Title: Quantum Field Theory for Cosmology - Achim Kempf - Lecture 19

Date: Mar 11, 2016 01:30 PM

URL: http://pirsa.org/16030018

Abstract:

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Note Title

QFT for Cosmology, Achim Kempf, Winter 16, Lecture 18

Time evolution and the fluctuation spectrum:

Recall:

- We assume the system is in the state $|0\rangle$ which is the vacuum at η_0.

  $\Rightarrow$ The system is always in the state $|0\rangle$ (Heisenberg picture).

- We solve the QFT with $\hat{\chi}_k(\eta) := a(\eta)\,\hat{\phi}_k(\eta)$ and the ansatz

$$\hat{\chi}_k(\eta) = \frac{1}{\sqrt{2}}\left(v_k^*(\eta)\, a_k + v_k(\eta)\, a_{-k}^+\right)$$

  where for convenience we choose the mode functions $\{v_k(\eta)\}_k$ so that $a_k|0\rangle = 0$.

- The technical challenge will be:

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where for convenience we choose the mode functions $\{v_k(\eta)\}_k$ so that $a_k|0\rangle = 0$.

- The technical challenge will be:
  - Identify $|0\rangle$, i.e., identify the initial conditions for the v_k at η_0.
  - Solve the K.G. eqn for the $v_k(\eta)$.

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Benefit:

- State $|0\rangle$ known
- Operators $\hat{\phi}_k(\eta)$ known $\forall \eta > \eta_0$

$\Rightarrow$ We can calculate all predictions for all times, even, e.g., at times of nonadiaticity or inverted potential!

In particular, we can calculate for all $\eta > \eta_0$:

$$\boxed{\delta\phi_k(\eta) = k^{3/2}\left|\frac{v_k(\eta)}{a(\eta)}\right|}$$

We observe: The dynamics of $|v_k(\eta)|$ crucially affects $\delta\phi_k(\eta)$.

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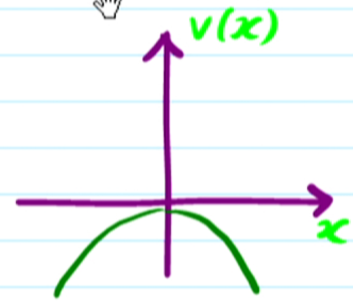
Q: In which circumstance does $v_k(\eta)$ grow most?

Answer: The most efficient mechanism to enlarge v_k occurs when the mode is nonadiabatically evolving in the sense that the mode oscillator is inverted:

$$\hat{\chi}_k''(\eta) + \overbrace{\omega_k^2(\eta)}^{\text{if } <0} \chi_k(\eta) = 0$$



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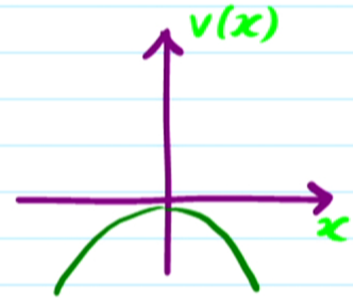

In such a time period, the Klein Gordon equation's solutions are not oscillatory because $\omega_k(\eta)$ is imaginary:

$$\omega_k(\eta) = \sqrt{k^2 + m^2 a^2(\eta) - \frac{a''(\eta)}{a(\eta)}}$$

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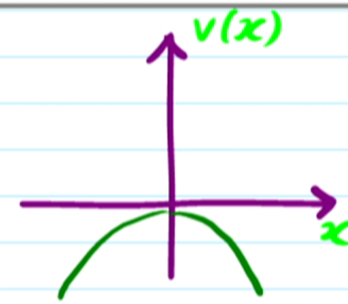

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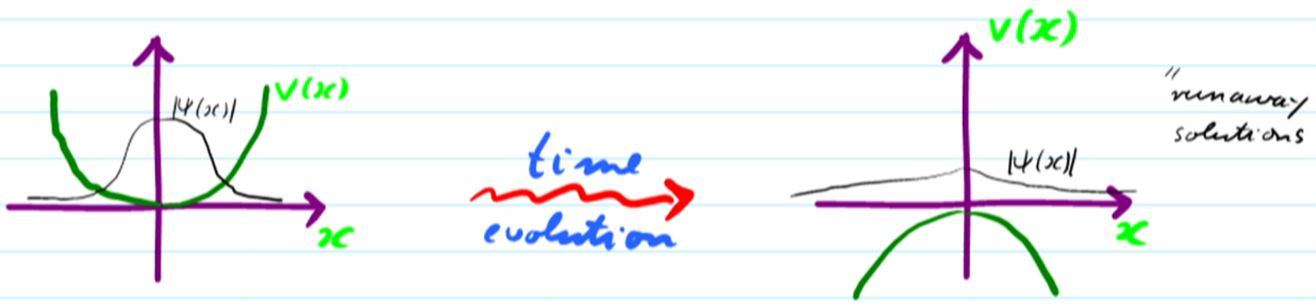

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Notice that this argument applies to x but $\phi = \frac{1}{a} x$.

