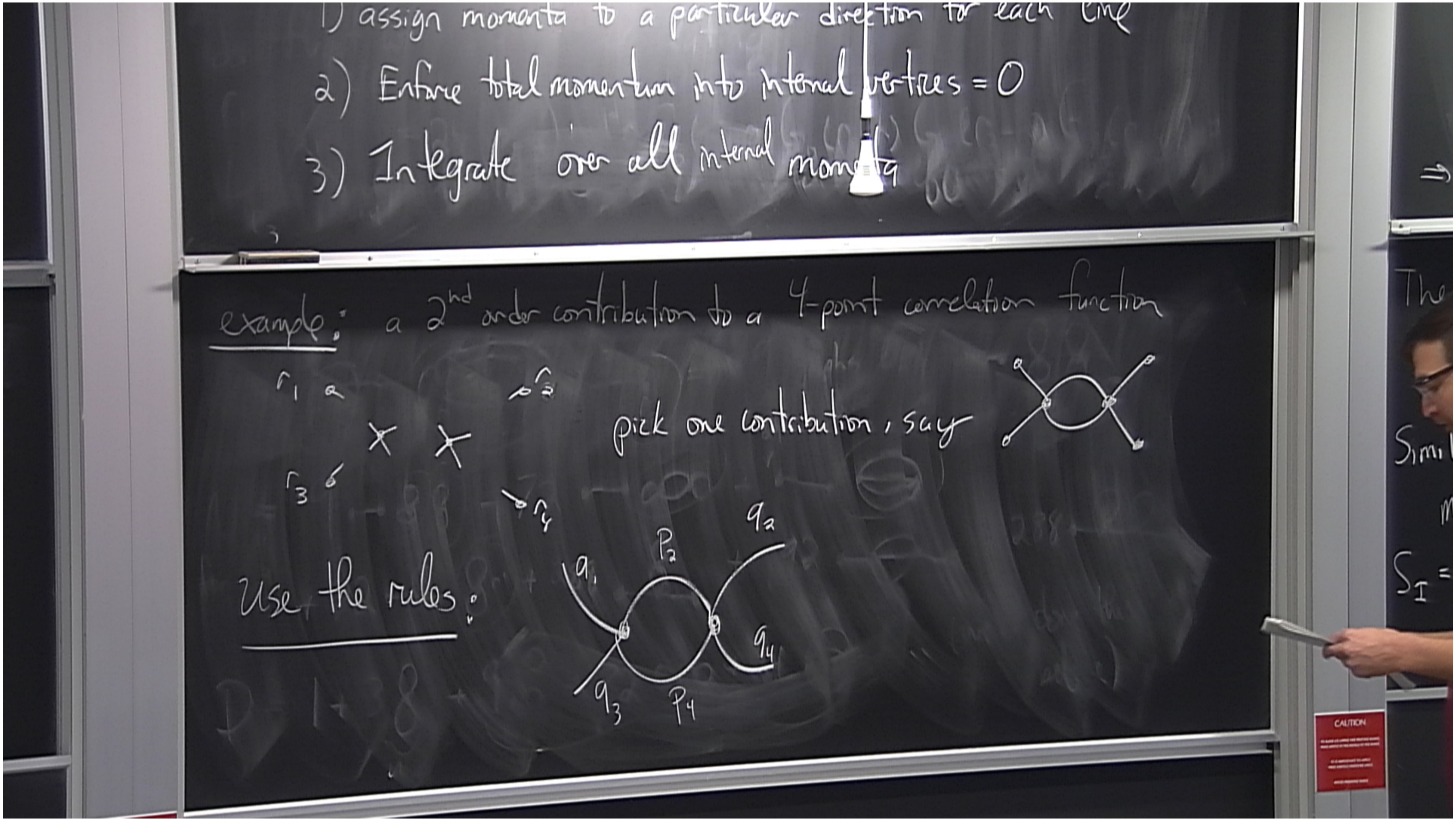
1) assign momenta to a particular direction for each line
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example: a 2nd order contribution to a 4-point correlation function
pick one contribution, say
Use the rules:
q1
q2
p2
q3
q4
p4

