Title: Quantum Field Theory for Cosmology - Achim Kempf - Lecture 22

Date: Mar 28, 2016 01:30 PM

URL: http://pirsa.org/16030023

Abstract:

Time line of standard model of cosmology:

A Λ, inducing a

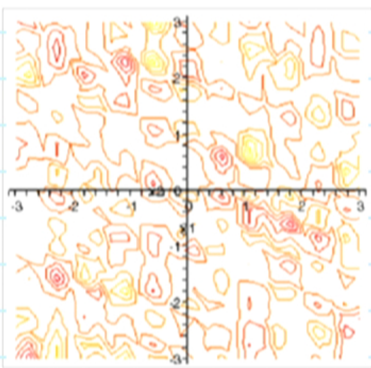
Planck era ? | Inflation | radiation dominates | matter dominates | "dark energy" dominates

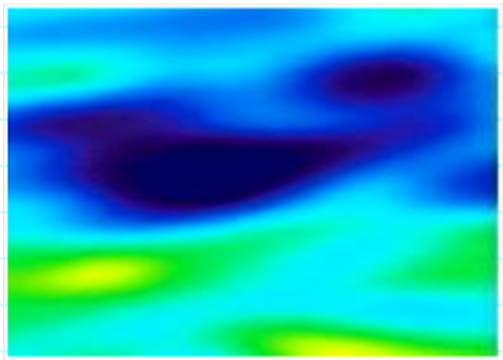
.... 10^{-43}s | 10^{-32}s | 10^{-11}s | 10^{0}s nuclei form | 10^{13}s (10^{5}ys) | 10^{17}s today | $\approx \log(t)$

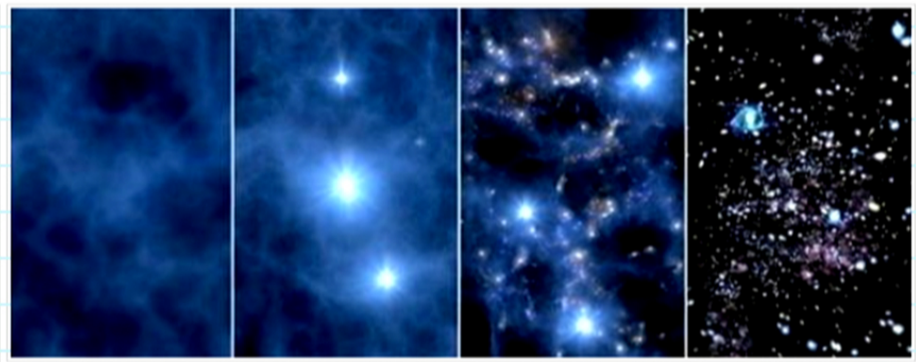
Re-heating

Reverberating expanding and cooling plasma

Atoms form: $p^+ + e^- \to H$
Universe becomes transparent
Photons travel freely $\to$ CMB

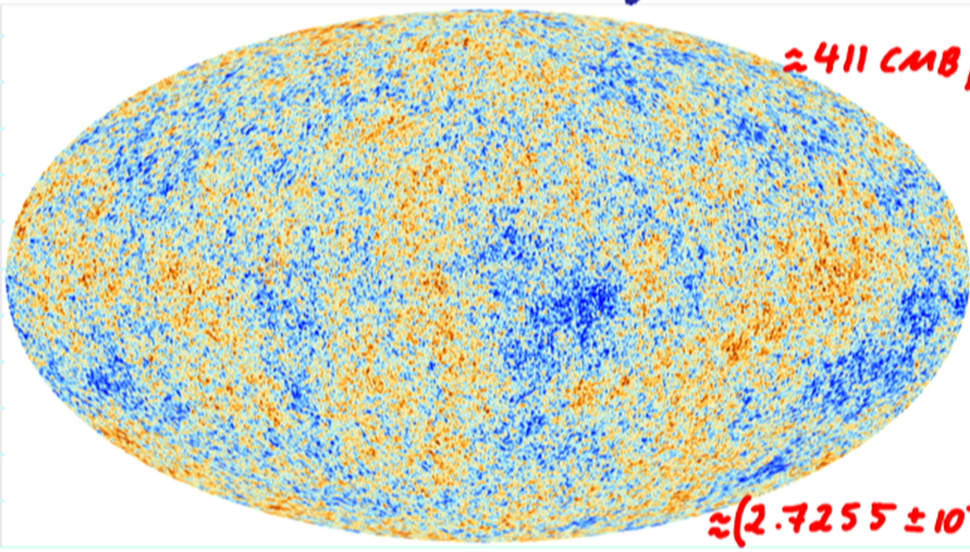






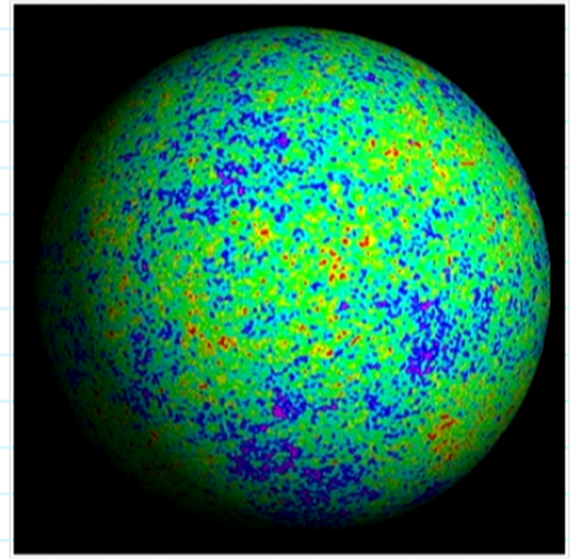
File Edit View Insert Actions Tools Help

