Title: PSI 2015/2016 Explorations in Condensed Matter - Guifre Vidal - 5

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Abstract:

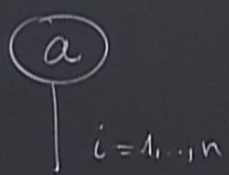
Week 2 Entanglement in many-body ground states

1. Basics of entanglement
2. Entanglement in quantum spin chains ($N \sim 20!$) — Tutorial 3
3. Entanglement in free fermion systems ($N \sim 1000$)
4. Area law — Tutorial 4
5. Entanglement as a theoretical tool — Homework 2

(A) Preliminaries

n-level system $\quad \mathbb{V} \cong \mathbb{C}^n \quad \{|1\rangle, |2\rangle, \dots |n\rangle\}$ orthonormal basis

$$\ni \; |\psi\rangle = \sum_{i=1}^{n} a_i |i\rangle$$



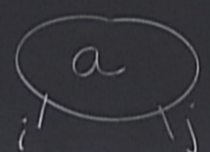
$n=2$ qubit, spin 1/2, fermion mode, pair Majorana modes

$\{|0\rangle, |1\rangle\} \quad \{|\uparrow\rangle, |\downarrow\rangle\} \quad \dots$

(B) 2 such systems A & B

$$\mathbb{V}^{AB} \cong \mathbb{V}^{A} \otimes \mathbb{V}^{B} \qquad \{ |i\, j\rangle \equiv |i^A\rangle \otimes |j^B\rangle \}$$ orthonormal basis

$$\ni \; |\psi\rangle = \sum_{i,j} a_{ij} |i\, j\rangle$$



5.2 Entanglement in bipartite systems

(A) Def

$|\psi\rangle \in \mathbb{V}^{AB}$

product state $|\psi\rangle = |\phi^A\rangle \otimes |\varphi^B\rangle$

entangled state $|\psi\rangle \neq |\phi^A\rangle \otimes |\varphi^B\rangle$

Examples: $|1^A\rangle \otimes |1^B\rangle = |11\rangle$

$|11\rangle + |22\rangle$

$|11\rangle + |12\rangle = |1\rangle(|1\rangle + |2\rangle)$

$|11\rangle + |12\rangle + |21\rangle + |22\rangle = (|1\rangle + |2\rangle)(|1\rangle + |2\rangle)$

$|11\rangle + |12\rangle + |21\rangle - |22\rangle$

$\frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$ EPR pair, ebit, singlet

$|\phi_n^+\rangle \equiv \frac{1}{\sqrt{n}}(|11\rangle + |22\rangle + \ldots + |nn\rangle)$ maximally entangled state of two n-level systems

