Title: Quantum Field Theory for Cosmology - Achim Kempf - Lecture 20

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Abstract:

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Note Title

QFT for Cosmology, Achim Kempf, Winter 16, Lecture 20

Realistic cosmic inflation

1. How can a period of near-exponential expansion be caused?

Recall the full action:

We neglect such terms by Occam's razor: there is no evidence for their existence as yet.

$$S = -\frac{1}{16\pi G}\int\Big[2\Lambda + R(x) + \underbrace{\cancel{\mathcal{O}(R\phi) + \mathcal{O}(R^2) + \dots}}\Big]\sqrt{|g|}\,d^4x$$

$\uparrow$ cosm. constant

Note: ϕ is now called the "Inflaton" field.

$$+\int\Big[\frac{1}{2}g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} - V(\phi)\Big]\sqrt{|g|}\,d^4x$$

$$+\cancel{S_{\text{other fields}}}$$

$\leftarrow$ We neglect this term because the contribution

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Note: ϕ is now called the "Inflaton" field.

$$+ \int \left[\frac{1}{2} g^{\mu\nu} \phi_{,\mu} \phi_{,\nu} - V(\phi) \right] \sqrt{|g|}\, d^4x$$

$$+ \cancel{S_{\text{other fields}}}$$

$\leftarrow$ We neglect this term because the contribution of the inflaton field ϕ and of $g_{\mu\nu}$ are assumed to have been dominant in the very early universe.

Example choice of V: $V(\phi) = m\,\phi^2 + \lambda\,\phi^4$

Equations of motion:

* $\frac{\delta S}{\delta \phi(x)} = 0$ yields the K.G. eqn.:

$$\frac{\partial}{\partial x^\nu}\left(g^{\mu\nu}(x)\, \phi_{,\mu}(x) \sqrt{|g(x)|} \right) + \frac{\partial V}{\partial \phi}(x) \sqrt{|g(x)|} = 0 \qquad (K.G.$$

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$\delta\phi(x)$

$$\frac{\partial}{\partial x^\mu}\left(g^{\mu\nu}(x)\,\phi_{,\nu}(x)\sqrt{|g(x)|}\right)+\frac{\partial V}{\partial\phi}(x)\sqrt{|g(x)|}=0 \qquad (KG)$$

* $\dfrac{\delta S}{\delta g_{\mu\nu}(x)}=0$ yields the Einstein eqn.:

$$R_{\mu\nu}(x)-g_{\mu\nu}(x)R(x)+\Lambda g_{\mu\nu}(x)=-8\pi G\,T_{\mu\nu}(x) \qquad (E)$$

where the energy-momentum tensor (for ϕ only) reads:

$$T_{\mu\nu}=\phi_{,\mu}\,\phi_{,\nu}-g_{\mu\nu}\left(g^{\alpha\beta}\phi_{,\alpha}\,\phi_{,\beta}-V(\phi)\right)+\underbrace{\cancel{T_{\mu\nu}^{(\text{other fields})}}}$$

We'll assume this small compared to the contribution of ϕ, during the very early universe.

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The important special case of homogeneity & isotropy

Eqns. (KG) and (E) are a set of coupled nonlinear partial differential equations which are even classically very hard.

⇝ As a lowest order approximation we assume perfect homogeneity & isotropy:

$$\phi(x,t) = \phi(t)$$

$$g_{\mu\nu}(x,t) = g_{\mu\nu}(t)$$

Note:

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→ As a lowest order approximation we assume perfect homogeneity & isotropy:

$$\phi(x,t) = \phi(t)$$

$$g_{\mu\nu}(x,t) = g_{\mu\nu}(t)$$

<u>Note:</u>

This may also be viewed as considering only the $k=0$ modes, neglecting all other modes.