Actual observations of the CMB:



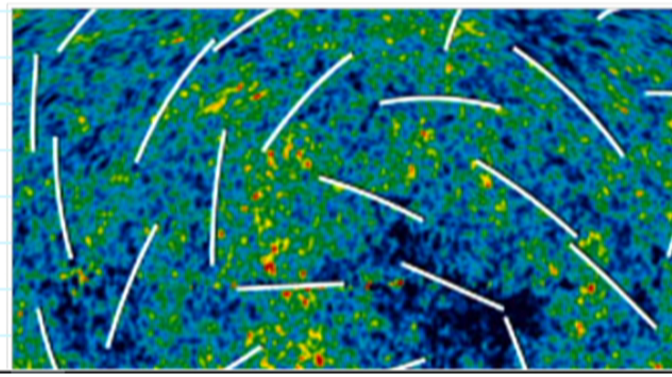
$\simeq 411$ CMB photons/cm^3

$\simeq (2.7255 \pm 10^{-5})^{\circ}$ K



Zoom-in,

with polarization:

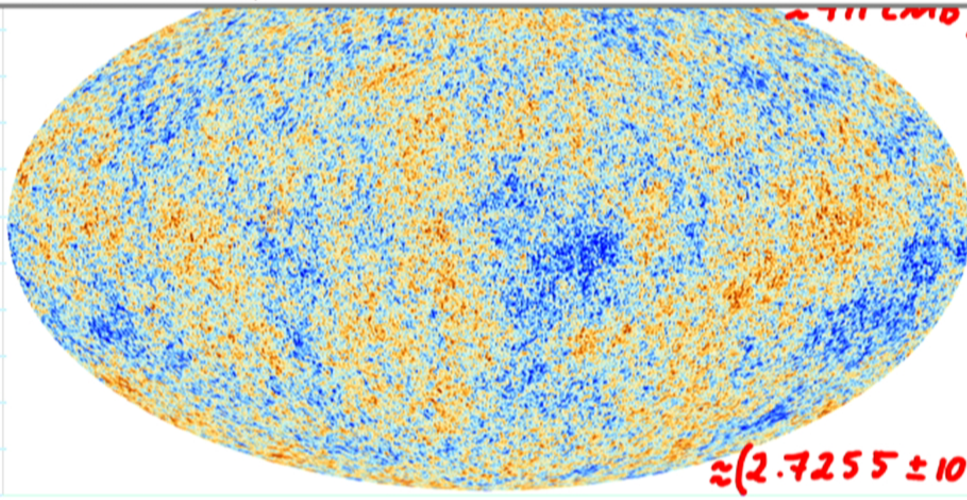


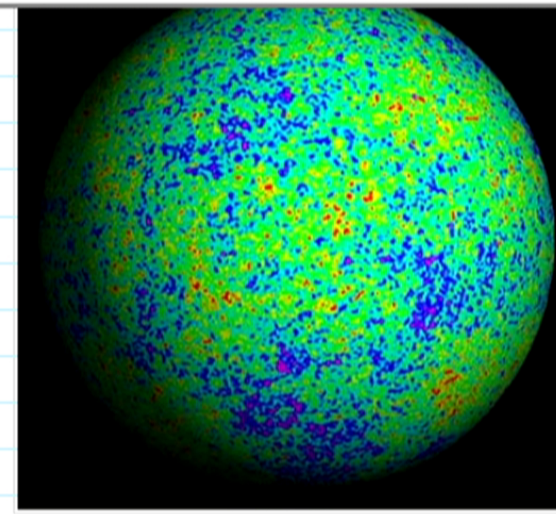
(avg. polarization $\simeq 10^{-6}$)

2 / 23

File Edit View Insert Actions Tools Help

≈ 411 CMB photons/cm³



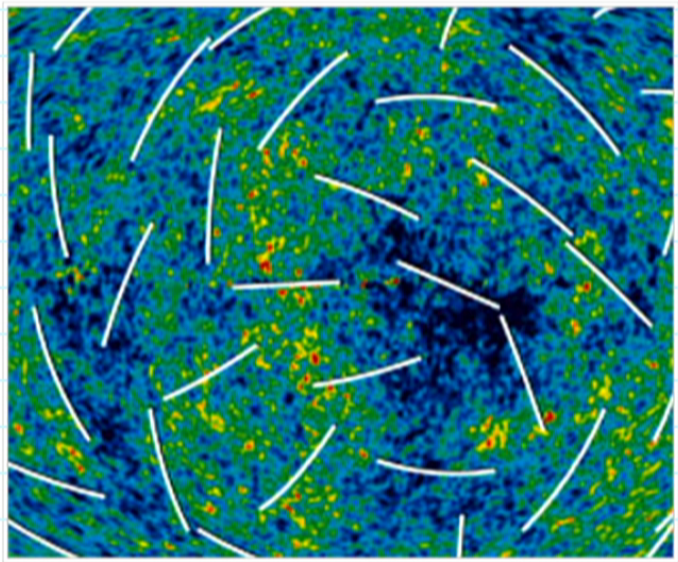


$\approx (2.7255 \pm 10^{-5})^{\circ}$K

Zoom-in,

with polarization:

(avg polarization ≈ 10^{-6})



2 / 23

File Edit View Insert Actions Tools Help

Recall:

$$\boxed{\phi(x,\eta) = \phi_0(\eta) + \varphi(x,\eta) \quad \text{with} \quad |\varphi(x,\eta)| \ll |\phi_0(\eta)|}$$

$$\boxed{g_{\mu\nu}(x,\eta) = a(\eta)\,\eta_{\mu\nu} + \gamma_{\mu\nu}(x,\eta) \quad \text{with} \quad |\gamma_{\mu\nu}(x,\eta)| \ll 1}$$

$$ds^2 = a^2(\eta)\left(d\eta^2 - \sum_{i=1}^{3}(dx^i)^2\right) + \underset{\text{scalar}}{ds_S^2} + \underset{\text{vector}}{ds_V^2} + \underset{\text{tensor}}{ds_T^2}$$

$$ds_S^2 = a^2(\eta)\Big[2\Phi(x,\eta)\,d\eta^2 - 2\sum_{i=1} \frac{\partial}{\partial x^i} B(x,\eta)\,dx^i d\eta$$
$$- \sum_{i,j=1}^{3}\left(2\Psi(x,\eta)\,\delta_{ij} - 2\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j} E(x,\eta)\right) dx^i dx^j\Big]$$

$$ds_V^2 = a^2(\eta)\Big[2\sum_{i=1}^{3} V_i(x,\eta)\,dx^i d\eta$$
$$- \sum_{i,j=1}^{3}\left(\frac{\partial}{\partial x^j} W_i(x,\eta) + \frac{\partial}{\partial x^i} W_j(x,\eta)\right) dx^i dx^j\Big]$$

3 / 23

The surviving gauge-invariant degrees of freedom are:

- The purely tensorial part of the metric: $h_{ij}(x,\eta)$
- A combination of a scalar part of the metric, $\Psi(x,\eta)$, and $\varphi(x,t)$:

$$r(x,\eta) := -\frac{a_0'}{a_0}\left(\phi_0(\eta)'\right)^{-1}\varphi(x,\eta) - \Psi(x,\eta)$$

They possess these actions:

$$\boxed{S_T = \frac{1}{64\pi G}\sum_{i,j=1}^{3}\int a^2(\eta)\,\frac{\partial}{\partial x^\mu}\left(h^i{}_j(x,\eta)\right)\frac{\partial}{\partial x^\nu}\left(h^i{}_j(x,\eta)\right)\eta^{\mu\nu}\,d^4x}$$

File Edit View Insert Actions Tools Help

$$r(x,\eta) := -\frac{a_0'}{a_0}\left(\phi_0(\eta)'\right)^{-1} \varphi(x,\eta) - \Psi(x,\eta)$$

They possess these actions:

$$\boxed{S_T = \frac{1}{64\pi G}\sum_{i,j=1}^{3}\int a^2(\eta)\frac{\partial}{\partial x^\mu}\left(h^i{}_j(x,\eta)\right)\frac{\partial}{\partial x^\nu}\left(h^i{}_j(x,\eta)\right)\eta^{\mu\nu}\, d^4x}$$

$$\boxed{S_s = \frac{1}{2}\int z^2(\eta)\left(\frac{\partial}{\partial x^\mu} r(x,\eta)\right)\left(\frac{\partial}{\partial x^\nu} r(x,\eta)\right)\eta^{\mu\nu}\, d^4x \quad \text{with} \quad z(\eta) := \frac{a_0^2(\eta)}{a_0'(\eta)}\phi_0'(\eta)}$$

To quantize without a friction term, change variable:

$$u(x,\eta) := -z(\eta)\, r(x,\eta)$$

4 / 23

To quantize without a friction term, change variable:

$$u(x,\eta) := -z(\eta)\, r(x,\eta)$$

convenient factors

$$p_{ij}(x,\eta) := \frac{1}{\sqrt{32\pi G}}\, a(\eta)\, h_{ij}(x,\eta)$$

Further, separate off polarization matrices:

$$p_{ij}(k,\eta) := \sum_{\lambda=1,2} v_{k,\lambda}(\eta)\, \varepsilon_{ij}(k,\lambda)$$

⇝ Equations of motion:

$$\hat{v}''_{k,\lambda}(\eta) + \left(k^2 - \frac{a''}{a}\right)\hat{v}_{k,\lambda}(\eta) = 0$$