(B) Schmidt decomposition & reduced density matrices

assume $m < n$

$m \times n$ matrix A

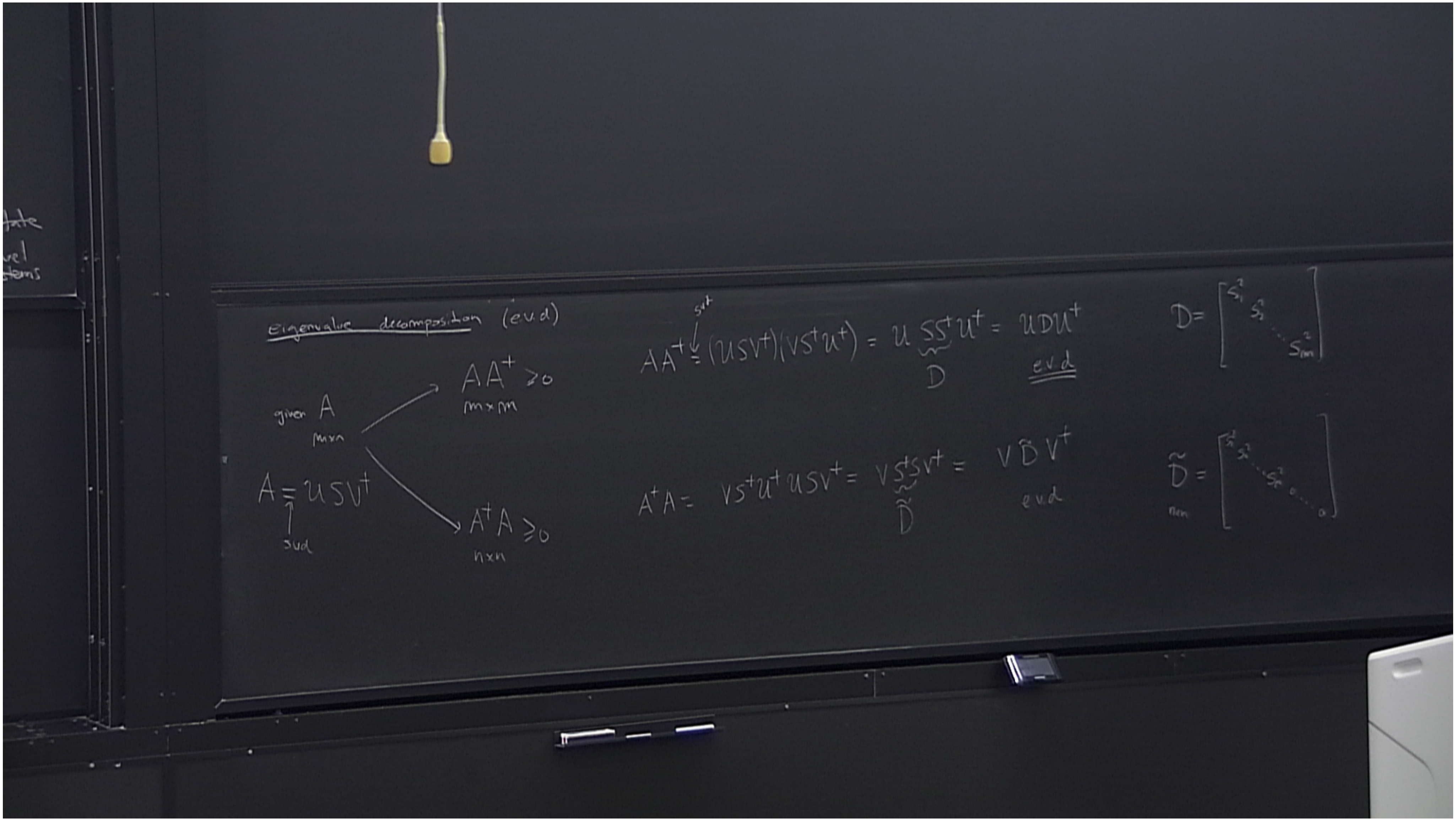
singular value decomposition (svd)