1) assign momenta to a particular direction for each line

2) Enforce total momentum into internal vertices = 0

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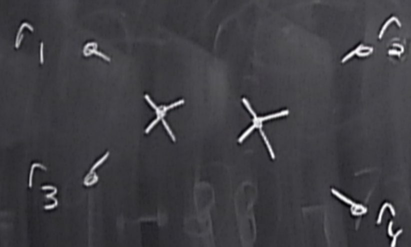
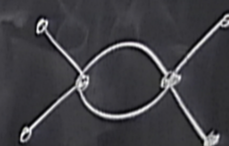
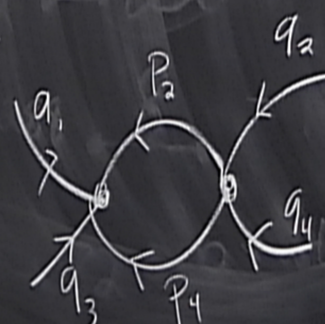
<u>example:</u> a 2^{nd} order contribution to a 4-point correlation function



pick one contribution, say



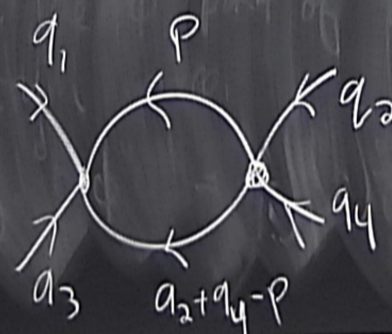
<u>Use the rules:</u>

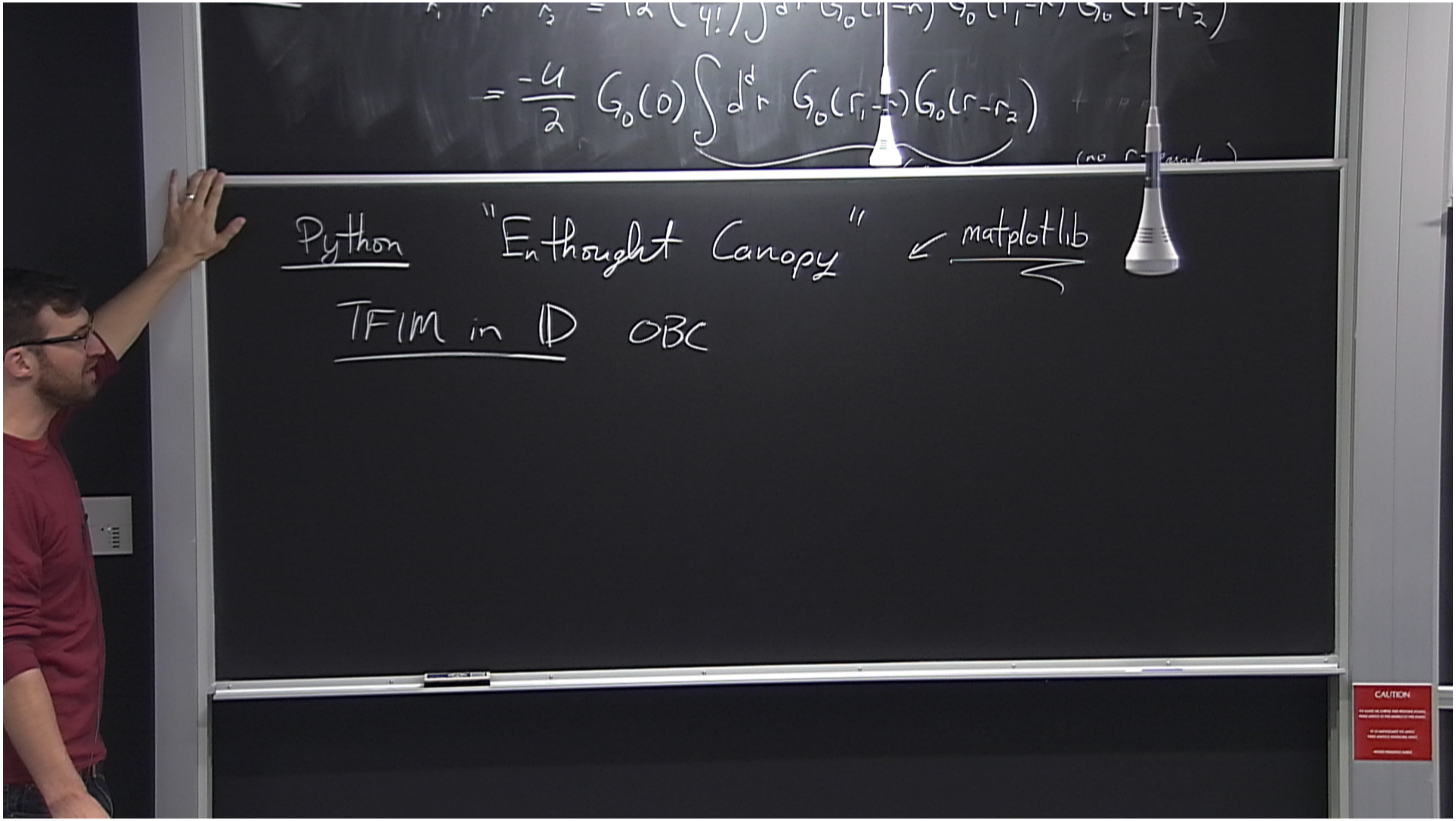


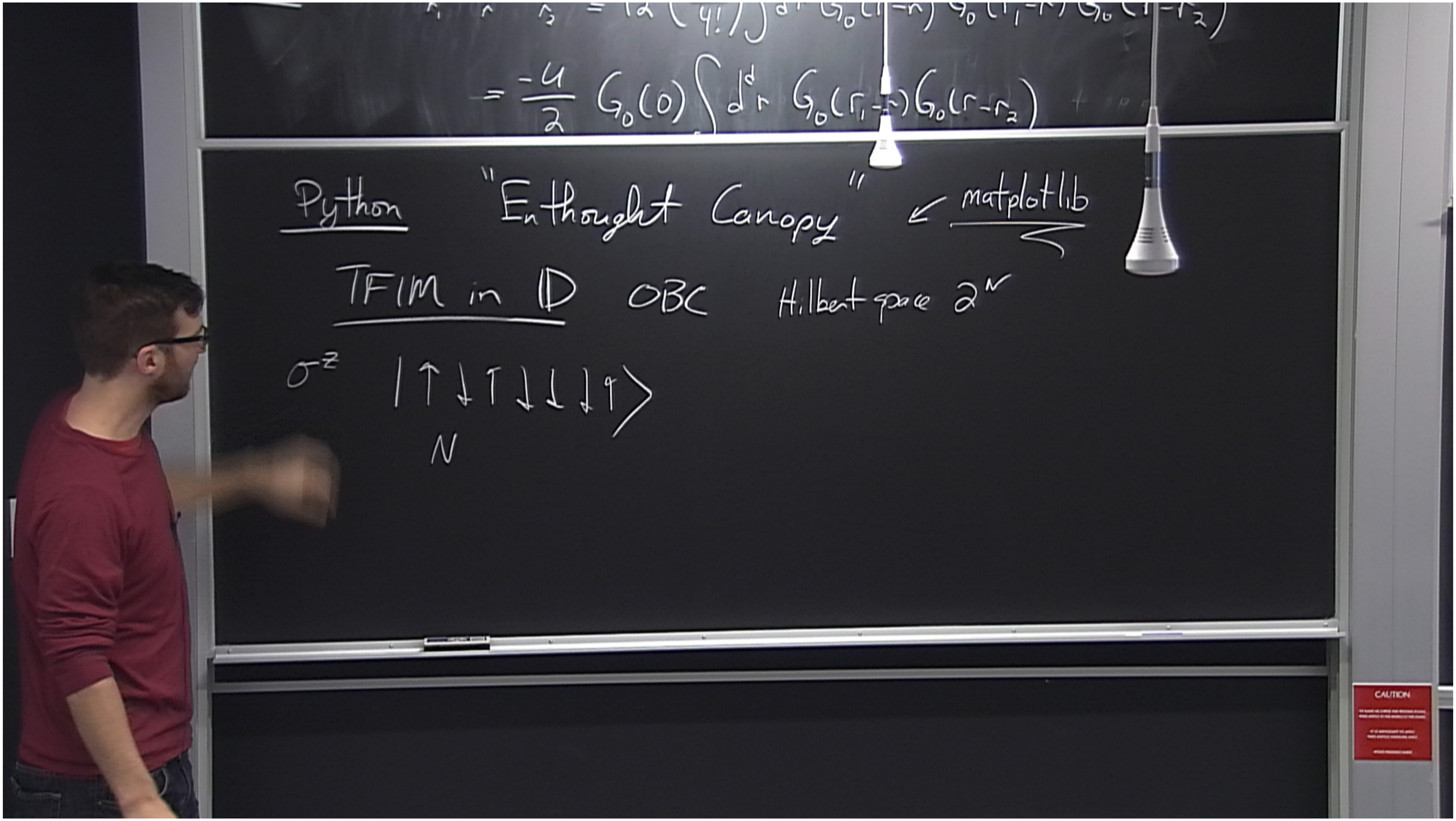
$$\delta(q_1+q_2+q_3+q_4) \qquad p_4 = q_2+q_4-p_2$$

