Title: PSI 2015/2016 Explorations in Condensed Matter - Guifre Vidal - 5

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Abstract:

Week 2 Entanglement in many-body ground states

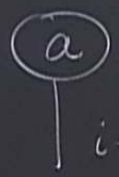
1. Basics of entanglement
2. Entanglement in quantum spin chains ($N \sim 20!$) Tutorial 3
3. Entanglement in free fermion systems ($N \sim 1000$)
4. Area law Tutorial 4
5. Entanglement as a theoretical tool Homework 2

(A) Preliminaries

n-level system $\quad \mathbb{V} \cong \mathbb{C}^n \quad \{|1\rangle, |2\rangle, \ldots |n\rangle\}$ orthonormal basis

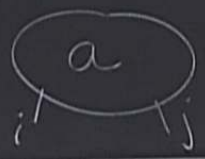
$\mathbb{V} \ni |\psi\rangle = \sum_{i=1}^{n} a_i |i\rangle$ $\quad n=2$ qubit, spin 1/2, fermion mode, pair Majorana modes

$\{|0\rangle, |1\rangle\}$ $\{|\uparrow\rangle, |\downarrow\rangle\}$ ..



(B) 2 such systems A & B $\quad \mathbb{V}^{AB} \cong \mathbb{V}^A \otimes \mathbb{V}^B \quad \{|i\ j\rangle \equiv |i^A\rangle \otimes |j^B\rangle\}$ orthonormal basis

$\mathbb{V} \ni |\psi\rangle = \sum_{i,j} a_{ij} |i\ j\rangle$



5.2 Entanglement in bipartite systems

A Def

$|\psi\rangle \in \mathbb{V}^{AB}$

- product state: $|\psi\rangle = |\phi^A\rangle \otimes |\varphi^B\rangle$
- entangled state: $|\psi\rangle \neq |\phi^A\rangle \otimes |\varphi^B\rangle$

examples:

$|1^A\rangle \otimes |1^B\rangle = |11\rangle$

$|11\rangle + |22\rangle$

$|11\rangle + |12\rangle = |1\rangle(|1\rangle + |2\rangle)$

$|11\rangle + |12\rangle + |21\rangle + |22\rangle = (|1\rangle + |2\rangle)(|1\rangle + |2\rangle)$

$|11\rangle + |12\rangle + |21\rangle - |22\rangle$

$\frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$ EPR pair ebit singlet

$|\phi_n^+\rangle \equiv \frac{1}{\sqrt{n}}(|11\rangle + |22\rangle + \ldots + |nn\rangle)$ maximally entangled state of two n-level systems

B Schmidt decomposition & reduced density matrices

assume $m \leq n$

$m \times n$ matrix A

$A = U S V^\dagger$ ($m \times m$, $m \times n$, $n \times n$)