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Thus, the eqns of motion simplify:

$\left(\dot{} = \frac{\partial}{\partial t} \right)$

$$\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} + \frac{dV}{d\phi} = 0$$ (K.G. eqn.)

$$3\left(\frac{\dot{a}}{a}\right)^2 = 8\pi G\, T^0_{\ 0} + \Lambda$$ (the 0,0 component of the Einstein equation)

no sum

$$-2\frac{\ddot{a}}{a} - \frac{\dot{a}}{a^2} = 8\pi G\, T^i_{\ i} - \Lambda$$ (the i,i components of the Einstein equation)

Here: $T^0_{\ 0} = \rho(t) = \frac{1}{2}\dot{\phi}^2 + V(\phi)$ (the energy density ρ of ϕ)

$T^i_{\ i} = p(t) = \frac{1}{2}\dot{\phi}^2 - V(\phi)$ (the pressure p of ϕ)

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$\left(\cdot = \frac{\partial}{\partial t} \right)$

$$\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} + \frac{dV}{d\phi} = 0 \qquad \text{(K.G. eqn.)}$$

$$3\left(\frac{\dot{a}}{a}\right)^2 = 8\pi G\, T^0_{\ 0} + \Lambda \qquad \left(\begin{array}{l}\text{the 0,0 component of}\\ \text{the Einstein equation}\end{array}\right)$$

no sum

$$-2\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} = 8\pi G\, T^i_{\ i} - \Lambda \qquad \left(\begin{array}{l}\text{the } i,i \text{ components of}\\ \text{the Einstein equation}\end{array}\right)$$

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$T^i_{\ i} = p(t) = \frac{1}{2}\dot{\phi}^2 - V(\phi)$ (the pressure p of ϕ)

Given any initial conditions and given any $V(\phi)$ one can now solve for $a(t)$, $\phi(t)$, at least numerically!

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Notice:

- The cosmological constant Λ contributes effectively a positive energy density ρ_Λ and effectively a negative pressure ρ_p.

- Vice versa, whenever $V(\phi) \gg \dot{\phi}^2/2$ then $V(\phi)$ temporarily plays the same rôle as Λ.

- How close we are to $V(\phi) \gg \dot{\phi}^2/2$ is described by the "Equation of state parameter"

$$w(t) := \frac{p(t)}{\rho(t)} = \frac{\dot{\phi}^2/2 - V(\phi)}{\dot{\phi}^2/2 + V(\phi)}$$

$-1 < w < 1$

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$$w(t) := \frac{p(t)}{\rho(t)} = \frac{\dot{\phi}^2/2 - V(\phi)}{\dot{\phi}^2/2 + V(\phi)} \qquad -1 < w < 1$$

$\Rightarrow$ Iff $w \approx -1$ then $V(\phi)$ acts like a cosm. constant.

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First attempt to get exponential expansion:

Assume that Λ dominates over $T_{\mu\nu}$ of all fields in nature.

Then, the 0,0 component of Einstein's equation,

$$\left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G}{3} T^0_0 + \frac{1}{3}\Lambda \quad \text{becomes} \quad \left(\frac{\dot a}{a}\right)^2 = \frac{1}{3}\Lambda$$

whose solution has the desired behavior:

$$a(t) = a_0 e^{Ht} \quad \text{with} \quad H = \sqrt{\Lambda/3} \;!$$

Problems: $\square$ Λ is too tiny! It is 122 orders of magnitude below the Planck scale. We'd need a Λ close to the Planck scale $10^{70} m^{-2}$. (just a few orders of magnitude below)

Note:
Λ manifests itself as dark energy and has

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- Λ is too tiny! It is 122 orders of magnitude below the Planck scale. We'd need a Λ close to the Planck scale $10^{70} m^{-2}$. (just a few orders of magnitude below)

- Since Λ is constant, such an inflation would never end!

Note:
Λ manifests itself as dark energy and has been measured:
$\Lambda \approx 10^{-52} m^{-2}$

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→ Realistic possibility:

$\boxed{V(\phi) \text{ temporarily very large}}$

→ One of the biggest ideas of science, ever:

- Consider a universe like ours.
  Everywhere, at all times, all fields quantum fluctuate.