File Edit View Insert Actions Tools Help

$$u(x,\eta) := - z(\eta)\, r(x,\eta)$$

convenient factors

$$p_{ij}(x,\eta) := \frac{1}{\sqrt{32\pi G}}\, a(\eta)\, h_{ij}(x,\eta)$$

Further, separate of polarization matrices:

$$p_{ij}(k,\eta) := \sum_{\lambda=1,2} v_{k,\lambda}(\eta)\, \mathcal{E}_{ij}(k,\lambda)$$

⇝ Equations of motion:

$$\hat{v}''_{k,\lambda}(\eta) + \left(k^2 - \frac{a''}{a}\right)\hat{v}_{k,\lambda}(\eta) = 0$$

$$\hat{u}''_k(\eta) + \left(k^2 - \frac{z''(\eta)}{z(\eta)}\right)\hat{u}_k(\eta) = 0$$

5 / 23

Quantum fluctuations

As before, this reduces to solving the eqns of motion for the mode functions, which are complex number-valued, say $\tilde{u}_k(\eta)$, $\tilde{v}_{k,\lambda}(\eta)$:

$$\tilde{u}_k''(\eta) + \left(k^2 - \frac{z''(\eta)}{z(\eta)}\right)\tilde{u}_k(\eta) = 0$$

$$\tilde{v}_{k,\lambda}''(\eta) + \left(k^2 - \frac{a''}{a}\right)\tilde{v}_{k,\lambda}(\eta) = 0$$

along with the Wronskian conditions.

Initial conditions?

We say we choose the "Bunch Davies vacuum".

At early times:

File Edit View Insert Actions Tools Help

Initial conditions?

We say we choose the "Bunch Davies vacuum".

At early times:

* The k^2 term dominates

$\Rightarrow$ Can choose Minkowski-like init. conds.

The mode fctns at late times?

At late times:

* The mode k crossed the Hubble horizon:

* The terms $\frac{z''}{z}$ and $\frac{a''}{a}$ dominate.

* The harmonic oscillator is inverted

6 / 23

File Edit View Insert Actions Tools Help

At late times:

* The mode k crossed the Hubble horizon:

* The terms $\frac{z''}{z}$ and $\frac{a''}{a}$ dominate.

* The harmonic oscillator is inverted

* Instead of 2 oscillatory basis sols we now expect one growing and one decaying basis solution.

* Soon after horizon crossing the mode function consists of essentially only the growing solution.

7 / 23

Which is the growing solution at late times?

Eqns of motion after horizon crossing:

$$\tilde{u}_k''(\eta) + \left(\cancel{k^2} - \frac{z''(\eta)}{z(\eta)}\right)\tilde{u}_k(\eta) = 0 \text{ , i.e., } \quad \frac{\tilde{u}_k(\eta)''}{\tilde{u}_k(\eta)} = \frac{z(\eta)''}{z(\eta)}$$

$$\tilde{v}_{k,\lambda}''(\eta) + \left(\cancel{k^2} - \frac{a''}{a}\right)\tilde{v}_{k,\lambda}(\eta) = 0 \text{ , i.e., } \quad \frac{\tilde{v}_{k,\lambda}(\eta)''}{\tilde{v}_{k,\lambda}(\eta)} = \frac{a(\eta)''}{a(\eta)}$$

$\Rightarrow$ Growing solution must behave as:

$$\tilde{u}_k(\eta) \sim z(\eta) \quad \text{at late } \eta$$

$$\tilde{v}_{k,\lambda}(\eta) \sim a(\eta) \quad \text{at late } \eta$$

$\Rightarrow$ The mode fctns $\tilde{r}_k(\eta) = -\frac{\tilde{u}_k(\eta)}{z(\eta)}$ and $\tilde{h}_{ij,k}(\eta) = 32\pi G \frac{\tilde{v}_{k,\lambda}(\eta)\epsilon_{ij}(k,\lambda)}{a(\eta)}$ beca...

File Edit View Insert Actions Tools Help

$$\tilde{u}_k''(\eta) + \left(\cancel{k^2} - \frac{z''(\eta)}{z(\eta)}\right)\tilde{u}_k(\eta) = 0 \text{ , i.e., } \quad \frac{\tilde{u}_k(\eta)''}{\tilde{u}_k(\eta)} = \frac{z(\eta)''}{z(\eta)}$$

$$\tilde{v}_{k,\lambda}''(\eta) + \left(\cancel{k^2} - \frac{a''}{a}\right)\tilde{v}_{k,\lambda}(\eta) = 0 \text{ , i.e., } \quad \frac{\tilde{v}_{k,\lambda}(\eta)''}{\tilde{v}_{k,\lambda}(\eta)} = \frac{a(\eta)''}{a(\eta)}$$

$\Rightarrow$ Growing solution must behave as:

$$\tilde{u}_k(\eta) \sim z(\eta) \quad \text{at late } \eta$$

$$\tilde{v}_{k,\lambda}(\eta) \sim a(\eta) \quad \text{at late } \eta$$

$\Rightarrow$ The mode fctns $\tilde{r}_k(\eta) = -\frac{\tilde{u}_k(\eta)}{z(\eta)}$ and $\tilde{h}_{ij\,k}(\eta) = 32\pi G \frac{\tilde{v}_{k,\lambda}(\eta)\epsilon_{ij}(k,\lambda)}{a(\eta)}$ become constant at late η, i.e., after the mode k crosses the horizon!