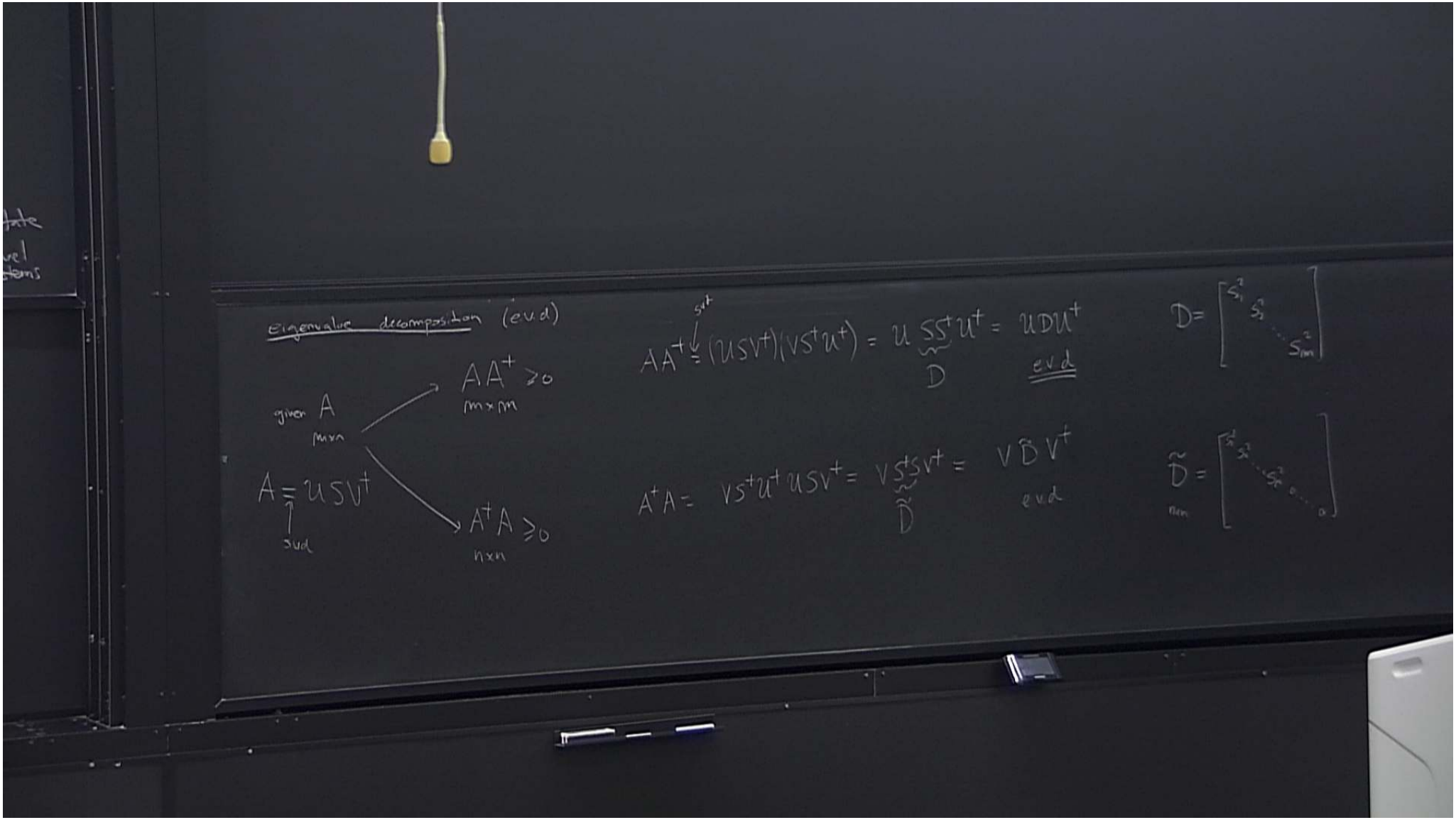
$= {}_m\!\left[\;\;\right]_m \begin{bmatrix} s_1 & & & 0 \\ & s_2 & & \\ & & \ddots & \\ 0 & & & s_m \end{bmatrix} \left[\begin{array}{c} \\ \\ \\ \end{array}\right]_n$