- As a rare fluke, the field ϕ quantum fluctuates in a patch a few Planck lengths in size

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One of the biggest ideas of science, ever:

- Consider a universe like ours.

  Everywhere, at all times, all fields quantum fluctuate.

- As a rare fluke, the field ϕ quantum fluctuates in a patch a few Planck lengths in size

to a ϕ value that makes $V(\phi)$ close to the Planck scale.

(Assume homogeneity in that patch, so that the $\partial_i \phi$ are small)

- In this patch, the equations above hold, with $V(\phi)$

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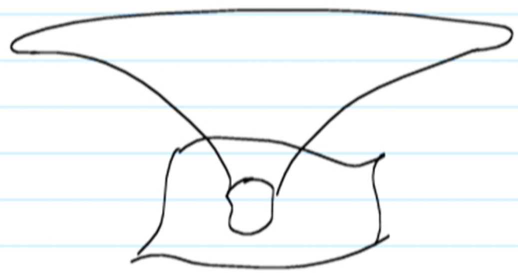
- In this patch, the equations above hold, with $V(\phi)$ dominant and inflating $a(t)$ like a large Λ would

$\Rightarrow$ Before the fluctuation can "snap back", general relativity will quasi-exponentially inflate this patch (potentially, e.g., by 10^5 orders of magnitude).



$\Rightarrow$ The mother universe spawns a daughter universe!

- $V(\phi)$ in the patch starts out high but will dynamicall..

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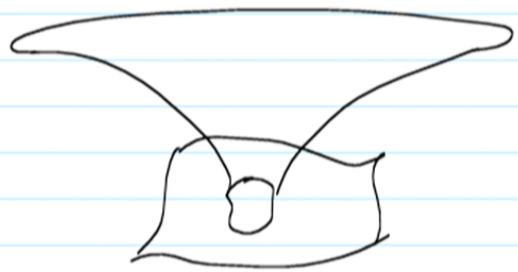

⇒ The mother universe spawns a daughter universe!

- $V(\phi)$ in the patch starts out high but will dynamically fall eventually to low value → Inflation ends.
- The energy in $V(\phi)$ turns into hot matter.

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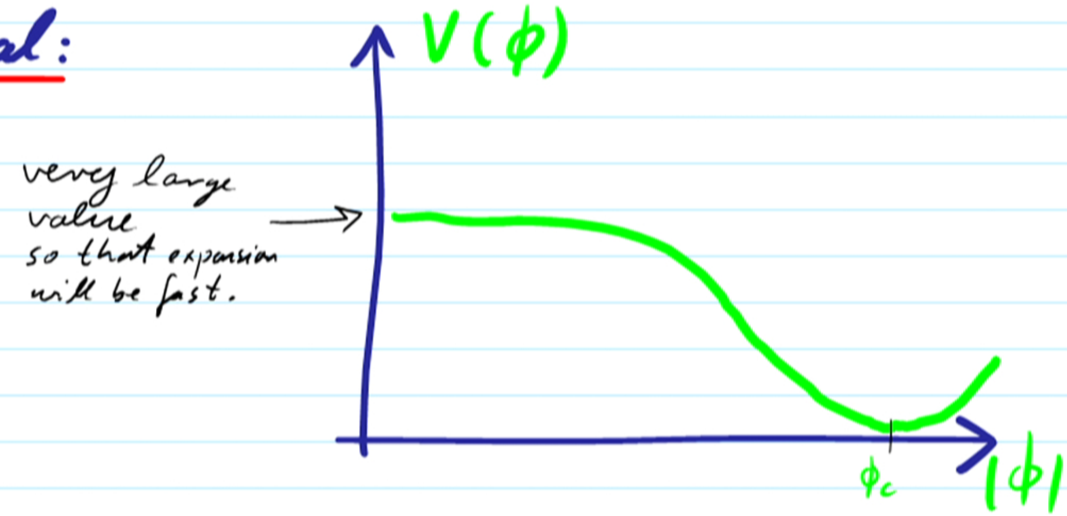
Example potential:



- Then, inflation starts when, in a patch, ϕ is very small, even though it is energetically expensive (a rare quantum fluctuation.)
- Then, after ϕ starts out at $\phi = 0$ and large $V(\phi)$, it will slowly evolve towards ϕ_c while the universe inflates, thus flattens, and the matter dilutes.
- Once $\phi = \phi_c$ is reached, $V(\phi) = 0$, and inflation has ended.

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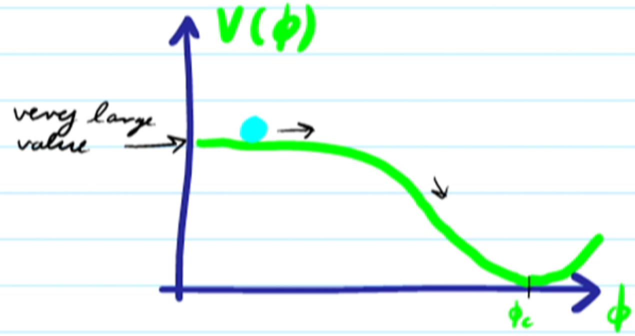
* **Concretely:**

The Klein Gordon equation reads:

friction term

$$\ddot{\phi} = -3\frac{\dot{a}}{a}\dot{\phi} - \frac{dV}{d\phi}$$

This is like the equation of motion of a ball rolling down a hill, with friction:



$\left(-\frac{dV}{d\phi}\right)$ acts to pull ϕ down the potential hill.

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* Definition:

If the initial value of $V(\phi)$ is very large and if the initial slope is very flat, i.e., if the ball for a period rolls slowly, with approximately constant $V(\phi)$, we call this a period of "Slow Roll Inflation".

* Observation:

During the slow roll period, we have, in particular, that $V(\phi)$ dominates over $\frac{1}{2}\dot{\phi}^2$.

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* Observation :

During the slow roll period, we have, in particular, that $V(\phi)$ dominates over $\frac{1}{2}\dot{\phi}^2$.

$$\Rightarrow \qquad w(t) = \frac{\frac{1}{2}\dot{\phi}^2 - V(\phi)}{\frac{1}{2}\dot{\phi}^2 + V(\phi)} \simeq -1 \quad (\text{temporarily})$$

* But, do we also get temporary exponential inflation?

Indeed, the 0,0 component of the Einstein equation

$$\left(\frac{\dot{a}}{a}\right)^2 - \frac{1}{3}\Lambda = \frac{8\pi G}{3} T^0_0$$

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Indeed, the 0,0 component of the Einstein equation

$$\left(\frac{\dot{a}}{a}\right)^2 - \frac{1}{3}\Lambda = \frac{8\pi G}{3} T^0_0$$

is during the slow roll period:

$$\left(\frac{\dot{a}}{a}\right)^2 \approx \frac{8\pi G}{3} V(\phi)$$

whose solution during slow roll is

$$a(t) \approx a_0 \, e^{t\sqrt{\frac{8\pi G}{3} V(\phi)}}$$

(ϕ and $V(\phi)$ change slowly over time in slow roll.)

* Thus, during slow roll, we have effectively a

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* Thus, during slow roll, we have effectively a slowly varying Hubble parameter!

Definition:

The function $H(t) := \frac{\dot{a}(t)}{a(t)}$ is called the Hubble parameter function.

* In the case $a(t) = e^{Ht}$ we recover $H \equiv H(t)$.