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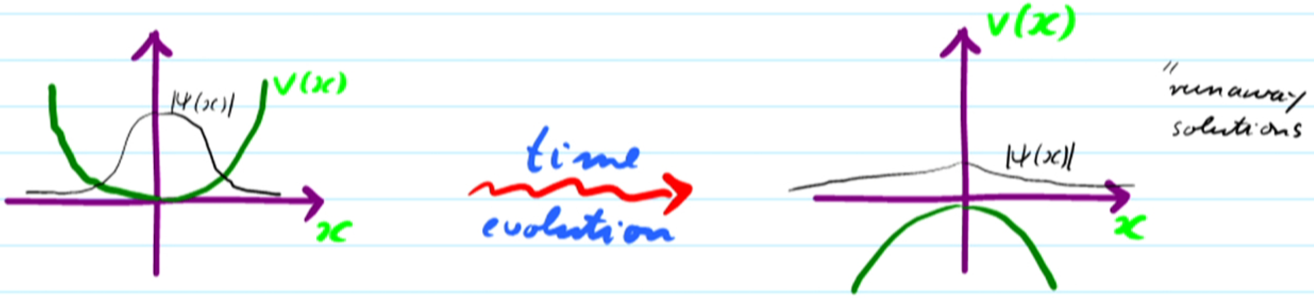

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Notice that this argument applies to x but $\phi = \frac{1}{a} x$.

Thus, ϕ's growth is slower than that of x.

Recall: The equation of motion of ϕ has a friction-type term.

Before we calculate the fluctuation amplification explicitly:

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$\Rightarrow$ We can identify $|vac_{\eta_1}\rangle$:

Using the adiabatic vacuum identification criterion, we find the mode function $\tilde{v}_k$ for which:

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- Case 1: The evolution of mode k was adiabatic from η_0 to η_1.

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□ **Case 1:** The evolution of mode k was adiabatic from η_0 to η_1.

* **Therefore:**

$$v_k = \tilde{v}_k \quad \text{and} \quad |0\rangle = |\tilde{0}\rangle$$

* **Therefore:**

The state of the system, $|\Omega\rangle = |0\rangle$, is still the vacuum state at time η_1:

$$|\Omega\rangle = |\tilde{0}\rangle$$

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* There is no particle creation.

* But since $v_k = \frac{1}{\sqrt{\omega_k(\eta)}} e^{i\int_{\eta_0}^{\eta_1} \omega(\eta')d\eta'}$, in general:

$$|v_k(\eta_1)| \neq |v_k(\eta_0)|$$ (namely $|v_k(\eta)| = \omega_k(\eta)^{-\frac{1}{2}}$)

$\Rightarrow$ the fluctuations, which depend on $|v_k(\eta)|$ can be affected even if there is no particle creation.

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II **Case 2**: The evolution was not always adiabatic between η_0 and η_1.

* Then,

$$v_k \neq \tilde{v}_k$$

* But since both are in the same 2 dimensional solution space to the K.G. equation, there exist α_k, β_k:

Recall: When particle concept applies, $|\beta_k|$ yields nonadiabatic particle production

$$v_k(\eta) = \alpha_k \tilde{v}_k(\eta) + \beta_k^* \tilde{v}_k^*(\eta)$$

* Substitute in the fluctuations equation:

$$\delta\phi_k(\eta)^2 = a^{-2}(\eta) k^3 \, |v_k(\eta)|^2$$

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* Substitute in the fluctuations equation:

$$\delta\phi_k(\eta)^2 = a^{-2}(\eta)\, k^3 \, |v_k(\eta)|^2$$

$$= a^{-2}(\eta)\, k^3 \, |\alpha_k \tilde{v}_k(\eta_1) + \beta_k^* \tilde{v}_k^*(\eta_1)|^2$$

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* For clarity, assume that the nonadiabatic period is over by η_1.