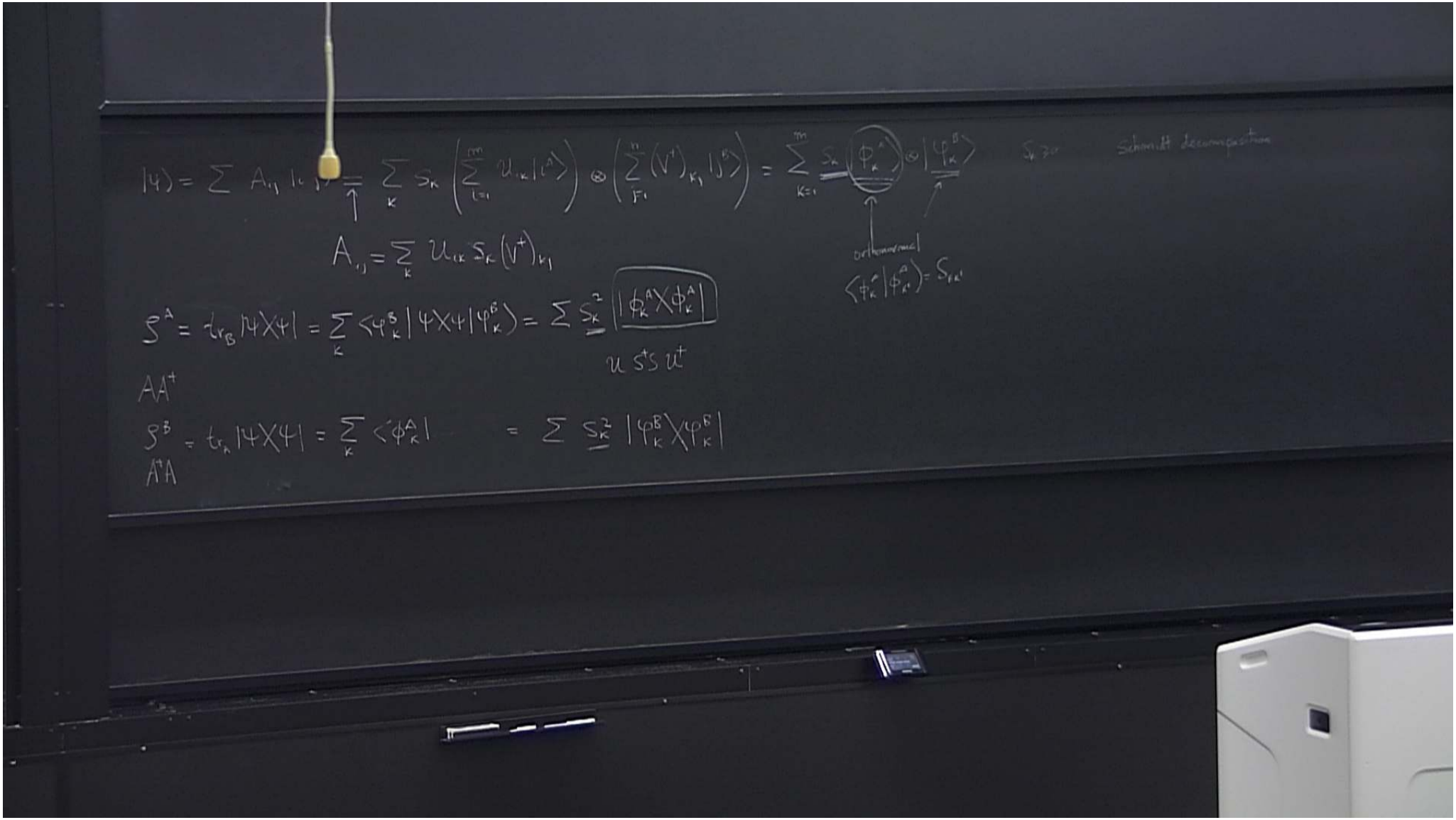
singular value decomposition (svd)