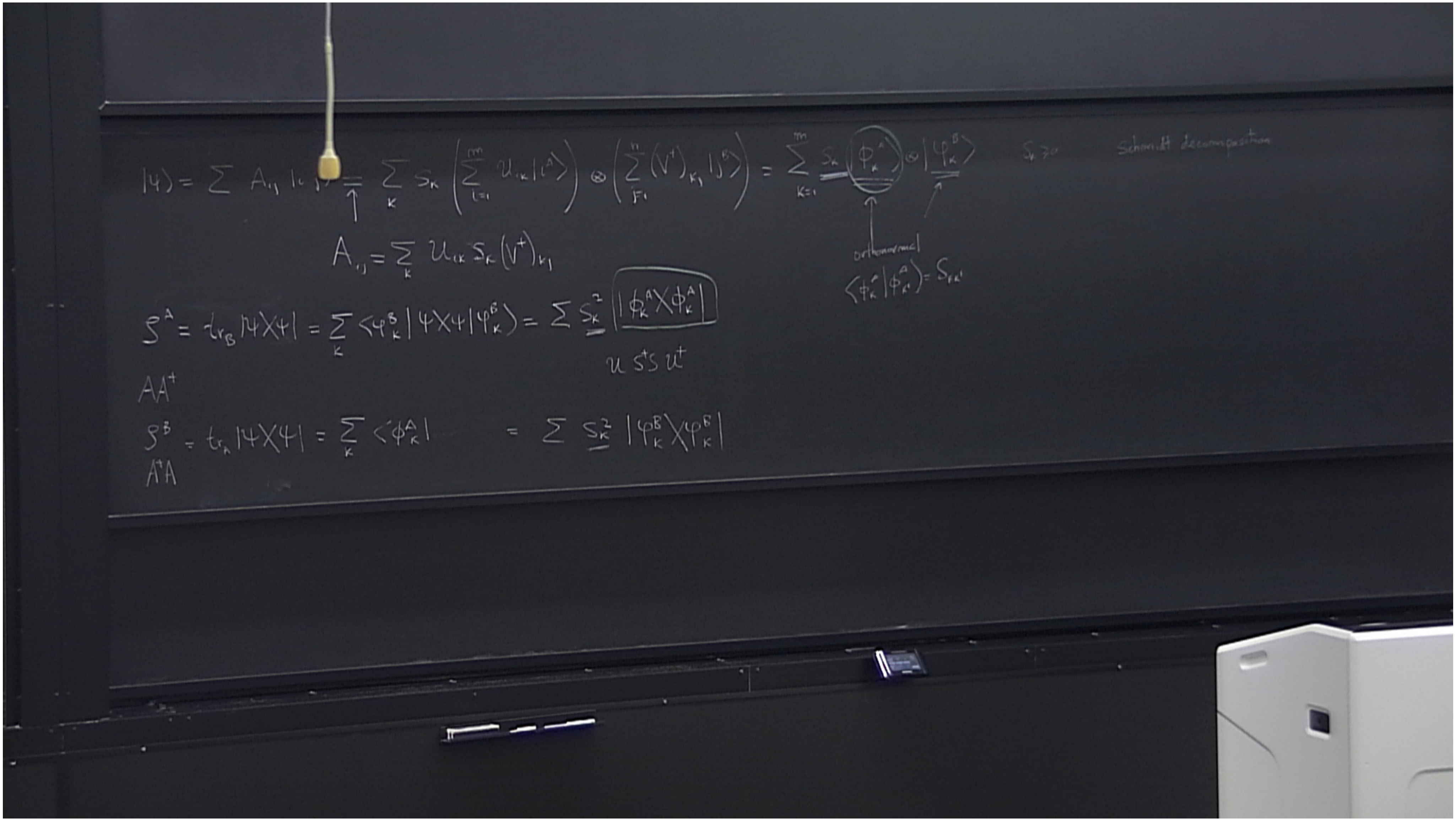
$$A = \underset{m\times m}{U}\ \underset{m\times n}{S}\ \underset{n\times n}{V^\dagger} = \begin{bmatrix} & \\ & \end{bmatrix}_{m} \begin{bmatrix} s_1 & & 0 & \\ & \ddots & & 0 \\ 0 & & s_m & \end{bmatrix} \begin{bmatrix} & \\ & \end{bmatrix}_{n}$$

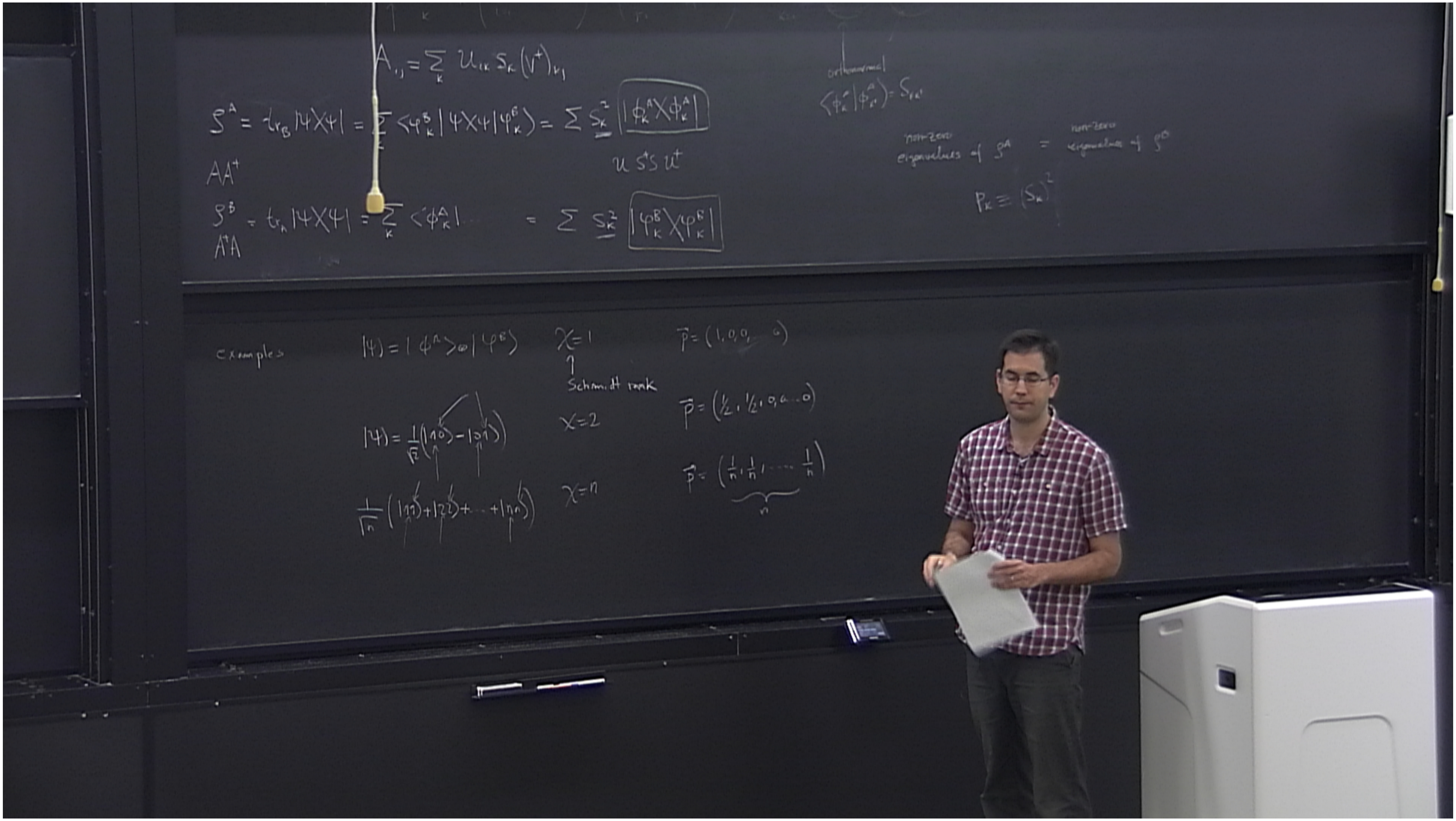
$UU^\dagger = U^\dagger U = \mathbb{1}_m$