8 / 23

File Edit View Insert Actions Tools Help

Check:

$$\tilde{v}_k(\eta) = \frac{1}{z(\eta)}\,\tilde{u}(\eta) \sim \frac{z(\eta)}{z(\eta)} \qquad \text{for late } \eta$$

$$\tilde{h}_{ij_k}(\eta) = \frac{1}{a(\eta)}\,\tilde{p}_{ij_k}(\eta) \sim \frac{1}{a(\eta)}\,\tilde{v}_{k,\lambda}(\eta) \sim \frac{a(\eta)}{a(\eta)} \qquad \text{for late } \eta$$

$\Rightarrow$ As expected, the magnitude of the mode k's quantum fluctuations

$$\underbrace{\delta r_k = k^{3/2}|\tilde{r}_k|^2}_{= z^{-1}k^{3/2}|\tilde{u}_k|} \quad \text{and} \quad \underbrace{\delta h_{ij_k} = k^{3/2}|\tilde{h}_{ij_k}|}_{= a^{-1}k^{3/2}|\tilde{v}|}$$

stay constant at the value that they possess when the mode

8 / 23

File Edit View Insert Actions Tools Help

⇒ As expected, the magnitude of the mode k's quantum fluctuations

$$\underbrace{\delta r_k = k^{3/2}|\tilde{r}_k|^2}_{= z^{-1} k^{3/2} |\tilde{u}_k|} \quad \text{and} \quad \underbrace{\delta h_{ij_k} = k^{3/2}|\tilde{h}_{ij_k}|}_{= a^{-1} k^{3/2} |\tilde{v}|}$$

stay constant at the value that they possess when the mode crosses the horizon, even as the mode's proper wavelength then continues to increase rapidly.

* Goal now: Calculate the magnitude of the fluctuations at horizon crossing!

Realistic example: "Power law inflation"

9 / 23

File Edit View Insert Actions Tools Help

Realistic example: "Power law inflation"

- We need an explicit potential $V(\phi)$ in order to be able to find explicit $a_0(\eta)$, $\phi_0(\eta)$ for which to calculate then the fluctuation spectrum.

- De Sitter is ruled out because:

  * $V(\phi)$, and therefore the temporary "cosmological constant $H \sim \sqrt{V(\phi)}$ must slowly decrease (slow roll).

  * In any case, our perturbation assumptions

10 / 23

- We need an explicit potential $V(\phi)$ in order to be able to find explicit $a_0(\eta)$, $\phi_0(\eta)$ for which to calculate then the fluctuation spectrum.

- De Sitter is ruled out because:

  * $V(\phi)$, and therefore the temporary "cosmological constant $H \sim \sqrt{V(\phi)}$ must slowly decrease (slow roll).

  * In any case, our perturbation assumptions don't allow exact de Sitter, as δr_k would diverge, invalidating the assumption that it is small

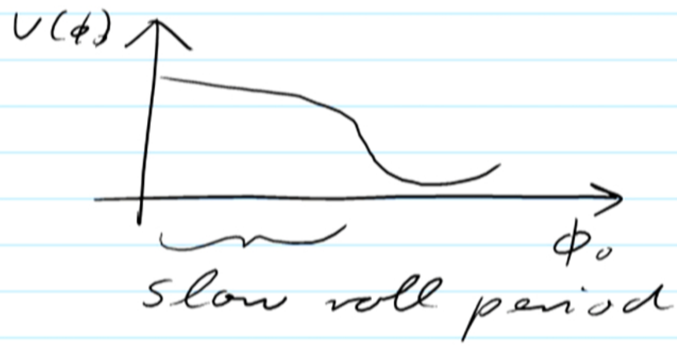
File Edit View Insert Actions Tools Help

The "slow roll parameters"

Idea:

* We do not know the exact slow roll potential:

E.g.



* For all values of ϕ_0 during the inflationary period we can parametrize the slope of the potential by its derivatives.

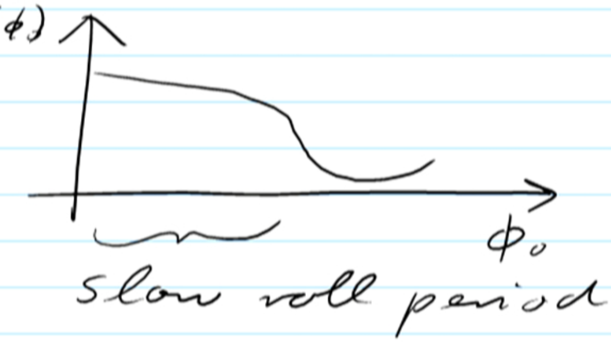
11 / 23

File Edit View Insert Actions Tools Help

Idea:

* We do not know the exact slow roll potential:

E.g.



* For all values of ϕ_0 during the inflationary period we can parametrize the slope of the potential by its derivatives.