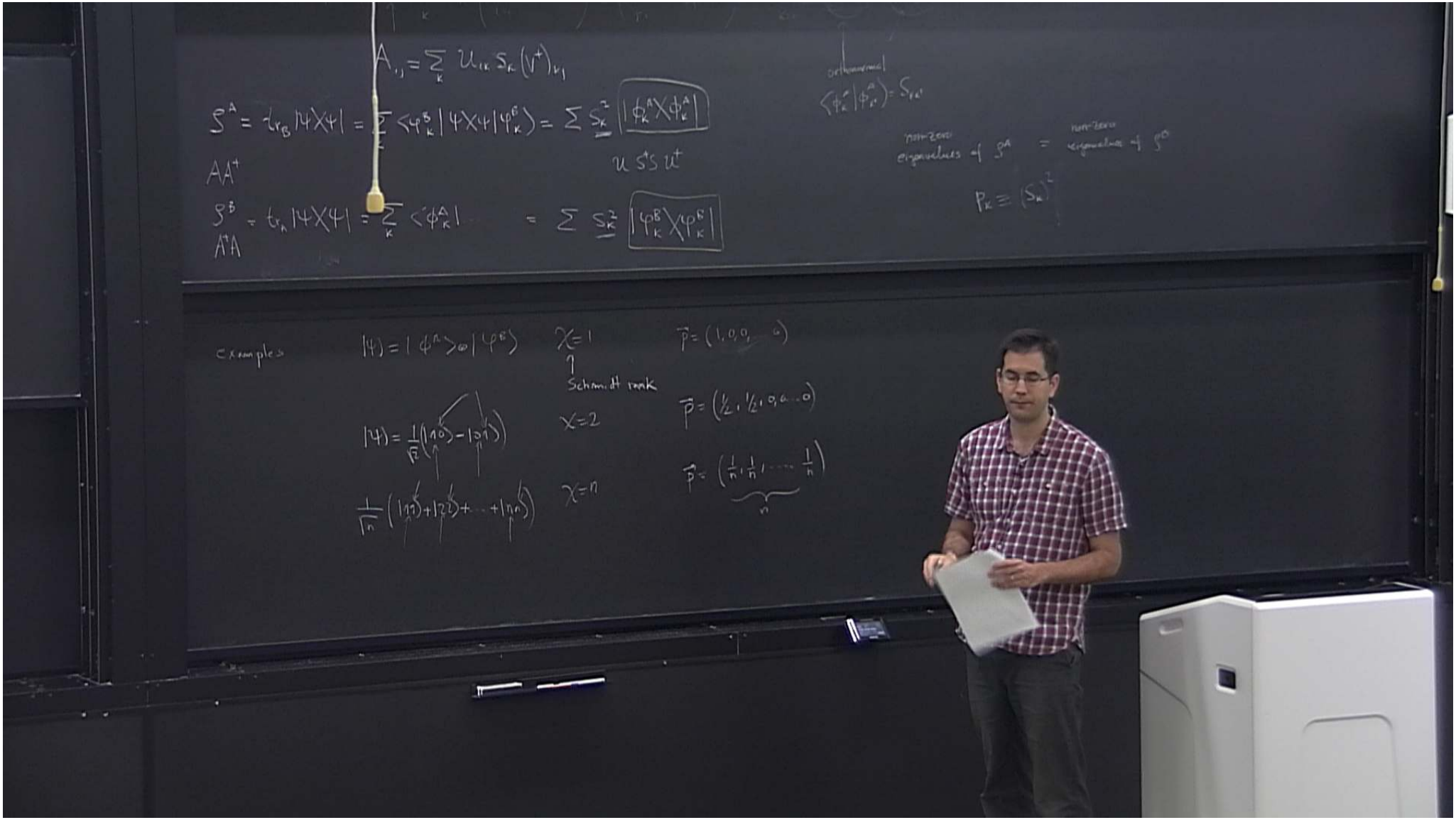
$UU^\dagger = U^\dagger U = \mathbb{1}_m$

$VV^\dagger = V^\dagger V = \mathbb{1}_n$

$s_i \geq 0$ singular values


Eigenvalue decomposition (evd)
given A
m×n
$AA^+ \geq 0$
m×m
$A = USV^+$
svd
$A^+A \geq 0$
n×n
$AA^+ = (USV^+)(VS^+U^+) = U\underbrace{SS^+}_{D}U^+ = UDU^+$
evd
$D = \begin{bmatrix} s_1^2 & & & \\ & s_2^2 & & \\ & & \ddots & \\ & & & s_m^2 \end{bmatrix}$
$A^+A = VS^+U^+USV^+ = V\underbrace{S^+S}_{\tilde{D}}V^+ = V\tilde{D}V^+$
evd
$\tilde{D} = \begin{bmatrix} s_1^2 & & & & \\ & s_2^2 & & & \\ & & \ddots & & \\ & & & s_n^2 & \\ & & & & 0 \end{bmatrix}$
n×n