* In the case of slow roll inflation, we have

$$H(t) \approx \sqrt{\frac{8\pi G}{3} V(\phi(t))}$$

Remark:

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Remark:

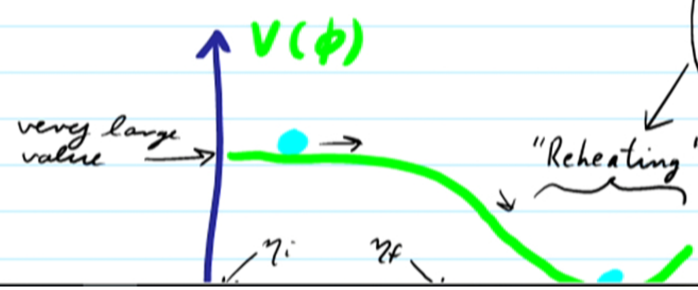
- As $V(t)$ decreases, also $H(t)$ decreases.

$\Rightarrow$ inflation predicts that $\delta\phi_k$ decreases for later and later horizon crossing modes, i.e., for smaller and smaller wavelength modes.

The WMAP satellite's CMB data show evidence for this!

The end of inflation:

(Note that this is a misnomer: We do not necessarily assume the existence of a prior hot phase)



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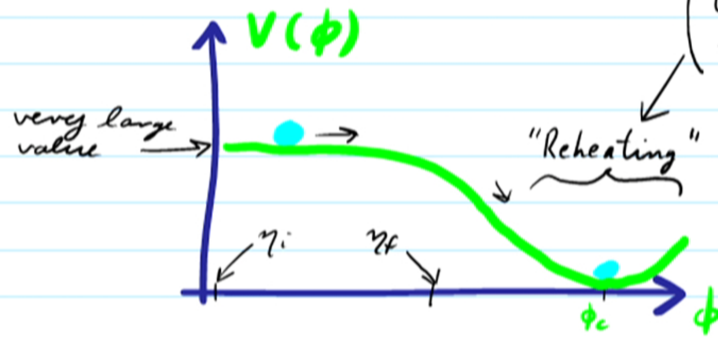

* In the period called "Re-heating", the energy of ϕ is transferred into the mode oscillators of the usual (low mass) particles, i.e., the inflaton particles decay and thereby create a high energetic, i.e. hot, plasma of literally all sorts of particles.

* From thereon, the usual big bang cosmology is

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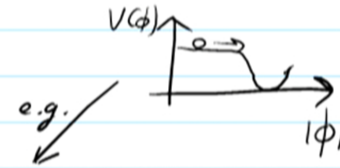
Quantum fluctuations during cosmic inflation



Strategy:

- We assume a suitable potential $V(\phi)$ and suitable initial conditions, as discussed before.

  $\Rightarrow$ Solutions $\phi_0(t)$ and $a_0(t)$ which exhibit slow roll inflation for a suitable finite time interval $[t_i, t_f]$, i.e., $[\eta_i, \eta_f]$.

- We consider the case of small inhomogeneities in the inflaton field:

$$\boxed{\phi(x,\eta) = \phi_0(\eta) + \varphi(x,\eta) \quad \text{with} \quad |\varphi(x,\eta)| \ll |\phi_0(\eta)|}$$

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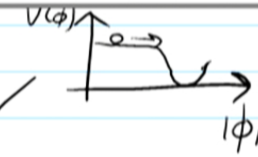
e.g.



Strategy:

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This means that we must also consider small fluctuations in the metric, because:

In inflationary theory we are always assuming that the largest contribution to $T_{\mu\nu}(x)$ stems from the inflaton field $\phi(x)$:

$$T^{\text{infl}}_{\mu\nu}(\eta,\vec{x}) = \phi_{,\mu}\phi_{,\nu} - g_{\mu\nu}\left[\frac{1}{2}g^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta} - \frac{1}{2}m^2\phi^2\right]$$

$\Rightarrow$ Thus, because of the Einstein equation,

$$R_{\mu\nu}(x) - \frac{1}{2}g_{\mu\nu}(x)R(x) + \Lambda g_{\mu\nu}(x) = 8\pi G\, T_{\mu\nu}(x)$$