$VV^\dagger = V^\dagger V = \mathbb{1}_n$

$s_i \geq 0$ singular values


eigenvalue decomposition (e.v.d)
given $\underset{m\times n}{A}$
$\underset{m\times m}{AA^\dagger} \geq 0$
$\underset{n\times n}{A^\dagger A} \geq 0$
$A = USV^\dagger$
svd
$AA^\dagger \overset{SV^\dagger}{=} (USV^\dagger)(VS^\dagger U^\dagger) = U \underbrace{SS^\dagger}_{D} U^\dagger = UDU^\dagger$
e.v.d
$D = \begin{bmatrix} s_1^2 & & & \\ & s_2^2 & & \\ & & \ddots & \\ & & & s_m^2 \end{bmatrix}$
$A^\dagger A = VS^\dagger U^\dagger U S V^\dagger = V \underbrace{S^\dagger S}_{\tilde{D}} V^\dagger = V\tilde{D}V^\dagger$
e.v.d
$\underset{n\times n}{\tilde{D}} = \begin{bmatrix} s_1^2 & & & & & \\ & s_2^2 & & & & \\ & & \ddots & & & \\ & & & s_m^2 & & \\ & & & & 0 & \\ & & & & & \ddots \; 0 \end{bmatrix}$


$|\psi\rangle = \sum A_{ij} |ij\rangle = \sum_k s_k \left(\sum_{i=1}^m U_{ik} |i^A\rangle\right) \otimes \left(\sum_{j=1}^n (V^\dagger)_{kj} |j^B\rangle\right) = \sum_{k=1}^m s_k |\phi_k^A\rangle \otimes |\varphi_k^B\rangle$
$s_k \geq 0$
Schmidt decomposition
$A_{ij} = \sum_k U_{ik} s_k (V^\dagger)_{kj}$
orthonormal
$\langle \phi_k^A | \phi_{k'}^A \rangle = \delta_{kk'}$
$\rho^A = tr_B |\psi\rangle\langle\psi| = \sum_k \langle \varphi_k^B | \psi\rangle\langle\psi | \varphi_k^B \rangle = \sum s_k^2 \boxed{|\phi_k^A\rangle\langle\phi_k^A|}$
$U S^\dagger S U^\dagger$
$AA^\dagger$
$\rho^B = tr_A |\psi\rangle\langle\psi| = \sum_k \langle \phi_k^A | \quad = \sum s_k^2 |\varphi_k^B\rangle\langle\varphi_k^B|$
$A^\dagger A$


examples
Schmidt rank
orthonormal
non-zero eigenvalues of ρ^A = non-zero eigenvalues of ρ^B