$|\psi\rangle = \sum A_{ij} |i\rangle|j\rangle = \sum_k s_k \left(\sum_{i=1}^{m} u_{ik}|i^A\rangle\right) \otimes \left(\sum_{j=1}^{n} (v^\dagger)_{kj}|j^B\rangle\right) = \sum_{k=1}^{m} s_k |\phi_k^A\rangle \otimes |\varphi_k^B\rangle$
$s_k \geq 0$
Schmidt decomposition
$A_{ij} = \sum_k u_{ik} s_k (v^\dagger)_{kj}$
orthonormal
$\langle \phi_k^a | \phi_{k'}^a \rangle = \delta_{kk'}$
$\rho^A = \mathrm{tr}_B |\psi\rangle\langle\psi| = \sum_k \langle \varphi_k^B | \psi\rangle\langle\psi | \varphi_k^B \rangle = \sum s_k^2 |\phi_k^A\rangle\langle\phi_k^A|$
$u\, s^2 u^\dagger$
$AA^\dagger$
$\rho^B = \mathrm{tr}_A |\psi\rangle\langle\psi| = \sum_k \langle \phi_k^A | \quad = \sum s_k^2 |\varphi_k^B\rangle\langle\varphi_k^B|$
$A^\dagger A$


$A_{ij} = \sum_k U_{ik} S_k (V^\dagger)_{kj}$
$\rho^A = tr_B |\psi\rangle\langle\psi| = \sum_k \langle \varphi_k^B | \psi\rangle\langle\psi | \varphi_k^B \rangle = \sum S_k^2 |\phi_k^A\rangle\langle\phi_k^A|$
$AA^\dagger$
$U S S U^\dagger$
$\rho^B = tr_A |\psi\rangle\langle\psi| = \sum_k \langle \phi_k^A | \ldots = \sum S_k^2 |\varphi_k^B\rangle\langle\varphi_k^B|$
$A^\dagger A$
orthonormal
$\langle \phi_k^A | \phi_{k'}^A \rangle = \delta_{kk'}$
non-zero eigenvalues of ρ^A = non-zero eigenvalues of ρ^B
$p_k \equiv (S_k)^2$
examples
$|\psi\rangle = |\phi^A\rangle \otimes |\varphi^B\rangle$
$\chi = 1$
Schmidt rank
$\vec{p} = (1, 0, 0, \ldots 0)$
$|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$
$\chi = 2$
$\vec{p} = (\frac{1}{2}, \frac{1}{2}, 0, 0 \ldots 0)$
$\frac{1}{\sqrt{n}}(|11\rangle + |22\rangle + \ldots + |nn\rangle)$
$\chi = n$
$\vec{p} = (\frac{1}{n}, \frac{1}{n}, \ldots \frac{1}{n})$
n

5.3 Measures of entanglement

$|\psi\rangle \longleftrightarrow \{\sqrt{p_k}\}$ Schmidt coefficients

$\mathbb{C}^m \otimes \mathbb{C}^n / U(m) \times U(n)$

$|\psi\rangle \sim |\psi'\rangle = U^A \otimes V^B |\psi\rangle$

$A \quad A'$

$E($

$\mathcal{E}(\psi)$ single number

- $\mathcal{E}(\psi) = f(\{\sqrt{p_k}\})$
- $\mathcal{E}(\psi) = 0 \iff \psi$ is a product state
- $\psi \xrightarrow{\text{LOCC}} \psi'$ $\quad \mathcal{E}(\psi) \geq \mathcal{E}(\psi')$

monotonicity under LOCC

Example 2 Entanglement entropy

$|\psi\rangle \to \rho^A \quad S(\rho^A)$

Von Neumann entropy of ρ

$$S(\rho) \equiv -\operatorname{tr} \rho \log_2 \rho = -\sum_x p_x \log_2 p_x$$

p_x: eigenvalues of ρ

examples

$|\psi\rangle = |\phi^A\rangle \otimes |\varphi^B\rangle \to \vec{p} = (1,0,0\ldots0) \quad S(\rho^A) = -1\log_2 1 - 0\log_2 0 \ldots = 0$