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In inflationary theory we are always assuming that the largest contribution to $T_{\mu\nu}(x)$ stems from the inflaton field $\phi(x)$:

$$T^{(\phi)}_{\mu\nu}(\eta,\vec{x}) = \phi_{,\mu}\phi_{,\nu} - g_{\mu\nu}\left[\frac{1}{2}g^{\sigma\rho}\phi_{,\sigma}\phi_{,\rho} - \frac{1}{2}m^2\phi^2\right]$$

$\Rightarrow$ Thus, because of the Einstein equation,

$$R_{\mu\nu}(x) - \frac{1}{2}g_{\mu\nu}(x)R(x) + \Lambda g_{\mu\nu}(x) = 8\pi G\, T_{\mu\nu}(x)$$

inhomogeneities of $\phi(x)$ induce inhomogeneities of $g_{\mu\nu}(x)$:

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⇝ Consider also small inhomogeneities in the metric, i.e., in the spacetime

$$\boxed{g_{\mu\nu}(x,\eta) = a(\eta)\,\eta_{\mu\nu} + \gamma_{\mu\nu}(x,\eta) \quad \text{with } |\gamma_{\mu\nu}(x,\eta)| \ll 1}$$

- We would like to solve the full quantum theory of $\hat{g}_{\mu\nu}(x)$ and $\hat{\phi}(x)$ but this is too hard, inconsistent so far.

- In lowest order perturbation theory we first find the classical solutions $g_{0\,\mu\nu}(\eta) = a(\eta)\,\eta_{\mu\nu}$ and $\phi_0(\eta)$ that are completely homogeneous and isotropic.

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- In lowest order perturbation theory we first find the classical solutions $g_{0\mu\nu}(\eta) = a(\eta)\,\eta_{\mu\nu}$ and $\phi_0(\eta)$ that are completely homogeneous and isotropic.

- Then, we quantize only the $\hat{\varphi}(x,\eta)$ and $\hat{h}_{\mu\nu}(x,\eta)$.

- Why does this approximation help?

  * For fields $\hat{\varphi}$, $\hat{h}_{\mu\nu}$ that are "small" the equations of motions are effectively linear in $\hat{\varphi}, \hat{h}_{\mu\nu}$.

  * This is because we can assume that in their equations of motion all terms that are quadratic

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* For fields $\hat{\epsilon}$, $\hat{\gamma}_{\mu\nu}$ that are "small" the equations of motions are effectively linear in $\hat{\epsilon}, \hat{\gamma}_{\mu\nu}$.

* This is because we can assume that in their equations of motion all terms that are quadratic or of higher power are negligible.

* This means that the quantum fields $\hat{\epsilon}$ and $\hat{\gamma}_{\mu\nu}$ have no potential terms, nor any mass terms.

$\Rightarrow$ We will obtain a free, i.e., non-interacting quantum field theory whose nontriviality only stems from the time-varying parameters $\phi_0(\eta)$, $a_0(\eta)$.

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▫ **Intuition:** We should expect two more sources of nontriviality:

1) Interdependence of φ and $g_{\mu\nu}$ inhomogeneities:

* Much of the inhomogeneities of $\hat{g}_{\mu\nu}(x,\eta)$ will be induced by the inhomogeneities of the inflaton, $\hat{\varphi}(x,\eta)$.

* Vice versa: we can also read the Einstein eqn from left to right $\Rightarrow$ these gravity inhomogeneities induce the inflaton's inhomogeneities.

* Thus, the inflaton's inhomogeneities' dynamics cannot be separated from that of the metric.

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Recall:

Gravity is a force with some similarity to electromagnetism:

- Some electromagnetic fields only exist because there are charges or currents.