* These are the so-called slow roll parameters: (Recall: $H(\phi) \sim \sqrt{V(\phi)}$)

11 / 23

File Edit View Insert Actions Tools Help

convenience factor

$$\varepsilon(\phi) := \frac{1}{4\pi G}\left(\frac{H'(\phi)}{H(\phi)}\right)^2 \qquad \left(= \frac{\frac{3}{2}\dot\phi^2}{V + \frac{1}{2}\dot\phi^2}\right)$$

$$\eta(\phi) := \frac{1}{4\pi G}\,\frac{H''(\phi)}{H(\phi)} \qquad \left(= \varepsilon - \frac{\varepsilon'}{\sqrt{16\pi G\varepsilon}}\right)$$

$$\xi(\phi) := \frac{1}{4\pi G}\sqrt{\frac{H'(\phi)\,H'''(\phi)}{H^2(\phi)}}$$

etc...

The simplest solvable case:

* The simplest case is that of

12 / 23

File Edit View Insert Actions Tools Help

The simplest solvable case:

* The simplest case is that of

$\varepsilon(\phi) = c$ where c is a constant.

* In this case:

$$c = \varepsilon(\phi) := \frac{1}{4\pi G}\left(\frac{H'(\phi)}{H(\phi)}\right)^2$$

Thus,

$$H(\phi) \sim e^{\sqrt{4\pi G c}\,\phi}$$

12 / 23

File Edit View Insert Actions Tools Help

Show that:

$$a(t) = a_0\, t^{1/c}$$ (t is proper time)

Exercise:

Show that, in terms of the conformal time η:

$$a(\eta) = \frac{-1}{\eta H} \frac{1}{1-\varepsilon}$$

Note: Still η is always negative and $t \to \infty$ means $\eta \to 0$.

14 / 23

File Edit View Insert Actions Tools Help

The mode equations:

Scalar: We can now calculate $z(\eta) = \frac{a^2(\eta)}{a'(\eta)} \phi_0'(\eta)$ and therefore also the mode equation's term z''/z explicitly, to obtain

A Bessel differential equation

$$\tilde{u}_k''(\eta) + \left(k^2 - \frac{(\nu^2 - 1/4)}{\eta^2}\right) \tilde{u}_k(\eta) = 0$$

where: $\nu := \frac{3}{2} + \frac{c}{1-c}$

* Solution for Bunch Davies initial conditions:

$$\tilde{u}_k(\eta) = \frac{\sqrt{\pi}}{2} e^{i(\nu + 1/2)\frac{\pi}{2}} (-\eta)^{1/2} H_\nu^{(1)}(-k\eta)$$

15 / 23

File Edit View Insert Actions Tools Help

Scalar: We can now calculate $z(\eta) = \frac{a\,\phi_0'}{a'(\eta)}\,\phi_0'(\eta)$ and

therefore also the mode equation's term z''/z explicitly,

to obtain

A Bessel differential equation

$$\tilde{u}_k''(\eta) + \left(k^2 - \frac{(\nu^2 - 1/4)}{\eta^2}\right)\tilde{u}_k(\eta) = 0$$

where: $\nu := \frac{3}{2} + \frac{c}{1-c}$

* Solution for Bunch Davies initial conditions:

$$\tilde{u}_k(\eta) = \frac{\sqrt{\pi}}{2}\, e^{i(\nu + 1/2)\frac{\pi}{2}} (-\eta)^{1/2} H_\nu^{(1)}(-k\eta)$$

Hankel fctn of 1st kind of order ν.

15 / 23

File Edit View Insert Actions Tools Help

* Solution for Bunch Davies initial conditions:

$$\tilde{u}_k(\eta) = \frac{\sqrt{\pi}}{2} e^{i(\nu+1/2)\frac{\pi}{2}} (-\eta)^{1/2} H_\nu^{(1)}(-k\eta)$$

Hankel fctn of 1st kind of order ν.

* Behavior after horizon crossing:

$$\tilde{u}_k(\eta) \rightarrow e^{i(\nu-\frac{1}{2})\frac{\pi}{2}} 2^{\nu-3/2} \frac{\Gamma(\nu)}{\Gamma(3/2)} \frac{1}{\sqrt{2k}} (-k\eta)^{-\nu+1/2}$$

* Thus, the magnitude of intrinsic curvature fluctuations after horizon crossing becomes:

15 / 23

File Edit View Insert Actions Tools Help

$$\tilde{u}_k(\eta) \to e^{\cdots}\, 2^{\nu - 3/2} \frac{\Gamma(\nu)}{\Gamma(3/2)} \frac{1}{\sqrt{2k}} (-k\eta)^{\cdots}$$

* Thus, the magnitude of intrinsic curvature fluctuations after horizon crossing becomes:

$$\boxed{\delta r_k\left(\eta > \eta_{hor}(k)\right) = G\, 2^{\nu - \frac{1}{2}} \frac{\Gamma(\nu)}{\Gamma(3/2)} \left(\nu - \tfrac{1}{2}\right)^{1/2 - \nu} \frac{H^2}{|H'|}}$$

at $k = aH$

horizon crossing

Exercise: verify

Notice: Measurement of δr_k in CMB can only tell us $\frac{H^2}{H'}$ (at horizon crossing) but not H or H' individually!