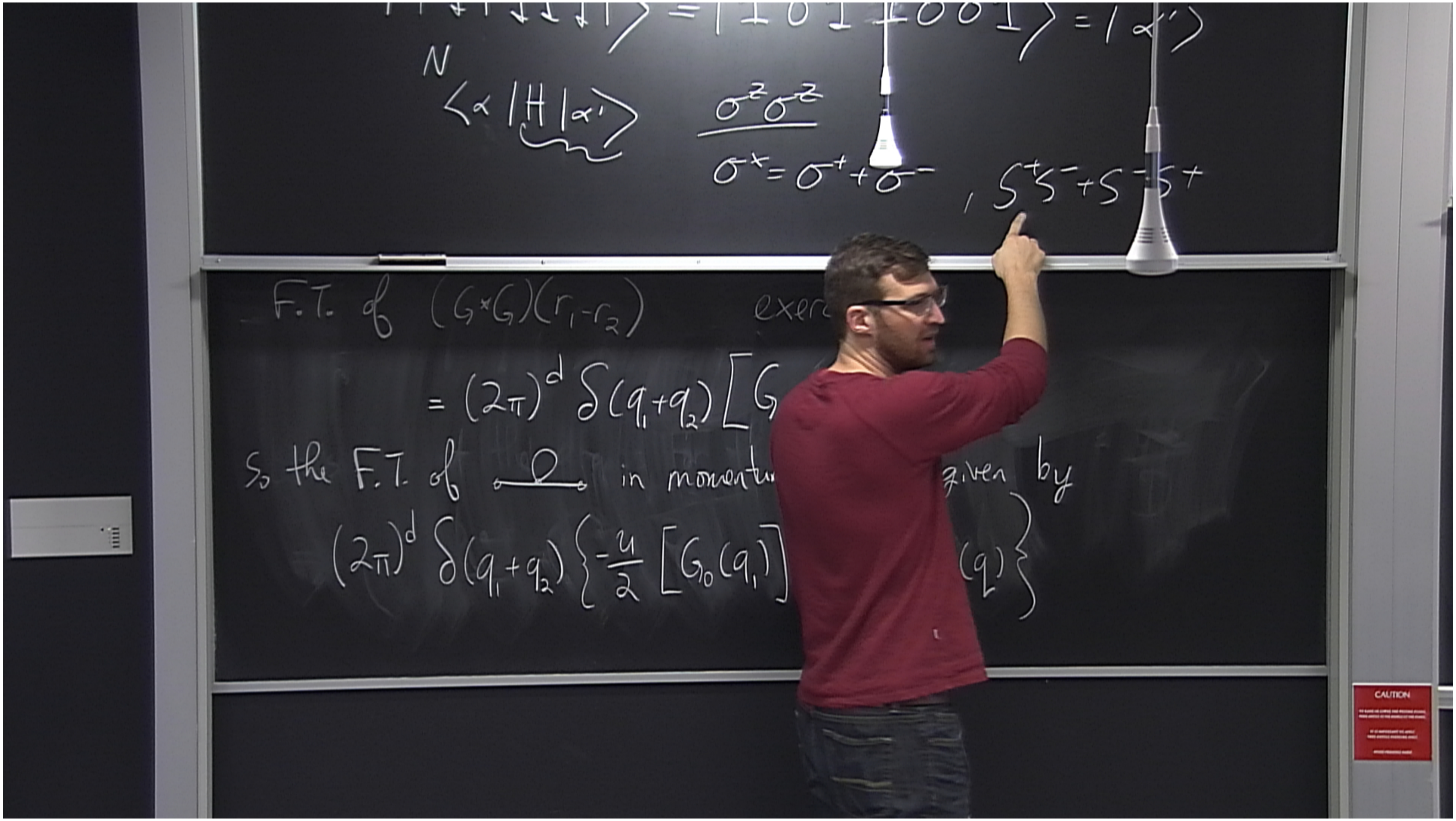
$$\propto u^2\, G_0(q_1)\, G_0(q_2)\, G_0(q_3)\, G_0(q_4)\; \delta(q_1+q_2+q_3+q_4)$$