Similarly: Some part of the metric will depend on ϕ, while some metric fluctuations (gravitational waves) will be self-sustaining, i.e. they are degrees of freedom independent of ϕ. $\rightarrow$

- But: also, some electromagnetic fields are self-sustaining, i.e., they exist independently, with their own dynamics.

Exercise: show this $\longrightarrow$

- Namely:

$\vec{E}: \mathbb{R}^3 \rightarrow \mathbb{R}^3$

Any vector field $\vec{E}(x)$ can be decomposed into:

$$\vec{E}(x) = \vec{E}_c(x) + \vec{E}_v(x)$$

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because there are charges or currents.

Similarly: Some part of the metric will depend on ϕ, while some metric fluctuations (gravitational waves) will be self-sustaining, i.e. they are degrees of freedom independent of ϕ. $\rightarrow$

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Exercise: show this $\longrightarrow$ Namely:

$\vec{E}: \mathbb{R}^3 \to \mathbb{R}^3$

Any vector field $\vec{E}(x)$ can be decomposed into:

$$\vec{E}(x) = \vec{E}_s(x) + \vec{E}_v(x)$$

"gradient part" or "scalar part"

"curl part" or "vector part"

Here, $\vec{E}_s$ and $\vec{E}_v$ derive from a scal

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Here, $\vec{E}_s$ and $\vec{E}_v$ derive from a scalar function Λ and a vector field $\vec{A}$ respectively:

$$\vec{E}_s = \vec{\nabla}\Lambda \text{ and } \vec{E}_v = \vec{\nabla}\times\vec{A}$$

They obey: $\vec{\nabla}\times\vec{E}_s = \vec{0}$ and $\vec{\nabla}\cdot\vec{E}_v = 0$ (A)

Exercise for physics students: verify →

According to the Maxwell equations, the scalar part, e.g., of the electric field, $\vec{E}$, is caused by (or causes) the electric charge density

$$\vec{\nabla}\cdot\vec{E}_s = \rho$$

[* An unusual but mathematically equivalent viewpoint.
* E.g. D-branes in string theory are charges that are defined from this viewpoint.

Thus, $\vec{E}$ and $\vec{B}$ fields can sustain each other, which makes possible

while the vector part is charge independent

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- Similarly, some curvature exists only where there is energy-momentum.
- But, also, some curvature is self-sustaining, with dynamics, e.g., gravitational waves.

$\Rightarrow$ We should expect that $\hat{g}_{\mu\nu}(x,\eta)$ contains:

* some curvature that is induced by (or induces) the inflaton inhomogeneities.

* some curvature inhomogeneities that are self-sustaining, i.e., that possess

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⇒ We should expect that $\hat{\mathcal{Y}}_{\mu\nu}(x,\eta)$ contains:

* some curvature that is induced by (or induces) the inflaton inhomogeneities.

* some curvature inhomogeneities that are self-sustaining, i.e., that possess their own dynamics - and therefore also their own quantum fluctuations.

How to seperate these inhomogeneities of $\mathcal{Y}_{\mu\nu}(x,\eta)$?

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How to seperate these inhomogeneities of $\gamma_{\mu\nu}(x,\eta)$?

Similar to vector fields $\vec{E}: \mathbb{R}^3 \to \mathbb{R}^3$ we have for the tensor field γ:

- The perturbations $\gamma_{\mu\nu}$ of the metric tensor can be decomposed into three types:

  a) The part of $\gamma_{\mu\nu}$ which can be derived from scalar functions.

  b) The part of $\gamma_{\mu\nu}$ which can be derived from vector fields.

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Similar to vector fields $\xi : \mathbb{R} \to \mathbb{R}$ we have for the tensor field γ:

□ The perturbations $\gamma_{\mu\nu}$ of the metric tensor can be decomposed into three types:

a) The part of $\gamma_{\mu\nu}$ which can be derived from scalar functions.

b) The part of $\gamma_{\mu\nu}$ which can be derived from vector fields.

c) The part of $\gamma_{\mu\nu}$ which is purely tensor.