* Also, assume that spacetime is again Minkowski around η_1. (Thus, we focus on nonadiabatic effects only)

* In this case:

$$\tilde{v}_k(\eta) = \frac{1}{\sqrt{\omega_k(\eta_1)}} e^{i\omega_k(\eta_1)\eta} \quad \text{for all } \eta \approx \eta_1$$

$$\Rightarrow \delta\phi_k^2(\eta) = a^{-2}(\eta)\frac{k^3}{\omega_k(\eta_1)}\left(|\alpha_k|^2+|\beta_k|^2-2\,\mathrm{Re}\underbrace{\left(\alpha_k\beta_k e^{2i\omega_k(\eta_1)\eta}\right)}_{\text{over a long enough time period this term averages}}\right)$$

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* We use: $|\alpha_k|^2 - |\beta_k|^2 = 1$ (W)

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$$\Rightarrow \delta\phi_k^2(\eta) = \underbrace{a^{-2}(\eta)\frac{k^3}{\sqrt{k^2+m^2a^2(\eta)}}}_{\text{This term is the same, whether or not the expansion of spacetime has been adiabatic.}}\left(1 + \underbrace{2|\beta_k|^2}_{\text{This term is only non zero if the evolution was non-adiabatic.}}\right) \quad (\mathcal{F})$$

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$$\Rightarrow \quad \delta\phi_k^2(\eta) = \underbrace{a^{-2}(\eta)\frac{k^3}{\sqrt{k^2+m^2a^2(\eta)}}}_{\text{This term is the same, whether or not the expansion of spacetime has been adiabatic.}}\Big(1 + 2\underbrace{|\beta_k|^2}_{\text{This term is only non zero if the evolution was non-adiabatic.}}\Big) \qquad (F)$$

* **Notice:** $|\beta_k|$ and $|\alpha_k|$ can both become very large. This is consistent with the Wronskian condition, (W).

* **Particle production:**

Recall that the expected number of created particles is also given by $|\beta_k|^2$:

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* Particle production:

Recall that the expected number of created particles is also given by $|\beta_k|^2$:

$$\overline{\tilde{N}}_k(\eta_1) = \langle \Omega | \hat{\tilde{N}}_k | \Omega \rangle = \dots$$

$$= |\beta_k|^2$$

$\tilde{N}_k = \tilde{a}_k^+ \tilde{a}_k$

* Remark:

But (F) holds even if $\omega_k^2 < 0$ at η_1! We know what we mean by field fluctuations even when we do not have a concept of vacuum and particles

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$N_k = \tilde{a}_k^\dagger \tilde{a}_k$

* Remark:

But (F) holds even if $\omega_k^2 < 0$ at η_0! We know what we mean by field fluctuations even when we do not have a concept of vacuum and particles.

$\Rightarrow$ 'Quantum fields more fundamental than quantum particles.'

Fluctuations in proper coordinates as opposed to comoving coordinates

* We have: $\underset{\text{proper}}{d} = a(\eta) \underset{\text{comoving}}{L}$, $\underset{\text{proper momenta}}{p} = \frac{1}{a(\eta)} \underset{\text{comoving moment}}{k}$

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Fluctuations in proper coordinates as opposed to comoving coordinates

* We have: $d = a(\eta)\, L$, $p = \frac{1}{a(\eta)}\, k$

d ↑ proper length; L ↑ comoving length; p ↑ proper momentum; k ↑ comoving momentum

* Therefore, $\delta\phi_k^2(\eta) = a^{-2}(\eta)\frac{k^3}{\sqrt{k^2+m^2a^2(\eta)}}\left(1+2|\beta_k|^2\right)$ becomes:

$$\delta\phi_p^2(\eta) = a^{-2}\frac{a^3p^3}{\sqrt{a^2p^2+a^2m^2}}\left(1+2|\beta_{ap}|^2\right)$$

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* **Note:** The nonadiabatic term depends on p.

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* Note: The nonadiabatic term depends on p.

* Note: This was the case where we end in a Minkowski space. We see that in this case we must get back the original Minkowski spectrum if the evolution from η_0 to η_1 was adiabatic.

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Application to specific cosmological models

The standard model of cosmology holds that the very early universe underwent a short period of almost exponential expansion, "inflation".

→ Begin by studying QFT in de Sitter spacetime:

The deSitter FRW spacetime can be defined through

$$\boxed{a(t) := e^{Ht} \text{ for all } t \in \mathbb{R}}$$

Notes: * t is the time on a comoving observer's wrist watch
* large $H \Leftrightarrow$ large acceleration

Here: $H > 0$ is a constant, the "Hubble constant".