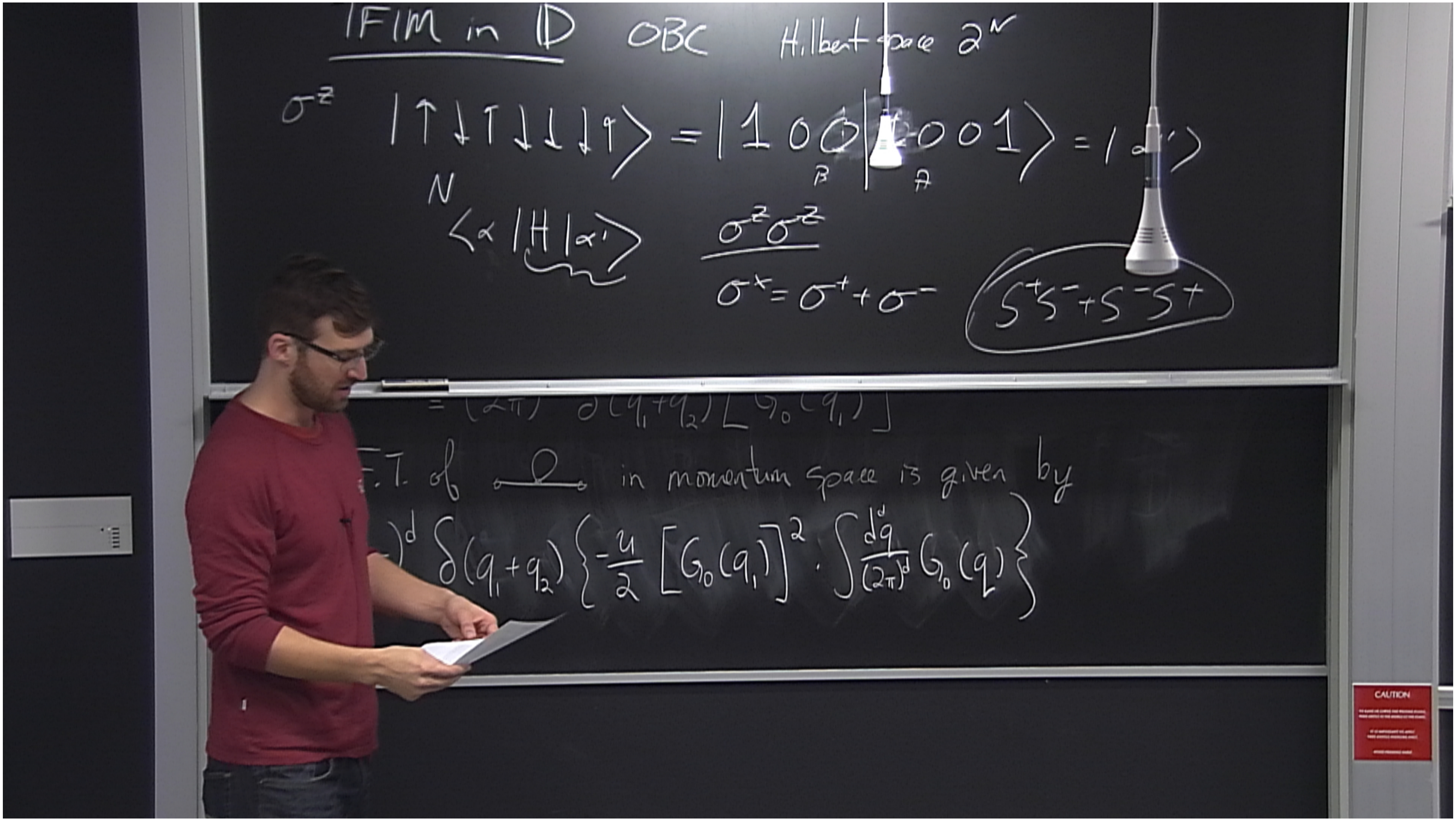
$$\cdot \int \frac{d^2 p_2}{(2\pi)^d}\, G_0(p_2)\, G_0(q_2+q_4-p_2)$$