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Decomposition of $g_{\mu\nu}(x,\eta)$, with respect to its spatial structure:

- One usually writes the "line element" ds^2, i.e., the infinitesimal proper distance (squared) from x to $x+dx$ as

$$ds^2 = g_{\mu\nu}(x,\eta)\, dx^\mu dx^\nu \quad \text{with} \quad dx^\mu = (d\eta, dx^1, dx^2, dx^3)$$

- Then, the decomposition takes the form:

$$ds^2 = \overbrace{a^2(\eta)\left(d\eta^2 - \sum_{i=1}^{3}(dx^i)^2\right)}^{\text{zero-mode, i.e., homogeneous and isotropic part}} + \underbrace{ds_S^2}_{\text{scalar}} + \underbrace{ds_V^2}_{\text{vector}} + \underbrace{ds_T^2}_{\text{tensor}}$$

- Here, the spatially "scalar" part of the inhomogeneities reads

$$ds_S^2 = a^2(\eta)\Big[\, 2\Phi(x,\eta)\, d\eta^2 - 2\sum^{3} \frac{\partial}{\ } B(x,\eta)\, dx^i d\eta$$

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$$ds^2 = \overbrace{a^2(\eta)\left(d\eta^2 - \sum_{i=1}^{3}(dx^i)^2\right)}^{\text{zero-mode, i.e., homogeneous and isotropic part}} + \underbrace{ds_S^2}_{\text{scalar}} + \underbrace{ds_V^2}_{\text{vector}} + \underbrace{ds_T^2}_{\text{tensor}}$$

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$$ds_S^2 = a^2(\eta)\Big[2\Phi(x,\eta)\,d\eta^2 - 2\sum_{i=1}^{3}\frac{\partial}{\partial x^i}B(x,\eta)\,dx^i d\eta - \sum_{i,j=1}^{3}\left(2\Psi(x,\eta)\,\delta_{ij} - 2\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j}E(x,\eta)\right)dx^i dx^j\Big]$$

where Φ, Ψ, B and E are scalar functions.

- The spatially "vector" part of the metric reads:

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- The spatially "vector" part of the metric reads:

$$ds_V^2 = a^2(\eta)\Big[2\sum_{i=1}^{3} V_i(x,\eta)\,dx^i d\eta - \sum_{i,j=1}^{3}\Big(\frac{\partial}{\partial x^j}W_i(x,\eta) + \frac{\partial}{\partial x^i}W_j(x,\eta)\Big)dx^i dx^j\Big]$$

where V_i and W_i are 3-vector fields.

- The spatially "tensor" part of the metric is the remainder, i.e., is what cannot be derived from a scalar or vector fields:

$$ds_T^2 = a^2(\eta)\sum_{i,j=1}^{3} h_{ij}(x,\eta)\; dx^i dx^j$$

Here, h_{ij} is a spatial tensor field.

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- Remark regarding the fields V^i, W^i and h_{ij}:

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Analogous to the equations (A) above in electromagnetism,

$$\vec{\nabla}\times\vec{E}_s = \vec{0} \text{ and } \vec{\nabla}\cdot\vec{E}_v = 0,$$

we now have:

* V_i, W_j obey:

$$\vec{\nabla}\cdot\vec{V} = 0, \quad \vec{\nabla}\cdot\vec{W} = 0 \qquad \left(\text{i.e. } \sum_{i=1}^{3} \frac{\partial}{\partial x^i} V^i = 0 \text{ etc.}\right)$$

* h_{ij} obeys:

$$h_{ij} = h_{ji}, \quad \sum_{i=1}^{3} h^i{}_i = 0, \quad \sum_{i=1}^{3} \frac{\partial}{\partial x_j} h_{ij} = 0$$

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Remark: This implies that h_{ij} describes "Weyl curvature" which is known to describe gravitational waves

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