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The de Sitter horizon

Proposition: (in particle picture) (Note: large $H \Leftrightarrow$ small horizon d_H)

Objects (or any observers) who are further apart than a proper distance of $d_H = 1/H$ can never meet, and cannot communicate.

Proof: * Consider an observer in a galaxy A. Let us choose the origin of the comoving coordinate system(s) to be where this observer sits.

* Now suppose that, at some arbitrary time, t_s, this observer sends a radio signal towards another galaxy, B.

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proper distance

our unit convention here

$$\frac{a(t)\,\Delta x}{\Delta t} = c = 1$$

speed of light

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Fix the integration constant C_1 so that $x(t_s) = 0 \Rightarrow C_1 = \frac{1}{H} e^{-Ht_s}$

$$\Rightarrow \quad x(t) = -\frac{1}{H} e^{-Ht} + \frac{e^{-Ht_s}}{H}$$

(Recall: The proper distance traveled is: $d(t) = a(t)\,x(t)$. Clearly: $d(t) \to \infty$ as $t \to \infty$)

e^{-Ht_s} Terminal comoving distance tra…

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$$\Rightarrow \quad x(t) = -\frac{1}{H} e^{-Ht} + \frac{e^{-Ht_s}}{H}$$

$\Rightarrow$ As $t \to \infty$ we have $x(t) \to \frac{e^{-Ht_s}}{H}$.

Terminal comoving distance traveled.

Thus, can reach galaxy B if comoving distance is at most $d_c = \frac{e^{-Ht_s}}{H}$

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Thus, can reach galaxy B if comoving distance is at most $d_c = \frac{e^{-Ht_s}}{H}$.

Q: Proper distance d_p of such B from A at t_s?

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Fix the integration constant C_1' so that $x(t_s) = 0 \Rightarrow C_1' = \frac{1}{H} e^{-Ht_s}$

Recall: The proper distance traveled is: $d(t) = a(t)\, x(t)$. Clearly: $d(t) \to \infty$ as $t \to \infty$

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Thus, can reach galaxy B if comoving distance is at most $d_c = \frac{e^{-Ht_s}}{H}$.

Q: Proper distance d_p of such B from A at t_s?

A: $d_s = a(t)\, d_c \Rightarrow d_s = e^{+Ht_s} \frac{e^{-Ht_s}}{H} = \frac{1}{H}$

Recall: This holds for arbitrary t_s.

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Recall: This holds for arbitrary t_s.

⇒ A signal sent by A at any time t_s can only ever reach B if at the time of sending, t_s, the proper distance between A and B is at most $\frac{1}{H}$.

⇒ Any two observers further apart than a proper distance of $1/H$ cannot communicate!

Interpretation: In the case where a deSitter exponential expansion lasts forever, between any objects of proper distance $> 1/H$, space is being created faster than

(Remark: Notice that the

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Interpretation: In the case where a deSitter exponential expansion lasts forever, between any objects of proper distance $> 1/H$, space is being created faster than what can be crossed when travelling with the speed of light.

(Remark: Notice that the proper size of the de Sitter horizon is constant in time.)

Proposition: (in wave picture)

Klein Gordon modes oscillate while their proper wavelength obeys $\lambda \ll \frac{1}{H}$ but stop oscillating and possess instead an imaginary frequency when their proper wavelength has grown beyond, i.e. when $\lambda \gg$

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Proposition: (in wave picture)

Klein Gordon modes oscillate while their proper wavelength obeys $\lambda \ll \frac{1}{H}$ but stop oscillating and possess instead an imaginary frequency when their proper wavelength has grown beyond, i.e., when $\lambda \gg \frac{1}{H}$, assuming that their mass is small: $m \ll H$.

Proof: 1) Let us switch to conformal time: (Thus, need $a(\eta)$!)

□ Recall: $\eta(t) := \int^t \frac{1}{a(t')} dt'$

The choices of the integration constant C_1 merely mean different fixed shifts

$\int^t e^{-Ht'} dt'$

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Proof: 1) Let us switch to conformal time: (Thus, need $a(\eta)$!)

- Recall: $\eta(t) := \int^t \frac{1}{a(t')} dt'$

here: $\eta(t) = \int^t e^{-Ht'} dt'$

$= -\frac{1}{H} e^{-Ht} + c_1$

The choices of the integration constant c_1 merely mean different fixed shifts in the time coordinate η relative to the time coordinate t.

- **Notice:**

  - As $t \to -\infty$ we have $\eta \to -\infty$.

  - But as $t \to +\infty$ we have $\eta \to c_1$.