16 / 23

File Edit View Insert Actions Tools Help

Intuition?

Earlier, for a K.G. field ϕ in a fixed background FRW universe, we found:

$$\delta\phi_k \sim H$$

Now, for the intrinsic curvature (the Mukhanov variable), we found:

$$\delta r_k \sim H^2/|H'| .$$

Recall: r_k is the slicing-independent combination of the scalar part of $\delta g_{\mu\nu}$ and ϕ.

The slower the roll ($|H'|$ small) the wider away from

17 / 23

File Edit View Insert Actions Tools Help

Analogous to: A river in a plain meanders the more widely, the flatter the plain is.

18 / 23

File Edit View Insert Actions Tools Help

Planck era | Inflation | radiation dominates | matter dominates | A Λ, inducing a "dark energy" dominates

$\dots$ $10^{-43}s$ — $10^{-32}s$ — $10^{-11}s$ — $10^{0}s$ nuclei form — $10^{13}s$ ($10^5 ys$) — $10^{17}s$ today — $\approx \log(t)$

Re-heating

Reverberating expanding and cooling plasma

known plasma physics there's code for that step.

Atoms form: $p^+ + e^- \rightarrow H$
Universe becomes transparent
Photons travel freely $\rightarrow$ **CMB**

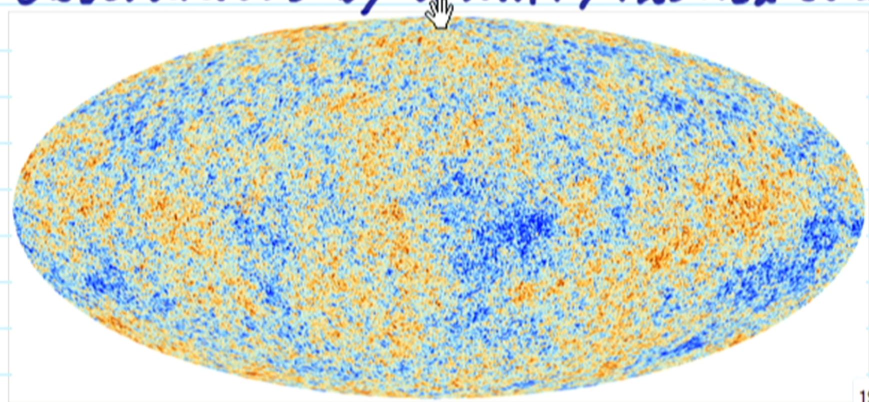
we calculated: δr_K

at horizon crossing,

then unchanged till end

of inflation: re-heating.

Observations by WMAP, Planck et al:



19 / 23

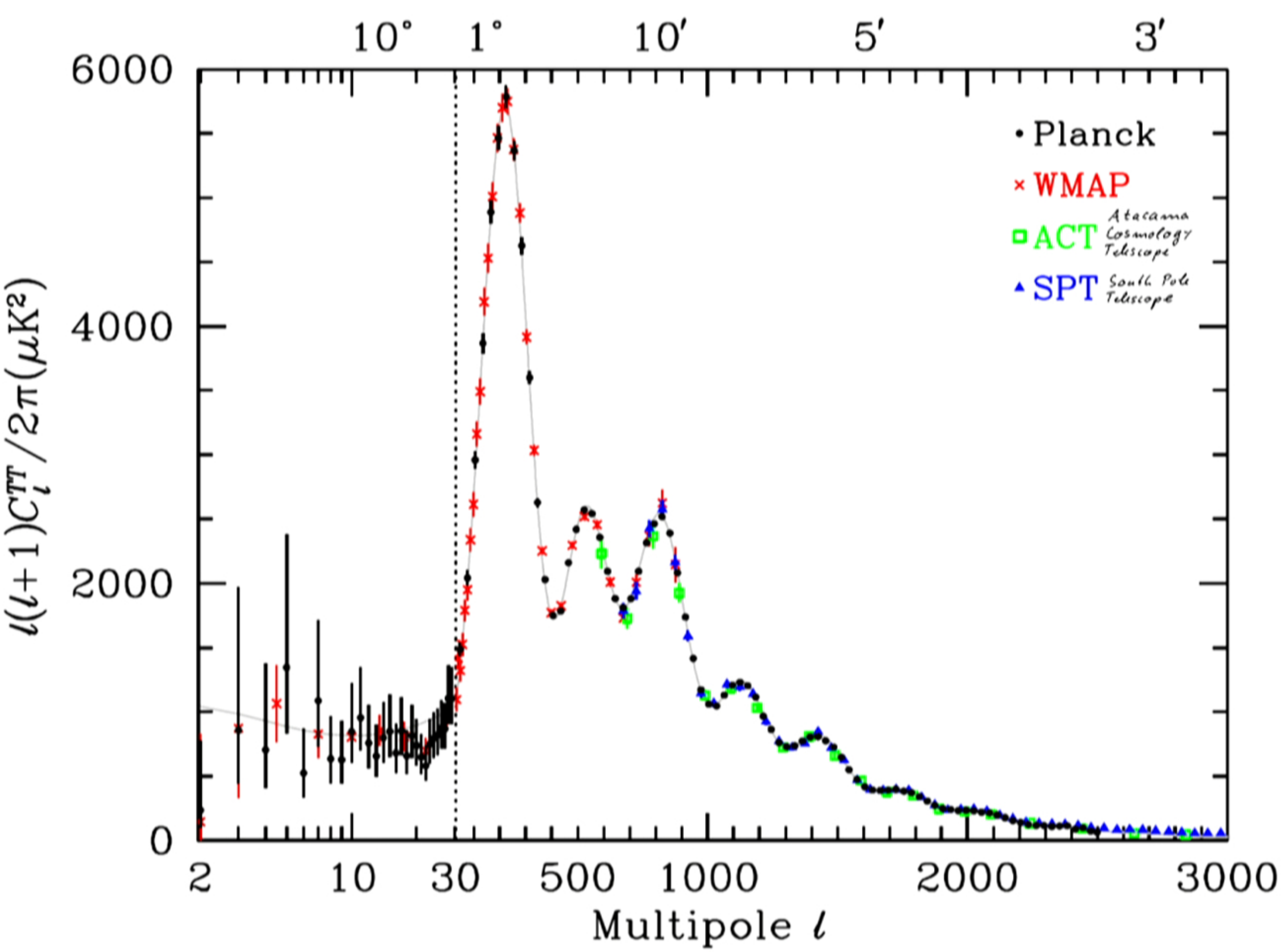
- δr_k is predicted to have seeded oscillations in the hot plasma after re-heating. The plasma decohered the quantum fluctuations of the intrinsic curvature r.