5.3 Measures of entanglement

$|\psi\rangle \longleftrightarrow \{\sqrt{p_k}\}$ Schmidt coefficients

$$\mathbb{C}^m \otimes \mathbb{C}^n / \mathcal{U}(m) \times \mathcal{U}(n)$$

$|\psi\rangle \sim |\psi'\rangle = \mathcal{U}^A \otimes V^B |\psi\rangle$

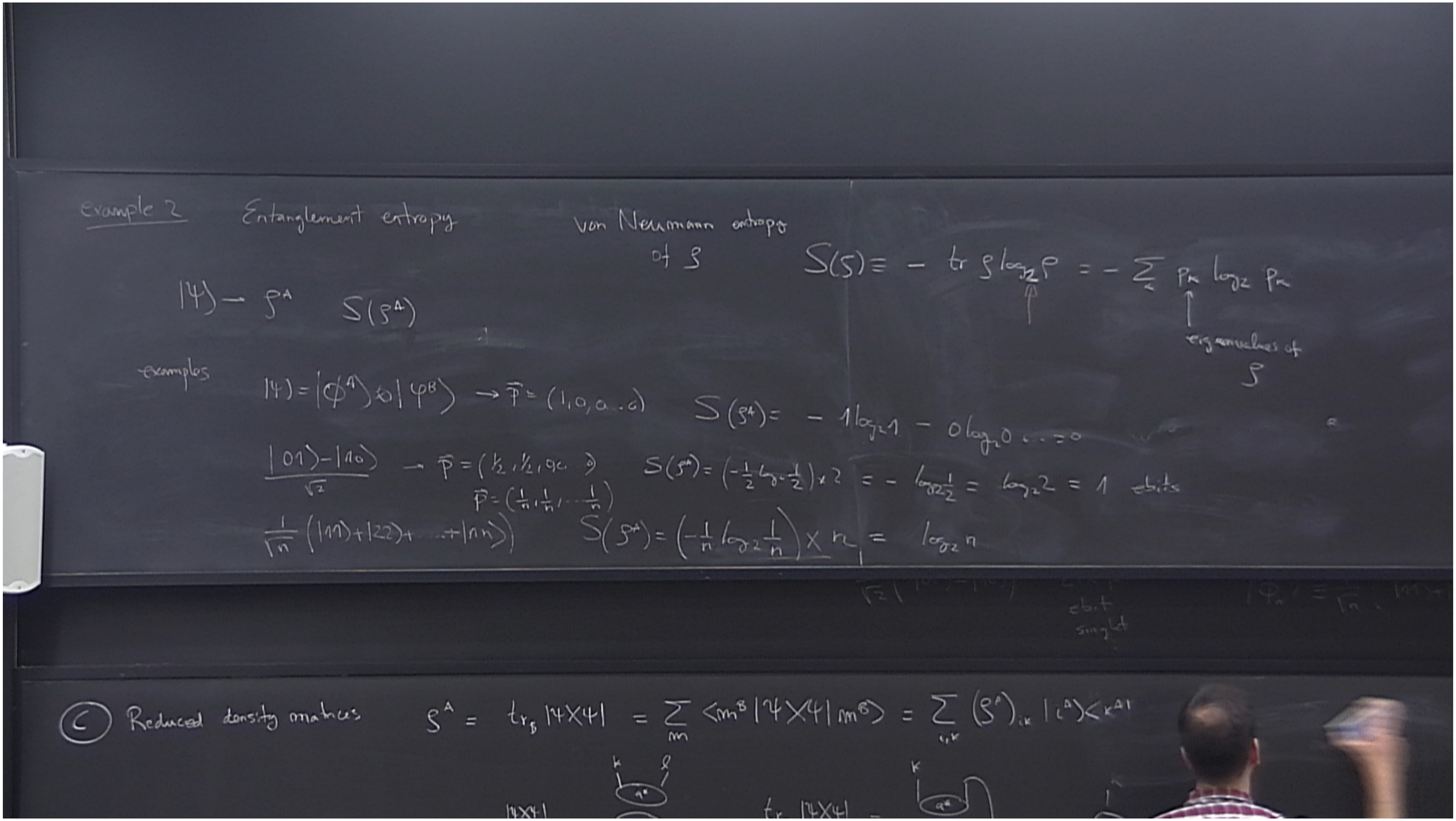
$A \qquad A'$

$\mathcal{E}($

$\mathcal{E}(\psi)$ single number

- $\mathcal{E}(\psi) = f(\{\sqrt{p_k}\})$
- $\mathcal{E}(\psi) = 0 \iff \psi$ is a product state
- $\psi \xrightarrow{LOCC} \psi'$ $\quad \underline{\mathcal{E}(\psi) \geq \mathcal{E}(\psi')}$