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But as $t \to +\infty$ we have $\eta \to C'$.

Choose $C' = 0$:

$$\Rightarrow \quad \eta(t) = -\frac{1}{H}\frac{1}{a(t)}$$

$$a(t) = -\frac{1}{H\eta(t)}$$

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- But as $t \to +\infty$ we have $\eta \to C_1'$.
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$\Rightarrow$

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$$a(t) = -\frac{1}{H\eta(t)}$$

i.e.:

$$\boxed{a(\eta) = -\frac{1}{H\eta}}$$

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2) Introduce $\hat{\chi}_k(\eta) := a(\eta)\,\hat{\phi}_k(\eta)$:

- We have: $\hat{\chi}_k(\eta) = -\frac{1}{H\eta}\,\hat{\phi}_k(\eta)$
- $\hat{\chi}_k$ obeys this Klein Gordon equation

$$\hat{\chi}_k''(\eta) + \omega_k^2(\eta)\,\hat{\chi}_k(\eta) = 0$$

with:

$$\omega_k^2(\eta) = k^2 + m^2 a^2(\eta) - \frac{a''(\eta)}{a(\eta)}$$

- Exercise: Show that in the de Sitter case this yields:

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- Exercise: Show that in the de Sitter case this yields:

$$\boxed{\omega_k^2(\eta) = k^2 + \frac{m^2}{H^2\eta^2} - \frac{2}{\eta^2}}$$

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$$\boxed{\omega_k^2(\eta) = k^2 + \frac{m^2}{H^2\eta^2} - \frac{2}{\eta^2}}$$

3.) Check for imaginary frequencies. $\frac{m^2}{H^2\eta^2} - \frac{2}{\eta^2} < 0$

- Recall: We are assuming $m \ll H$.

- Thus, in $\omega_k^2(\eta) = k^2 + \underbrace{\frac{m^2}{H^2\eta^2} - \frac{2}{\eta^2}}_{\text{we have: } < 0}$

- Therefore: For each mode k there comes a time when ω_k^2 becomes negative!

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The case relevant in cosmology: $m = 0$ (we'll assume this)

$\Longrightarrow$ The time when a mode k crosses the horizon is given by:

$$\boxed{\eta_{hor}(k) \approx -\frac{\sqrt{2}}{k}}$$

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4) Conclusion:

- A mode oscillates as long as:

Recall: $\eta \in (-\infty, 0)$ i.e. $|\eta| \gg 1/k$ means early times.

$$|\eta| \gg \frac{1}{k} \quad \text{i.e., while } |\eta| k \gg 1 \qquad \text{(a)}$$

(Used that $\sqrt{2}$ and 1 are of same order of magnitude)

- A mode has imaginary frequency from, when

This is late times, i.e. when $\eta \approx 0$.

$$|\eta| \ll \frac{1}{k} \quad \text{i.e., from when } |\eta| k \ll 1 \qquad \text{(b)}$$

Re-expressed in terms of proper wavelength?

Noting $|\eta| = \frac{1}{Ha}$ and multiplying it with $k = 2\pi/L$ we obtain:

comoving wavele...

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$$|\eta| k = \frac{1}{Ha}\,\frac{2\pi}{L}$$

Transforming to the proper wavelength, $\lambda = a(\eta) L$, we obtain:

$$|\eta| k = \frac{2\pi}{H\lambda}$$

(Thus, the proper wavelength, λ, of a fixed comoving mode, k, obeys: $\lambda(\eta) = \frac{2\pi}{Hk|\eta|}$)

Thus, finally, the two cases, ⓐ and ⓑ become:

A mode oscillates as long as: $|\eta| k \gg 1$

i.e., as long as $\frac{2\pi}{H\lambda} \gg 1$ i.e.: $\boxed{\lambda \ll \frac{1}{H}}$ ⓐ'

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(Thus, the proper wavelength, λ, of a fixed comoving mode, k, obeys: $\lambda(\eta) = \frac{2\pi}{Hk|\eta|}$)

Thus, finally, the two cases, (a) and (b) become:

- A mode oscillates as long as: $|\eta| k \gg 1$

  i.e., as long as $\frac{2\pi}{H\lambda} \gg 1$ i.e.: $\boxed{\lambda \ll \frac{1}{H}}$ (a')

- A mode has imaginary frequency from when:

  $\frac{2\pi}{H\lambda} = |\eta| k \ll 1$, i.e., from when $\boxed{\lambda \gg \frac{1}{H}}$ (b')

This is what we had set out to show.

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