- Standard plasma physics allows one to calculate the propagation and dispersion for the $\approx 10^{[illegible]}$ ys until hydrogen formed.

- The temperature fluctuation spectrum in the CMB is from gravitational blue and redshifts due to these curvature fluctuations.

- Theory matches experiment closely, while fixing cosmological parameters, including indications that $\xi \neq 0$, namely that $\delta r_k \neq$ const.

File Edit View Insert Actions Tools Help



$K = 0$

$\Lambda \approx 0.7\, \rho_{critical}$

$\rho_{matter} \approx 0.3\, \rho_{critical}$

$\rho_{dark\ matter} \approx 0.9\, \rho_{matter}$

$\rho_{visible\ matter} \approx 0.1\, \rho_{matter}$

$\rho_{CMB} \approx 5 \cdot 10^{-5}\, \rho_{critical}$

$v_{peculiar}$ of earth ≈ 370 km/s

21 / 23

File Edit View Insert Actions Tools Help

Tensor modes: $$\tilde{v}_{k,\lambda}'' + \left(k^2 - \frac{a''}{a}\right)\tilde{v}_{k,\lambda} = 0$$

we obtain for the term a''/a:

$$\frac{a''}{a} = 2a^2H^2\left(1 - \varepsilon/2\right)$$

which comes out to be (verify):

$$\frac{a''}{a} = \frac{1}{\eta^2}\left(\nu^2 - \frac{1}{4}\right)$$ recall: $\nu = \frac{3}{2} - \frac{c}{1-c}$

$\Rightarrow$ The mode eqn is again solved by the Hankel function.

$\Rightarrow$ The tensor fluctuation spectrum: horizon crossing.

22 / 23

File Edit View Insert Actions Tools Help

$$\frac{a''}{a} = \frac{1}{\eta^2}\left(\nu^2 - \frac{1}{4}\right) \qquad \text{recall: } \nu = \frac{3}{2} - \frac{c}{1-c}$$

⇒ The mode eqn is again solved by the Hankel function.

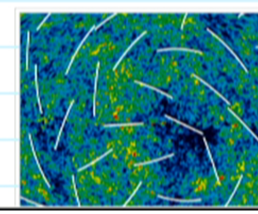
⇒ The tensor fluctuation spectrum:

horizon crossing.

$$\boxed{\delta h_{ij\,k} = \frac{2}{\sqrt{\pi}}\, 2^{\nu-\frac{1}{2}} \frac{\Gamma(\nu)}{\Gamma(3/2)} (\nu - 1/2)^{1/2-\nu} \sqrt{G}\, H \Big|_{k=aH}}$$

δh_{ij} should have left curl ("B") polarization in the CMB

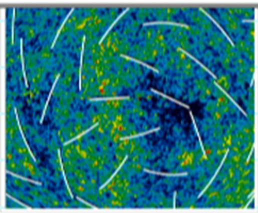
Experiments show polarization in the CMB:



22 / 23

File Edit View Insert Actions Tools Help

Experiments show polarization in the CMB:



- But most is gradient ("E") polarization that originated in δv_k or in foreground.

see BICEP2 (2014)

- So far, h_{ij}-originated B-polarization cannot be distinguished from foreground.

- Observation of h_{ij} polarization:
  - * Would show quantized gravitational waves!
  - * Would determine the scale of H, and therefore of H'!
  - * This would tell the slope of the spectra

$\Rightarrow$ Nontrivial consistency conditions to check inflation.

23 / 23