$\frac{|01\rangle - |10\rangle}{\sqrt{2}} \to \vec{p} = (\frac{1}{2}, \frac{1}{2}, 0 \ldots 0) \quad S(\rho^A) = \left(-\frac{1}{2}\log\frac{1}{2}\right)\times 2 = -\log_2\frac{1}{2} = \log_2 2 = 1$ ebits

$\vec{p} = (\frac{1}{n}, \frac{1}{n}, \ldots \frac{1}{n})$

$\frac{1}{\sqrt{n}}\left(|11\rangle + |22\rangle + \ldots + |nn\rangle\right) \quad S(\rho^A) = \left(-\frac{1}{n}\log_2\frac{1}{n}\right) \times n = \log_2 n$

ebit
singlet

(c) Reduced density matrices $\rho^A = \operatorname{tr}_B |\psi\rangle\langle\psi| = \sum_m \langle m^B|\psi\rangle\langle\psi|m^B\rangle = \sum_{i,k} (\rho^A)_{ik} |i^A\rangle\langle k^A|$

$|\psi\rangle\langle\psi|$

$\operatorname{tr}_B |\psi\rangle\langle\psi| =$

$|\psi\rangle \to \rho^A \quad S(\rho^A)$

$S(\rho) = -\operatorname{tr} \rho \log_2 \rho = -\sum_n p_n \log_2 p_n$

p_n: eigenvalues of ρ

examples

$|\psi\rangle = |\phi^A\rangle \otimes |\varphi^B\rangle \to \vec{p} = (1,0,0\ldots 0) \quad S(\rho^A) = -1\log_2 1 - 0\log_2 0 \ldots = 0$

$\frac{|01\rangle - |10\rangle}{\sqrt{2}} \to \vec{p} = (\tfrac{1}{2}, \tfrac{1}{2}, 0 \ldots) \quad \vec{p} = (\tfrac{1}{n}, \tfrac{1}{n}, \ldots \tfrac{1}{n}) \quad S(\rho^A) = \left(-\tfrac{1}{2}\log_2 \tfrac{1}{2}\right) \times 2 = -\log_2 \tfrac{1}{2} = \log_2 2 = 1$ bits

$\frac{1}{\sqrt{n}}(|11\rangle + |22\rangle + \ldots + |nn\rangle) \quad S(\rho^A) = \left(-\tfrac{1}{n}\log_2 \tfrac{1}{n}\right) \times n = \log_2 n$

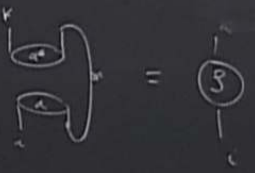
ebit

singlet

(C) Reduced density matrices $\rho^A = \operatorname{tr}_B |\psi\rangle\langle\psi| = \sum_m \langle m^B|\psi\rangle\langle\psi|m^B\rangle = \sum_{c,c'} (\rho^A)_{cc'} |c^A\rangle\langle c'^A|$

$|\psi\rangle\langle\psi|$

$\operatorname{tr}_B |\psi\rangle\langle\psi| =$



Example 3 Renyi entropies

$S_\alpha(\rho^A) = \frac{1}{1-\alpha}\log_2 (\operatorname{tr} \rho^\alpha) = \frac{1}{1-\alpha}\log_2 \sum_n (p_n)^\alpha$

$\alpha \in [0, \infty]$

$\lim_{\alpha \to 1} S_\alpha = S$

$\lim_{\alpha \to 0} S_\alpha = \log \chi$