monotonicity under LOCC


example 2
Entanglement entropy
von Neumann entropy of ρ
$S(\rho) \equiv -\mathrm{tr}\,\rho \log_2 \rho = -\sum_\alpha p_\alpha \log_2 p_\alpha$
eigenvalues of ρ
$|\psi\rangle \to \rho^A \quad S(\rho^A)$
examples
$|\psi\rangle = |\phi^A\rangle \otimes |\varphi^B\rangle \to \vec{p} = (1, 0, 0 \ldots 0)$
$S(\rho^A) = -1 \log_2 1 - 0 \log_2 0 \ldots = 0$
$\frac{|01\rangle - |10\rangle}{\sqrt{2}} \to \vec{p} = (\frac{1}{2}, \frac{1}{2}, 0, \ldots 0)$
$S(\rho^A) = (-\frac{1}{2}\log_2 \frac{1}{2}) \times 2 = -\log_2 \frac{1}{2} = \log_2 2 = 1$ ebits
$\vec{p} = (\frac{1}{n}, \frac{1}{n}, \ldots \frac{1}{n})$
$\frac{1}{\sqrt{n}}(|11\rangle + |22\rangle + \ldots + |nn\rangle)$
$S(\rho^A) = (-\frac{1}{n}\log_2 \frac{1}{n}) \times n = \log_2 n$
(c) Reduced density matrices
$\rho^A = \mathrm{tr}_B |\psi\rangle\langle\psi| = \sum_m \langle m^B|\psi\rangle\langle\psi|m^B\rangle = \sum_{i,k} (\rho^A)_{ik} |i^A\rangle\langle k^A|$
$|\psi\rangle\langle\psi|$
$\mathrm{tr}\,|\psi\rangle\langle\psi| =$

$|\psi\rangle \to \rho^A \quad S(\rho^A)$

$S(\rho) = -\mathrm{tr}\,\rho\log_2\rho = -\sum_k p_k \log_2 p_k$

eigenvalues of ρ

examples

$|\psi\rangle = |\phi^A\rangle\otimes|\varphi^B\rangle \to \vec p = (1,0,0\ldots 0) \quad S(\rho^A) = -1\log_2 1 - 0\log_2 0 \ldots = 0$

$\frac{|01\rangle-|10\rangle}{\sqrt2} \to \vec p = (\tfrac12, \tfrac12, 0 \ldots 0) \quad S(\rho^A) = (-\tfrac12\log_2\tfrac12)\times 2 = -\log_2\tfrac12 = \log_2 2 = 1$ ebits

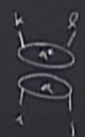
$\vec p = (\tfrac1n, \tfrac1n, \ldots \tfrac1n)$

$\frac{1}{\sqrt n}(|11\rangle+|22\rangle+\ldots+|nn\rangle) \quad S(\rho^A) = (-\tfrac1n\log_2\tfrac1n)\times n = \log_2 n$

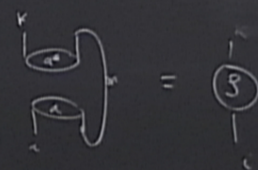
(c) Reduced density matrices

$\rho^A = \mathrm{tr}_B|\psi\rangle\langle\psi| = \sum_m \langle m^B|\psi\rangle\langle\psi|m^B\rangle = \sum_{i,k}(\rho^A)_{ik}|i^A\rangle\langle k^A|$

$|\psi\rangle\langle\psi|$



$\mathrm{tr}_B|\psi\rangle\langle\psi| =$



Example 3 Renyi entropies

$S_\alpha(\rho^A) = \frac{1}{1-\alpha}\log_2(\mathrm{tr}\,\rho^\alpha) = \frac{1}{1-\alpha}\log_2\sum_k (p_k)^\alpha$

$\alpha\in[0,\infty]$

$\lim_{\alpha\to1} S_\alpha = S$

$\lim_{\alpha\to0} S_\alpha = \log_2 \chi$