$= \frac{-u}{2} G_0(0) \int d^d r \, G_0(r_1 - r) G_0(r - r_2)$
Python
"Enthought Canopy"
matplotlib
TFIM in 1D OBC


$= \frac{-u}{2} G_0(0) \int d^d r \, G_0(r_1 - r) G_0(r - r_2)$
Python
"Enthought Canopy"
matplotlib
TFIM in 1D
OBC
Hilbert space 2^N
σ^z $|\uparrow\downarrow\uparrow\downarrow\downarrow\downarrow\uparrow\rangle$
N


N
⟨α|H|α'⟩
σ^z σ^z
σ^x = σ^+ + σ^-
, S^+S^- + S^-S^+
F.T. of (G*G)(r1-r2)
exerc
= (2π)^d δ(q1+q2) [G
So the F.T. of
in momentu
given by
(2π)^d δ(q1+q2) {-u/2 [G0(q1)]
(q)}


TFIM in 1D OBC Hilbert space 2^N
σ^z $|\uparrow\downarrow\uparrow\downarrow\downarrow\downarrow\uparrow\rangle = |1000|1001\rangle = |\alpha'\rangle$
N
$\langle\alpha|H|\alpha'\rangle$ $\sigma^z\sigma^z$
$\sigma^x = \sigma^+ + \sigma^-$ $S^+S^- + S^-S^+$
F.T. of in momentum space is given by
$\delta(q_1+q_2)\left\{-\frac{u}{2}\left[G_0(q_1)\right]^2 \cdot \int\frac{d^dq}{(2\pi)^d}G_0(q)\right\}$